SG Fourier Integral Operators with Closure under Composition

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Abstract

This thesis is concerned with SG Fourier Integral Operators (FIOs). In particular we define two principal classes of Fourier Integral Operator which we call Type $\mathcal{P}$ and Type $\mathcal{Q}$ operators. Our main results are that generalised versions of these operator classes are closed under composition.

Contents

Notation

Spaces of differentiable functions:

Let $U \subset \mathbb{R}^n$. $C^r(U, \mathbb{C})$ denotes the space of functions $f : U \to \mathbb{C}$ with the property that all partial derivatives of order $\leq r$ exist and are continuous. We will often abbreviate $C^r(U, \mathbb{C})$ to $C^r(U)$. We define $C^r(U, \mathbb{R})$ in the obvious way. $C^r(U, \mathbb{C}^n)$ is the space of functions $f = (f_1, \ldots, f_n)$ with $f_i \in C^r(U, \mathbb{C})$ for $i = 1, \ldots, n$.

$C^\infty(U) := \bigcap_{r=1}^\infty C^r(U)$. If $f \in C^\infty(U)$ we will say that f is "smooth." Define $C_0^\infty(U)$ as the space of smooth functions $f : U \to \mathbb{C}$ such that the support of f is compact.

Derivative notation: A multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n tuple of non-negative integers. For a multi-index α and $x \in \mathbb{R}^n$, we define

$|\alpha| := \sum_{i=1}^n \alpha_i$,

$\partial_x^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$,

$\partial_{x_i} := \frac{\partial}{(}\partial x_i)$,

$x^\alpha := \prod_{i=1}^n x_i^{\alpha_j}$.

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be such that $X \times Y$ is open. Suppose that $f(x, y) \in C^r(X \times Y)$. Let α and β be n and m dimensional multi-indices respectively. Let ω be the $n + m$ dimensional multi-index obtain by taking the cartesian product of α and β. That is, $\omega = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ For $(x, y) \in X \times Y$ we will write $\partial_x^\alpha \partial_y^\beta f(x, y)$ to mean $\partial^\omega f$. For such functions f, x will be called the "first variable" and y the "second variable." We will also write $\partial_1^\alpha \partial_2^\beta f$ to mean $\partial_x^\alpha \partial_y^\beta f(x, y)$. ∇_x and $\triangle_x$ denote the gradient and Laplacian operators respectively. (in the x variable.) For example, given a function $f(x, y) \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, we have $\nabla_x f := (\partial_{x_1} f, \ldots, \partial_{x_n} f)$.

Miscellaneous: For $x \in \mathbb{R}^n$, define $\langle x \rangle := \sqrt{1 + |x|^2}$. $\mathcal{S}(\mathbb{R}^n)$ denotes the space of smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ such that for any non-negative integer N and multi-index α we have $\sup_{x \in \mathbb{R}^n} \langle x \rangle^N \partial^\alpha f < \infty$. $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$. For $s = (s_1, s_2)$ with $s_1, s_2 \in \mathbb{R}$ define Π_s as the pseudo with symbol $\langle x \rangle^{s_2} \langle \xi \rangle^{s_1}$. The weighted Sobolev space H^s is the following set:

$\{u \in \mathcal{S}' : \Pi_s u \in \mathbb{L}^2(\mathbb{R}^n)\}$

Let $f, g : \mathbb{R}^n \to \mathbb{C}$. We write $f \prec g$ if there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all $x \in \mathbb{R}^n$. We write $f \sim g$ if $f \prec g$ and $g \prec f$.

Chapter 1

Introduction

This thesis is devoted to the study of Fourier Integral Operators with amplitudes in the SG symbol class. In particular, our goal was to define a class of SG Fourier Integral Operators which is closed under composition.

We begin this introductory chapter with an informal introduction in which we present some basic notation and terminology. We'll then discuss some existing work in the field of SG Fourier Integral Operators as well as the possible applications of a class of SG Fourier Integral Operators which is closed under composition. The introduction finishes with an outline of the content of the later chapters.

Given $u \in \mathcal{S}(\mathbb{R}^n)$ define the Fourier Transform $\hat{u}$ of u by the following integral:

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \mathrm{d}x.$$

The Fourier transformation $\mathcal{F} : u \mapsto \hat{u}$ sends $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and it's invertible with

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi) \mathrm{d}\xi.$$

Now, by integration by parts, for $u \in \mathcal{S}(\mathbb{R}^n)$ we have $\widehat{\partial_x^\alpha u} = (i\xi)^\alpha \hat{u}$. This property is useful when trying to solve certain partial differential equations with constant coefficients; loosely speaking we apply the Fourier Transform, "convert" derivatives into multiplications by $i\xi$, "divide" and invert. As Shubin and Egorov remark in [9], Pseudodifferential Operator Theory developed when people tried to apply Fourier Transform methods to non-constant coefficient PDEs.

We now show how to represent a differential operator in integral form using the Fourier Transform. This is the standard way to introduce Pseudodifferential Operators (ψdos), see [25]. Define $D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha$ and consider the operator

$$A := \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

where $a_\alpha(x) \in C^\infty(\mathbb{R}^n, \mathbb{C})$. For $u \in \mathcal{S}(\mathbb{R}^n)$ we have $u = \mathcal{F}^{-1} \circ \mathcal{F} u$. Writing the inverse Fourier transform as an integral we have

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi) \mathrm{d}\xi. \tag{1.1}$$

Applying the differential operator A to both sides of (1.1) we have

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi) \hat{u}(\xi) \mathrm{d}\xi. \tag{1.2}$$

where $a(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$. So, when we restrict the general differential operator A to $\mathcal{S}(\mathbb{R}^n)$, we can represent A as an integral. The function $a(x,\xi)$ is called the "symbol" of the operator A. Writing the Fourier transfrom in (1.2) as an integral gives the basic form of a ψdo:

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} a(x,\xi) u(y) \mathrm{d}y \mathrm{d}\xi, \qquad (1.3)$$

with the usual distributional interpretation. By allowing the symbol to depend on y, we obtain the general form of a Pseudodifferential Operator. These operators have been defined and studied for symbols chosen from various spaces, perhaps the most well known of which is the class $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, introduced by Hörmander [18]. The space $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, defined for $m \in \mathbb{R}$, is the class of functions $a(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ such that for all multi-indices α, β, there exists $c_{\alpha,\beta} > 0$ with $|\partial_x^\alpha \partial_y^\gamma \partial_\xi^\beta a(x, y, \xi)| \leq c_{\alpha,\beta,\gamma} \langle \xi \rangle^{m-|\beta|}$ for all $x, y, \xi \in \mathbb{R}^n$.

Pseudodifferential Operators are useful in the study of elliptic equations, see [2], [22], [26]. For hyperbolic problems we use a generalisation of (1.3); loosely speaking, we replace $(x - y) \cdot \xi$ with a function $\phi(x, y, \xi)$ (with carefully chosen properties) to obtain a "Fourier Integral Operator" (FIO). Informally, a general Fourier Integral Operator A acting on $u \in \mathcal{S}$ is an integral operator of the form

$$Au(x) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\{i\phi(x, y, \xi)\} a(x, y, \xi) u(y) \mathrm{d}y \mathrm{d}\xi,$$

where $\phi(x, y, \xi)$ is called the "phase" and $a(x, y, \xi)$ is called the "amplitude" of the operator, again with the usual distributional interpretation of the integral.

The symbol space S^m and its variants are most useful in the analysis of differential operators where the spatial variable x is restricted to some compact subset of $\mathbb{R}^n$. In order to study problems which are global in the spatial variable the "SG " symbol space was introduced, which we now define. For real numbers m_1, m_2 and m_3, we say that $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ if $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$ and for all multi-indices α, β, γ there exists some constant $c_{\alpha,\beta,\gamma}$ such that $|\partial_x^\alpha \partial_y^\gamma \partial_\xi^\beta a| \leq c_{\alpha,\beta,\gamma} \langle x \rangle^{m_1-|\alpha|} \langle y \rangle^{m_2-|\gamma|} \langle \xi \rangle^{m_3-|\beta|}$, for all $x, y, \xi \in \mathbb{R}^n$.

We will now introduce the SG Fourier Integral Operators defined by Coriasco in [3]. A real valued function $\phi \in SG^{1,1}_{x,\xi}$, belongs to the class $\mathcal{P}_\epsilon$ if there exist constants $C, c, \epsilon > 0$ such that $\forall x, \xi \in \mathbb{R}^n$ we have

$$c\langle x \rangle \leq \langle \nabla_\xi \phi(x,\xi) \rangle \leq C \langle x \rangle,$$

$$c\langle \xi \rangle \leq \langle \nabla_x \phi(x,\xi) \rangle \leq C \langle \xi \rangle,$$

$$|\det \left(\partial_{x_i} \partial_{\xi_j} \phi(x,\xi)\right)_{i,j=1}^n | \geq \epsilon, \tag{1.4}$$

where $\langle x \rangle := \sqrt{1+|x|^2}$ for $x \in \mathbb{R}^n$. For $a(x,\xi) \in SG^{m_1,m_2}_{x,\xi}$ with m_1, m_2 arbitrary and $\phi \in \mathcal{P}_\epsilon$ Coriasco defines his Type 1 Fourier Integral Operator $A_{\phi,a}$ acting on $u \in \mathcal{S}$ as follows:

$$A_{\phi,a} u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp\{i\phi(x,\xi)\} a(x,\xi) \hat{u}(\xi) \mathrm{d}\xi.$$

For $b(x,\xi) \in SG^{m_1,m_2}_{x,\xi}$ and $\phi \in \mathcal{P}_\epsilon$ he also defines a Type 2 operator $B_{\phi,b}$ as

$$\widehat{B_{\phi,b} u}(\xi) = \int_{\mathbb{R}^n} \exp\{-i\phi(x,\xi)\} \overline{b(x,\xi)} u(x) \mathrm{d}x.$$

In [3] Coriasco establishes the following composition structure for Type 1 operators. If $A_{\phi,a}$ is any Type 1 operator and P is an arbitrary SG ψdo, then (modulo operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ - defined in Chapter 2) we have:

- $A_{\phi,a} \circ P$ and $P \circ A_{\phi,a}$ are Type 1 operators with the same phase ϕ and some amplitudes,
- $A^\star_{\phi,a} \circ A_{\phi,a}$ and $A_{\phi,a} \circ A^\star_{\phi,a}$ are ψdos,

where $A^\star_{\phi,a}$ is the adjoint operator of $A_{\phi,a}$. (see Chapter 2.)A similar composition structure is also established for Type 2 operators.

In [4], Coriasco applies his calculus to the study of Hyperbolic PDEs. See also [7]. He examines systems of the form

$$\partial_t u(t,x) - iK(t)u(t,x) = f(t,x), \qquad \text{for } t \in J = (T_0, T_1), \text{ with } T_0 < 0 < T_1$$
$$u(0,x) = u_0(x) \tag{1.5}$$

- K is a $\nu \times \nu$ matrix of ψdos with symbols $k_{i,j} = k_{i,j}(t; x, \xi) \in C^\infty(J, SG^{1,1}_{x,\xi})$
- $u_0(x)$ is a ν dimensional vector valued function in the weighted Sobolev space H^s (see Notation section for the definition),
- $f = f(t,x) \in C^\infty(J, H^s)$
- The matrix of symbols of k has the form $k = k_1 + k_0$ where
  - $k_1 = diag(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\mu)$ where $\tilde{\lambda}_j = diag(\lambda_j, \ldots, \lambda_j)$ is a $l_j \times l_j$ diagonal matrix with $\nu \geq l_j \geq 1$. The number l_j is the multiplicity of λ_j and $\sum_{j=1}^{\mu} l_j = \nu$,

- $\lambda_j \in C^\infty(J, SG^{1,1}_{x,\xi})$,
- $k_{0i,j} \in C^\infty(J, SG^{0,0}_{x,\xi})$,

Coriasco also assumes that the λ_j are real valued and satisfy the following separation condition: For $j = 1, \dots, \mu - 1$ there exists $c_{j+1,j} > 0$ such that

$$|\lambda_{j+1}(t; x, \xi) - \lambda_j(t; x, \xi)| \geq c_{j+1,j}\langle x\rangle\langle\xi\rangle, \tag{1.6}$$

for $t \in J$ and $x, \xi \in \mathbb{R}^n$. In [4] it is shown that systems of the form (1.5), have a unique solution $u \in C(J', H^s)$ on a subinterval $J' \subset J$ and that for homogeneous systems, the solution operator is a matrix of Type 1 Fourier Integral Operators. (modulo operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.) Coriasco also studies scalar Cauchy problems

$$\begin{aligned} Lu(t, x) &= f(t, x), \quad t \in J, \\ D_t^k u(0, x) &= u_x^k, \quad k = 1, \dots, \nu - 1 \end{aligned} \tag{1.7}$$

where $L = D_t^\nu + P_1(t)D_t^{\nu-1} + \ldots + P_\nu(t)$ and the symbols p_j of the ψdos P_j are such that $p_j = p_j(t; x, \xi) \in C^\infty(J, SG^{j,j}_{x,\xi})$. The operator L is assumed to be "hyperbolic with constant multiplicities," meaning that the roots of the characteristic equation are real and satisfy the condition (1.6). For homogeneous problems of the form (1.7) which satisfy a certain factorisation condition (called the Levi condition), Coriasco shows that solution operator is a sum of Type 1 FIOs. (modulo operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.) These higher order scalar equations satisfying the Levi condition written as can be reduced to equivalent systems of the form (1.5), see [4].

The main aim of this thesis is to define an SG FIO class which is closed under composition and which generalises the Coriasco class. This is interesting for its own sake, but there are also possible applications of such a closed SG FIO class to hyperbolic PDEs of the form (1.7). If we follow methods of Treves [26], it should be possible to obtain the solution operator to the homogeneous problem (1.7) as a sum of compositions of Coriasco Type 1 FIOs, without the need for the Levi condition. By our results, the solution operator would then be a sum of generalised FIOs. This is a desirable situation as the following example shows.

Define $I^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ as the class of Fourier Integral Operators with amplitude $a(x, y, \xi)$ in $S^m(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, with compact support in x and y and phase $x \cdot \xi - \phi(y, \xi)$ where $\phi(y, \xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \backslash 0))$ with $\phi(y, \lambda\xi) = \lambda\phi(x, \xi)$ (for $\lambda > 0$ and $\xi \neq 0$) and $\det\left(\partial_{y_i}\partial_{\xi_j}\phi\right)_{i,j=1}^n \neq 0$. Given an operator $T_a \in I^m$, we can extend its definition to $\mathcal{S}'$ in the standard way, as discussed in chapter 3. The operator class I^m is well understood and there exist results about the action of operators $T_a \in I^m$ on singularities of u. In particular, in [24], in is proved that if $T_a \in I^0$ then

$$T_a : L^p_{\alpha+\gamma(p)} \to L^p_\alpha, \tag{1.8}$$

continuously, where $\gamma(p) = (n-1)|\frac{1}{p} - \frac{1}{2}|$ and $1 < p < \infty$. Suppose that $T_a, T_b \in I^0$, then by (1.8)

$$T_a \circ T_b : L^p_{\alpha+2\gamma(p)} \to L^p_\alpha.$$

Each application of an operator in I^0 leads to a loss of $\gamma(p)$ derivatives. However, by results in [8] the composition $T_a \circ T_b$ is a generalised FIO which can be reduced to a FIO in I^0 by the Hörmander equivalence of phase function theorem. [8] So, we only lose derivatives $\gamma(p)$ once when we apply operators in I^0.

If similar results to (1.8) about losses of derivatives are proved my generalised FIOs are applied are proved, then there would be applications of my work to the study of regularity of solutions to equations of the form (1.7) which do not satisfy the Levi factorisation condition. I had hoped to investigate these possible applications but due to time restrictions I was not able to do so.

We also seek to define FIO classes with the properties that

- the composition of a FIO with a ψdo gives a FIO with the same phase
- the composition of a FIO with its adjoint is a ψdo,

by placing as few restrictions on the phase as possible. In this thesis, I say an operator class has a "calculus " structure if it has the two properties above.

Now we outline what is to come in the subsequent chapters. In the next chapter we give the basic definitions and set up the machinery which we will use in later chapters.

In chapter 3, the Type $\mathcal{P}$ Fourier Integral Operator is defined. This is the most general operator class I could define with the property that composition of a FIO with a ψdo gives a FIO with the same phase. For the Type $\mathcal{P}$ operator we choose phases from the class $\mathcal{P}$ defined as the collection of functions $\Phi(x, y, \xi) \in C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ satisfying the following criteria:

$$\forall j, \quad \partial_{x_j}\Phi(x, y, \xi) \in \mathrm{SG}^{0,0,1}_{x,y,\xi}, \tag{1.9}$$

$$\forall j, \quad \partial_{y_j}\Phi(x, y, \xi) \in \mathrm{SG}^{0,0,1}_{x,y,\xi}, \tag{1.10}$$

$$\langle \nabla_x \Phi(x, y, \xi)\rangle \succ \langle \xi \rangle, \tag{1.11}$$

$$\langle \nabla_y \Phi(x, y, \xi)\rangle \succ \langle \xi \rangle, \tag{1.12}$$

$$\exists c_\Phi > 0 : |x - y| \geq c_\Phi \langle y \rangle \Rightarrow |\nabla_\xi \Phi(x, y, \xi)| \succ \langle x \rangle + \langle y \rangle, \tag{1.13}$$

$$\exists c_\Phi > 0 : |x - y| \geq c_\Phi \langle x \rangle \Rightarrow |\nabla_\xi \Phi(x, y, \xi)| \succ \langle x \rangle + \langle y \rangle, \tag{1.14}$$

$$\forall \gamma, \quad \partial_\xi^\gamma \Phi(x, y, \xi) \prec (\langle x \rangle + \langle y \rangle)\langle \xi \rangle^{1-|\gamma|}, \tag{1.15}$$

where for $f, g : \mathbb{R}^{3n} \to \mathbb{C}$, the notation $f(x, y, \xi) \prec g(x, y, \xi)$ means that there exists $c > 0$ such that $|f(x, y, \xi)| \leq c|g(x, y, \xi)|$ for all $x, y, \xi \in \mathbb{R}^n$. For $a(x, y, \xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with m_1, m_2 and m_3 arbitrary real numbers we define the Type $\mathcal{P}$ FIO $A_{\Phi,a}$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$ as follows:

$$A_{\Phi,a}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\{i\Phi(x, y, \xi)\} a(x, y, \xi)\gamma(\epsilon\xi)u(y) \mathrm{d}y \mathrm{d}\xi,$$

where $\gamma(\epsilon\xi)$ is a mollifier (as defined in Chapter 2). The basic properties of these operators are presented in Chapter 3.

We present several technical results in Chapter 4. These are mostly very small generalisations of proofs of corresponding results in Coriasco. [3]. We follow the arguments there exactly. I have tried to avoid repetition where possible and results in Chapter 4 are often used in later chapters.

We show in Chapter 5 that composition of Type $\mathcal{P}$ FIOs with pseudodifferential operators gives a Type $\mathcal{P}$ operator with the same phase and modified amplitude, modulo operators with kernel in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

In Chapter 6 we present some results about SG structure preserving changes of variables. These are used in Chapters 7, 8 and 9.

Chapter 7 sees us define the Type $\mathcal{Q}$ FIO, which for a long time was the most general operator I could define with a calculus structure. This operator is defined as for the Type $\mathcal{P}$ FIO but with phases $\Phi(x, y, \xi) = f(x, \xi) + g(y, \xi)$ where f and

g have the following properties:

$$f(x,\xi) \in SG^{1,1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), \tag{1.16}$$

$$g(y,\xi) \in SG^{1,1}_{y,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), \tag{1.17}$$

$$\langle \nabla_x f(x,\xi) \rangle \sim \langle \xi \rangle, \tag{1.18}$$

$$\langle \nabla_x g(x,\xi) \rangle \sim \langle \xi \rangle, \tag{1.19}$$

$$\langle \nabla_\xi f(x,\xi) \rangle \sim \langle x \rangle, \tag{1.20}$$

$$\langle \nabla_\xi g(x,\xi) \rangle \sim \langle x \rangle, \tag{1.21}$$

$$\left| \det \partial_{x_i} \partial_{\xi_j} f(x,\xi) \right| \succ 1, \tag{1.22}$$

$$\left| \det \partial_{y_i} \partial_{\xi_j} g(y,\xi) \right| \succ 1, \tag{1.23}$$

where for $f, g : \mathbb{R}^{3n} \to \mathbb{C}$ the notation $f(x,y,\xi) \sim g(x,y,\xi)$ means $f(x,y,\xi) \prec g(x,y,\xi)$ and $g(x,y,\xi) \prec f(x,y,\xi)$. The Type $\mathcal{Q}$ operator class is a sub-class of the Type $\mathcal{P}$ FIO class. We shall sometimes call the function f the first component of the phase and call g the second component. Note that a Coriasco Type 1 operator is the sub-class of Type $\mathcal{Q}$ FIOs where the second component is $-y \cdot \xi$ and the amplitude is independent of y. In addition to establishing the calculus structure, we show that if we compose two Type $\mathcal{Q}$ FIOs A and B with the second component of the phase of A equal to minus one times the first component of B's phase, then the composition is a Type $\mathcal{Q}$ FIO.

Chapter 8 is the main part of the thesis in which we generalise the definitions of the Type $\mathcal{P}$ and Type $\mathcal{Q}$ operators and show that each of these generalised operator classes are closed under composition. For both classes, we modify the conditions on the phase to allow the frequency variable ξ to have dimension greater that of the spatial variables x and y. As shown in [8], when we compose two operators in I^m we obtain a generalised operator with dimension of the frequency variable equal to $3n$. In this case, the dimension of the frequency variable can be reduced back to n by the Hörmander equivalence of phase function theorem. There is no version of this in the global setting, so we developed our closed composition structure for operators with different frequency variable dimensions. For the Type $\mathcal{P}$ operator, the generalisation was obvious. The situation for the Type $\mathcal{Q}$ operator was more delicate as the Type $\mathcal{Q}$ phase has additional structure; we can make certain changes of variables involving the phase which preserve the SG structure of the amplitude. The main difficulty was generalising the conditions on the phase whilst retaining the capacity to make the natural SG structure preserving changes of variables. This is discussed in more detail in Chapter 8.

In Chapter 9, we define the Type $\mathcal{R}$ FIO. This is an operator class with a calculus structure without the somewhat unnatural assumption that the mixed spatial derivatives of the phase are identically zero, i.e. we do not assume that $\partial_{x_i}\partial_{y_j}\Phi(x,y,\xi) \equiv 0$ for all i,j as we did for the Type $\mathcal{Q}$ operator. It should be possible to generalise the Type $\mathcal{R}$ class and obtain closedness under composition.

Due to time restrictions, I was not able to do this.

Chapter 2

SG Function Space

In this chapter we give a brief introduction to the SG function space and present some definitions and notation which we will use later.

2.1 SG Function Space Definition

For $x \in \mathbb{R}^n$, define $\langle x \rangle := \sqrt{1 + |x|^2}$. The function $\langle \cdot \rangle$ behaves like $| \cdot |$ for large arguments but it's smooth everywhere. We collect a few simple properties of $\langle \cdot \rangle$ in a Proposition.

Proposition 2.1.1. *For all $x, y \in \mathbb{R}^n$, the following inequalities hold:*

1. $\langle x+y\rangle \leq \langle x\rangle + \langle y\rangle$,

2. $\langle x+y\rangle \geq \langle x\rangle - \langle y\rangle$,

3. $\langle x+y\rangle \geq |x| - \langle y\rangle$,

4. $\langle x+y\rangle \geq |x| - |y|$.

Proof. Statement 1 follows from the triangle inequality. The first statement implies the second and the second implies the third since $\langle x\rangle \geq |x|$ for all $x \in \mathbb{R}^n$. For statement 4, we use the fact that $\langle x+y\rangle \geq |x+y|$ and $|x+y| \geq |x| - |y|$ for all $x, y \in \mathbb{R}^n$. □

Notation. Let $f, g : \mathbb{R}^n \to \mathbb{C}$. We write $f \prec g$ if there exists a constant C such that $|f(x)| \leq C|g(x)|$ for all $x \in \mathbb{R}^n$. We write $f \sim g$ if $f \prec g$ and $g \prec f$.

Definition 2.1.2. *For $m_x, m_\xi \in \mathbb{R}$, let $SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ denote the space of all functions $f \in C^\infty(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C})$ satisfying the following estimates for all multi-indices α, β:*

$$\partial_x^\alpha \partial_\xi^\beta f(x,\xi) \prec \langle x\rangle^{m_x-|\alpha|}\langle \xi\rangle^{m_\xi-|\beta|}. \tag{2.1}$$

Remarks. Definition 2.1.2 just means that for every pair of multi-indices (α, β) there exists a constant $c_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta f(x,\xi)\right| \leq c_{\alpha,\beta} \langle x\rangle^{m_x - |\alpha|} \langle \xi \rangle^{m_\xi - |\beta|}, \quad \forall x \in \mathbb{R}^{n_x}, \xi \in \mathbb{R}^{n_\xi}.$$

It is clear that $C_0^\infty(\mathbb{R}^n) \subset SG^m(\mathbb{R}^n)$ for any $m \in \mathbb{R}$. So, all SG classes are non-empty and we observe that whether or not a C^∞ function is in an SG class depends on its behaviour at infinity.

$SG_{x,\xi}^{m_x,m_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ is called the SG class of order (m_x, m_ξ).

$SG_{x,\xi}^{m_x,m_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C}^n)$ denotes the space of functions $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \longrightarrow \mathbb{C}^n$ with each component $f_i(x,\xi) \in SG_{x,\xi}^{m_x,m_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$, $1 \leq i \leq n$.

Let $f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \longrightarrow \mathbb{C}$. Suppose that for some $m_x, m_\xi \in \mathbb{R}$ the derivatives $\partial_x^\alpha \partial_\xi^\beta f(x,\xi)$ exist for all α, β and satisfy (2.1) for all (x, y) in some set $W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$. Then we say " f satisfies $SG_{x,\xi}^{m_x,m_\xi}$ estimates on W."

Examples of SG functions supported on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, define $f(x) := x$ and $g(x) := \langle x \rangle$. Then $f \in SG_x^1(\mathbb{R}^n, \mathbb{R}^n)$ and $g(x) \in SG_x^1(\mathbb{R}^n, \mathbb{R})$.

We now collect some basic consequences of the SG class definition. These facts will be used frequently.

Proposition 2.1.3. *Suppose* $f \in SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ *and* $g \in SG^{s_x,s_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$. *Then:*

1. *For any* $\lambda \in \mathbb{C}$, *we have* $\lambda f \in SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$,

2. $fg \in SG^{m_x+s_x,m_\xi+s_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$,

3. $f + g \in SG^{\max(m_x,s_x),\max(m_\xi,s_\xi)}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$,

4. *If* $s_x \leq m_x$ *and* $s_\xi \leq m_\xi$ *then* $SG^{s_x,s_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}) \subset SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$.

Proof. These statements follow from the triangle inequality and the product rule.

□

Remark. Proposition 2.1.3 changes in the obvious way when the functions f and g depend on more or fewer variables. The results of Proposition 2.1.3 will often be referred to as " basic facts about SG functions."

By parts one and three of Proposition 2.1.3, SG classes are vector spaces over $\mathbb{C}$. For $f \in SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$, define

$$||f||_{\alpha,\beta} = \sup_{x\in\mathbb{R}^{n_x},\xi\in\mathbb{R}^{n_\xi}} \langle x\rangle^{|\alpha|-m_x}\langle\xi\rangle^{|\beta|-m_\xi}\left|\partial_x^\alpha\partial_\xi^\beta f(x,\xi)\right|.$$

The collection $\{||\cdot||_{\alpha\beta} : \alpha, \beta \text{ multi-indices}\}$ is a family of semi-norms on $SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$. The space $SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ is given the topology generated by these semi-norms. Indeed, $SG^{m_x,m_\xi}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ is a Frechet space. See [2].

2.2 Ellipticity

Let $f(x,\xi) \in SG_{x,\xi}^{r_x,r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C})$. We now present standard sufficient conditions under which $\frac{1}{f} \in SG_{x,\xi}^{-r_x,-r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C})$.

Proposition 2.2.1. *Let $f(x,\xi) \in SG_{x,\xi}^{r_x,r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C})$ and suppose that $f(x,\xi) \succ \langle x\rangle^{r_x}\langle\xi\rangle^{r_\xi}$. Then*

$$\frac{1}{f(x,\xi)} \in SG_{x,\xi}^{-r_x,-r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C}).$$

A function satisfying the conditions of Proposition (2.2.1) will be called "globally md-elliptic of order (r_x, r_ξ)" or simply "elliptic" and we'll write $f(x,\xi) \in ESG_{x,\xi}^{r_x,r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$.

Remark. If $f(x,\xi) \in SG_{x,\xi}^{r_x,r_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{C})$ and $f(x,\xi) \succ \langle x\rangle^{r_x}\langle\xi\rangle^{r_\xi}$ for $(x,\xi) \in W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$. Then $\frac{1}{f}$ satisfies $SG_{x,\xi}^{-r_x,-r_\xi}$ estimates for $(x,\xi) \in W \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ and we say " f is md-elliptic on W" or just "f is elliptic on W."

Example. The function $f(x) = \langle x\rangle$ belongs to $ESG_x^1(\mathbb{R}^n, \mathbb{R})$.

2.3 Integral Operators with Schwartz Kernel

Definition 2.3.1. *Let $\mathcal{K}$ denote the space of integral operators with kernel in $\mathcal{S}\left(\mathbb{R}^{2n}\right)$ i.e. the set of integral operators K acting on $\mathcal{S}\left(\mathbb{R}^{n}\right)$ such that*

$$Kf(x) = \int_{\mathbb{R}^n} k(x,y)f(y)\mathrm{d}y, \tag{2.2}$$

with $k(x,y) \in \mathcal{S}(\mathbb{R}^{2n})$.

Proposition 2.3.2. *Let $K \in \mathcal{K}$. Then,*

1. *$K : \mathcal{S}\left(\mathbb{R}^n\right) \to \mathcal{S}\left(\mathbb{R}^n\right)$ continuously.*

2. *$K : \mathcal{S}'\left(\mathbb{R}^n\right) \to \mathcal{S}\left(\mathbb{R}^n\right)$.*

For the proof of Proposition 2.3.2, see [9].

Remarks. Operators of the form (2.2) are called “globally smoothing” or simply “smoothing” because of Proposition 2.3.2 part 2.

2.4 Mollifiers.

Definition 2.4.1. *A mollifier is a real valued function $\gamma(\epsilon\xi)$ depending on a parameter $\epsilon > 0$ such that $\gamma \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with $\gamma = 1$ in some neighborhood of the origin. We will always restrict ϵ to the open interval $(0,1)$.*

Proposition 2.4.2. *Let $\gamma(\epsilon\xi)$ be a mollifier. Then:*

1. *For all $\xi \in \mathbb{R}^n$, $\quad \gamma(\epsilon\xi) \to 1$ as $\epsilon \to 0$,*
2. *For $|\alpha| \geq 1$, as $\epsilon \to 0$ we have $\partial_\xi^\alpha [\gamma(\epsilon\xi)] \to 0$ for all $\xi \in \mathbb{R}^n$,*
3. *$\partial_\xi^\alpha [\gamma(\epsilon\xi)] \prec \langle\xi\rangle^{-|\alpha|}$ with the implicit constant independent of ϵ.*

Proof. The first two statements are obvious. For the third statement note that as γ is compactly supported, for any α there exists a constant c_α such that

$$\langle\epsilon\xi\rangle^{|\alpha|} (\partial_1^\alpha \gamma)(\epsilon\xi) \leq c_\alpha, \tag{2.3}$$

for all $\xi \in \mathbb{R}^n$. As $\epsilon \in (0,1)$, we have $\langle\xi\rangle^{|\alpha|}\epsilon^{|\alpha|} \leq \langle\epsilon\xi\rangle^{|\alpha|}$ for any non-negative integer $|\alpha|$. So, by (2.3), we have

$$\langle\xi\rangle^{|\alpha|}\epsilon^{|\alpha|} (\partial_1^\alpha \gamma)(\epsilon\xi) \leq c_\alpha, \tag{2.4}$$

for all $\epsilon \in (0,1)$ and $\xi \in \mathbb{R}^n$. Since $\epsilon^{|\alpha|} (\partial_1^\alpha \gamma)(\epsilon\xi) = \partial_\xi^\alpha [\gamma(\epsilon\xi)]$, we're done. □

2.5 Some Operators

In later arguments we will establish estimates of oscillatory integrals by applying differential operators which send the exponential term to itself and then integrating by parts. We introduce our two main differential operators in this section.

Definition 2.5.1. *For functions $u, v \in \mathcal{S}$, define $\langle u, v\rangle = \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x$. Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. The transpose A^T of A is the operator such that*

$$\langle Au, v\rangle = \langle u, A^T v\rangle \qquad \forall u, v \in \mathcal{S}(\mathbb{R}^n).$$

Definition 2.5.2. *For $w \in C^\infty(\mathbb{R}^n, \mathbb{R})$, define*

$$L_{y,w(y)} = \frac{1 - \triangle_y}{\langle \nabla_y w\rangle^2 - i\,\triangle_y\, w}.$$

By construction, $L_{y,w(y)}\left(e^{i\{w(y)\}}\right) = e^{i\{w(y)\}}$. We also note that

$$L^T_{y,w(y)} = (1 - \triangle_y)\frac{1}{\langle \nabla_y w\rangle^2 - i\,\triangle_y\, w}.$$

Definition 2.5.3. *Let $w \in C^\infty(\mathbb{R}^n, \mathbb{R})$ be such that $|\nabla_y w| \neq 0$ for all $y \in \mathbb{R}^n$. Define*

$$U_{y,w(y)} = \frac{-i}{|\nabla_y w|^2}\sum_{k=1}^{n}\left(\partial_{y_k} w\right)\partial_{y_k}.$$

By construction, $Ue^{iw} = e^{iw}$. Also, for any natural number r, we have

$$(U^T_{y,w(y)})^r = \frac{1}{|\nabla_y w|^{4r}}\sum_{|\alpha|\leq r} P_{\alpha,r}\partial_y^\alpha.$$

In the above, $P_{\alpha,r}$ is a linear combination terms of the form $(\nabla_y w)^\gamma \partial_y^{\delta_1} w \ldots \partial_y^{\delta_r} w$ with, $|\gamma| = 2r$, $|\delta_j| \geq 1$ and $|\alpha| + \sum_{j=1}^r |\delta_j| = 2r$.

The operators introduced in Definitions 2.5.2 and 2.5.3 are taken from Coriasco, [3].

2.6 SG Cut-Off Functions

In later proofs we'll establish estimates of integrals using different arguments in different parts of the domain of integration. We will use SG cut-off functions to divide up the region of integration. The class of cut-off functions defined below is taken from Coriasco [3].

Definition 2.6.1. *For $k > 0$, let $\Xi^{\Delta}(k)$ denote the set of all functions $\chi(x, y) \in SG^{0,0}_{x,y}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ such that:*

$$|x - y| \leq \frac{k}{2}\langle y \rangle \Rightarrow \chi(x, y) = 1,$$

$$|x - y| > k\langle y \rangle \Rightarrow \chi(x, y) = 0.$$

The notation $\Xi^{\Delta}(k)$ is as in Coriasco [3]. As an example, let $k > 0$ and $f \in C_0^{\infty}(\mathbb{R}, \mathbb{R})$ with

$$f(x) = \begin{cases} 1 & \text{if} \quad |x| \leq \frac{k}{2}, \\ 0 & \text{if} \quad |x| > k. \end{cases}$$

Then, $f\left(\frac{|x-y|^2}{\langle y \rangle^2}\right) \in \Xi^{\Delta}(k)$.

We also use the following facts, stated in [3].

Proposition 2.6.2. *Let* $\chi \in \Xi^{\Delta}(k)$.

1. *If* $k < 1$, *then* $\langle x \rangle \sim \langle y \rangle$ *on* $Supp(\chi(x, y))$,

2. *For any* $k > 0$, *on* $Supp(1 - \chi(x, y))$ *we have* $|x - y| \succ \langle x \rangle + \langle y \rangle$.

2.7 Asymptotic Expansions

In proving composition results, we will need some results which tell us when a sum of SG functions plus a remainder is of SG type.

Definition 2.7.1. *An infinite sum* $\sum_{j=1}^{\infty} a_j(x, y, \xi)$ *is an asymptotic expansion if it satisfies the following three conditions.*

1. $\forall j \in \mathbb{N}, a_j \in SG_{x,y,\xi}^{m_{x,j}, m_{y,j}, m_{\xi,j}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_\xi})$.

2. $\forall j \in \mathbb{N}, m_{x,j+1} \leq m_{x,j}, \quad m_{y,j+1} \leq m_{y,j}$ *and* $m_{\xi,j+1} \leq m_{\xi,j}$.

3. $\lim_{j \to \infty}(m_{x,j}, m_{y,j}, m_{\xi,j}) = (-\infty, -\infty, -\infty)$.

Further, we write $a \sim \sum_{j=1}^{\infty} a_j$ *if* $\forall N \in \mathbb{N}$ *we have,*

$$a - \sum_{j=1}^{N} a_j \in SG_{x,y,\xi}^{m_{x,N+1}, m_{y,N+1}, m_{\xi,N+1}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_\xi}).$$

The definition changes in the obvious way when the a_j *depend on more or fewer variables.*

Remark. Asymptotic expansions do not necessarily converge for all points $x, \xi \in \mathbb{R}^n$. For example, consider $f_n(x,\xi) = n\langle x\rangle^{-n}\langle \xi\rangle^{-n}$. Clearly $\sum_{n=1}^{\infty} f_n$ is an asymptotic expansion but $\sum_{n=1}^{\infty} f_n(0,0)$ does not converge.

We shall need the following results about asymptotic expansions.

Proposition 2.7.2. *Let $\sum_{j=1}^{\infty} a_j(x,y,\xi)$ be any asymptotic expansion. Then:*

1. *There exists $a \in SG_{x,y,\xi}^{m_x,1,m_y,1,m_\xi,1}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_\xi}\right)$ such that $a \sim \sum_{j=1}^{\infty} a_j$.*

2. *If $a, a' \sim \sum_{j=1}^{\infty} a_j$, then $a - a' \in \mathcal{S}\left(\mathbb{R}^{n_x+n_y+n_\xi}\right)$.*

Proposition 2.7.2 above is proved with fewer variables in Cordes [2].

The following Proposition 2.7.3, stated in [3], will be useful.

Proposition 2.7.3. *Let $\sum_{j=1}^{\infty} p_j$ be an asymptotic expansion and suppose that $p \in C^\infty(\mathbb{R}^{3n})$ satisfies the following two conditions:*

1. *For all α, β, γ there exist $k_1(\alpha), k_2(\beta), k_3(\gamma) \in \mathbb{R}$ such that*

   $\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma p(x,y,\xi) \prec \langle x\rangle^{k_1(\alpha)}\langle y\rangle^{k_2(\beta)}\langle \xi\rangle^{k_3(\gamma)}$.

2. *There exists a sequence $\{l_r\}$ of real numbers with $l_r \to -\infty$ such that*

   $p(x,y,\xi) - \sum_{j=1}^{r} p_j \prec \langle x\rangle^{l_r}\langle y\rangle^{l_r}\langle \xi\rangle^{l_r}$.

Then, we have $p \sim \sum_{j=1}^{\infty} p_j$.

Chapter 3

Type $\mathcal{P}$ SG Fourier Integral Operator

3.1 Definition

Definition 3.1.1. ***(Phase Assumptions)*** *Denote by $\mathcal{P}$ the collection of functions $\Phi(x, y, \xi) \in C^\infty(\mathbb{R}^{3n}, \mathbb{R})$ satisfying the following criteria:*

$$\forall j = 1, \dots, n \quad \partial_{x_j}\Phi(x, y, \xi) \in SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n), \tag{3.1}$$

$$\forall j = 1, \dots, n, \quad \partial_{y_j}\Phi(x, y, \xi) \in SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n), \tag{3.2}$$

$$\langle \nabla_x \Phi(x, y, \xi) \rangle \succ \langle \xi \rangle, \tag{3.3}$$

$$\langle \nabla_y \Phi(x, y, \xi) \rangle \succ \langle \xi \rangle, \tag{3.4}$$

$$\exists c_\Phi > 0 : |x - y| \geq c_\Phi \langle y \rangle \Rightarrow |\nabla_\xi \Phi(x, y, \xi)| \succ \langle x \rangle + \langle y \rangle, \tag{3.5}$$

$$\exists c_\Phi > 0 : |x - y| \geq c_\Phi \langle x \rangle \Rightarrow |\nabla_\xi \Phi(x, y, \xi)| \succ \langle x \rangle + \langle y \rangle, \tag{3.6}$$

$$\forall \gamma, \quad \partial_\xi^\gamma \Phi(x, y, \xi) \prec (\langle x \rangle + \langle y \rangle)\langle \xi \rangle^{1-|\gamma|}. \tag{3.7}$$

Remark The variables x and y will sometimes be referred to as the "spatial variables" and ξ may be called the "frequency variable." Later we will allow the frequency variable to have dimension $n_\xi \geq n$.

Definition 3.1.2. *Let $\Phi(x,y,\xi) \in \mathcal{P}$, $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ with m_1, m_2, m_3 arbitrary and let $\gamma(\epsilon\xi)$ be a mollifier. Define the Type $\mathcal{P}$ operator $A_{\Phi,a}$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$ as follows:*

$$A_{\Phi,a}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi \tag{3.8}$$

Notation The operator $A_{\Phi,a}$ will also be denoted by $FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\right)$.

Remarks It will be shown later that the operator $A_{\Phi,a}$ is independent of the choice of mollifier $\gamma(\epsilon\xi)$. Also, the operators defined by Coriasco in [3] are of type $\mathcal{P}$.

3.2 Continuity on $\mathcal{S}(\mathbb{R}^n)$.

We start with a result about the form of the transpose of the operator $L_{y,\Phi}$ (see Definition 2.5.2) when $\Phi \in \mathcal{P}$. Again we employ the notation of Coriasco [3].

Lemma 3.2.1. *Let* $\Phi(x,y,\xi) \in \mathcal{P}$ *and let*

$$L_{y,\Phi(x,y,\xi)} := \frac{1 - \triangle_y}{\langle \nabla_y \Phi \rangle^2 - i \triangle_y \Phi}.$$

Then for $r \in \mathbb{N}$,

$$\left(L^T_{y,\Phi(x,y,\xi)}\right)^r = \sum_{|\delta| \le 2r} f_\delta(x,y,\xi) \partial_y^\delta,$$

where $f_\delta(x,y,\xi) \in SG^{0,0,-2r}_{x,y,\xi}$

Proof. Given that Φ is real valued, it follows from assumption (3.4) that $L_{y,\Phi(x,y,\xi)}$ is well-defined for all $x, y, \xi \in \mathbb{R}^n$. By integration by parts, we have

$$L^T_{y,\Phi(x,y,\xi)} = (1 - \triangle_y) \frac{1}{\langle \nabla_y \Phi \rangle^2 - i \triangle_y \Phi}.$$

Define $g(x,y,\xi) := \langle \nabla_y \Phi \rangle^2 - i \triangle_y \Phi$. As $\Phi \in \mathcal{P}$, it follows from the basic facts about SG functions that $g(x,y,\xi) \in SG^{0,0,2}_{x,y,\xi}$. As Φ is real valued and $\langle \nabla_y \Phi \rangle \succ \langle \xi \rangle$ we have $g \succ \langle \xi \rangle^2$. So, $g \in ESG^{0,0,2}_{x,y,\xi}$ and therefore

$$\frac{1}{g} \in SG^{0,0,-2}_{x,y,\xi}.$$

The result follows by induction. □

The following result will shorten many subsequent proofs.

Theorem 3.2.2. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi}\right)$ *for any* $m_x, m_y, m_\xi \in \mathbb{R}$ *and let* $\gamma(\epsilon\xi)$ *be a mollifier. Suppose also that*

$\Phi(x,y,\xi) \in C^\infty\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R}\right)$ *has the following properties on Supp(a):*

$$\nabla_\xi \Phi \succ \langle x\rangle + \langle y\rangle, \tag{3.9}$$

$$\forall \text{ multi-indices } \alpha,\beta,\gamma, \quad \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \Phi \prec \left(\langle x\rangle + \langle y\rangle\right)\langle \xi\rangle^{1-|\gamma|} \tag{3.10}$$

Then the integral operator

$$Bu(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^{n_\xi}}\int_{\mathbb{R}^n} \exp\{i\Phi(x,y,\xi)\}a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,$$

acting on functions $u \in \mathcal{S}\left(\mathbb{R}^n\right)$*, has kernel in* $\mathcal{S}\left(\mathbb{R}^{2n}\right)$.

Proof. By definition,

$$Bu(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^{n_\xi}}\int_{\mathbb{R}^n} \exp\{i\Phi(x,y,\xi)\}a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

For fixed ϵ, the integral $\int\int \exp\{i\Phi(x,y,\xi)\}a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$ is absolutely convergent for any $x \in \mathbb{R}^n$. Therefore the order of integration can be changed, so that

$$Bu(x) = \lim_{\epsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}^{n_\xi}} \exp\{i\Phi(x,y,\xi)\}a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}\xi\mathrm{d}y.$$

We assumed that $|\nabla_\xi\Phi(x,y,\xi)| \succ \langle x\rangle + \langle y\rangle$ on *Supp*(a). Therefore the operator $U_{\xi,\Phi(x,y,\xi)} = \frac{-i}{|\nabla_\xi\Phi(x,y,\xi)|^2}\sum_{k=1}^{n_\xi}\partial_{\xi_k}\Phi(x,y,\xi)\partial_{\xi_k}$ is well-defined on the support of the integrand.

By construction $U_{\xi,\Phi(x,y,\xi)}e^{i\Phi} = e^{i\Phi}$. Applying this operator s times to the exponential term and integrating by parts s times gives

$$Bu(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \xi} \exp\{i\Phi(x,y,\xi)\} \left(U^T_{\xi,\Phi}\right)^s [a(x,y,\xi)\gamma(\epsilon\xi)]\, u(y) \mathrm{d}\xi \mathrm{d}y.$$

Recall that

$$(U^T_{\xi,\Phi})^s = \frac{1}{|\nabla_\xi \Phi|^{4s}} \sum_{|\theta| \le s} P_{\theta,s}(x,y,\xi)\partial^\theta_\xi.$$

In the above, $P_{\theta,s}$ is a finite sum of terms of the form $(\nabla_\xi \Phi)^\gamma \partial_\xi^{\delta_1}\Phi \ldots \partial_\xi^{\delta_s}\Phi$, with, $|\gamma| = 2s$, $|\delta_j| \ge 1$ for all $j = 1, \ldots, s$ and $|\theta| + \sum_{j=1}^s |\delta_j| = 2s$ for each term in the sum.

We have assumed that, for all multi-indices α, β and θ we have $\partial^\alpha_x \partial^\beta_y \partial^\theta_\xi \Phi(x,y,\xi) \prec (\langle x \rangle + \langle y \rangle) \langle \xi \rangle^{1-|\theta|}$ on $Supp(a)$. Therefore,

$$\partial^\alpha_x \partial^\beta_y \left(\nabla_\xi \Phi\right)^\gamma \prec \left(\langle x \rangle + \langle y \rangle\right)^{|\gamma|}, \tag{3.11}$$

on $Supp(a)$.

Recalling the definition of $P_{\theta,s}(x,y,\xi)$ and using (3.10) and (3.11) we obtain,

$$\partial^\alpha_x \partial^\beta_y P_{\theta,s}(x,y,\xi) \prec \left(\langle x \rangle + \langle y \rangle\right)^{3s} \langle \xi \rangle^{|\theta|-s}. \tag{3.12}$$

Since by assumption $|\nabla_\xi \Phi(x,y,\xi)| \succ \langle x \rangle + \langle y \rangle$ on the support of a, it follows that

$$\frac{1}{|\nabla_\xi \Phi(x,y,\xi)|^{4s}} \prec \left(\langle x \rangle + \langle y \rangle\right)^{-4s},$$

on *Supp*(a). It follows from assumption (3.10) that for any multi-indices α, β we have

$$\partial_x^\alpha \partial_y^\beta \frac{1}{|\nabla_\xi \Phi(x,y,\xi)|^{4s}} \prec (\langle x \rangle + \langle y \rangle)^{-4s}, \tag{3.13}$$

on *Supp*(a). As $\gamma(\epsilon\xi)$ is a mollifier and $a \in SG_{x,y,\xi}^{m_x,m_y,m_\xi}$, we have,

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\theta \left[a(x,y,\xi)\gamma(\epsilon\xi)\right] \prec \langle x \rangle^{m_x} \langle y \rangle^{m_y} \langle \xi \rangle^{m_\xi - |\theta|}. \tag{3.14}$$

where the constant implicit in (3.14) is independent of ϵ. To obtain (3.14) we used part 3 of Proposition 2.4.2. Recalling the form $(U_{\xi,\Phi}^T)^s$ and using (3.12) , (3.13) and (3.14) we see that

$$\partial_x^\alpha \partial_y^\beta \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\gamma(\epsilon\xi)\right] \prec \langle x \rangle^{m_x - \frac{s}{2}} \langle y \rangle^{m_y - \frac{s}{2}} \langle \xi \rangle^{m_\xi - s} \tag{3.15}$$

with implicit constant independent of ϵ. For large enough s, we can apply the Lebesgue Dominated Convergence Theorem to see that

$$Bu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n_\xi}} \exp(i\Phi(x,y,\xi)) \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\right] u(y) \mathrm{d}\xi \mathrm{d}y.$$

To obtain the above we used the fact that $\lim_{\epsilon \to 0} \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\gamma_\epsilon(x,y,\xi)\right] = \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\right]$. So, the kernel $k(x,y)$ of B is

$$k(x,y) := \int_{\mathbb{R}^{n_\xi}} \exp(i\Phi(x,y,\xi)) \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\right] \mathrm{d}\xi.$$

By similar work to that which produced (3.15), we have

$$\partial_x^\alpha \partial_y^\beta \left(U_{\xi,\Phi}^T\right)^s \left[a(x,y,\xi)\right] \prec \langle x \rangle^{m_x - \frac{s}{2}} \langle y \rangle^{m_y - \frac{s}{2}} \langle \xi \rangle^{m_\xi - s}. \tag{3.16}$$

To show that $k(x,y) \in \mathcal{S}\left(\mathbb{R}^{2n}\right)$ it suffices to show that for any non-negative integer N and multi-indices α, β we have $\langle x\rangle^N \langle y\rangle^N \partial_x^\alpha \partial_y^\beta k(x,y) \prec 1$. This follows easily from the definition of $k(x,y)$, assumption (3.10) and the estimates (3.16), by choosing s to be sufficiently large. $\square$

Theorem 3.2.3. *Let $\Phi(x,y,\xi) \in \mathcal{P}$ and let $a \in SG^{m_1,m_2,m_3}_{x,y,\xi}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\right)$ with m_1, m_2, m_3 arbitrary real numbers. Then $FIO(\Phi(x,y,\xi), a(x,y,\xi))$ is a continuous mapping from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Consider

$$A_\epsilon u(x) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

By definition, $Au(x) = \lim_{\epsilon\to 0} A_\epsilon u(x)$. Let $\chi(x,y) \in \Xi^\Delta(k)$, where $k > 2c_\Phi$. This means that that on $Supp(1-\chi(x,y))$, we have $\nabla_\xi \Phi \succ \langle x\rangle + \langle y\rangle$. Define

$$\begin{aligned}
A_{\epsilon,1}u(x) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}\chi(x,y)\chi(y,x)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,\\
A_{\epsilon,2}u(x) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}\left(1-\chi(x,y)\right)\chi(y,x)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,\\
A_{\epsilon,3}u(x) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}\left(1-\chi(x,y)\right)\left(1-\chi(y,x)\right)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,\\
A_{\epsilon,4}u(x) &:= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}\left(1-\chi(y,x)\right)\chi(x,y)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi.
\end{aligned}$$

By construction, $A_\epsilon u(x) = A_{\epsilon,1}u(x) + A_{\epsilon,2}u(x) + A_{\epsilon,3}u(x) + A_{\epsilon,4}u(x)$. Define

$$A_1 u(x) := \lim_{\epsilon \to 0} A_{\epsilon,1}u(x),$$

$$A_2 u(x) := \lim_{\epsilon \to 0} A_{\epsilon,2}u(x),$$

$$A_3 u(x) := \lim_{\epsilon \to 0} A_{\epsilon,3}u(x),$$

$$A_4 u(x) := \lim_{\epsilon \to 0} A_{\epsilon,4}u(x). \tag{3.17}$$

By Theorem 3.2.2, the operators A_2, A_3 and A_4 belong to $\mathcal{K}$ and therefore they send $\mathcal{S}$ to $\mathcal{S}$ continuously. So if we can show that for any (N, α) the semi-norm $||A_1 u||_{N,\alpha}$ is bounded by a linear combination of semi-norms of u, the proof will be complete. To prove this we will use the operator $L_{y,\Phi(x,y,\xi)} := \frac{1-\triangle_y}{(\nabla_y \Phi)^2 - i\triangle_y \Phi}$. (We will abbreviate $L_{y,\Phi(x,y,\xi)}$ to L_y.) Consider $A_{\epsilon,1}$. By construction,

$$L_y : e^{i\Phi} \mapsto e^{i\Phi}$$

So, for $r \in \mathbb{N}$,

$$A_{\epsilon,1} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} L_y^r (e^{i\Phi(x,y,\xi)}) \chi(x,y)\chi(y,x)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$$

Integrating by parts in y r times gives

$$A_{\epsilon,1} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} (L_y^T)^r \left[\chi(x,y)\chi(y,x)a(x,y,\xi)u(y)\right] \gamma(\epsilon\xi)\mathrm{d}y\mathrm{d}\xi \tag{3.18}$$

By Lemma 3.2.1, we have,

$$(L_y^T)^r \left[a(x,y,\xi)\chi(x,y)\chi(y,x)u(y)\right] =$$

$$\sum_{|\delta| \le 2r} f_\delta(x,y,\xi)\partial_y^\delta \left[\chi(x,y)\chi(y,x)a(x,y,\xi)\gamma(\epsilon\xi)u(y)\right],$$

where $f_\delta(x,y,\xi) \in SG^{0,0,-2r}_{x,y,\xi}$. It follows from the product rule and the basic facts about SG functions, that

$$(L_y^T)^r(\chi(x,y)\chi(y,x)a(x,y,\xi)u(y)) = \sum_{|\delta|\le 2r} b_\delta(x,y,\xi)\partial_y^\delta u(y) \tag{3.19}$$

where $b_\delta(x,y,\xi) \in SG^{m_1,m_2,m_3-2r}_{x,y,\xi}$ and we have $\langle x\rangle \sim \langle y\rangle$ on $Supp(b_\delta)$ for all δ. Since $u \in \mathcal{S}(\mathbb{R}^n)$ and $\langle x\rangle \sim \langle y\rangle$ on $Supp(b_\delta)$ it follows that for $r > \frac{m_3+n+1}{2}$ we have,

$$(L_y^T)^r\left[\chi(x,y)\chi(y,x)a(x,y,\xi)u(y)\right] \prec \sum_{|\delta|\le 2r} \langle y\rangle^{-(n+1)}\langle\xi\rangle^{-(n+1)}||u||_{|m_1+m_2|+n+1,\delta}, \tag{3.20}$$

where $||\cdot||_{N,\delta}$ is the N,δ semi-norm on $\mathcal{S}$. Therefore we can apply the Lebesgue dominated convergence theorem and conclude that

$$A_1u(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}(L_y^T)^r(\chi(x,y)\chi(y,x)a(x,y,\xi)u(y))\mathrm{d}y\mathrm{d}\xi.$$

Inserting the expression (3.19) for $(L_y^T)^r\left[\chi(x,y)\chi(y,x)a(x,y,\xi)u(y)\right]$, we have

$$A_1u(x) = \sum_{|\delta|\le 2r}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}b_\delta(x,y,\xi)\partial_y^\delta u(y)\mathrm{d}y\mathrm{d}\xi. \tag{3.21}$$

For a non - negative integer N, consider

$$\langle x\rangle^N\partial_x^\alpha\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)}b_\delta(x,y,\xi)\partial_y^\delta u(y)\mathrm{d}y\mathrm{d}\xi. \tag{3.22}$$

By similar arguments to those which produced (3.20), taking derivatives of any order inside the integral can be justified. So,

$$\langle x\rangle^N \partial_x^\alpha \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} b_\delta(x,y,\xi)\partial_y^\delta u(y)\mathrm{d}y\mathrm{d}\xi =$$

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle x\rangle^N \partial_x^\alpha \left[e^{i\Phi(x,y,\xi)} b_\delta(x,y,\xi)\right] \partial_y^\delta u(y)\mathrm{d}y\mathrm{d}\xi. \tag{3.23}$$

Since $\Phi \in \mathcal{P}$, $b_\delta(x,y,\xi) \in SG^{m_1,m_2,m_3-2r}_{x,y,\xi}$ and $\langle x\rangle \sim \langle y\rangle$ on $Supp(b_\delta)$, we have

$$\langle x\rangle^N \partial_x^\alpha \left[e^{i\Phi(x,y,\xi)} b_\delta(x,y,\xi)\right] \prec \langle y\rangle^{m_1+m_2+N}\langle \xi\rangle^{m_3+|\alpha|-2r}. \tag{3.24}$$

For any $m_1, m_2 \in \mathbb{R}$ and any non-negative integer N we have $\langle y\rangle^{m_1+m_2+N} \prec \langle y\rangle^{|m_1+m_2|+N}$. So, by (3.24) we have

$$\langle x\rangle^N \partial_x^\alpha \left[e^{i\Phi(x,y,\xi)} b_\delta(x,y,\xi)\right] \prec \langle y\rangle^{|m_1+m_2|+N}\langle \xi\rangle^{m_3+|\alpha|-2r}. \tag{3.25}$$

Using this fact, it follows from (3.23) that for $r > \frac{m_3+|\alpha|+n+1}{2}$ we have

$$\langle x\rangle^N \partial_x^\alpha \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} b_\delta(x,y,\xi)\partial_y^\delta u(y)\mathrm{d}y\mathrm{d}\xi \prec$$

$$||u||_{|m_1+m_2|+N+n+1,\delta} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \langle y\rangle^{-(n+1)}\langle \xi\rangle^{-(n+1)}\mathrm{d}y\mathrm{d}\xi \prec ||u||_{|m_1+m_2|+N+n+1,\delta}. \tag{3.26}$$

So, by (3.21) and (3.26) we have

$$\langle x\rangle^N \partial_x^\alpha A_1 u(x) \prec \sum_{|\delta|\le 2r} ||u||_{|m_1+m_2|+N+n+1,\delta}.$$

Therefore A_1 sends $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ continuously.

□

In the proof of this Theorem and of Theorem (3.2.2) we saw that the functions $A_1u(x), A_2u(x), A_3u(x)$ and $A_4u(x)$ were independent of the choice of mollifier

γ. Since $Au(x) = A_1u(x) + A_2u(x) + A_3u(x) + A_4u(x)$ it is clear that $Au(x)$ is independent of γ and the following proposition is also proved.

Proposition 3.2.4. *For $a(x, y, \xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ with m_1, m_2, m_3 arbitrary real numbers, and $\Phi \in \mathcal{P}$, the function $A_{\Phi,a}u(x)$ is independent of the choice of mollifier.*

Recall that for functions $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, v\rangle := \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x$$

Clearly this integral converges for any pair of functions in $\mathcal{S}(\mathbb{R}^n)$. Suppose $B : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Then $\langle Bu, v\rangle$ is well defined. The transpose of B, denoted by B^T, is the operator sending $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ such that $\langle Bu, v\rangle = \langle u, B^Tv\rangle$ for all $u, v \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 3.2.5. *Let $a(x, y, \xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ with m_1, m_2, m_3 arbitrary real numbers, let $\Phi \in \mathcal{P}$ and define $A = FIO(\Phi(x, y, \xi), a(x, y, \xi))$. Then we have;*

1. $A^Tu(x) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp i\Phi(y, x, \xi)a(y, x, \xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$,
2. *A^T is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$,*
3. *$A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ continuously.*

Proof. The first two statements are straightforward given that the phase assumptions are symmetrical in x and y. See Definition 3.1.1.

Given that statement 2 of the Theorem is true we can extend the definition of A to $\mathcal{S}'(\mathbb{R}^n)$ in the usual way. That is, for $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the operator Au acting on $v \in \mathcal{S}(\mathbb{R}^n)$ by

$$Au(v) := u(A^T v).$$

To show that $Au \in \mathcal{S}'(\mathbb{R}^n)$ we need to show that Au is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. As A^T is linear, it is obvious that Au is a linear functional. We have to show that it is continuous. Suppose $\{v_k\}$ is a sequence such that $v_k \to v$ in $\mathcal{S}(\mathbb{R}^n)$. Since A^T is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we have

$$Au(v_k) = u(A^T v_k) \to u(A^T v) = Au(v),$$

in $\mathbb{C}$. So, $Au \in \mathcal{S}'(\mathbb{R}^n)$. Having established that A sends $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, we now need to check that A is continuous from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

Let $\{u_k\}$ be a sequence in $\mathcal{S}'(\mathbb{R}^n)$ such that $u_k \to u$ in $\mathcal{S}'(\mathbb{R}^n)$. This means that for all $v \in \mathcal{S}(\mathbb{R}^n)$, we have $u_k(v) \to u(v)$ in $\mathbb{C}$. Clearly then, for any $v \in \mathcal{S}$, we have,

$$Au_k(v) = u_k(A^T v) \to u(A^T v) = Au(v),$$

in $\mathbb{C}$. So, $u_k \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ implies that $Au_k \to Au$ in $\mathcal{S}'(\mathbb{R}^n)$. So the map A is continuous from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$

□

A FIO with phase $\Phi \in \mathcal{P}$ and amplitude $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ will be called a "Type $\mathcal{P}$ FIO."

3.3 Reduced form of the Type $\mathcal{P}$ FIO modulo $\mathcal{K}$.

Proposition 3.3.1. *Let $\Phi(x,y,\xi) \in \mathcal{P}$, $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ for m_1, m_2, m_3 arbitrary real numbers and let $\chi(x,y) \in \Xi^{\Delta}(k)$, where $k > 2c_\Phi$. Then, modulo $\mathcal{K}$*

$$FIO(\Phi(x,y,\xi), a(x,y,\xi)) = FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\chi(x,y)\chi(y,x)\right).$$

Proof. This was proved in the process of proving Theorem 3.2.3. □

Let $A = FIO(\Phi(x,y,\xi), a(x,y,\xi))$ with $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ and $\Phi(x,y,\xi) \in \mathcal{P}$. For any $\chi(x,y) \in \Xi^{\Delta}(k)$, with $k > 2c_\Phi$, we will call

$FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\chi(x,y)\chi(y,x)\right)$ the "reduced form" of A modulo $\mathcal{K}$.

Chapter 4

Some Technical Lemmas.

In this chapter we present a series of technical results which will shorten the proofs of later composition theorems.

The first result tells us that when considering compositions of Type $\mathcal{P}$ operators modulo $\mathcal{K}$, we only need to look at the composition of their reduced forms modulo $\mathcal{K}$.

Proposition 4.0.1. *Let A, A_1, B, B_1 be Type $\mathcal{P}$ operators. Suppose that*

$$A = A_1 \text{ modulo } \mathcal{K} \text{ and}$$
$$B = B_1 \text{ modulo } \mathcal{K}. \tag{4.1}$$

Then we have

$$A \circ B = A_1 \circ B_1 \text{ modulo } \mathcal{K}.$$

Proof. The proof is straightforward by integration by parts. □

The next result will be used frequently.

Theorem 4.0.2. *For any $m_1, m_2, m_3, m_4, m_5 \in \mathbb{R}$, let*

$a(x, y, z, \xi, \eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}\right)$ *be such that* $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ *on the support of a. Let* $\gamma(\epsilon\xi), \gamma(\delta\eta)$ *be mollifiers. Suppose also that* $\Phi \in C^\infty\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R}\right)$ *has the following properties on Supp(a):*

$$\nabla_y \Phi \succ \langle\xi\rangle + \langle\eta\rangle, \tag{4.2}$$

$$\forall \text{ multi-indices } \alpha, \beta, \gamma, \quad \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \Phi \prec \left(\langle\xi\rangle + \langle\eta\rangle\right) \langle y\rangle^{1-|\beta|} \tag{4.3}$$

Then the integral operator

$$Bu(x) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^n} \exp\{i\Phi(x, y, z, \xi, \eta)\} a(x, y, z, \xi, \eta) \times$$
$$\times \gamma(\epsilon\xi)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi, \tag{4.4}$$

acting on functions $u \in \mathcal{S}$, has kernel in $\mathcal{S}\left(\mathbb{R}^{2n}\right)$.

Proof. For fixed ϵ and δ, the integral in (4.4) is absolutely convergent. So we can change the orders of integration around. Doing so, we have,

$$Bu(x) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\{i\Phi(x, y, z, \xi, \eta)\} a(x, y, z, \xi, \eta) \times$$
$$\times \gamma(\epsilon\xi)\gamma(\delta\eta)u(z)\mathrm{d}y\mathrm{d}z\mathrm{d}\eta\mathrm{d}\xi. \tag{4.5}$$

We assumed that $\nabla_y \Phi \succ \langle\xi\rangle + \langle\eta\rangle$ on $Supp(a)$. So, the operator $U_{y,\Phi} = \frac{-i}{|\nabla_y \Phi|^2} \sum_{k=1}^n \partial_{y_k} \Phi(x, y, z, \xi, \eta) \partial_{y_k}$ is well-defined on the support of the integrand. We will abbreviate $U_{y,\Phi}$ to U_y. Applying U_y to the exponential term s times and integrating by parts s times gives

$$Bu(x) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\{i\Phi(x,y,z,\xi,\eta)\} \left(U_y^T\right)^s [a(x,y,z,\xi,\eta)] \times$$
$$\times \gamma(\epsilon\xi)\gamma(\delta\eta)u(z)\mathrm{d}y\mathrm{d}z\mathrm{d}\eta\mathrm{d}\xi. \tag{4.6}$$

Recall that

$$(U_{y,\Phi}^T)^s = \frac{1}{|\nabla_y \Phi|^{4s}} \sum_{|\theta| \le s} P_{\theta,s}(x,y,z,\xi,\eta)\partial_y^\theta.$$

In the above, $P_{\theta,s}$ is a finite sum of terms of the form $(\nabla_y\Phi)^\gamma \partial_y^{\delta_1}\Phi \dots \partial_y^{\delta_s}\Phi$, with, $|\gamma| = 2s$, $|\delta_j| \ge 1$ for all $j = 1, \dots, s$ and $|\theta| + \sum_{j=1}^s |\delta_j| = 2s$ for each term in the sum.

By assumption (4.2), we have, $\frac{1}{|\nabla_y\Phi|^{4s}} \prec (\langle\xi\rangle + \langle\eta\rangle)^{-4s}$ on $Supp(a)$. It follows from assumption (4.3) that differentiating with respect to x and z does not destroy the improvement of these estimates with s. Precisely, for any multi-indices α, β, we have

$$\partial_x^\alpha \partial_z^\beta \frac{1}{|\nabla_y \Phi|^{4s}} \prec (\langle\xi\rangle + \langle\eta\rangle)^{-4s}, \tag{4.7}$$

on $Supp(a)$.

Using the definition of $P_{\theta,s}$, and assumption (4.3) we see that for any multi-indices α, β, we have

$$\partial_x^\alpha \partial_z^\beta P_{\theta,s} \prec (\langle\xi\rangle + \langle\eta\rangle)^{3s} \langle y\rangle^{|\theta|-s}, \tag{4.8}$$

on $Supp(a)$.

As $a(x,y,z,\xi,\eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}$, we obviously have

$$\partial_x^\alpha \partial_z^\beta \partial_y^\theta a(x,y,z,\xi,\eta) \prec \langle x\rangle^{m_1}\langle y\rangle^{m_2-|\theta|}\langle z\rangle^{m_3}\langle \xi\rangle^{m_4}\langle \eta\rangle^{m_5}. \tag{4.9}$$

Using the estimates (4.7) , (4.8) and (4.9) it follows from the form of U^T that

$$\partial_x^\alpha \partial_z^\beta \left((U^T_{y,\Phi})^s[a(x,y,z,\xi,\eta)]\right) \prec \left(\langle \xi\rangle + \langle \eta\rangle\right)^{-s}\langle x\rangle^{m_1}\langle y\rangle^{m_2-s}\langle z\rangle^{m_3}\langle \xi\rangle^{m_4}\langle \eta\rangle^{m_5}. \tag{4.10}$$

By the Cauchy- Schwartz inequality, we have $\left(\langle \xi\rangle + \langle \eta\rangle\right)^{-s} \prec \langle \xi\rangle^{-\frac{s}{2}}\langle \eta\rangle^{-\frac{s}{2}}$ for any $s \in \mathbb{N}$. Using this fact and since $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ on $Supp(a)$, we have

$$\langle x\rangle^N\langle z\rangle^N \partial_x^\alpha \partial_z^\beta \left((U^T_{y,\Phi})^s[a(x,y,z,\xi,\eta)]\right) \prec \langle y\rangle^{m_1+m_2+m_3+2N-s}\langle \xi\rangle^{m_4-\frac{s}{2}}\langle \eta\rangle^{m_5-\frac{s}{2}}. \tag{4.11}$$

for any $N \in \mathbb{N}$.

The result follows from the estimates (4.11), assumption (4.3) and the Lebesgue Dominated Convergence Theorem.

□

The following Theorem is proved by similar arguments.

Theorem 4.0.3. *For any $m_1, m_2, m_3, m_4, m_5 \in \mathbb{R}$, let $a(x,y,z,\xi,\eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ be such that $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle \xi\rangle \sim \langle \eta\rangle$on the support of a. Let $\gamma(\epsilon\xi), \gamma(\delta\eta)$ be mollifiers. Suppose also that*

$\Phi \in C^\infty\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R}\right)$ *has the following properties on Supp(a):*

$$\nabla_\xi \Phi \succ \langle x \rangle, \tag{4.12}$$

$$\forall \text{ multi-indices } \alpha, \beta, \gamma, \quad \partial_x^\alpha \partial_z^\beta \partial_\xi^\gamma \Phi \prec \langle x \rangle \langle \xi \rangle^{1-|\gamma|} \tag{4.13}$$

Then the integral operator

$$Bu(x) = \lim_{\epsilon \to 0} \lim_{\delta \to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^n} \exp\{i\Phi(x,y,z,\xi,\eta)\} a(x,y,z,\xi,\eta) \times$$

$$\times \gamma(\epsilon\xi)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi, \tag{4.14}$$

acting on functions $u \in \mathcal{S}$, has kernel in $\mathcal{S}\left(\mathbb{R}^{2n}\right)$.

The next Proposition is the main result of this chapter. For its proof and for the proofs of the necessary Lemmas, we follow the corresponding results in Coriasco [3] exactly.

Proposition 4.0.4. *Let $\Phi \in \mathcal{P}$ and $a(x,y,z,\xi,\eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\right)$ be such that on Supp(a) we have $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$, $\langle \eta \rangle \sim \langle \xi \rangle$ and $|x-y| \leq k\langle y \rangle$. Also let $\gamma(\epsilon\xi)$ be a mollifier. Define*

$$h(x,z,\eta) := \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\{\Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\xi\}} a(x,y,z,\xi,\eta)\gamma(\epsilon\xi)\mathrm{d}y\mathrm{d}\xi.$$

Then, for small enough k, the function $h(x,z,\eta) \in SG^{m_1+m_2,m_3,m_4+m_5}_{x,z,\eta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\right)$ and

$$h(x,z,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\left[D_y^\alpha\left[e^{i\psi(x,y,z,\eta)}\left(\partial_4^\alpha a\right)(x,y,z,\nabla_x \Phi(x,z,\eta),\eta)\right]\right]_{y=x},$$

where $\psi(x, y, z, \eta) := \Phi(y, z, \eta) - \Phi(x, z, \eta) + (x - y) \cdot \nabla_x \Phi(x, z, \eta)$.

Remark. The notation ∂_i denotes differentiation in the *ith* variable.

Remark. Since $a(x, y, z, \xi, \eta) \in SG^{m_x, m_y, m_z, m_\xi, m_\eta}_{x,y,z,\xi,\eta} (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ on *Supp*(a), it follows that $a \in SG^{p,q,r,m_\xi,m_\eta}_{x,y,z,\xi,\eta} (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ where p, q, r are real numbers such that $p+q+r = m_x+m_y+m_z$. So, if we can prove the Theorem we will actually also have $h(x, z, \eta) \in SG^{s,t,m_\xi+m_\eta}_{x,z,\eta} (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ where s and t are any real numbers such that $s + t = m_x + m_y + m_z$.

The proof of Proposition 4.0.4 will require several Lemmas which we now present.

Lemma 4.0.5. *Let* $\Phi \in \mathcal{P}$ *and*
$a(x, y, z, \xi, \eta) \in SG^{m_x, m_y, m_z, m_\xi, m_\eta}_{x,y,z,\xi,\eta} (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *with* $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ *and* $\langle \eta \rangle \sim \langle \xi \rangle$ *on Supp*(a). *Define*

$$c_\alpha(x, z, \eta) := \frac{1}{\alpha!} \left[D^\alpha_y \{ e^{i\psi(x,y,z,\eta)} \left(\partial^\alpha_4 a \right) (x, y, z, \nabla_x \Phi(x, z, \eta), \eta) \} \right]_{y=x}.$$

where $\psi(x, y, z, \eta) := \Phi(y, z, \eta) - \Phi(x, z, \eta) + (x - y) \cdot \nabla_x \Phi(x, z, \eta)$. *Then, we have,*

$$c_\alpha \in SG^{m_x+m_y-\frac{|\alpha|}{4}, m_z-\frac{|\alpha|}{4}, m_\xi+m_\eta-\frac{|\alpha|}{2}}_{x,z,\eta}$$

That is, $\sum_\alpha c_\alpha$ *is an asymptotic expansion.*

Proof. All variables are n dimensional, so we will write $SG^{m_x, m_y, m_z, m_\xi, m_\eta}_{x,y,z,\xi,\eta}$ to mean $SG^{m_x, m_y, m_z, m_\xi, m_\eta}_{x,y,z,\xi,\eta} (\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ and similarly when we have fewer variables. Also, as always, $\int$ means $\int_{\mathbb{R}^n}$.

We assumed that $a(x,y,z,\xi,\eta) \in SG^{m_x,m_y,m_z,m_\xi,m_\eta}_{x,y,z,\xi,\eta}$. So, we obviously have

$$(\partial_4^\alpha a)(x,y,z,\xi,\eta) \in SG^{m_x,m_y,m_z,m_\xi-|\alpha|,m_\eta}_{x,y,z,\xi,\eta}.$$

Now, since $\langle \nabla_x \Phi(x,z,\xi)\rangle \sim \langle \xi \rangle$ and $\partial_{x_j}\Phi(x,z,\xi) \in SG^{0,0,1}_{x,z,\xi}$ for $j=1,\ldots,n$, we have

$$(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\xi),\eta) \in SG^{m_x,m_y,m_z,m_\xi-|\alpha|,m_\eta}_{x,y,z,\xi,\eta}.$$

Putting $\xi=\eta$, it follows that

$$(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta) \in SG^{m_x,m_y,m_z,m_\xi+m_\eta-|\alpha|}_{x,y,z,\eta}.$$

By similar arguments

$$\partial_y^\beta(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)\big|_{y=x} \in SG^{m_x+m_y-|\beta|,m_z,m_\xi+m_\eta-|\alpha|}_{x,z,\eta}. \tag{4.15}$$

Since $\langle x\rangle \sim \langle z\rangle$ on $Supp(a)$, it follows from (4.15) that

$$\partial_y^\beta(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)\big|_{y=x} \in SG^{m_x+m_y-\frac{|\beta|}{2},m_z-\frac{|\beta|}{2},m_\xi+m_\eta-|\alpha|}_{x,z,\eta}. \tag{4.16}$$

Now let's consider $\partial_y^\beta e^{i\psi(x,y,z,\eta)}\big|_{y=x}$. Recall that by definition $\psi(x,y,z,\eta) := \Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot\nabla_x\Phi(x,z,\eta)$. Define

$$f_\beta(x,y,z,\eta) := e^{-i\psi(x,y,z,\eta)}\partial_y^\beta e^{i\psi(x,y,z,\eta)}.$$

For $|\beta|\geq 1$ we have

$$f_\beta(x,y,z,\eta)\big|_{y=x} = \sum_{j_1}\prod_{j_2=1}^{n_{j_1}} c_{j_1,j_2}\partial_y^{\gamma_{j_1,j_2}}\Phi(y,z,\eta)\big|_{y=x}, \tag{4.17}$$

where the sum over j_1 is finite , $c_{j_1,j_2} \in \mathbb{C}$ and $n_{j_1} \in \mathbb{N}$ for all j_1. Note that for any j_1 we have

$$\sum_{j_2=1}^{n_{j_1}} |\gamma_{j_1,j_2}| = |\beta|. \tag{4.18}$$

Observe that for all β such that $|\beta| = 1$,

$$\partial_y^\beta \psi(x, y, z, \eta)\big|_{y=x} \equiv 0. \tag{4.19}$$

Therefore in the expression (4.17) we have

$$|\gamma_{j_1,j_2}| \geq 2, \quad \forall j_1, j_2. \tag{4.20}$$

It follows that

$$\sum_{j_2=1}^{n_{j_1}} |\gamma_{j_1,j_2}| \geq 2n_{j_1}. \tag{4.21}$$

In order to satisfy (4.18), we conclude from (4.21) that

$$2n_{j_1} \leq |\beta|. \tag{4.22}$$

for all j_1. Now, as $\Phi \in \mathcal{P}$, and since $|\gamma_{j_1,j_2}| \geq 2$, (we assumed $\partial_y^\beta \Phi(x, y, \xi) \in SG^{0,0,1}_{x,y,\xi}$ for $|\beta| \geq 1$) we have

$$\partial_y^{\gamma_{j_1,j_2}} \Phi(y, z, \eta)\big|_{y=x} \in SG^{1-|\gamma_{j_1,j_2}|,0,1}_{x,z,\eta}.$$

Therefore, by the basic facts about SG functions,

$$\prod_{j_2=1}^{n_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y, z, \eta)\big|_{y=x} \in SG^{n_{j_1}-\sum_{j_2=1}^{n_{j_1}} |\gamma_{j_1,j_2}|,0,n_{j_1}}_{x,z,\eta}.$$

By (4.18), $\sum_{j_2}^{n_{j_1}} |\gamma_{j_1,j_2}| = |\beta|$, so

$$\prod_{j_2=1}^{n_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y,z,\eta)\big|_{y=x} \in SG_{x,z,\eta}^{n_{j_1}-|\beta|,0,n_{j_1}}.$$

We have $n_{j_1} \le \frac{|\beta|}{2}$ for all j_1, so

$$\sum_{j_1} \prod_{j_2=1}^{n_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y,z,\eta)\big|_{y=x} \in SG_{x,z,\eta}^{\frac{|\beta|}{2}-|\beta|,0,\frac{|\beta|}{2}}.$$

Simplifying the SG order of the x variable gives

$$\sum_{j_1} \prod_{j_2}^{n_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y,z,\eta)\big|_{y=x} \in SG_{x,z,\eta}^{\frac{-|\beta|}{2},0,\frac{|\beta|}{2}}.$$

Recalling that $f_\beta\big|_{y=x} := \sum_{j_1} \prod_{j_1}^{n_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y,z,\eta)\big|_{y=x}$, we have

$$f_\beta(x,y,z,\eta)\big|_{y=x} \in SG_{x,z,\eta}^{-\frac{|\beta|}{2},0,\frac{|\beta|}{2}}.$$

By definition,

$$\partial_y^\beta e^{i\psi(x,y,z,\eta)} = f_\beta(x,y,z,\eta) e^{i\psi(x,y,z,\eta)}$$

Noting that $e^{i\psi}|_{y=x} = 1$, it is clear that

$$\partial_y^\beta e^{i\psi(x,y,z,\eta)}\big|_{y=x} \in SG_{x,z,\eta}^{-\frac{|\beta|}{2},0,\frac{|\beta|}{2}}. \tag{4.23}$$

The result follows from (4.16) and (4.23) by the product rule. We also use the fact that $\langle x\rangle \sim \langle z\rangle$ on the support of $\partial_y^\beta(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)\big|_{y=x}$.

□

Proof of Proposition 4.0.4 By definition

$$h(x,z,\eta) := \lim_{\epsilon\to 0}\iint e^{i\{\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot\xi\}}a(x,y,z,\xi,\eta)\gamma(\epsilon\xi)\mathrm{d}y\mathrm{d}\xi$$

We have, $\langle\xi\rangle \sim \langle\eta\rangle$ and $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ on $Supp(a)$. So for fixed x, z, η we have

$$e^{i\{\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot\xi\}}a(x,y,z,\xi,\eta)\gamma(\epsilon\xi) \prec \langle y\rangle^{-r}\langle\xi\rangle^{-r}, \tag{4.24}$$

for any $r \in \mathbb{N}$ with the implicit constants independent of ϵ. Therefore, by the Lebesgue Dominated Convergence Theorem,

$$h(x,z,\eta) = \iint e^{i\{\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot\xi\}}a(x,y,z,\xi,\eta)\mathrm{d}y\mathrm{d}\xi.$$

Expanding $a(x,y,z,\xi,\eta)$ in a Taylor series about $\xi = \nabla_x\Phi(x,z,\eta)$ gives

$$\begin{aligned} a(x,y,z,\xi,\eta) &= \sum_{|\alpha|<M}\frac{(\xi-\nabla_x\Phi)^\alpha}{\alpha!}(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi,\eta) \\ &+ \sum_{|\alpha|=M}\frac{M(\xi-\nabla_x\Phi)^\alpha}{\alpha!}r_\alpha(x,z,\eta,\xi-\nabla_x\Phi), \\ \text{where } r_\alpha(x,y,z,\xi-\nabla_x\Phi,\eta) &:= \int_0^1(1-t)^{M-1}(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi+t(\xi-\nabla_x\Phi),\eta)\mathrm{d}t. \end{aligned}$$

Inserting the Taylor series for a and defining the new variable $\theta := \xi - \nabla_x\Phi$, we have

$$\begin{aligned} h(x,z,\eta) &= \sum_{|\alpha|<M}\iint e^{i[\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot(\theta+\nabla_x\Phi(x,z,\eta))]}\frac{\theta^\alpha}{\alpha!}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi,\eta)\mathrm{d}y\mathrm{d}\theta \\ &+ \sum_{|\alpha|=M}\iint e^{i[\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot(\theta+\nabla_x\Phi(x,z,\eta))]}\frac{M\theta^\alpha}{\alpha!}r_\alpha(x,y,z,\theta,\eta)\mathrm{d}y\mathrm{d}\theta \end{aligned}$$

Define

$$c_\alpha(x,z,\eta) := \iint e^{i[\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot(\theta+\nabla_x\Phi(x,z,\eta))]}\frac{\theta^\alpha}{\alpha!}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi,\eta)\mathrm{d}y\mathrm{d}\theta,$$
$$R_\alpha := \iint e^{i[\Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot(\theta+\nabla_x\Phi(x,z,\eta))]}\frac{M(\theta)^\alpha}{\alpha!}r_\alpha(x,y,z,\theta,\eta)\mathrm{d}y\mathrm{d}\theta.$$

$\sum_\alpha c_\alpha(x,z,\eta)$ is an asymptotic expansion. As before, define $\psi(x,y,z,\eta) = \Phi(y,z,\eta)-\Phi(x,z,\eta)+(x-y)\cdot\nabla_x\Phi(x,z,\eta)$. Letting $\mathcal{F}_{x\to\xi}$ denote the Fourier transform sending $f(x)$ to $\hat{f}(\xi)$, we have, (up to a multiplicative constant)

$$c_\alpha = \frac{1}{\alpha!}\mathcal{F}^{-1}_{\theta\to x}\left[\theta^\alpha\mathcal{F}_{y\to\theta}[e^{i\psi(x,y,z,\eta)}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)]\right]$$

Note that the (x,z,η) section of $e^{i\psi(x,y,z,\eta)}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)$ belongs to $\mathcal{S}(\mathbb{R}^n)$. Converting multiplications by θ into y derivatives in the standard way gives

$$c_\alpha(x,z,\eta) = \frac{1}{\alpha!}\left[D_y^\alpha(e^{i\psi(x,y,z,\eta)}\left(\partial_\xi^\alpha a\right)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta))\right]_{y=x}.$$

By Lemma 4.0.5, $\sum_\alpha c_\alpha(x,z,\eta)$ is an asymptotic expansion.

Define $R_M := \sum_{|\alpha|=M} R_\alpha$. By Proposition 2.7.3, we will have $h \sim \sum_\alpha c_\alpha$ if it can be shown that

$$R_M \prec \langle x\rangle^{l_M}\langle z\rangle^{l_M}\langle\eta\rangle^{l_M} \tag{4.25}$$

where $l_M \to -\infty$ as $M \to \infty$. (It is easy to show that the first condition of Proposition 2.7.3 is satisfied.)

Then, by Proposition 2.7.2 part 1, there will exist a function $r(x,z,\eta) \in SG^{m_1+m_2,m_3,m_4+m_5}_{x,z,\eta}$ with $r(x,z,\eta) \sim \sum_\alpha c_\alpha(x,z,\eta)$. If $h = r$ we'll be done. If

$h \neq r$ then we'll have $h = r$ up to an additive element of $\mathcal{S}(\mathbb{R}^{3n})$ by Proposition 2.7.2 part 2. This will imply that h belongs to the same SG class as r.

that $h(x, z, \eta)$ belongs to the stated SG class.

We will prove the estimates (4.25) in two steps. We introduce the cut - off $\chi^\star(\frac{\theta}{\langle\eta\rangle})$, where $\chi^\star \in C_0^\infty(\mathbb{R}^n)$ such that $\chi^\star(x) = 1$ when $|x| \leq \frac{c}{4}$ and $\chi^\star(x) = 0$ when $|x| \geq \frac{c}{2}$.

Define

$$I_\alpha(x, z, \eta) := \iint e^{i(x-y)\cdot\theta} \chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right) \frac{M(\theta)^\alpha}{\alpha!} e^{i\psi(x,y,z,\eta)} r_\alpha(x, y, z, \theta, \eta) \mathrm{d}y\mathrm{d}\theta \quad (4.26)$$

$$K_\alpha(x, z, \eta) := \iint e^{i(x-y)\cdot\theta} (1 - \chi^\star)\left(\frac{\theta}{\langle\eta\rangle}\right) \frac{M(\theta)^\alpha}{\alpha!} e^{i\psi(x,y,z,\eta)} r_\alpha(x, y, z, \theta, \eta) \mathrm{d}y\mathrm{d}\theta$$

(4.27)

By Lemmas which we prove later, for small enough $c > 0$, we have

$$I_\alpha \prec \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}} \langle z\rangle^{m_3-\frac{|\alpha|}{4}} \langle\eta\rangle^{m_4+m_5+2n-\frac{|\alpha|}{2}}, \quad (4.28)$$

and given any $c > 0$, we can choose k to be small enough so that

$$K_\alpha \prec \langle x\rangle^{-|\alpha|} \langle z\rangle^{-|\alpha|} \langle\eta\rangle^{-|\alpha|}. \quad (4.29)$$

Recall that c is the constant in the definition of the cut-off function $\chi^\star$ and we assumed that $|x - y| \leq k\langle y\rangle$ on $Supp(a)$.

Using the estimates (4.28) and (4.29) , we see that $R_\alpha = I_\alpha + K_\alpha$ and that

$$R_\alpha \prec \langle x\rangle^{-|\alpha|} \langle z\rangle^{-|\alpha|} \langle\eta\rangle^{-|\alpha|} + \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}} \langle z\rangle^{m_3-\frac{|\alpha|}{4}} \langle\eta\rangle^{m_4+m_5+2n-\frac{|\alpha|}{2}}.$$

So

$$R_\alpha \prec \langle x\rangle^{\max\{0,m_1+m_2\}-\frac{|\alpha|}{4}}\langle z\rangle^{\max\{0,m_3\}-\frac{|\alpha|}{4}}\langle \eta\rangle^{\max\{0,m_4+m_5+2n\}-\frac{|\alpha|}{4}}.$$

As $R_M := \sum_{|\alpha|=M} R_\alpha$ we have

$$R_M \prec \langle x\rangle^{\max\{0,m_1+m_2\}-\frac{M}{4}}\langle z\rangle^{\max\{0,m_3\}-\frac{M}{4}}\langle \eta\rangle^{\max\{0,m_4+m_5+2n\}-\frac{M}{4}}.$$

□

Lemma 4.0.6. *Let* $a(x,y,z,\xi,\eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *with* $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ *and* $\langle \eta\rangle \sim \langle \xi\rangle$ *on Supp(a). Let* $\Phi \in \mathcal{P}$ *and define* $r_\alpha(x,y,z,\theta,\eta) := \int_0^1 (1-t)^{M-1}(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta)+t\theta,\eta)\mathrm{d}t$. *Let* $\chi^\star \in C_0^\infty(\mathbb{R}^n)$ *be such that* $\chi^\star(x) = 1$ *when* $|x| \leq \frac{c}{4}$ *and* $\chi^\star(x) = 0$ *when* $|x| \geq \frac{c}{2}$ *Define,*

$$f_{\alpha,\beta}(x,y,z,\eta) := \int \mathrm{d}\theta e^{i(x-y)\cdot\theta}\left(\partial_y^\beta r_\alpha(x,y,z,\theta,\eta)\right)\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right).$$

Then we can choose c to be small enough so that for any $L \in \mathbb{N}$, *we have*

$$f_{\alpha,\beta} \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}}\langle z\rangle^{m_3-\frac{|\beta|}{2}}\langle\eta\rangle^{m_4+m_5+n-|\alpha|}(1+|x-y|\langle\eta\rangle)^{-L}.$$

Proof. We start by estimating

$$r_\alpha(x,y,z,\theta,\eta) := \int_0^1 (1-t)^{M-1}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta)+t(\theta),\eta)\mathrm{d}t.$$

We can take y,θ derivatives inside, so

$$\partial_\theta^\gamma\partial_y^\beta r_\alpha(x,y,z,\theta,\eta) = \int_0^1 (1-t)^{M-1}(\partial_2^\beta\partial_4^{\alpha+\gamma}a)(x,y,z,\nabla_x\Phi+t(\theta),\eta)t^{|\gamma|}\mathrm{d}t$$

We have

$$\left|\int_0^1 (1-t)^{M-1}(\partial_2^\beta\partial_4^{\alpha+\gamma}a)(x,y,z,\nabla_x\Phi(x,z,\eta)+t(\theta),\eta)t^{|\gamma|}\mathrm{d}t\right|$$
$$\leq \sup_{t\in[0,1]}\left|(\partial_2^\beta\partial_4^{\alpha+\gamma}a)(x,y,z,\nabla_x\Phi(x,z,\eta)+t(\theta),\eta)\right|\int_0^1 t^{|\gamma|}(1-t)^{M-1}\mathrm{d}t. \quad (4.30)$$

As $\langle\nabla_x\Phi(x,z,\eta)\rangle \sim \langle\eta\rangle$, we can choose c small enough so that on $Supp(\chi^\star)$ we have,

$$\langle\nabla_x\Phi(x,z,\eta)+t\theta\rangle \sim \langle\eta\rangle. \quad (4.31)$$

More precisely, (4.31) means that there exist constants c_1, c_2 such that

$$\langle \nabla_x \Phi(x,z,\eta) + t\theta \rangle \leq c_1 \langle \eta \rangle \quad \forall t \in [0,1], x, z \in \mathbb{R}^n \text{ and } \eta, \theta \in Supp(\chi^\star)$$

$$c_2 \langle \eta \rangle \leq \langle \nabla_x \Phi(x,z,\eta) + t\theta \rangle \quad \forall t \in [0,1], x, z \in \mathbb{R}^n \text{ and } \eta, \theta \in Supp(\chi^\star)$$

The integral $\int_0^1 (1-t)^{M-1} t^{|\gamma|}$ converges for all $M \in \mathbb{N}$ and multi-indices γ, so it follows from (4.30) that

$$\partial_\theta^\gamma \partial_y^\beta r_\alpha(x,y,z,\theta,\eta) \leq \sup_{t\in[0,1]} \left| (\partial_2^\beta \partial_4^{\alpha+\gamma} a)(x,y,z,\nabla_x\Phi + t(\theta),\eta) \right|. \tag{4.32}$$

As $a \in SG^{m_x,m_y,m_z,m_\xi,m_\eta}_{x,y,z,\xi,\eta}$ and $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ on $Supp(a)$, it follows from (4.31) and (4.32) that,

$$\partial_\theta^\gamma \partial_y^\beta r_\alpha(x,y,z,\theta,\eta) \prec \langle x \rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z \rangle^{m_3-\frac{|\beta|}{2}} \langle \eta \rangle^{m_4+m_5-|\alpha|-|\gamma|}, \tag{4.33}$$

on $Supp(\chi^\star)$. We now use the estimates (4.33) to obtain a bound on the y, θ derivatives of $f_{\alpha,\beta}$. Recall that by definition

$$f_{\alpha,\beta}(x,y,z,\eta) := \int \mathrm{d}\theta e^{i(x-y)\cdot\theta} \partial_y^\beta r_\alpha(x,y,z,\theta,\eta) \chi^\star\left(\frac{\theta}{\langle \eta \rangle}\right).$$

Set $u = x - y$. Using this notation, we have

$$f_{\alpha,\beta}(x,x-u,z,\eta) = \mathcal{F}^{-1}_{\theta\to u}\left[\partial_y^\beta r_\alpha(x,x-u,z,\theta,\eta)\chi^\star(\frac{\theta}{\langle\eta\rangle})\right].$$

We remark that for fixed η, the function $\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)$ has compact support in θ. By integration by parts, we have,

$$u^\omega f_{\alpha,\beta}(x,x-u,z,\eta) = \mathcal{F}^{-1}_{\theta\to u}\left[D_\theta^\omega\left(\partial_y^\beta r_\alpha(x,x-u,z,\theta,\eta)\chi^\star(\frac{\theta}{\langle\eta\rangle})\right)\right].$$

Let $E_{\frac{c}{2},\eta} = \{\theta \in \mathbb{R}^n : |\theta| \leq \frac{c}{2}\langle\eta\rangle\}$. We have, $Supp(\chi^\star) \subset E_{\frac{c}{2},\eta}$. Letting $\mu_\theta(E_{\frac{c}{2},\eta})$ denote the measure of $E_{\frac{c}{2},\eta}$, we have

$$u^\omega f_{\alpha,\beta}(x, x-u, z, \eta) \prec \mu_\theta(E_{\frac{c}{2},\eta}) \sup_{\theta \in E_{\frac{c}{2},\eta}} \left| D_\theta^\omega \left(\partial_y^\beta r_\alpha(x, y, z, \theta, \eta) \chi^\star(\frac{\theta}{\langle\eta\rangle}) \right) \right|. \quad (4.34)$$

As $\chi^\star$ is compactly supported, it is clear that

$$\partial_\theta^\omega \chi^\star \left(\frac{\theta}{\langle\eta\rangle} \right) \prec \langle\eta\rangle^{-|\omega|} \quad \text{for all multi-indices } \omega.$$

Using this fact and (4.33), we have

$$D_\theta^\omega \left(\partial_y^\beta r_\alpha(x, x-u, z, \theta, \eta) \chi^\star(\frac{\theta}{\langle\eta\rangle}) \right) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5-|\alpha|-|\omega|} \quad (4.35)$$

We also have

$$\mu_\theta(E_{\frac{c}{2},\eta}) \prec \langle\eta\rangle^n. \quad (4.36)$$

To obtain (4.36), note that

$$\mu_\theta(E_{\frac{c}{2},\eta}) = \int_{|\theta| \leq \frac{c}{2}\langle\eta\rangle} \mathrm{d}\theta.$$

Define $s := \frac{\theta}{\langle\eta\rangle}$. Then $\mu_\theta(E_{\frac{c}{2},\eta}) = \langle\eta\rangle^n \int_{s \leq \frac{c}{2}} \mathrm{d}s \prec \langle\eta\rangle^n$.

By (4.34) (4.35) and (4.36), we conclude that

$$u^\omega f_{\alpha,\beta}(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|-|\omega|}. \quad (4.37)$$

Multiplying both sides of (4.37) by $\langle\eta\rangle^{|\omega|}$ we see that

$$\langle\eta\rangle^{|\omega|} u^\omega f_{\alpha,\beta}(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|} \quad (4.38)$$

It will now be shown that (4.38) implies that for any $r \in \mathbb{N}$,

$$(\langle\eta\rangle|u|)^r f_{\alpha,\beta}(x, x-u, z, \eta)(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|}. \tag{4.39}$$

To this end, note that (4.38) holds for any multi-index ω. Since (4.38) holds for all ω, (4.39) will be proved if we can show that $(|u|\langle\eta\rangle)^r$ can be bounded by a finite sum $\sum_i u^{\beta^i}\langle\eta\rangle^r$ with $|\beta^i| = r$ for all i.(The β^i are multi-indices) If $|u|\langle\eta\rangle \leq 1$, (4.39) follows by taking $\omega = 0$ in (4.38). If $|u|\langle\eta\rangle > 1$, then we have $(|u|\langle\eta\rangle)^r \leq (|u|\langle\eta\rangle)^{2r}$ and expanding we have the desired bound. So we have shown that for any $r \in \mathbb{N}$,

$$(\langle\eta\rangle|u|)^r f_{\alpha,\beta}(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|}.$$

It follows easily that for any non-negative integer L,

$$(1 + |u|\langle\eta\rangle)^L f_{\alpha,\beta}(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|}. \tag{4.40}$$

Hence

$$f_{\alpha,\beta}(x, x-u, z, \eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}} \langle z\rangle^{m_3-\frac{|\beta|}{2}} \langle\eta\rangle^{m_4+m_5+n-|\alpha|}(1 + |u|\langle\eta\rangle)^{-L}.$$

Recalling that $u = x - y$ completes the proof. □

Lemma 4.0.7. *Let* $\Phi \in \mathcal{P}$ *and let*

$\psi(x, y, z, \eta) := \Phi(y, z, \eta) - \Phi(x, z, \eta) + (x - y) \cdot \nabla_x \Phi(x, z, \eta)$. *If* $|x - y| \leq k\langle y \rangle$ *with* $k < 1$, *then we have*

$$\partial_y^\alpha e^{i\psi(x,y,z,\eta)} \prec (1 + |y - x|\langle \eta \rangle)^{|\alpha|} \langle \eta \rangle^{\frac{|\alpha|}{2}} \langle x \rangle^{-\frac{|\alpha|}{2}}.$$

Proof. We start by determining estimates of y derivatives of $\psi(x, y, z, \eta)$. Let $W := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq k\langle y \rangle\}$. We note first that $|x - y| \leq k\langle x \rangle$ with $k < \frac{1}{2}$ implies that $\langle x \rangle \sim \langle y \rangle$. Also, as $\Phi \in \mathcal{P}$, for $|\alpha| \geq 1$ we have $\partial_y^{|\alpha|} \Phi(y, z, \eta) \in SG^{1-|\alpha|,0,1}_{y,z,\eta}$.

First we show that for any $j = 1, \ldots, n$, we have

$$\partial_{y_j} \psi(x, y, z, \xi, \eta) \prec (1 + |y - x|) \frac{\langle \eta \rangle}{\langle x \rangle} \text{ for } (x, y) \in W.$$

By the definition of ψ, we have

$$\partial_{y_j} \psi = \partial_{y_j} \Phi(y, z, \eta) - \partial_{x_j} \Phi(x, z, \eta).$$

By the Mean Value Theorem, we have

$$\partial_{y_j} \Phi(y, z, \eta) - \partial_{x_j} \Phi(x, z, \eta) = \sum_{k=1}^{n} \int_0^1 dt \partial_{1_k} \partial_{1_j} \Phi(x + t(y - x), z, \eta)(y_k - x_k).$$

Taking moduli, we have

$$\partial_{y_j} \Phi(y, z, \eta) - \partial_{x_j} \Phi(x, z, \eta) \prec |y - x| \sup_{t \in [0,1]} \langle \eta \rangle \langle x + t(y - x) \rangle^{-1}$$

For $(x, y) \in W$, we have $\sup_{t \in [0,1]} \langle \eta \rangle \langle x + t(y - x) \rangle^{-1} \leq \langle \eta \rangle \langle x \rangle^{-1}$. (We used the fact that $k < \frac{1}{2}$ which means that on W we have $|x - y| < c\langle x \rangle$ for some $c < 1$.)

Therefore

$$\partial_{y_j}\Phi(y,z,\eta) - \partial_{x_j}\Phi(x,z,\eta) \prec |y-x|\frac{\langle\eta\rangle}{\langle x\rangle} \prec (1+|y-x|)\frac{\langle\eta\rangle}{\langle x\rangle},$$

for $(x, y) \in W$. To obtain the second inequality we used the fact that $|y-x| \leq 1+|y-x|$ for all $x, y \in \mathbb{R}^n$. Recalling that $\partial_{y_j}\psi = \partial_{y_j}\Phi(y,z,\eta) - \partial_{x_j}\Phi(x,z,\eta)$ we have established that

$$\partial_{y_j}\psi(x,y,z,\eta) \prec (1+|y-x|\langle\eta\rangle)\,\langle x\rangle^{-1}. \tag{4.41}$$

for $(x, y) \in W$. As $\Phi \in \mathcal{P}$, it follows that for $|\alpha| \geq 2$ we have,

$$\partial_y^\alpha\psi(x,y,z,\eta) \prec \langle\eta\rangle\langle y\rangle^{1-|\alpha|}. \tag{4.42}$$

Since $\langle x\rangle \sim \langle y\rangle$ on W we have

$$\partial_y^\alpha\psi(x,y,z,\eta) \prec \langle\eta\rangle\langle x\rangle^{1-|\alpha|}, \tag{4.43}$$

for $(x, y) \in W$.

Case $|\alpha| = 0$. Obvious since $e^{i\psi} \prec 1$.

Case $|\alpha| = 1$. Follows immediately by (4.41).

Case $|\alpha| = 2$. Let $\alpha = e_i+e_j$ for i, j arbitrary. Recall that e_i is the multi-index with 1 in the *ith* place and zeroes elsewhere. Then

$$\partial_y^\alpha e^{i\psi(x,y,z,\eta)} = \left[(\nabla_y\psi)^\alpha + \partial_y^{e_i}\psi\partial_y^{e_j}\psi\right]e^{i\psi(x,y,z,\eta)}.$$

The result for $|\alpha| = 2$ follows from (4.41) and (4.43).

Case $|\alpha| \geq 3$ Now, by induction, we have

$$\partial_y^\alpha e^{i\psi(x,y,z,\eta)} = \sum_{j_1} \Big[(\nabla_y \psi(x,y,z,\eta))^{\theta_{j_1}} \prod_{j_2=1}^{n_{j_1}} \partial_y^{\beta_{j_1,j_2}} \Phi(y,z,\eta) + \\ + (\nabla_y \psi(x,y,z,\eta))^\alpha + \\ + \prod_{j_2=1}^{m_{j_1}} \partial_y^{\gamma_{j_1,j_2}} \Phi(y,z,\eta) \Big] e^{i\psi(x,y,z,\eta)}, \tag{4.44}$$

where the sum over j_1 is finite and we have the following statements holding:

$$|\beta_{j_1,j_2}| \geq 2 \quad \forall j_1, j_2$$

$$|\gamma_{j_1,j_2}| \geq 2 \quad \forall j_1, j_2$$

$$|\theta_{j_1}| + \sum_{j_2=1}^{n_{j_1}} \beta_{j_1,j_2} = \alpha \quad \forall j_1$$

$$\sum_{j_2=1}^{m_{j_1}} |\gamma_{j_1,j_2}| = |\alpha| \forall j_1$$

$$m_{j_1}, n_{j_1} \leq \frac{|\alpha|}{2} \tag{4.45}$$

The Lemma follows from the above statement about the structure of derivatives of $e^{i\psi(x,y,z,\eta)}$ using the estimates (4.41) and (4.43).

□

Lemma 4.0.8. *Let* $\Phi(x,y,\xi) \in \mathcal{P}$ *and define*

$\psi(x,y,z,\eta) := \Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\nabla_x\Phi(x,z,\eta).$

Let $a(x,y,z,\xi,\eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n)$ *with* $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$, $\langle\eta\rangle \sim \langle\xi\rangle$ *and* $|x-y| \le k\langle y\rangle$ *for* $k<\frac{1}{2}$ *on Supp(a). Define* $r_\alpha(x,y,z,\theta,\eta) := \int_0^1 (1-t)^{M-1}(\partial_4^\alpha a)(x,y,z,\nabla_x\Phi+t(\theta),\eta)\mathrm{d}t$. *Let* $\chi^\star \in C_0^\infty(\mathbb{R}^n)$ *be such that* $\chi^\star(x)=1$ *when* $|x|\le\frac{c}{4}$ *and* $\chi^\star(x)=0$ *when* $|x|\ge\frac{c}{2}$ *Define,*

$$I_\alpha(x,z,\eta) := \iint e^{i(x-y)\cdot\theta}\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)\frac{M\theta^\alpha}{\alpha!}e^{i\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)\mathrm{d}y\mathrm{d}\theta.$$

Then, if we choose the constant c in the definition of $\chi^\star$ *to be small enough, we have*

$$I_\alpha(x,z,\eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}}\langle z\rangle^{m_3-\frac{|\alpha|}{4}}\langle\eta\rangle^{m_4+m_5+2n-\frac{|\alpha|}{2}}.$$

Proof. By definition

$$I_\alpha = \int e^{ix\cdot\theta}\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)\frac{M(\theta)^\alpha}{\alpha!}\mathcal{F}_{y\to\theta}\left[e^{i\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)\right]\mathrm{d}\theta$$

The (x,z,η,θ) section of $e^{i\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)$ is in $\mathcal{S}(\mathbb{R}^n)$. Converting θ mutliplications into y derivatives, we have, up to multiplicative constants,

$$I_\alpha = \int e^{ix\cdot\theta}\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)\mathcal{F}_{y\to\theta}\left(\partial_y^\alpha\left[e^{i\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)\right]\right)\mathrm{d}\theta.$$

Inserting the definition of the Fourier transform gives

$$I_\alpha = \int\int e^{i(x-y)\cdot\theta}\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)\partial_y^\alpha\left[e^{i\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)\right]\mathrm{d}y\mathrm{d}\theta. \qquad (4.46)$$

Define

$$f_{\alpha,\beta}(x,y,z,\eta) := \int \mathrm{d}\theta e^{i(x-y)\cdot\theta}\partial_y^\beta r_\alpha(x,y,z,\theta,\eta)\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right).$$

Using this notation, (4.46) becomes

$$I_\alpha = \sum_{\beta\leq\alpha}\int f_{\alpha,\beta}(x,y,z,\eta)\partial_y^{\alpha-\beta}e^{i\psi(x,y,z,\eta)},$$

up to multiplicative constants. By Lemma 4.0.6, we have

$$f_{\alpha,\beta}(x,y,z,\eta) \prec \langle x\rangle^{m_1+m_2-\frac{|\beta|}{2}}\langle z\rangle^{m_3-\frac{|\beta|}{2}}\langle\eta\rangle^{m_4+m_5+n-|\alpha|}(1+|x-y|\langle\eta\rangle)^{-L}, \quad (4.47)$$

for sufficiently small c. Now, recall that

$$I_\alpha(x,z,\eta) = \sum_{\beta\leq\alpha}\int f_{\alpha,\beta}(x,y,z,\eta)\partial_y^{\alpha-\beta}e^{i\psi(x,y,z,\eta)}\mathrm{d}y$$

On $Supp(f_{\alpha,\beta})$, we have $|x-y|\leq k\langle x\rangle$ with $k<\frac{1}{2}$. So, by Lemma 4.0.7 we have

$$\partial_y^{\alpha-\beta}e^{i\psi(x,y,z,\eta)} \prec (1+|y-x|\langle\eta\rangle)^{|\alpha-\beta|}\langle\eta\rangle^{\frac{|\alpha-\beta|}{2}}\langle x\rangle^{-\frac{|\alpha-\beta|}{2}}, \quad (4.48)$$

with (4.48) holding on $Supp(f_{\alpha,\beta})$. By (4.47) and (4.48) (and since $\langle x\rangle\sim\langle z\rangle$ on $Supp(f_{\alpha,\beta})$) we have

$$f_{\alpha,\beta}(x,y,z,\eta)\partial_y^{\alpha-\beta}e^{i\psi(x,y,z,\eta)} \prec \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}}\langle z\rangle^{-\frac{|\alpha|}{4}-\frac{|\beta|}{2}}\langle\eta\rangle^{m_4+m_5+n-\frac{|\alpha|}{2}-\frac{|\beta|}{2}}.$$
$$.(1+|y-x|\langle\eta\rangle)^{|\alpha-\beta|}(1+|x-y|\langle\eta\rangle)^{-L}. \quad (4.49)$$

Therefore, taking the maximum in each exponent when β ranges over its possible values (i.e. $\beta \leq \alpha$)) we have

$$I_\alpha \prec \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}} \langle z\rangle^{m_3-\frac{|\alpha|}{4}} \langle \eta\rangle^{m_4+m_5+n-\frac{|\alpha|}{2}} \int (1+|y-x|\langle\eta\rangle)^{|\alpha|}(1+|x-y|\langle\eta\rangle)^{-L} \mathrm{d}y. \tag{4.50}$$

Let $L = |\alpha| + L_2$, where $L_2 \in \mathbb{N}$. Then, (4.50) becomes

$$I_\alpha \prec \langle x\rangle^{m_1+m_2-\frac{|\alpha|}{4}} \langle z\rangle^{m_3-\frac{|\alpha|}{4}} \langle \eta\rangle^{m_4+m_5+n-\frac{|\alpha|}{2}} \int (1+|x-y|\langle\eta\rangle)^{-L_2} \mathrm{d}y. \tag{4.51}$$

Once we show that for sufficiently large L_2

$$\int (1+|x-y|\langle\eta\rangle)^{-L_2} \mathrm{d}y \prec \langle\eta\rangle^n, \tag{4.52}$$

we'll be done by (4.51). To see this, define the new variable $u := x - y$ and choose L_2 even. Setting $L_2 = 2s$, observe that

$$(1+|u|\langle\eta\rangle)^{2s} \geq (|u|\langle\eta\rangle)^{2s} = \left(\sum_{j=1}^n u_j^2 \langle\eta\rangle^2\right)^s.$$

Define $v := \langle\eta\rangle u$. Changing variables again we see that

$$\int (1+|x-y|\langle\eta\rangle)^{-L_2} \mathrm{d}y \leq \langle\eta\rangle^n \int_{\mathbb{R}^n} \frac{1}{|v|^{-2s}} \mathrm{d}v.$$

For $s \geq \frac{n+1}{2}$, the integral $\int_{\mathbb{R}^n} \frac{1}{|v|^{-2s}} \mathrm{d}v$ converges, so

$$\int (1+|x-y|\langle\eta\rangle)^{-L_2} \mathrm{d}y \prec \langle\eta\rangle^n.$$

□

Lemma 4.0.9. *Let $\Phi(x,y,\xi) \in \mathcal{P}$ and define $\psi(x,y,z,\eta) := \Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\xi$. Let $a(x,y,z,\xi,\eta) \in SG^{m_x,m_y,m_z,m_\xi,m_\eta}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$, $\langle\eta\rangle \sim \langle\xi\rangle$ and $|x-y| \le k\langle y\rangle$ for $k<1$ on Supp(a). Define $\theta := \xi - \nabla_x\Phi(x,z,\eta)$ and $r_\alpha(x,y,z,\theta,\eta) := \int_0^1 (1-t)^{M-1}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi(x,z,\eta) + t\theta,\eta)\mathrm{d}t$. Let $\chi^\star \in C_0^\infty(\mathbb{R}^n)$ be such that $\chi^\star(x) = 1$ when $|x| \le \frac{c}{4}$ and $\chi^\star(x) = 0$ when $|x| \ge \frac{c}{2}$ Define,*

$$K_\alpha(x,z,\eta) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta}\left(1-\chi^\star\left(\frac{\theta}{\langle\eta\rangle}\right)\right)\frac{(\theta)^\alpha}{\alpha!}e^{\psi(x,y,z,\eta)}r_\alpha(x,y,z,\theta,\eta)\mathrm{d}y\mathrm{d}\theta$$

Then, given any $c>0$, we can choose k to be small enough so that we have the following estimates for large $|\alpha|$;

$$K_\alpha \prec \langle x\rangle^{-|\alpha|}\langle z\rangle^{-|\alpha|}\langle\eta\rangle^{-|\alpha|}$$

Proof. We assumed that $|x-y| \le k\langle y\rangle$ for $k<1$ and $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ on *Supp*(a). It follows from the definition of r_α that these statements also hold of the support of r_α. We remark that the implicit constants do not depend on α.

We now show that, given any c (the constant in the definition of $\chi^\star$), we can choose k small enough so that on the support of r_α $(1-\chi^\star)$, we have $\nabla_y[\psi(x,y,z,\eta) + (x-y)\cdot\theta] \succ \langle\theta\rangle + \langle\eta\rangle$. Clearly

$$\nabla_y[\psi(x,y,z,\eta) + (x-y)\cdot\theta] = \nabla_y\psi(x,y,z,\eta) - \theta.$$

By definition, $\psi(x,y,z,\eta) := \Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\nabla_x\Phi(x,z,\eta)$. So,

$$\nabla_y\psi(x,y,z,\eta) = \nabla_y\Phi(y,z,\eta) - \nabla_x\Phi(x,z,\eta).$$

We assumed that $\partial_1^\alpha \Phi(y,z,\eta) \in SG^{0,0,1}_{z,y,\eta}$ for $|\alpha| = 1$. Since $|x-y| \leq k\langle y\rangle$ on the support of r_α, it follows from the mean value theorem that there exists some constant C such that

$$\nabla_y \Phi(y,z,\eta) - \nabla_x \Phi(x,z,\eta) \leq kC\langle\eta\rangle,$$

on the support of r_α. Therefore, on the support of the integrand of r_α we have

$$|\nabla_y[\psi(x,y,z,\eta) + (x-y)\cdot\theta]| \geq |\theta| - kC\langle\eta\rangle. \tag{4.53}$$

Obviously

$$|\theta| - kC\langle\eta\rangle = \frac{|\theta|}{2} + \left(\frac{|\theta|}{2} - kC\langle\eta\rangle\right). \tag{4.54}$$

On the support of $(1-\chi^\star)$, we have $|\theta| \geq \frac{c}{4}\langle\eta\rangle$. So given any c, we can choose k small enough so that

$$\left(\frac{|\theta|}{2} - kC\langle\eta\rangle\right) \succ \langle\eta\rangle, \tag{4.55}$$

on the support of $r_\alpha(1-\chi^\star)$. Also, $|\theta| \geq \frac{c}{4}\langle\eta\rangle$ implies that $|\theta| \geq \epsilon\langle\theta\rangle$, for some $\epsilon > 0$. So, it follows from (4.55) and (4.54) that given any c, we can choose k to be sufficiently small so that $|\nabla_y[\psi(x,y,z,\eta) + (x-y)\cdot\theta]| \succ \langle\theta\rangle + \langle\eta\rangle$ on the support of the integrand of $r_\alpha(1-\chi^\star)$.

Therefore, the operator $U_{y,w}$ is well defined (with $w := \psi(x,y,z,\eta) + (x-y)\cdot\theta$.) We will abbreviate $U_{y,w}$ to U_y. Applying U_y s times and integrating by parts gives,

$$K_\alpha = \iint e^{i\{w(x,y,z,\eta,\theta)\}}(1-\chi^\star)\left(\frac{\theta}{\langle\eta\rangle}\right)\times$$
$$\times\left(U_y^T\right)^s\{r_\alpha(x,y,z,\theta,\eta)\}\,\mathrm{d}y\mathrm{d}\theta \tag{4.56}$$

Recall the form of $(U_y^T)^s$;

$$(U_y^T)^r = \frac{1}{|\nabla_y w|^{4s}}\sum_{\varsigma\leq r} P_{\varsigma,s}\partial_y^\varsigma.$$

In the above, up to multiplicative constants, $P_{\varsigma,s}$ is a sum of terms of the form $(\nabla_y w)^\gamma \partial_y^{\delta_1} w \ldots \partial_y^{\delta_s} w$, with, $|\gamma| = 2s$, $|\delta_j| \geq 1$ and $|\varsigma| + \sum_{j=1}^s |\delta_j| = 2s$. As $\Phi \in \mathcal{P}$ we have

$$\partial_y^\delta w \prec (\langle\theta\rangle + \langle\eta\rangle)\,\langle y\rangle^{1-|\delta|}, \tag{4.57}$$

on the support of r_α. Recalling the definition of $P_{\varsigma,r}$, we can use (4.57) to see that

$$P_{\varsigma,r} \prec (\langle\theta\rangle + \langle\eta\rangle)^{3s}\,\langle y\rangle^{s-\sum_1^s|\delta_i|},$$

on the support of r_α. Since $\sum_1^s |\delta_i| = 2s - |\varsigma|$, we have

$$P_{\varsigma,s} \prec (\langle\theta\rangle + \langle\eta\rangle)^{3s}\,\langle y\rangle^{|\varsigma|-s}, \tag{4.58}$$

on the support of r_α. We showed earlier that

$$|\nabla_y w| \succ \langle\theta\rangle + \langle\eta\rangle, \tag{4.59}$$

on the support of r_α. Now we need to estimate

$$\partial_y^\varsigma r_\alpha(x,y,z,\theta,\eta).$$

Recall that by definition

$$r_\alpha(x,y,z,\theta,\eta) := \int_0^1 (1-t)^{M-1}(\partial_\xi^\alpha a)(x,y,z,\nabla_x\Phi + t(\theta),\eta)\mathrm{d}t.$$

Differentiating under the integral sign we have

$$\partial_y^\zeta r_\alpha(x,y,z,\theta,\eta) = \int_0^1 (1-t)^{M-1}(\partial_2^\zeta\partial_4^\alpha a)(x,y,z,\nabla_x\Phi + t(\theta),\eta)t^{|\gamma|}\mathrm{d}t$$

Recall that $a(x,y,z,\xi,\eta) \in SG_{x,y,z,\xi,\eta}^{m_1,m_2,m_3,m_4,m_5}$. We are interested in the behaviour for large $|\alpha|$, so we may assume that $m_4 - |\alpha| < 0$. For $|\alpha| > m_4$, we have

$$\partial_y^\beta r_\alpha(x,y,z,\theta,\eta) \prec \langle x\rangle^{m_1}\langle y\rangle^{m_2-|\zeta|}\langle z\rangle^{m_3}\langle\eta\rangle^{m_5}. \tag{4.60}$$

Using (4.58),(4.59) and (4.60), we have

$$\begin{aligned}\left(U_y^T\right)^s r_\alpha(x,y,z,\theta,\eta) \prec \\ \langle x\rangle^{m_1}\langle y\rangle^{m_2-s}\langle z\rangle^{m_3}\langle\eta\rangle^{m_4}\left(\langle\eta\rangle+\langle\theta\rangle\right)^{-s}.\end{aligned} \tag{4.61}$$

The proof is complete when we recall that $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ on $Supp(r_\alpha)$ and $(\langle\eta\rangle+\langle\theta\rangle)^{-r} \prec \langle\eta\rangle^{-\frac{r}{2}}\langle\theta\rangle^{-\frac{r}{2}}$ (by choosing s to be sufficiently large). $\square$

Chapter 5

Composition of Type $\mathcal{P}$ operator with Pseudo.

This will be a short chapter as most of the results we need were proved in chapter 4. Again, we follow the proof of the corresponding result in [3].

Theorem 5.0.1. *Let* $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *for any* $m_1, m_2, m_3 \in \mathbb{R}$ *and* $\Phi(x,y,\xi) \in \mathcal{P}$. *Define* $A := FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\right)$*.Let* $p(x,y,\xi) \in SG^{t_1,t_2,t_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *for any* $t_1, t_2, t_3 \in \mathbb{R}$ *and let* $P := Op(p(x,y,\xi))$. *Then, modulo* $\mathcal{K}$,

$$P \circ A = FIO\left(\Phi(x,z,\eta), c(x,z,\eta)\right)$$

where $c(x,z,\eta) \in SG^{m_1+m_2+t_1,t_2,m_3+t_3}_{x,z,\eta}$. *We also obtain an asymptotic expansion for* $c(x,z,\eta)$.

Proof. In this proof, all integrals are over $\mathbb{R}^n$. Let $\chi(x,y) \in \Xi^{\Delta}(k)$, with $k > 2c_\Phi$ and let $\chi_1 \in \Xi^{\Delta}(k_1)$ with $k_1 \in (0,1)$. Define A_{Red} and P_{Red} as follows:

$$A_{Red} := FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\chi(x,y)\chi(y,x)\right),$$
$$P_{Red} := Op(p(x,y,\xi)\chi_1(x,y)\chi_1(y,x)). \tag{5.1}$$

We have $A = A_{Red}$ and $P = P_{Red}$ modulo $\mathcal{K}$. So, by Proposition 4.0.1 we have $A \circ P = A_{Red} \circ P_{Red}$ modulo $\mathcal{K}$.

Define

$$A_{Red,\delta}u(y) := \iint \exp\{i\Phi(y,z,\eta)\}a(y,z,\eta)\chi(z,y)\chi(y,z)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta. \tag{5.2}$$

By definition

$$A_{Red}u(y) = \lim_{\delta\to 0} A_{Red,\delta}u(y). \tag{5.3}$$

Now, $A_{Red,\delta}u$ tends to $A_{Red}u$ in $\mathcal{S}(\mathbb{R}^n)$. So, as P_{Red} sends $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ continuously, we have

$$\left(P_{Red} \circ A_{Red}\right)u(x) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \iiiint \exp\{i\left(\Phi(y,z,\eta) + (x-y)\cdot\xi\right)\}p(x,y,\xi)a(y,z,\eta)\times$$
$$\chi(z,y)\chi(y,z)\chi_1(x,y)\chi_1(y,x)\gamma(\delta\eta)\gamma(\epsilon\xi)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi. \tag{5.4}$$

For convenience define $\tilde{a}(x,y,z,\xi,\eta) := p(x,y,\xi)a(y,z,\eta)\chi(z,y)\chi(y,z)\chi_1(x,y)\chi_1(y,x)$.

So, using this notation, we have

$$(P_{Red} \circ A_{Red})\, u(x) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \iiiint \exp\{i\,(\Phi(y,z,\eta) + (x-y)\cdot\xi)\} \times$$
$$\times \tilde{a}(x,y,z,\xi,\eta)\gamma(\delta\eta)\gamma(\epsilon\xi)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi. \qquad (5.5)$$

Let $\chi_2 \in \Xi^{\Delta}(k_2)$ where $k_2 \in (0,1)$. By making a partition of unity using $\chi_2\left(\nabla_y \Phi(y,z,\eta), \xi\right)$, we can use Theorem 4.0.2 to see that

$$(P_{Red} \circ A_{Red})\, u(x) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \iiiint \exp\{i\,(\Phi(y,z,\eta) + (x-y)\cdot\xi)\} \times$$
$$\times \tilde{a}(x,y,z,\xi,\eta)\chi_2(\nabla_y \Phi(y,z,\eta),\xi)\gamma(\delta\eta)\gamma(\epsilon\xi)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi, \qquad (5.6)$$

modulo Ku where $K \in \mathcal{K}$. Define $b(x,y,z,\xi,\eta) := \tilde{a}(x,y,z,\xi,\eta)\chi_2(\nabla_y\Phi(y,z,\eta),\xi)$ and let $g_{\delta,\epsilon}(x,y,z,\xi,\eta)$ denote the integrand in (5.6). Now, on $Supp(b)$, we have $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle \xi\rangle \sim \langle \eta\rangle$. Also, for fixed δ we have $\langle \eta\rangle \prec 1$ on $Supp(g_{\delta,\epsilon})$. Using these facts and as $u \in \mathcal{S}$, it follows that for fixed δ we have

$$g_{\delta,\epsilon}(x,y,z,\xi,\eta) \prec \langle x\rangle^{-r}\langle y\rangle^{-r}\langle z\rangle^{-r}\langle \xi\rangle^{-r}\langle \eta\rangle^{-r}, \qquad (5.7)$$

for any $r \in \mathbb{N}$, with the implicit constant independent of ϵ. So, we can apply the Lebesgue Dominated Convergence Theorem to obtain

$$(P_{Red} \circ A_{Red})\, u(x) = \lim_{\delta \to 0} \iiiint \exp\{i\,(\Phi(y,z,\eta) + (x-y)\cdot\xi)\} \times$$
$$\times b(x,y,z,\xi,\eta)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi. \qquad (5.8)$$

By the estimates (5.7) we can re-order the integrals in (5.8) to get

$$(P_{Red} \circ A_{Red})\, u(x) = \lim_{\delta\to 0} \iiiint \exp\{i\,(\Phi(y,z,\eta) + (x-y)\cdot\xi)\}\times$$
$$\times b(x,y,z,\xi,\eta)\gamma(\delta\eta)u(z)\mathrm{d}y\mathrm{d}\xi\mathrm{d}z\mathrm{d}\eta. \qquad (5.9)$$

Multiplying by $\exp\{i\Phi(x,z,\eta)\}\exp\{-i\Phi(x,z,\eta)\}$ which is just 1 we have

$$(P_{Red} \circ A_{Red})\, u(x) = \lim_{\delta\to 0} \iint \exp\{i\Phi(x,z,\eta)\}\times$$
$$\left[\iint \exp\{i\,(\Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\xi)\}b(x,y,z,\xi,\eta)\mathrm{d}y\mathrm{d}\xi\right]\times$$
$$\times\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta. \qquad (5.10)$$

Define

$$c(x,z,\eta) := \iint \exp\{i\,(\Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\xi)\}b(x,y,z,\xi,\eta)\mathrm{d}y\mathrm{d}\xi.$$

The proof will be complete it we can show that $c(x,z,\eta) \in SG^{m_1+m_2+t_1,t_2,m_3+t_3}_{x,z,\eta}$. The function b in the definition of $c(x,z,\eta)$ satisfies the conditions of Proposition 4.0.4. Therefore, for small enough k_1, (the constant in cut-off χ_1) we have $c(x,z,\eta) \in SG^{m_1+m_2+t_1,t_2,m_3+t_3}_{x,z,\eta}$ and

$$c(x,z,\eta) \sim \sum_\alpha \frac{1}{\alpha!}\left[D^\alpha_y\left[e^{i\psi(x,y,z,\eta)}\left(\partial^\alpha_4 b\right)(x,y,z,\nabla_x\Phi(x,z,\eta),\eta)\right]\right]_{y=x},$$

where $\psi(x,y,z,\eta) := \Phi(y,z,\eta) - \Phi(x,z,\eta) + (x-y)\cdot\nabla_x\Phi(x,z,\eta)$. Recall that $b := p(x,y,\xi)a(y,z,\eta)\chi(z,y)\chi(y,z)\chi_1(x,y)\chi_1(y,x)\chi_2(\nabla_y\Phi(y,z,\eta),\xi)$ □

Remark By the remarks following Proposition 4.0.4, we actually have $c(x,z,\eta) \in SG^{p,q,m_3+t_3}_{x,z,\eta}$ where p and q are any real numbers such that $p+q = m_2+m_1+t_2+t_1$.

Let A be a Type $\mathcal{P}$ FIO and let P be a pseudodifferential Operator. As we shall see, the fact that the composition $A \circ P$ is a FIO follows from the previous Theorem.

Theorem 5.0.2. *Let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for any $m_1, m_2, m_3 \in \mathbb{R}$ and $\Phi(x,y,\xi) \in \mathcal{P}$. Define $A := FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\right)$. Let $p(x,y,\xi) \in SG^{t_1,t_2,t_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for any $t_1, t_2, t_3 \in \mathbb{R}$ and let $P := Op(p(x,y,\xi))$. Then, modulo $\mathcal{K}$, the composition*

$$A \circ P = FIO\left(\Phi(x,z,\eta), c(x,z,\eta)\right)$$

where $c(x,z,\eta) \in SG^{m_2+m_1+t_2,t_1,m_3+t_3}_{x,z,\eta}$. We also obtain an asymptotic expansion for $c(x,z,\eta)$.

Proof. Consider $(A \circ P)^T$. This is $P^T \circ A^T$. Now,

$$A^T u(x) = \lim_{\delta \to 0} \iint \exp\{i\Psi(x,y,\xi)\} a_0(x,y,\xi)\gamma(\delta\xi)u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $\Psi(x,y,\xi) := \Phi(y,x,\xi)$ and $a_0(x,y,\xi) = a(y,x,\xi)$. This is a FIO because of the symmetry of the phase assumptions in x and y. Also,

$$P^T u(x) = \lim_{\epsilon \to 0} \iint \exp\{i(y-x)\cdot\xi\} p(y,x,\xi)\gamma(\delta\xi)u(y)\mathrm{d}y\mathrm{d}\xi,$$

Define $\tilde{\xi} = -\xi$ and changing variables we have

$$P^T u(x) = \lim_{\epsilon \to 0} \iint \exp\{i(x-y)\cdot\tilde{\xi}\} p(y,x,-\tilde{\xi})\gamma(-\delta\tilde{\xi})u(y)\mathrm{d}y\mathrm{d}\tilde{\xi}.$$

Since $\gamma(-\delta\tilde{\xi})$ is also a mollifier, P^T is a ψdo with symbol $p(y, x, -\xi)$. Define $p_0(x, y, \xi) := p(y, x, -\xi)$. We can now apply the previous Theorem 5.0.1 to see that

$$P^T \circ A^T = FIO(\Psi(x, z, \eta), c(x, z, \eta))$$

where $c(x, z, \eta) \in SG^{m_2+m_1+t_2,t_1,m_3+t_3}_{x,z,\eta}$ and

$$c(x, z, \eta) \sim \sum_\alpha \frac{1}{\alpha!} \left[D^\alpha_y \left[e^{i\psi(x,y,z,\eta)} \left(\partial^\alpha_4 b\right)(x, y, z, \nabla_x\Psi(x, z, \eta), \eta)\right]\right]_{y=x}, \qquad (5.11)$$

where $\psi(x, y, z, \eta) := \Psi(y, z, \eta) - \Psi(x, z, \eta) + (x - y) \cdot \nabla_x\Psi(x, z, \eta)$. with $b := p_0(x, y, \xi)a_0(y, z, \eta)\chi(z, y)\chi(y, z)\chi_1(x, y)\chi_1(y, x)\chi_2(\nabla_y\Psi(y, z, \eta), \xi)$. Taking transposes again, we have

$$\left(P^T \circ A^T\right)^T = FIO(\Phi(x, z, \eta), c_0(x, z, \eta))$$

where $c_0(x, z, \eta) = c(z, x, \eta)$. (Recall that $\Psi(x, y, \xi) := \Phi(y, x, \xi)$) Since $\left(P^T \circ A^T\right)^T = A \circ P$ we have $A \circ P = FIO(\Phi(x, z, \eta), c_0(x, z, \eta))$ with $c_0(x, z, \eta) \in SG^{t_1,m_2+m_1+t_2,m_3+t_3}_{x,z,\eta}$. We also have

$$c_0(x, z, \eta) \sim \sum_\alpha \frac{1}{\alpha!} \left[D^\alpha_y \left[e^{i\psi(z,y,x,\eta)} \left(\partial^\alpha_4 b\right)(z, y, x, \nabla_2\Phi(x, z, \eta), \eta)\right]\right]_{y=z},$$

where $\psi(z, y, x, \eta) := \Phi(x, y, \eta) - \Phi(x, z, \eta) + (z - y) \cdot \nabla_2\Phi(x, z, \eta)$. with $b(z, y, x, \xi, \eta) := p(y, z, -\xi)a(x, y, \eta)\chi(x, y)\chi(y, x)\chi_1(z, y)\chi_1(y, z)\chi_2(\nabla_y\Phi(x, y, \eta), \xi)$. We obtained the above asymptotic expansion from (5.11) by inserting our definitions of Ψ, p_0 and a_0 and interchanging z and x.

$\square$

Remark In the above Theorem, we actually have $c(x, z, \eta) \in SG^{p,q,m_3+t_3}_{x,z,\eta}$ where p and q are any real numbers such that $p + q = m_2 + m_1 + t_2 + t_1$.

Chapter 6

SG Diffeomorphisms

In this chapter we shall discuss changes of variable which preserve the SG structure.

6.1 Global Inverse Function Theorem

Let $V \subset \mathbb{R}^n$ be open and let $f \in C^1(V, \mathbb{R}^n)$. The Jacobian matrix of f at $x_0 \in V$ is the $n \times n$ matrix with i, j entry $\frac{\partial f_i}{\partial x_j}\big|_{x=x_0}$. The "*jacobian*" of f at $x_0 \in V$ is the determinant of the Jacobian matrix of f at x_0.

Theorem 6.1.1. ***(Inverse Function Theorem)*** *Let $f \in C^1(E, \mathbb{R}^n)$ for some open set $E \subset \mathbb{R}^n$. Suppose that the Jacobian of f is non-zero at some point $x_0 \in E$. Then there exist open sets $U, V \subset \mathbb{R}^n$ with $x_0 \in U$ such that:*

- *f is a bijection from U to V,*

- *The inverse map* $f^{-1} : V \to U$ *belongs to* $C^1(V, \mathbb{R}^n)$

We now state a global version of the Inverse Function Theorem.

Theorem 6.1.2. ***(Global Inverse Function Theorem)*** *Let* $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. *The function* f *is a diffeomorphism if and only if the Jacobian of* f *is non-zero for all* $x \in \mathbb{R}^n$ *and* $|f(x)| \to \infty$ *as* $|x| \to \infty$.

Theorem 6.1.2 is proved in [12]. For more details about global diffeomorphisms see [1] and [21]. There are also results, due to Hadamard, about global diffeomorphisms between more general manifolds. These are discussed in [21].

We state the local Implicit Function Theroem for completeness.

Theorem 6.1.3. ***(Implicit Function Theorem)***
Let $f(x, y) \in C^1(U, \mathbb{R}^n)$ *where* U *is an open neighborhood in* $\mathbb{R}^m \times \mathbb{R}^n$. *Suppose that for some point* $(x_0, y_0) \in U$ *we have* $f(x_0, y_0) = 0$ *and* $\left(\partial_{y_j} f_i\right)_{i,j=1}^n$ *has non-zero determinant at* (x_0, y_0). *Then there exists a unique function* $g : V \to \mathbb{R}^n$ *defined in an open neighborhood* $V \subset \mathbb{R}^m$ *of* x_0, *such that* $g(x_0) = y_0$ *and* $f(x, g(x)) = 0$ *for all* $x \in V$. *Furthermore,* $g \in C^1(V, \mathbb{R}^n)$.

6.2 SG Changes of Variable.

In this section we prove some results about SG structure preserving changes of variable. All of the results presented here generalise to cases involving more variables.

We start with a standard result. See [3].

Lemma 6.2.1. *Let* $a \in SG^{m_x,m_\xi}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}\right)$ *and let the function* $h = (h_1, h_2)$ *be such that* $h_1 \in SG^{1,0}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_x}\right)$ *with* $\langle h_1 \rangle \sim \langle x \rangle$ *and* $h_2 \in SG^{0,1}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_\xi}\right)$ *with* $\langle h_2 \rangle \sim \langle \xi \rangle$. *Then the composition* $a \circ h \in SG^{m_x,m_\xi}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}\right)$.

Proof. By induction. □

Lemma 6.2.2. *Let* $V \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ *be open and let* $F(x,\xi) = (F_1(x,\xi), F_2(x,\xi))$ *belong to* $C^\infty\left(V, \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}\right)$ *with* $F_1 \in C^\infty\left(V, \mathbb{R}^{n_x}\right)$ *and*

$F_2 \in C^\infty\left(V, \mathbb{R}^{n_\xi}\right)$. *Assume that the following statements hold on* $W \subset V$:

1. *For* $i = 1, \ldots n_x$, $(F_1)_i$, *the ith component of* F_1, *satisfies* $SG^{1,0}_{x,\xi}$ *estimates,*

2. *For* $i = 1, \ldots n_\xi$, $(F_2)_i$, *the ith component of* F_2, *satisfies* $SG^{0,1}_{x,\xi}$ *estimates.*

Let $JM(F)$ *denote the Jacobian matrix of* F *and let* $Adj\,(JM(F))$ *denote its adjugate matrix. Then,*

$$JM(F) = \begin{pmatrix} A(x,\xi) & B(x,\xi) \\ C(x,\xi) & D(x,\xi), \end{pmatrix}, Adj\,(JM(F)) = \begin{pmatrix} \tilde{A}(x,\xi) & \tilde{B}(x,\xi) \\ \tilde{C}(x,\xi) & \tilde{D}(x,\xi), \end{pmatrix},$$

where

1. $A(x,\xi)$ *and* $\tilde{A}(x,\xi)$ *are* $n_x \times n_x$ *matrices of functions in* $C^\infty(V, \mathbb{R})$ *satisfying* $SG^{0,0}_{x,\xi}$ *estimates on* W,

2. *$B(x,\xi)$ and $\tilde{B}(x,\xi)$ are $n_x \times n_\xi$ matrices of functions in $C^\infty(V,\mathbb{R})$ satisfying $SG^{1,-1}_{x,\xi}$ estimates on W,*

3. *$C(x,\xi)$ and $\tilde{C}(x,\xi)$ are $n_\xi \times n_x$ matrices of functions in $C^\infty(V,\mathbb{R})$ satisfying $SG^{-1,1}_{x,\xi}$ estimates on W,*

4. *$D(x,\xi)$ and $\tilde{D}(x,\xi)$ are $n_\xi \times n_\xi$ matrices of functions in $C^\infty(V,\mathbb{R})$ satisfying $SG^{0,0}_{x,\xi}$ estimates on W.*

Proof. By definition, for the Jacobian matrix, $A_{i,j} = \partial_{x_j}(F_1)_i$, $B_{i,j} = \partial_{\xi_j}(F_1)_i$, $C_{i,j} = \partial_{x_j}(F_2)_i$, and $D_{i,j} = \partial_{\xi_j}(F_2)_i$. So, by assumptions 1 and 2, the Jacobian matrix has the stated form. Now consider $Adj\,(JM\,(F))$. (We will shorten $Adj\,(JM\,(F))$ to Adj.) By definition, the i,j entry of Adj is

$$Adj_{i,j} = (-1)^{i+j}\det M^{i,j}, \tag{6.1}$$

where $M^{i,j}$ is the matrix obtained by deleting row j and column i from $JM\,(F)$. We will show that each entry of $\tilde{B}$ satisfies $SG^{1,-1}_{x,\xi}$ estimates on W. That $\tilde{A}$, $\tilde{C}$ and $\tilde{D}$ have the stated properties follows in exactly the same way. By (6.1), the r,s entry of $\tilde{B}$ is

$$\tilde{B}_{r,s} = (-1)^{r+s+n_x}\det M^{r,n_x+s}.$$

It then follows from the structure of the Jacobian matrix $JM\,(F)$ that

$$M^{r,n_x+s} = \begin{pmatrix} E & F \\ G & H, \end{pmatrix}$$

where

1. $E(x, \xi)$ is an $n_x \times (n_x - 1)$ matrix of functions in $C^\infty(V, \mathbb{R})$ satisfying $SG^{0,0}_{x,\xi}$ estimates on W,

2. $F(x, \xi)$ is an $n_x \times n_\xi$ matrix of functions $C^\infty(V, \mathbb{R})$ satisfying $SG^{1,-1}_{x,\xi}$ estimates on W,

3. $G(x, \xi)$ is an $(n_\xi - 1) \times (n_x - 1)$ matrix of functions $C^\infty(V, \mathbb{R})$ satisfying $SG^{-1,1}_{x,\xi}$ estimates on W,

4. $H(x, \xi)$ is an $(n_\xi - 1) \times n_\xi$ matrix of functions $C^\infty(V, \mathbb{R})$ satisfying $SG^{0,0}_{x,\xi}$ estimates on W.

We are required to show that $\det M^{r,n_x+s}$ satisfies $SG^{1,-1}_{x,\xi}$ estimates on W. To do this, we will multiply rows and columns by SG functions to reduce M^{r,n_x+s} to a matrix of functions satisfying $SG^{0,0}_{x,\xi}$ estimates on W. To do so, we carry out the following steps:

1. Multiply the first n_x rows by $\frac{1}{\langle x \rangle}$,

2. Multiply the last $n_\xi - 1$ rows by $\frac{1}{\langle \xi \rangle}$,

3. Multiply the first $n_x - 1$ columns by $\langle x \rangle$,

4. Multiply the last n_ξ columns by $\langle \xi \rangle$.

After cancellation, we have $\frac{\langle\xi\rangle}{\langle x\rangle}\det M^{r,n_x+s}$ is equal to the determinant of a matrix of functions satisfying $SG^{0,0}_{x,\xi}$ estimates on W. So,

$$\frac{\langle\xi\rangle}{\langle x\rangle}\det M^{r,n_x+s} \text{ satisfies } SG^{0,0}_{x,\xi} \text{ estimates on W.}$$

Since $\frac{\langle\xi\rangle}{\langle x\rangle} \in ESG^{-1,1}_{x,\xi}\left(\mathbb{R}^n \times \mathbb{R}^n\right)$, it follows that

$$\det M^{r,n_x+s} \text{ satisfies } SG^{1,-1}_{x,\xi} \text{ estimates on W.}$$

As $\tilde{B}_{r,s} = (-1)^{r+s+n_x}\det M^{r,n_x+s}$ we're done. □

Proposition 6.2.3. *Let $F(x,\xi) = (F_1(x,\xi), F_2(x,\xi))$ be a C^∞ diffeomorphism from $V \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ onto its range with $F_1 \in C^\infty\left(V, \mathbb{R}^{n_x}\right)$ and $F_2 \in C^\infty\left(V, \mathbb{R}^{n_\xi}\right)$. The variables in V are denoted by x, ξ and those in $F(V)$ are denoted by $\tilde{x}, \tilde{\xi}$. We also write*

$$F^{-1}(\tilde{x},\tilde{\xi}) = \left(\left(F^{-1}\right)_1(\tilde{x},\tilde{\xi}), \left(F^{-1}\right)_2(\tilde{x},\tilde{\xi})\right).$$

We assume that

1. *$\langle F_1\rangle \sim \langle x\rangle$ and F_1 satisfies $SG^{1,0}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_x}\right)$ estimates on $W \subset V$,*

2. *$\langle F_2\rangle \sim \langle\xi\rangle$ and F_2 satisfies $SG^{0,1}_{x,\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_\xi}\right)$ estimates on $W \subset V$,*

3. *the Jacobian of F satisfies $ESG^{0,0}_{x,\xi}$ estimates on $W \subset V$.*

Then, we have

1. $\langle (F^{-1})_1 \rangle \sim \langle \tilde{x} \rangle$ *and* $(F^{-1})_1$ *satisfies* $SG^{1,0}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_x})$ *estimates on* $F(W)$,

2. $\langle (F^{-1})_2 \rangle \sim \langle \tilde{\xi} \rangle$ *and* $(F^{-1})_2$ *satisfies* $SG^{0,1}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_\xi})$ *estimates on* $F(W)$,

3. *The Jacobian of* $F^{-1} \succ 1$ *and satisfies* $SG^{0,0}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R})$ *estimates on* $F(W)$.

Proof. As F is a diffeomorphism from V onto $F(V)$, we have the following equality for all $\tilde{x}, \tilde{\xi} \in F(V)$:

$$\begin{pmatrix} \tilde{x} \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} F_1\left((F^{-1})_1(\tilde{x},\tilde{\xi}), (F^{-1})_2(\tilde{x},\tilde{\xi})\right) \\ F_2\left((F^{-1})_1(\tilde{x},\tilde{\xi}), (F^{-1})_2(\tilde{x},\tilde{\xi})\right). \end{pmatrix} \tag{6.2}$$

As $F_1(x,\xi) \sim \langle x \rangle$ and $F_2(x,\xi) \sim \langle \xi \rangle$ on W, it follows from (6.2) that

$$\langle (F^{-1})_1 \rangle \sim \langle \tilde{x} \rangle \text{ on } F(W)$$

$$\langle (F^{-1})_2 \rangle \sim \langle \tilde{\xi} \rangle \text{ on } F(W). \tag{6.3}$$

All functions are smooth so we can differentiate (6.2). Define $\tilde{w} := (\tilde{x}, \tilde{\xi})$. Differentiating (6.2) with respect to $\tilde{w}_i$ gives

$$e_i = JM(F)(x,\xi)\mid_{x=(F^{-1})_1(\tilde{x},\tilde{\xi}),\xi=(F^{-1})_2(\tilde{x},\tilde{\xi})} \partial_{\tilde{w}_i}\left(F^{-1}\right), \tag{6.4}$$

where $JM(F)$ is the Jacobian matrix of F and e_i is the $n_x + n_\xi$ dimensional vector with ith entry equal to 1 and zero otherwise. So,

$$\partial_{\tilde{w}_i}\left(F^{-1}\right) = (JM(F))^{-1}\mid_{x=(F^{-1})_1(\tilde{x},\tilde{\xi}),\xi=(F^{-1})_2(\tilde{x},\tilde{\xi})} e_i \tag{6.5}$$

By assumption 3, $\frac{1}{\det JM(F)(x,\xi)}$ satisfies $SG^{0,0}_{x,\xi}$ estimates on W. So, using Lemma 6.2.2, we have that for $(x,\xi) \in W$

$$(JM(F))^{-1}(x,\xi) = \begin{pmatrix} A(x,\xi) & B(x,\xi) \\ C(x,\xi) & D(x,\xi), \end{pmatrix}$$

where

1. $A(x,\xi)$ is an $n_x \times n_x$ matrix of functions in $C^\infty(V,\mathbb{R})$ satisfying $SG^{0,0}_{x,\xi}$ estimates on W,

2. $B(x,\xi)$ is an $n_x \times n_\xi$ matrix of functions $C^\infty(V,\mathbb{R})$ satisfying $SG^{1,-1}_{x,\xi}$ estimates on W,

3. $C(x,\xi)$ is an $n_\xi \times n_x$ matrix of functions $C^\infty(V,\mathbb{R})$ satisfying $SG^{-1,1}_{x,\xi}$ estimates on W,

4. $D(x,\xi)$ is an $n_\xi \times n_\xi$ matrix of functions $C^\infty(V,\mathbb{R})$ satisfying $SG^{0,0}_{x,\xi}$ estimates on W.

By (6.4) and the structure of the inverse, we have

$$\begin{aligned}
\partial_{\tilde{x}_j}\left(F_1^{-1}\right)_i &= A_{i,j}(F_1^{-1}(\tilde{x},\tilde{\xi}),F_2^{-1}(\tilde{x},\tilde{\xi})),\\
\partial_{\tilde{x}_j}\left(F_2^{-1}\right)_i &= C_{i,j}(F_1^{-1}(\tilde{x},\tilde{\xi}),F_2^{-1}(\tilde{x},\tilde{\xi})),\\
\partial_{\tilde{\xi}_j}\left(F_1^{-1}\right)_i &= B_{i,j}(F_1^{-1}(\tilde{x},\tilde{\xi}),F_2^{-1}(\tilde{x},\tilde{\xi})),\\
\partial_{\tilde{\xi}_j}\left(F_2^{-1}\right)_i &= D_{i,j}(F_1^{-1}(\tilde{x},\tilde{\xi}),F_2^{-1}(\tilde{x},\tilde{\xi})),
\end{aligned} \tag{6.6}$$

where $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j}$ satisfy $SG^{0,0}_{x,\xi}, SG^{1,-1}_{x,\xi}, SG^{-1,1}_{x,\xi}$ and $SG^{0,0}_{x,\xi}$ estimates respectively on W

Recall that

$$\begin{aligned} \langle (F^{-1})_1 (\tilde{x}, \tilde{\xi})\rangle &\sim \langle \tilde{x}\rangle \text{ on } F(W), \\ \langle (F^{-1})_2 (\tilde{x}, \tilde{\xi})\rangle &\sim \langle \tilde{\xi}\rangle \text{ on } F(W). \end{aligned} \tag{6.7}$$

It follows from (6.6) and (6.7) by induction that $(F^{-1})_1$ and $(F^{-1})_2$ satisfy $SG^{1,0}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_x})$ and $SG^{0,1}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_\xi})$ estimates respectively on $F(W)$.

Part 3 follows from parts 1 and 2 using similar arguments to those used in the proof of Lemma 6.2.2. (We multiply rows and columns by $\langle x\rangle, \langle\xi\rangle, \langle x\rangle^{-1}\langle\xi\rangle^{-1}$ to reduce to a matrix of functions satisfying $SG^{0,0}$ estimates and everything we multiply by cancels.) We also use the fact that the jacobian of $F \sim 1$ on W.

□

Proposition 6.2.4. *Let V be an open subset of $\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$ and let $a(x,\xi) \in C^\infty(V)$. Suppose that $a(x,\xi)$ satisfies $SG^{m_x,m_\xi}_{x,\xi}$ estimates on $W \subset V \subset \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$. Let $G(x,\xi) := (G_1(x,\xi), G_2(x,\xi))$ be a C^∞ diffeomorphism from V to $G(V)$, with $G_1(x,\xi) \in C^\infty(V, \mathbb{R}^{n_x})$ and $G_2(x,\xi) \in C^\infty(V, \mathbb{R}^{n_\xi})$. Assume that*

1. *$G_1(x,\xi)$ satisfies $SG^{1,0}_{x,\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_x})$ estimates on W,*
2. *$\langle G_1(x,\xi)\rangle \sim \langle x\rangle$ on W,*
3. *$G_2(x,\xi)$ satisfies $SG^{0,1}_{x,\xi}\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}, \mathbb{R}^{n_\xi}$ estimates on W,*
4. *$\langle G_2(x,\xi)\rangle \sim \langle \xi\rangle$ on W,*
5. *The Jacobian of G satisfies $ESG^{0,0}_{x,\xi}$ estimates on W.*

Then the function $a((G^{-1})_1(\tilde{x},\tilde{\xi}), (G^{-1})_2(\tilde{x},\tilde{\xi}),)$ belongs to $C^\infty(G(V))$ and satisfies $SG^{m_x,m_\xi}_{\tilde{x},\tilde{\xi}}$ estimates on $G(W)$.

Proof. The fact that $a((G^{-1})_1(\tilde{x},\tilde{\xi}), (G^{-1})_2(\tilde{x},\tilde{\xi}),)$ belongs to $C^\infty(G(V))$ is obvious. By Proposition 6.2.3, the assumptions 1,2,3,4 and 5 imply that

1. $\langle (G^{-1})_1(\tilde{x},\tilde{\xi})\rangle \sim \langle \tilde{x}\rangle$ on $G(W)$,
2. $\langle (G^{-1})_2(\tilde{x},\tilde{\xi})\rangle \sim \langle \tilde{\xi}\rangle$ on $G(W)$,
3. $(G^{-1})_1$ satisfies $SG^{1,0}_{\tilde{x},\tilde{\xi}}$ estimates on $G(W)$,
4. $(G^{-1})_2$ satisfies $SG^{0,1}_{\tilde{x},\tilde{\xi}}$ estimates on $G(W)$.

Therefore $a((G^{-1})_1(\tilde{x},\tilde{\xi}),(G^{-1})_2(\tilde{x},\tilde{\xi}))$ satisfies SG^{m_x,m_ξ} estimates on $G(W)$. (by an obvious variant of Lemma 6.2.1.) □

The next result is important when considering changes of variables in integrals.

Proposition 6.2.5. *Let $a(x,\xi) \in SG^{m_x,m_\xi}(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi})$ and suppose that Supp$(a) \subset V$, where V is an open subset of $\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}$. Let $G = (G_1(x,\xi), G_2(x,\xi))$ be a C^∞ diffeomorphism from V onto its range $G(V)$. (with $G_1(x,\xi) \in C^\infty(V,\mathbb{R}^{n_x})$ and $G_2(x,\xi) \in C^\infty(V,\mathbb{R}^{n_\xi})$) Assume that*

1. *$G_1(x,\xi)$ satisfies $SG^{1,0}_{x,\xi}$ estimates on Supp(a),*
2. *$\langle G_1(x,\xi)\rangle \sim \langle x\rangle$ on Supp(a),*
3. *$G_2(x,\xi)$ satisfies $SG^{0,1}_{x,\xi}$ estimates on Supp(a),*
4. *$\langle G_2(x,\xi)\rangle \sim \langle \xi\rangle$ on Supp(a),*
5. *The Jacobian of G satisfies $ESG^{0,0}_{x,\xi}$ estimates on Supp(a).*

Then the function

$$a\left(\left(G^{-1}\right)_1(\tilde{x},\tilde{\xi}),\left(G^{-1}\right)_2(\tilde{x},\tilde{\xi})\right)|\det \partial_{(\tilde{x},\tilde{\xi})}G^{-1}|,$$

(where $\det \partial_{(\tilde{x},\tilde{\xi})}G^{-1}$ is the Jacobian of G^{-1}) satisfies $SG^{m_x,m_\xi}_{\tilde{x},\tilde{\xi}}$ estimates on $G(V)$. Further, the extended function

$$a_E(\tilde{x},\tilde{\xi}) = \begin{cases} a\left((G^{-1})_1(\tilde{x},\tilde{\xi}),(G^{-1})_2(\tilde{x},\tilde{\xi})\right), & \tilde{x},\tilde{\xi} \in G(V) \\ 0, & \textit{otherwise} \end{cases}$$

belongs to $SG^{m_x,m_\xi}_{\tilde{x},\tilde{\xi}}\left(\mathbb{R}^{n_x}\times\mathbb{R}^{n_\xi}\right)$.

Proof. The fact that $a\left((G^{-1})_1(\tilde{x},\tilde{\xi}),(G^{-1})_2(\tilde{x},\tilde{\xi})\right)$ is smooth on $G(V)$ is obvious. The SG estimates of $a\left((G^{-1})_1(\tilde{x},\tilde{\xi}),(G^{-1})_2(\tilde{x},\tilde{\xi})\right)$ on $G(Supp(a))$ follow from Proposition 6.2.4. The Jacobian of G^{-1} is smooth on $G(V)$ and satisfies $ESG^{0,0}_{\tilde{x},\tilde{\xi}}$ estimates on $G(Supp(a))$ by Proposition 6.2.3. So,

$$a\left(\left(G^{-1}\right)_1(\tilde{x},\tilde{\xi}),\left(G^{-1}\right)_2(\tilde{x},\tilde{\xi})\right)|\det\partial_{(\tilde{x},\tilde{\xi})}G^{-1}|,$$

satisfies $SG^{m_x,m_\xi}_{\tilde{x},\tilde{\xi}}$ estimates on $G(Supp(a))$.

The only real issue is the extension. If $G(V)=\mathbb{R}^{n_x}\times\mathbb{R}^{n_\xi}$, then there is nothing to check - the extended function is just the function. So we assume that $G(V)\neq\mathbb{R}^{n_x}\times\mathbb{R}^{n_\xi}$.

We claim that $G(Supp(a))$ is closed in $\mathbb{R}^{n_x+n_\xi}$. If we can show this then it's clear that the extension a_E is smooth on $\mathbb{R}^{n_x+n_\xi}$.

We assume that $G(Supp(a))$ is not closed in $\mathbb{R}^{n_x+n_\xi}$. This means that there exists a point $(x_0,\xi_0)\in\mathbb{R}^{n_x+n_\xi}\backslash G(Supp(a))$ and a sequence $\{(x_j,\xi_j)\}$ of points belonging to $G(Supp(a))$ which tends to (x_0,ξ_0) in $\mathbb{R}^{n_x+n_\xi}$. We will use the properties of G to show that this sequence has a subsequence tending to a different limit.

Since $\{(x_j,\xi_j)\}$ is a convergent sequence of points in $G(Supp(a))$ then the sequence $\{(x_j,\xi_j)\}$ is contained in a bounded subset of $G(Supp(a))$. It follows from assumptions 1,2,3 and 4 that $\langle G_1^{-1}(\tilde{x},\tilde{\xi})\rangle\sim\langle\tilde{x}\rangle$ and $\langle G_2^{-1}(\tilde{x},\tilde{\xi})\rangle\sim\langle\tilde{\xi}\rangle$. It follows that the pre-image of $\{(x_j, \xi_j)\}$ is contained in a bounded subset of $Supp(a)$. So the sequence $\{G^{-1}(x_j, \xi_j)\}$ is contained in a compact set and therefore it has a convergent subsequence $\{G^{-1}(x_{n_j}, \xi_{n_j})\}$, tending to a limit $(x, \xi) \in Supp(a)$, by the Bolzano - Weierstrass Theorem. Given that G is continuous, the image of this convergent subsequence under the map G, i.e. $\{(x_{n_j}, \xi_{n_j})\}$ tends to $G(x, \xi) \in G(Supp(a))$. Since the limit of $\{(x_j, \xi_j)\}$ belongs to $\mathbb{R}^{n_x+n_\xi} \backslash G(Supp(a))$ we have shown that $\{(x_j, \xi_j)\}$ has a subsequence tending to a different limit.

We have shown that the function a_E is smooth everywhere and satisfies SG^{m_x,m_ξ} estimates on its support. So, $a_E \in SG^{m_x,m_\xi}\left(\mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi}\right)$.

□

Chapter 7

Type $\mathcal{Q}$ Fourier Integral Operator

We now introduce a new class of SG FIO. The basic structure of the operator is the same as for the Type $\mathcal{P}$ Operator but we place more restrictions on the phase $\Phi(x, y, \xi)$.

7.1 Definition

Definition 7.1.1. *We denote by $\mathcal{Q}$, the set of real-valued smooth functions $\Phi(x,y,\xi) = f(x,\xi) + g(y,\xi)$ where f and g have the following properties;*

$$\begin{aligned}
f(x,\xi) &\in SG^{1,1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), && (7.1)\\
g(y,\xi) &\in SG^{1,1}_{y,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), && (7.2)\\
\langle \nabla_x f(x,\xi)\rangle &\sim \langle \xi \rangle, && (7.3)\\
\langle \nabla_y g(y,\xi)\rangle &\sim \langle \xi \rangle, && (7.4)\\
\langle \nabla_\xi f(x,\xi)\rangle &\sim \langle x \rangle, && (7.5)\\
\langle \nabla_\xi g(y,\xi)\rangle &\sim \langle y \rangle, && (7.6)\\
\left|\det\left(\partial_{x_i}\partial_{\xi_j} f\right)_{i,j=1}^{n}(x,\xi)\right| &\succ 1, && (7.7)\\
\left|\det\left(\partial_{y_i}\partial_{\xi_j} g\right)_{i,j=1}^{n}(y,\xi)\right| &\succ 1. && (7.8)
\end{aligned}$$

Remark We will call a real valued function f satisfying (7.1) , (7.3) , (7.5) and (7.7) a "phase component."

Definition 7.1.2. *Let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for arbitrary $m_1, m_2, m_3 \in \mathbb{R}$ and let $f(x,\xi) + g(y,\xi) \in \mathcal{Q}$. The Type $\mathcal{Q}$ FIO A is defined for $u \in \mathcal{S}(\mathbb{R}^n)$ by the following integral;*

$$Au(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\{f(x,\xi)+g(y,\xi)\}} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $\gamma(\epsilon\xi)$ is a mollifier.

We collect some properties of Type $\mathcal{Q}$ operators in a Theorem.

Theorem 7.1.3. *Let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for arbitrary $m_1, m_2, m_3 \in \mathbb{R}$, let $f(x,\xi) + g(y,\xi) \in \mathcal{Q}$ and define $A := FIO\left(f(x,\xi) + g(y,\xi), a(x,y,\xi)\right)$. Then:*

1. *$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ continuously,*

2. *A is independent of the choice of mollifier,*

3. *$A^T = FIO\left(g(x,\xi) + f(y,\xi), a(y,x,\xi)\right)$,*

4. *$A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ continuously,*

Proof. We have $\mathcal{Q} \subset \mathcal{P}$ and all the statements hold for arbitrary Type $\mathcal{P}$ operators. □

Remark. The Type 1 FIO introduced by Coriasco in [3] is the subclass of the Type $\mathcal{Q}$ FIO where we always take $g(y,\xi) = -y \cdot \xi$ and we take the amplitude a to be independent of y. His Type 2 operator corresponds to the subclass with $f = x \cdot \xi$.

Let $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Recall that for functions $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle u, v \rangle := \int_{\mathbb{R}^n} u(x)v(x)\mathrm{d}x$$

The adjoint of A, denoted by $A^\star$, is defined as the unique operator $A^\star : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ such that

$$\langle Au, \bar{v} \rangle = \langle u, \overline{A^\star v} \rangle \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n).$$

In the above, the overbar denotes the complex conjugate.

Theorem 7.1.4. *Let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for arbitrary $m_1, m_2, m_3 \in \mathbb{R}$, let $f(x,\xi)+g(y,\xi) \in \mathcal{Q}$ and define $A := FIO\,(f(x,\xi)+g(y,\xi), a(x,y,\xi))$. Then*

$$A^\star = FIO\left(-g(x,\xi) - f(y,\xi), \overline{a(y,x,\xi)}\right).$$

Proof. Straightforward recalling the definition of the adjoint operator and using the symmetry of the phase assumptions in the spatial variables x and y as well as the fact that $\Phi \in \mathcal{Q}$ implies that $-\Phi \in \mathcal{Q}$. □

We conclude this section by presenting the reduced form of a Type $\mathcal{Q}$ operator.

Theorem 7.1.5. *Let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ for arbitrary $m_1, m_2, m_3 \in \mathbb{R}$, let $f(x,\xi)+g(y,\xi) \in \mathcal{Q}$ and define $A := FIO\,(f(x,\xi)+g(y,\xi), a(x,y,\xi))$. Let $\chi \in \Xi^\Delta(k)$ with k arbitrary. Then if we define*

$$A_{Red} = FIO(f(x,\xi)+g(y,\xi), a(x,y,\xi)\chi(\nabla_\xi f, \nabla_\xi g))\chi(\nabla_\xi g, \nabla_\xi f)),$$

we have $A = A_{Red}$ modulo $\mathcal{K}$.

Proof. Use Theorem 3.2.2. □

7.2 Composition Results

We now present the main result of this chapter.

Theorem 7.2.1. *Let* $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *for arbitrary* $m_1, m_2, m_3 \in \mathbb{R}$*and let* $f(x,\xi) + g(y,\xi) \in \mathcal{Q}$*. Similarly, let* $b(x,y,\xi) \in SG^{t_1,t_2,t_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *for arbitrary* $t_1, t_2, t_3 \in \mathbb{R}$ *and let* $r(x,\xi) + s(y,\xi) \in \mathcal{Q}$*. Define:*

$$A = FIO(f(x,\xi) + g(y,\xi), a(x,y,\xi)),$$

$$B = FIO(r(x,\xi) + s(y,\xi), b(x,y,\xi)).$$

Then:

1. *If* $g(y,\xi) = -r(y,\xi) \quad \forall y, \xi$*, we have* $A \circ B = FIO(f(x,\xi) + s(y,\xi), a(x,y,\xi))$ *modulo* $\mathcal{K}$ *with* $c(x,y,\xi) \in SG^{p,q,m_3+t_3}_{x,y,\xi}$ *for any* $p, q \in \mathbb{R}$ *such that* $p + q = m_1 + m_2 + t_1 + t_2$*. We also obtain an asymptotic expansion for* $c(x,y,\xi)$*.*

2. *If* $g(y,\xi) = -r(y,\xi) \quad \forall y, \xi$ *and* $f(x,\xi) = -s(x,\xi) \quad \forall x, \xi$*, we have* $A \circ B = Op(\tilde{c}(x,y,\xi))$ *modulo* $\mathcal{K}$ *with* $\tilde{c}(x,y,\xi) \in SG^{p,q,m_3+t_3}_{x,y,\xi}$ *for any* $p, q \in \mathbb{R}$ *such that* $p + q = m_1 + m_2 + t_1 + t_2$*. That is, the composition* $A \circ B$ *is a* ψdo*. Again we obtain an asymptotic expansion for* $\tilde{c}(x,y,\xi)$*.*

In the proof of Theorem 7.2.1 will use results from Chapters 3,4 and 6 as well as some lemmas which we now present. The following result is proved in Coriasco [3]. We include the proof for completeness. Note that all the changes of variable in this chapter are of the type used by Coriasco in [3].

Lemma 7.2.2. *Let* $f(x,\xi) \in SG^{1,1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ *be such that* $\langle \nabla_x f(x,\xi) \rangle \sim \langle \xi \rangle$ *and* $\det \left(\partial_{x_i} \partial_{\xi_j} f \right)^n_{i,j=1} (x,\xi) \succ 1$. *Let* $W_k := \{(x,y) \in \mathbb{R}^{2n} : |y-x| < k\langle x \rangle \text{ with } k < 1\}$ *Suppose that we define a real-valued function* $h = (h_1(x,y,\xi), \ldots, h_n(x,y,\xi))$ *on* $W_k \times \mathbb{R}^n$ *as follows:*

$$h_i(x,y,\xi) := \int_0^1 \partial_{1_i} f(x + t(y-x), \xi) \mathrm{d}t$$

Then we can choose a constant $k < 1$ *to be small enough to ensure that*

1. *Each component* $h_i(x,y,\xi)$ *satisfies* $SG^{0,0,1}_{x,y,\xi}$ *estimates on* $W_k \times \mathbb{R}^n$,
2. $\det \left(\partial_{\xi_j} h_i \right)^n_{i,j=1} (x,y,\xi) \succ 1$ *on* $W_k \times \mathbb{R}^n$ *and*
3. $\langle h(x,y,\xi) \rangle \sim \langle \xi \rangle$ *on* $W_k \times \mathbb{R}^n$.

Proof. **Condition 1.** By definition $h_i(x,y,\xi) := \int_0^1 \partial_{1_i} f(x+t(y-x),\xi)\mathrm{d}t$. Clearly we can differentiate under the integral sign so that

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma h_i(x,y,\xi) = \int_0^1 \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \left[\partial_{1_i} f(x + t(y-x), \xi) \right] \mathrm{d}t.$$

This is just

$$\int_0^1 (1-t)^{|\alpha|} t^{|\beta|} \partial_1^{\alpha+\beta+e_j} \partial_2^\gamma f(x + t(y-x), \xi) \mathrm{d}t.$$

As usual, ∂_1 means "derivative in the first variable, counting from the left." Now, $k < 1$ means that for $(x,y) \in W_k$ we have $\langle x \rangle \sim \langle y \rangle \sim \langle x + t(y-x) \rangle$ for $t \in [0,1]$. Since $f(x,\xi) \in SG^{1,1}_{x,\xi}$ we have

$$\int_0^1 (1-t)^{|\alpha|} t^{|\beta|} \partial_1^{\alpha+\beta+e_j} \partial_2^\gamma f(x + t(y-x), \xi) \mathrm{d}t. \prec \langle x \rangle^{-|\alpha|} \langle y \rangle^{|\beta|} \langle \xi \rangle^{1-|\gamma|}.$$

on $W_k \times \mathbb{R}^n$. So Condition 1 is satisfied.

Condition 2 Note that we can write

$$\partial_{1_i} f(x,\xi) = \int_0^1 \partial_{1_i} f(x,\xi) \mathrm{d}t_1.$$

So,

$$h_i(x,y,\xi) - \partial_{1_i} f(x,\xi) = \int_0^1 \left[\partial_{1_i} f(x + t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\right] \mathrm{d}t_1. \tag{7.9}$$

By the Mean Value Theorem, the right hand side of the above equation is

$$\int_0^1 \int_0^1 \sum_{r=1}^n \partial_{1_r} \partial_{1_i} f(x + t_1 t_2(y-x),\xi) t_1(y_r - x_r) \mathrm{d}t_2 \mathrm{d}t_1.$$

So, (7.9) becomes

$$h_i(x,y,\xi) - \partial_{1_i} f(x,\xi) = \int_0^1 \int_0^1 \sum_{r=1}^n \partial_{1_r} \partial_{1_i} f(x + t_1 t_2(y-x),\xi) t_1(y_r - x_r) \mathrm{d}t_2 \mathrm{d}t_1.$$

Differentiating this expression with respect to ξ_j and rearranging gives

$$\partial_{\xi_j} h_i(x,y,\xi) = \partial_{\xi_j} \partial_{1_i} f(x,\xi) - \int_0^1 \int_0^1 \sum_{r=1}^n \partial_{1_r} \partial_{1_i} \partial_{\xi_j} f(x + t_1 t_2(y-x),\xi) t_1(y_r - x_r) \mathrm{d}t_2 \mathrm{d}t_1.$$

Consider

$$\partial_{1_r} \partial_{1_i} \partial_{\xi_j} f(x + t_1 t_2(y-x),\xi) t_1(y_r - x_r) \mathrm{d}t_2 \mathrm{d}t_1$$

Recall that $f(x,\xi) \in SG^{1,1}_{x,\xi}$. Also, for $(x,y) \in W_k$ we have $\langle x + t_1 t_2(y-x)\rangle \sim \langle x \rangle \sim \langle y \rangle$ with implicit constants independent of $t_1, t_2 \in [0,1]$. We also have $\langle x + t_1 t_2(y-x)\rangle \geq (1-k)\langle x \rangle$ for $(x,y) \in W_k$ and $t_1, t_2 \in [0,1]$. Therefore

$$\int_0^1 \int_0^1 \partial_{1_r}\partial_{1_i}\partial_{\xi_j} f(x + t_1t_2(y-x), \xi)t_1(y_r - x_r)\mathrm{d}t_2\mathrm{d}t_1 \prec \frac{k}{1-k}.$$

So, we have shown that $\partial_{\xi_j}h_i(x,y,\xi) = \partial_{\xi_j}\partial_{1_i}f(x,\xi) + M_{i,j}$ where each entry of the matrix $M_{i,j} \prec \frac{k}{1-k}$. It follows that we have

$$\det\left(\partial_{\xi_j}h_i(x,y,\xi)\right)_{i,j=1}^n = \det\left(\partial_{\xi_j}\partial_{1_i}f(x,\xi)\right)_{i,j=1}^n + \sum_l g_l(x,y,\xi) \tag{7.10}$$

where the sum is finite and each function $g_l(x,y,\xi)$ is a product of elements of $\left(\partial_{\xi_j}\partial_{1_i}f(x,\xi)\right)_{i,j=1}^n$ and elements of M with at least one of the terms in the product coming from M. Since each entry of $\partial_{\xi_j}\partial_{1_i}f(x,\xi) \prec 1$ for all i,j and $M_{i,j} \prec \frac{k}{1-k}$ for all i,j, we have

$$g_l(x,y,\xi) \prec \frac{k}{1-k}$$

on $W_k \times \mathbb{R}^n$, for all l in the sum. Since the sum is finite we then have

$$\sum_l g_l(x,y,\xi) \prec \frac{k}{1-k}, \text{ on } W_k \times \mathbb{R}^n. \tag{7.11}$$

Recall that $\det\left(\partial_{\xi_j}\partial_{1_i}f(x,\xi)\right)_{i,j=1}^n \succ 1$ on $W_k \times \mathbb{R}^n$. So, by (7.10) and (7.11) we can choose the constant k small enough so that $\det\left(\partial_{\xi_j}h_i(x,y,\xi)\right)_{i,j=1}^n \succ 1$ on $W_k \times \mathbb{R}^n$.

Condition 3. Showing that $\langle h_i(x,y,\xi)\rangle \prec \langle\xi\rangle$ is straightforward. For the lower bound note that by (7.9), we have

$$h_i(x,y,\xi) - \partial_{1_i}f(x,\xi) = \int_0^1 \left[\partial_{1_i}f(x+t_1(y-x),\xi) - \partial_{1_i}f(x,\xi)\right]\mathrm{d}t_1.$$

So,

$$\langle h(x,y,\xi)\rangle^2 = \langle \nabla_1 f(x,\xi)\rangle^2 + 2\sum_{i=1}^{n} \partial_{1_i} f(x,\xi) \int_0^1 \partial_{1_i} f(x+t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\mathrm{d}t_1 + \\ + \sum_{i=1}^{n} \left[\int_0^1 \partial_{1_i} f(x+t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\mathrm{d}t_1\right]^2. \tag{7.12}$$

As $\langle \nabla_1 f(x,\xi)\rangle^2 \succ \langle\xi\rangle^2$ and $\partial_{1_i} f(x,\xi) \prec \langle\xi\rangle$ for all i, it is clear from (7.12) that the lower bound will be established if we can show that $\int_0^1 \partial_{1_i} f(x+t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\mathrm{d}t_1 \prec r(k)\langle\xi\rangle$ on $W_k \times \mathbb{R}^n$, for some real valued function $r(k)$ with $r(k) \to 0$ as $k \to 0$. By the Mean Value Theorem

$$\int_0^1 \partial_{1_i} f(x+t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\mathrm{d}t_1 = \\ \int_0^1\int_0^1 \sum_{r=1}^{n} \partial_{1_k}\partial_{1_i} f(x+t_1t_2(y-x),\xi)t_1(y_r-x_r)\mathrm{d}t_2\mathrm{d}t_1. \tag{7.13}$$

Recall that $f(x,\xi) \in SG^{1,1}_{x,\xi}$. Also, for $(x,y) \in W$ we have $\langle x+t_1t_2(y-x)\rangle \sim \langle x\rangle \sim \langle y\rangle$ with implicit constants independent of $t_1, t_2 \in [0,1]$. We also have $\langle x+t_1t_2(y-x)\rangle \geq (1-k)\langle x\rangle$ for $(x,y) \in W_k$ and $t_1,t_2 \in [0,1]$. Using these facts it follows from (7.13) that

$$\int_0^1 \partial_{1_i} f(x+t_1(y-x),\xi) - \partial_{1_i} f(x,\xi)\mathrm{d}t_1 \prec \frac{k}{1-k}\langle\xi\rangle.$$

on $W_k \times \mathbb{R}^n$. □

Lemma 7.2.3. *Let* $f(x,\xi) \in SG^{1,1}_{x,\xi}(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$ *be such that* $\langle\nabla_\xi f(x,\xi)\rangle \sim \langle x\rangle$ *and* $\det \partial_{x_i}\partial_{\xi_j} f(x,\xi) \succ 1$. *Then we have*

$$\nabla_\xi\left[f(x,\xi) - f(y,\xi)\right] \succ |x-y|.$$

Proof. Consider the map $F(x,\xi) := (\nabla_\xi f(x,\xi),\xi)$. Since $\langle \nabla_\xi f(x,\xi)\rangle \sim \langle x\rangle$ and $\det\left(\partial_{x_i}\partial_{\xi_j} f\right)_{i,j=1}^n (x,\xi) \succ 1$, it follows from the Global Inverse Function Theorem (Theorem 6.1.2) that $F(x,\xi)$ is a diffeomorphism from $\mathbb{R}^n\times\mathbb{R}^n$ to $\mathbb{R}^n\times\mathbb{R}^n$. We can apply Proposition 6.2.3 to see that if we write $F^{-1} = \left((F^{-1})_1, (F^{-1})_2\right)$ we have $(F^{-1})_1$ satisfies $SG^{1,0}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ estimates. We have $(F^{-1})_1 = (\nabla_\xi f)^{-1}(\tilde{x},\tilde{\xi})$ where the inverse is in the first variable with the second fixed.

So, as in Coriasco [3], set $v = \nabla_\xi f(x,\xi)$ and $w = \nabla_\xi f(y,\xi)$. So,

$$|x-y| = |\,(\nabla_\xi f)^{-1}(v,\xi) - (\nabla_\xi f)^{-1}(w,\xi)|. \tag{7.14}$$

By the mean value theorem and since $(\nabla_\xi f)^{-1} \in SG^{1,0}_{\tilde{x},\tilde{\xi}}(\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$ we have

$$|x-y| \prec |v-w|$$

Since $|v-w| = |\nabla_\xi\left[f(x,\xi)-f(y,\xi)\right]|$ we're done. □

The following result is proved in [3]:

Lemma 7.2.4. *Let $\Phi(x,y,\xi)\in\mathcal{Q}$, with $\Phi = f(x,\xi)-f(y,\xi)$. Then, for any $a(x,y,\xi)\in SG^{m_1,m_2,m_3}_{x,y,\xi}$ with m_1,m_2,m_3 arbitrary , there exists $\tilde{a}(x,y,\xi)\in SG^{m_1,m_2,m_3}$ such that*

$$FIO(f(x,\xi)-f(y,\xi),a(x,y,\xi)) = Op(\tilde{a}(x,y,\xi)) \quad \textit{modulo } \mathcal{K}.$$

Proof. By Lemma 7.2.3 and Theorem 3.2.2 we see that if $\chi\in\Xi^\Delta(k)$ with $k<1$ we have

$$FIO(f(x,\xi)-f(y,\xi),a(x,y,\xi)) = FIO(f(x,\xi)-f(y,\xi),a(x,y,\xi)\chi(x,y)\chi(y,x)).$$

modulo $\mathcal{K}$. Let $A := FIO(f(x,\xi) - f(y,\xi), a(x,y,\xi)\chi(x,y)\chi(y,x)$. By definition,

$$Au(x) = \lim_{\epsilon \to 0} \iint \exp\{i(f(x,\xi) - f(y,\xi))\} a(x,y,\xi)\chi(x,y)\chi(y,x)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$$

The support of the amplitude is contained in $W_k \times \mathbb{R}^n$ where $W_k := \{(x,y) \in \mathbb{R}^{2n} : |y-x| < 2k\langle x \rangle\}$. The integral is absolutely convergent so we can write the repeated integral as one integral over the x section of $W_k \times \mathbb{R}^n$ without changing the value of $Au(x)$. So,

$$Au(x) = \lim_{\epsilon \to 0} \int_{(W_k \times \mathbb{R}^n)_x} \exp\{i(f(x,\xi) - f(y,\xi))\} a(x,y,\xi)\chi(x,y)\chi(y,x)\gamma(\epsilon\xi)u(y)\mathrm{d}m,$$

where m is the two-fold product of Lebesgue measure on $\mathbb{R}^n$.

Let $h_i := \int_0^1 \partial_{1_i} f(y + t(x-y), \xi)\mathrm{d}t$. By Lemma 7.2.2 we can choose k to be small enough so that

1. Each component $h_i(x,y,\xi)$ satisfies $SG^{0,0,1}_{x,y,\xi}$ estimates on $W_k \times \mathbb{R}^n$,
2. $\det\left(\partial_{\xi_j} h_i\right)_{i,j=1}^n (x,y,\xi) \succ 1$ on $W_k \times \mathbb{R}^n$ and
3. $\langle h(x,y,\xi)\rangle \sim \langle \xi \rangle$ on $W_k \times \mathbb{R}^n$.

Define $F(x,y,\xi) := (x,y,h(x,y,\xi))$. For k small enough, F is a C^∞ diffeomorphism from $W_k \times \mathbb{R}^n$ to $W_k \times \mathbb{R}^n$. by the above facts and the Global Inverse Function Theorem (Theorem 6.1.2). We have $F^{-1}(x,y,\xi) = (x,y,h^{-1}(x,y,\xi))$ where $h^{-1}(x,y,\xi)$ denotes the inverse of the x,y section of h.

Define new variables $(\tilde{x}, \tilde{y}, \tilde{\xi}) := F(x, y, \xi)$. By Proposition 6.2.3 $h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})$ satisfies $SG^{0,0,1}_{\tilde{x},\tilde{y},\tilde{\xi}}$ estimates on $W_k \times \mathbb{R}^n$. We also have that the Jacobian $\det \partial_\eta h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})$ satisfies $ESG^{0,0,1}_{\tilde{x},\tilde{y},\tilde{\xi}}$ estimates on $W_k \times \mathbb{R}^n$ by the same Proposition.

Making the change of variables we have

$$Au(\tilde{x}) = \lim_{\epsilon \to 0} \int_{(W_k \times \mathbb{R}^n)_{\tilde{x}}} \exp\{i(\tilde{x} - \tilde{y}) \cdot \tilde{\xi}\} a_0(\tilde{x}, \tilde{y}, h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})) \gamma(\epsilon h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})) \times \\ \times |\det \partial_{\tilde{\xi}} h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})| u(\tilde{y}) \mathrm{d}m. \tag{7.15}$$

By Proposition 6.2.5 we can extend $a_0(x, y, h^{-1}(\tilde{x}, \tilde{y}, \tilde{\eta}))|\det \partial_{\tilde{\xi}} h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})|$ by zero outside $W_k \times \mathbb{R}^n$ to give a function in $SG^{m_1,m_2,m_3}_{x,y,\eta}$.

We also extend $\gamma(\epsilon h^{-1}(x, y, \eta))$ by zero outside $W_k \times \mathbb{R}^n$.

Re-writing the integral 7.15 as repeated integrals over $\mathbb{R}^n$ we have

$$Au(\tilde{x}) = \lim_{\epsilon \to 0} \iint \exp\{i(\tilde{x} - \tilde{y}) \cdot \tilde{\xi}\} a_0(\tilde{x}, \tilde{y}, h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})) \gamma(\epsilon h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})) \times \\ \times |\det \partial_{\tilde{\xi}} h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})| u(\tilde{y}) \mathrm{d}\tilde{y} \mathrm{d}\tilde{\xi}. \tag{7.16}$$

We can replace $\gamma(\epsilon h^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi}))$ by $\gamma(\epsilon\tilde{\xi})$ by integration by parts. This completes the proof. □

Proof of Composition Theorem 7.2.1

Proof. By definition

$$A = FIO(f(x,\xi) + g(y,\xi), a(x,y,\xi)),$$

$$B = FIO(-g(x,\xi) + s(y,\xi), b(x,y,\xi)). \tag{7.17}$$

Let $\chi(x,y) \in \Xi^{\Delta}(c)$, with $c \in (0,1)$.

$$A_{Red} = FIO(f(x,\xi) + g(y,\xi), a(x,y,\xi)\chi(\nabla_\xi f, -\nabla_\xi g))\chi(\nabla_\xi g, -\nabla_\xi f)),$$

$$B_{Red} = FIO(-g(x,\xi) + s(y,\xi), b(x,y,\xi)\chi(\nabla_\xi g, \nabla_\xi s))\chi(\nabla_\xi s, \nabla_\xi g)). \tag{7.18}$$

As for the Type $\mathcal{P}$ operator we only need to consider composition of the reduced forms A_{Red} and B_{Red} of A and B since we have $A \circ B = A_{Red} \circ B_{Red}$ modulo $\mathcal{K}$.

For convenience, define

$$a_r(x,y,\xi) := a(x,y,\xi)\chi(\nabla_\xi f(x,\xi), -\nabla_\xi g(y,\xi)))\chi(\nabla_\xi g(y,\xi), -\nabla_\xi f(x,\xi))$$

$$b_r(x,y,\xi) := b(x,y,\xi)\chi(\nabla_\xi g(x,\xi), \nabla_\xi s(y,\xi)))\chi(\nabla_\xi s(y,\xi), \nabla_\xi g(x,\xi)) \tag{7.19}$$

By the now standard arguments, we have the following equality, modulo Ku for $K \in \mathcal{K}$:

$$(A_{Red} \circ B_{Red})\,u(x) = \lim_{\delta\to 0}\lim_{\epsilon\to 0}\iiiint \exp\{i\,(f(x,\xi) + g(y,\xi) - g(y,\eta) + s(z,\eta),)\}\times$$
$$\times a_r(x,y,\xi)b_r(y,z,\eta)\gamma(\delta\eta)\gamma(\epsilon\xi)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi. \tag{7.20}$$

Let $\chi_2(\xi,\eta) \in \Xi^{\Delta}(k)$, with $c_2 \in (0,1)$. By constructing a partition of unity with χ_2 and using Theorem 4.0.2 and Lemma 7.2.3 we have

$$(A_{Red} \circ B_{Red})\, u(x) = \lim_{\delta \to 0} \lim_{\epsilon \to 0} \iiiint \exp\{i\,(f(x,\xi) + g(y,\xi) - g(y,\eta) + s(z,\eta),)\} \times$$
$$\times a_r(x,y,\xi) b_r(y,z,\eta) \chi_2(\xi,\eta) \gamma(\delta\eta) \gamma(\epsilon\xi) u(z) \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi, \tag{7.21}$$

modulo an integral operator with Schwartz kernel applied to u. Again, by the same argument as used in the proof of Theorem 5.0.1 we can use the Lebesgue Dominated Convergence Theorem to remove the ϵ limit to give,

$$(A_{Red} \circ B_{Red}) u(x) = \lim_{\delta \to 0} \iiiint \exp\{i\,(f(x,\xi) + g(y,\xi) - g(y,\eta) + s(z,\eta),)\} \times$$
$$\times a_r(x,y,\xi) b_r(y,z,\eta) \chi_2(\xi,\eta) \gamma(\delta\eta) u(z) \mathrm{d}z \mathrm{d}\eta \mathrm{d}y \mathrm{d}\xi. \tag{7.22}$$

We will now make an SG structure preserving change of variable in order to replace $g(y,\xi) - g(y,\eta)$ with $f(\tilde{y},\eta) - f(\tilde{y},\xi)$, for some new variable $\tilde{y}$. Define $W_{c_2} := \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi - \eta| < 2c_2\langle\eta\rangle\}$. The support of the integrand in (7.22) is contained in $V_{c_2} = \{(x,y,z,\xi,\eta) : x,y,z \in \mathbb{R}^n, (\xi,\eta) \in W_{c_2}\}$. We remark that V_{c_2} is open in $\mathbb{R}^{5n}$.

Let $G(x,y,z,\xi,\eta) := (x, \int_0^1 \nabla_2 g(y, \eta + t(\xi-\eta)), z, \xi, \eta)$ and define

$$G_1(x,y,z,\xi,\eta) := x, \tag{7.23}$$

$$G_2(x,y,z,\xi,\eta) := \int_0^1 \nabla_2 g(y, \eta + t(\xi-\eta))\mathrm{d}t, \tag{7.24}$$

$$G_3(x,y,z,\xi,\eta) := z, \tag{7.25}$$

$$G_4(x,y,z,\xi,\eta) := \xi, \tag{7.26}$$

$$G_5(x,y,z,\xi,\eta) := \eta. \tag{7.27}$$

Using this notation, $G = (G_1, G_2, G_3, G_4, G_5)$. We can apply Lemma 7.2.2 to see that for c_2 small enough we have:

$$\text{Each component } (G_2)_i(x,y,z,\xi,\eta) \text{ satisfies } SG^{0,0,1}_{x,y,\xi} \text{ estimates on } \mathbb{R}^n \times V_{c_2} \tag{7.28}$$

$$, \det\left(\partial_{y_j}(G_2)_i\right)_{i,j=1}^n (x,y,z,\xi,\eta) \succ 1 \text{ on } \mathbb{R}^n \times V_{c_2} \tag{7.29}$$

$$\langle G_2(x,y,z,\xi,\eta)\rangle \sim \langle y\rangle \text{ on } \mathbb{R}^n \times V_{c_2}. \tag{7.30}$$

It follows from the above facts and the Global Inverse Function Theorem 6.1.2 that G is a global diffeomorphism from V_{c_2} to V_{c_2}. Further, G satisfies the conditions of Proposition 6.2.3 (taking V and W to be V_{c_2} therein) as we now show. Writing

$G = (G_1, G_2, G_3, G_4, G_5)$, we therefore have the following holding on V_{c_2}.

$$\langle G_1 \rangle \sim \langle x \rangle, \tag{7.31}$$

$$\langle G_2 \rangle \sim \langle y \rangle, \tag{7.32}$$

$$\langle G_3 \rangle \sim \langle z \rangle, \tag{7.33}$$

$$\langle G_4 \rangle \sim \langle \xi \rangle, \tag{7.34}$$

$$\langle G_5 \rangle \sim \langle \eta \rangle, \tag{7.35}$$

We also have the following statements holding on V_{c_2}.

$$G_1 \text{ satisfies } SG^{1,0,0,0,0}_{x,y,z,\xi,\eta} \text{ estimates,} \tag{7.36}$$

$$G_2 \text{ satisfies } SG^{0,1,0,0,0}_{x,y,z,\xi,\eta} \text{ estimates,} \tag{7.37}$$

$$G_3 \text{ satisfies } SG^{0,0,1,0,0}_{x,y,z,\xi,\eta} \text{ estimates,} \tag{7.38}$$

$$G_4 \text{ satisfies } SG^{0,0,0,1,0}_{x,y,z,\xi,\eta} \text{ estimates,} \tag{7.39}$$

$$G_5 \text{ satisfies } SG^{0,0,0,0,1}_{x,y,z,\xi,\eta} \text{ estimates.} \tag{7.40}$$

The Jacobian of G is $\det \left(\partial_{y_j} h_i(y, \xi, \eta)\right)_{i,j=1}^{n}$. So the Jacobian of $G \succ 1$ on V_{c_2} by (7.29). So all the conditions of Proposition 6.2.3 are satisfied. Therefore, if we define $\left(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}\right) := (G_1, G_2, G_3, G_4, G_5)$ for $(x, y, z, \xi, \eta) \in V_{c_2}$ and write $G^{-1} = \left((G^{-1})_1, (G^{-1})_2, (G^{-1})_3, (G^{-1})_4, (G^{-1})_5\right)$, where $(G^{-1})_i = (G^{-1})_i\,(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta})$ for $\left(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}\right) \in G(V_{c_2}) = V_{c_2}$ we have

$$\langle (G^{-1})_1 \rangle \sim \langle \hat{x} \rangle, \tag{7.41}$$

$$\langle (G^{-1})_2 \rangle \sim \langle \hat{y} \rangle, \tag{7.42}$$

$$\langle (G^{-1})_3 \rangle \sim \langle \hat{z} \rangle, \tag{7.43}$$

$$\langle (G^{-1})_4 \rangle \sim \langle \hat{\xi} \rangle, \tag{7.44}$$

$$\langle (G^{-1})_5 \rangle \sim \langle \hat{\eta} \rangle \tag{7.45}$$

$$(G^{-1})_1 \text{ satisfies } SG^{1,0,0,0,0}_{\hat{x},\hat{y},\hat{z},\hat{\xi},\hat{\eta}} \text{ estimates,} \tag{7.46}$$

$$(G^{-1})_2 \text{ satisfies } SG^{0,1,0,0,0}_{\hat{x},\hat{y},\hat{z},\hat{\xi},\hat{\eta}} \text{ estimates,} \tag{7.47}$$

$$(G^{-1})_3 \text{ satisfies } SG^{0,0,1,0,0}_{\hat{x},\hat{y},\hat{z},\hat{\xi},\hat{\eta}} \text{ estimates,} \tag{7.48}$$

$$(G^{-1})_4 \text{ satisfies } SG^{0,0,0,1,0}_{\hat{x},\hat{y},\hat{z},\hat{\xi},\hat{\eta}} \text{ estimates,} \tag{7.49}$$

$$(G^{-1})_5 \text{ satisfies } SG^{0,0,0,0,1}_{\hat{x},\hat{y},\hat{z},\hat{\xi},\hat{\eta}} \text{ estimates.} \tag{7.50}$$

holding on $G(V_{c_2}) = V_{c_2}$. Now, since $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ and $\langle \xi \rangle \sim \langle \eta \rangle$, on V_{c_2}, it follows that

$$\begin{aligned} &\langle (G^{-1})_1 \rangle \sim \langle (G^{-1})_2 \rangle \sim \langle (G^{-1})_3 \rangle \\ &\qquad \langle (G^{-1})_4 \rangle \sim \langle (G^{-1})_5 \rangle \end{aligned} \tag{7.51}$$

on $G(V_{c_2})$. Therefore, by (7.41), (7.42), (7.43) ,(7.44) ,(7.45) we have $\langle \hat{x} \rangle \sim \langle \hat{y} \rangle \sim \langle \hat{z} \rangle$ and $\langle \hat{\xi} \rangle \sim \langle \hat{\eta} \rangle$ on $G(V_{c_2}) = V_{c_2}$. After we make the change the amplitude will go over to a function r say, with support inside $G(V_{c_2}) = V_{c_2}$. So, we will have $\langle \hat{x} \rangle \sim \langle \hat{y} \rangle \sim \langle \hat{z} \rangle$ and $\langle \hat{\xi} \rangle \sim \langle \hat{\eta} \rangle$ on $Supp(r)$.

Define new variables $(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}) := G(x, y, z, \xi, \eta)$. We write the absloutely convergent integral in (7.22) as one $4n$ dimensional integral over the x section of V_{c_2}. Then we change variables $(x, y, z, \xi, \eta) \to (\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta})$ to obtain

$$(A_{Red} \circ B_{Red})\, u(\hat{x}) = \lim_{\delta \to 0} \int_{\left(V_{c_2}\right)_{\hat{x}}} \exp\{i \left(f(\hat{x}, \hat{\xi}) + \hat{y} \cdot (\hat{\xi} - \hat{\eta}) + s(\hat{z}, \hat{\eta}), \right)\} \times$$
$$\times a_r(\hat{x}, h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta}), \hat{\xi}) b_r(h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta}), \hat{z}, \hat{\eta}) \chi_2(\hat{\xi}, \hat{\eta}) \gamma(\delta\hat{\eta}) u(\hat{z}) |\det \partial_{\hat{y}} h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta})| \mathrm{d}l. \tag{7.52}$$

where l is the four-fold product of Lebesgue measure on $\mathbb{R}^n$. (By $h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta})$ we mean the inverse taken in the first variable with the other variables fixed.)

By Proposition 6.2.5 the function

$a_r(\hat{x}, h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta}), \hat{\xi}) b_r(h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta}), \hat{z}, \hat{\eta}) \chi_2(\hat{\xi}, \hat{\eta}) |\det \partial_{\hat{y}} h^{-1}(\hat{y}, \hat{\xi}, \hat{\eta})|$ extends by zero outside $G(V_{c_2}) = V_{c_2}$ to a function in $SG^{m_1, m_2+t_1, t_2, m_3, t_3}_{\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}}$.

Now, define $M(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}) = (\hat{x}, -\int_0^1 \nabla_2 f(\hat{y}, \hat{\eta} + t(\hat{\xi} - \hat{\eta}))\mathrm{d}t, \hat{z}, \hat{\xi}, \hat{\eta}$ and set $m := -\int_0^1 \nabla_2 f(\hat{y}, \hat{\eta}+t(\hat{\xi}-\hat{\eta}))\mathrm{d}t$ By similar arguments to those used for the previous change of variable, we see that M is a global diffeomorphism from V_{c_2} to V_{c_2} for small enough c_2. We define $\left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) := M^{-1}(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta})$. Since M satisfies the conditions of Proposition 6.2.3 (taking V and W to be V_{c_2} therein) it follows that M^{-1} does also. Making the change of variable $\left(\hat{x}, \hat{y}, \hat{z}, \hat{\xi}, \hat{\eta}\right) \to \left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right)$ (7.52) becomes

$$(A_{Red} \circ B_{Red})\, u(\tilde{x}) = \lim_{\delta \to 0} \int_{(V_{c_2})_{\tilde{x}}} \exp\{i \Big(f(\tilde{y}, \tilde{\eta}) + f(\tilde{x}, \tilde{\xi}) - f(\tilde{y}, \tilde{\xi}) + s(\tilde{z}, \tilde{\eta}), \Big)\} \times$$
$$\times a_r(\tilde{x}, h^{-1}(m(\tilde{y}, \tilde{\xi}, \tilde{\eta}), \tilde{\xi}, \tilde{\eta}), \tilde{\xi}) b_r(h^{-1}(m(\tilde{y}, \tilde{\xi}\tilde{\eta}), \tilde{\xi}, \tilde{\eta}), \tilde{z}, \tilde{\eta}) \chi_2(\tilde{\xi}, \tilde{\eta}) \gamma(\delta\tilde{\eta}) u(\tilde{z}) \times$$
$$\times |\det \partial_{\tilde{y}} h^{-1}(m(\tilde{y}, \tilde{\xi}, \tilde{\eta}), \tilde{\xi}, \tilde{\eta})| |\det \partial_1 m(\tilde{y}, \tilde{\xi}, \tilde{\eta})| \mathrm{d}l. \tag{7.53}$$

Recall that l is the four-fold product of Lebesgue measure on $\mathbb{R}^n$.

By Proposition 6.2.5 the function

$$a_r(\tilde{x}, h^{-1}(m(\tilde{y}, \tilde{\xi}\tilde{\eta}), \tilde{\xi}, \tilde{\eta}), \tilde{\xi}) b_r(h^{-1}(m(\tilde{y}, \tilde{\xi}\tilde{\eta}), \tilde{\xi}, \tilde{\eta}), \tilde{z}, \tilde{\eta}) \chi_2(\tilde{\xi}, \tilde{\eta}) \gamma(\delta\tilde{\eta}) u(\tilde{z}) \times$$
$$\times |\det \partial_{\tilde{y}} h^{-1}(m(\tilde{y}, \tilde{\xi}\tilde{\eta}), \tilde{\xi}, \tilde{\eta})| |\det \partial_1 m(\tilde{y}, \tilde{\xi}, \tilde{\eta})| \tag{7.54}$$

extends by zero outside V_{c_2} to a function in $SG^{m_1, m_2+t_1, t_2, m_3, t_3}_{\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}}$.

Re-writing the integral in (7.53) as repeated integrals over $\mathbb{R}^n$ and dropping the $\tilde{}$s on the variables, we have

$$(A_{Red} \circ B_{Red})\, u(x) = \lim_{\delta \to 0} \iiiint \exp\{i \Big(f(y, \eta) + f(x, \xi) - f(y, \tilde{\xi}) + s(z, \eta), \Big)\} \times$$
$$\times a_r(x, h^{-1}(m(y, \xi\eta), \xi, \eta), \xi) b_r(h^{-1}(m(y, \xi, \eta), \xi, \eta), z, \eta) \chi_2(\xi, \eta) \gamma(\delta\eta) u(z) \times$$
$$\times |\det \partial_1 h^{-1}(m(y, \xi, \eta), \xi, \eta)| |\partial_1 m(y, \xi, \eta)| \mathrm{d}l.. \tag{7.55}$$

For convenience define

$$\tilde{a}(x,y,z,\xi,\eta) := a_r(x, h^{-1}(m(y,\xi\eta),\xi,\eta),\xi) b_r(h^{-1}(m(y,\xi\eta),\xi,\eta),z,\eta)\chi_2(\xi,\eta)\gamma(\delta\eta)\times$$

$$\times|\det\partial_1 h^{-1}(m(y,\xi\eta),\xi,\eta)||\partial_1 m(y,\xi,\eta)|.$$

$$\tilde{a}_E(\tilde{x},\tilde{\xi}) := \begin{cases} \tilde{a}(x,y,z,\xi,\eta) & \text{on } V_{c_2} \\ 0, & \text{otherwise} \end{cases} \tag{7.56}$$

We remark that on the support of $\tilde{a}_E(x,y,z,\xi,\eta)$ we have $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle\xi\rangle \sim \langle\eta\rangle$.

Let $\chi_3 \in \Xi^{\Delta}(c_3)$ for $c_3 \in (0,1)$. By making a partition of unity with $\chi_3(x,y)$ we can use Lemma 4.0.3 to reduce to the following (modulo an operator with Schwartz kernel applied to u)

$$(A_{Red}\circ B_{Red})\, u(x) = \lim_{\delta\to 0}\iiiint \exp\{i\,(f(y,\eta)+f(x,\xi)-f(y,\xi)+s(z,\eta),)\}\times$$

$$\times\tilde{a}_E(x,y,z,\xi,\eta)\chi_3(x,y)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\eta\mathrm{d}y\mathrm{d}\xi\times$$

$$. \tag{7.57}$$

We will make yet another change of variables. Define

$W_{c_3} = \{(x,y,z,\xi,\eta) : x,y,z,\xi,\eta \in \mathbb{R}^n : |x-y| < 2c_3\}$ and define

$Q(x,y,z,\xi,\eta) := (x,y,z,\int_0^1 \nabla_1 f(y+t(x-y),\xi),\eta)$ and set $q(x,y,\xi) := \int_0^1 \nabla_1 f(y+t(x-y),\xi)$. Arguing as before, we can choose c_3 small enough so that the change of variables is structure preserving. Defining $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) := Q(x, y, z, \xi, \eta)$, making the change of variables $(x, y, z, \xi, \eta) \longrightarrow (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta})$ and arguing as before we have

$$(A_{Red} \circ B_{Red})\, u(\tilde{x}) = \lim_{\delta \to 0} \iiiint \exp\{i \left(f(\tilde{y}, \tilde{\eta}) + (\tilde{x} - \tilde{y}) \cdot \tilde{\xi} + s(\tilde{z}, \tilde{\eta}), \right) \} \times$$
$$\times h_E(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta} u(\tilde{z}) \gamma(\delta \tilde{\eta}) \mathrm{d}\tilde{z} \mathrm{d}\tilde{\eta} \mathrm{d}\tilde{y} \mathrm{d}\tilde{\xi}. \tag{7.58}$$

where h_E (defined below) belongs to $SG^{m_1, m_2+t_1, t_2, m_3, t_3}_{\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}}$ and by $q^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})$ we mean the inverse in the third variable with the other variables fixed. As ever, equality is modulo Ku with $K \in \mathcal{K}$.

$$h(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) := \tilde{a}_E(\tilde{x}, \tilde{y}, \tilde{z}, q^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi}), \tilde{\eta}) \chi_3(\tilde{x}, \tilde{y}) |\det \partial_3 q^{-1}(\tilde{x}, \tilde{y}, \tilde{\xi})|$$
$$h_E(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) := \begin{cases} h(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) & \text{on } W_{c_3} \\ 0, & \text{otherwise} \end{cases} \tag{7.59}$$

We also have

$$\langle \tilde{x} \rangle \sim \langle \tilde{y} \rangle \sim \langle \tilde{z} \rangle \text{ on } Supp(h_E)$$
$$\langle \tilde{\xi} \rangle \sim \langle \tilde{\eta} \rangle \text{ on } Supp(h_E) \tag{7.60}$$

Multiplying by $e^{if(\tilde{x},\tilde{\eta})}.e^{-if(\tilde{x},\tilde{\eta})} = 1$ and rearranging we have

$$(A_{Red} \circ B_{Red})\, u(\tilde{x}) = \lim_{\delta \to 0} \iint \exp\{i\,(f(\tilde{x},\tilde{\eta}) + s(\tilde{z},\tilde{\eta}),)\}\times$$

$$\times\Big[\iint \exp\{i\left(f(\tilde{y},\tilde{\eta}) - f(\tilde{x},\tilde{\eta}) + (\tilde{x}-\tilde{y})\cdot\tilde{\xi},\right)\}\times$$

$$\times h_E(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta})\mathrm{d}\tilde{y}\mathrm{d}\tilde{\xi}\Big]\times$$

$$\times\gamma(\delta\tilde{\eta})u(\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}\tilde{\eta}. \tag{7.61}$$

Define

$$c(\tilde{x},\tilde{z},\tilde{\eta}) := \Big[\iint \exp\{i\left(f(\tilde{y},\tilde{\eta}) - f(\tilde{x},\tilde{\eta}) + (\tilde{x}-\tilde{y})\cdot\tilde{\xi},\right)\}\times$$

$$\times h_E(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta})\mathrm{d}\tilde{y}\mathrm{d}\tilde{\xi}\Big] \tag{7.62}$$

The proof of part 1 will be complete if we can show that $c(\tilde{x},\tilde{z},\tilde{\eta})$ belongs to the appropriate SG class. We will prove this by using Proposition 4.0.4.

Observe that $f(\tilde{y},\tilde{\eta}) - f(\tilde{x},\tilde{\eta}) = f(\tilde{y},\tilde{\eta}) + g(\tilde{z},\tilde{\eta}) - g(\tilde{z},\tilde{\eta}) - f(\tilde{x},\tilde{\eta})$. (Recall that g is a phase component.)

If we define $\Phi(\tilde{y},\tilde{z},\tilde{\eta}) := f(\tilde{y},\tilde{\eta}) + g(\tilde{z},\tilde{\eta})$ we have $\Phi \in \mathcal{Q} \subset \mathcal{P}$ and $f(\tilde{y},\tilde{\eta}) - f(\tilde{x},\tilde{\eta}) = \Phi(\tilde{y},\tilde{z},\tilde{\eta}) - \Phi(\tilde{x},\tilde{z},\tilde{\eta})$. Using this notation, $c(\tilde{x},\tilde{z},\tilde{\eta})$ becomes

$$c(\tilde{x},\tilde{z},\tilde{\eta}) := \Big[\iint \exp\{i\left(\Phi(\tilde{y},\tilde{z},\tilde{\eta}) - \Phi(\tilde{x},\tilde{z},\tilde{\eta}) + (\tilde{x}-\tilde{y})\cdot\tilde{\xi},\right)\}\times$$

$$\times h_E(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta})\mathrm{d}\tilde{y}\mathrm{d}\tilde{\xi}\Big] \tag{7.63}$$

We also have the following holding on $Supp(h_E)$:

1. $\langle\tilde{x}\rangle \sim \langle\tilde{y}\rangle \sim \langle\tilde{z}\rangle$

2. $\langle\tilde{\xi}\rangle \sim \langle\tilde{\eta}\rangle$ and

3. $|\tilde{x} - \tilde{y}| \leq c_3\langle\tilde{y}\rangle$ where $c_3 < 1$.

So, we can apply Proposition 4.0.4 to see that for c_3 sufficiently small, $c(\tilde{x}, \tilde{z}, \tilde{\eta})$ belongs to $SG^{p,q,m_3+t_3}_{\tilde{x},\tilde{z},\tilde{\eta}}$ for any $p, q \in \mathbb{R}$ such that $p + q = m_1 + m_2 + t_1 + t_2$. By Proposition 4.0.4 we also obtain an asymptotic expansion for $c(\tilde{x}, \tilde{z}, \tilde{\eta})$.

The second part of the proof follows from the first part and Lemma 7.2.4.

□

Remarks All the changes of variables made in this proof are of the type used by Coriasco in [3]. In view of Theorem 7.2.1 we can think of Type $\mathcal{Q}$ operators as compositions of arbitrary Coriasco Type 1 and Type 2 operators (see Introduction). We also note that in view of Lemma 7.2.4, we could use a similar argument to show that the composition of a Type $\mathcal{Q}$ operator with a ψdo is again a Type $\mathcal{Q}$ operator with the same phase and a modified amplitude.

Chapter 8

Closedness under Composition

8.1 Generalised Type $\mathcal{P}$ FIO.

We will now modify the definition of the Type $\mathcal{P}$ operator to allow the frequency variable ξ to have dimension greater than that of the spatial variables x and y.

Definition 8.1.1. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *where* $n_\xi \geq n$. *We denote by* $\mathcal{P}(a)$ *the set of all functions* $\Phi(x,y,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R})$ *having the following properties on the support of the amplitude* $a(x,y,\xi)$.

$$\text{For } j = 1,\ldots,n, \text{ the function } \partial_{x_j}\Phi(x,y,\xi) \text{ satisfies}$$

$$SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi}) \text{ estimates}, \tag{8.1}$$

$$\text{For } j = 1,\ldots,n, \text{ the function } \partial_{y_j}\Phi(x,y,\xi) \text{ satisfies}$$

$$SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi}) \text{ estimates}, \tag{8.2}$$

$$\langle \nabla_x \Phi(x,y,\xi) \rangle \succ \langle \xi \rangle, \tag{8.3}$$

$$\langle \nabla_y \Phi(x,y,\xi) \rangle \succ \langle \xi \rangle, \tag{8.4}$$

$$\exists C_\Phi > 0 : |x-y| \geq C_\Phi \langle y \rangle \Rightarrow |\nabla_\xi \Phi(x,y,\xi)| \succ \langle x \rangle + \langle y \rangle, \tag{8.5}$$

$$\exists C_\Phi > 0 : |x-y| \geq C_\Phi \langle x \rangle \Rightarrow |\nabla_\xi \Phi(x,y,\xi)| \succ \langle x \rangle + \langle y \rangle, \tag{8.6}$$

$$\text{For all multi-indices } \gamma \text{ we have }, \partial_\xi^\gamma \Phi(x,y,\xi) \prec (\langle x \rangle + \langle y \rangle)\langle \xi \rangle^{1-|\gamma|}. \tag{8.7}$$

Definition 8.1.2. *Given* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *and* $\Phi \in \mathcal{P}(a)$, *we define the generalised Type* $\mathcal{P}$ *operator* A *in the following way:*

$$Au(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \exp\{i\Phi(x,y,\xi)\} a(x,y,\xi) \gamma(\epsilon\xi) u(y) \mathrm{d}y \mathrm{d}\xi. \tag{8.8}$$

As ever, $\gamma(\epsilon\xi)$ *is a mollifier.*

Remark. For any $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, we have $\mathcal{P} \subset \mathcal{P}(a)$.

Notation. Given $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ and $\Phi \in \mathcal{P}(a)$, we will write $FIO(\Phi(x,y,\xi), a(x,y,\xi))$ to mean the operator defined in (8.8).

We now present some basic properties of the generalised Type $\mathcal{P}$ operator.

Theorem 8.1.3. *Let $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$, for any $m_x, m_y, m_\xi \in \mathbb{R}$ and let $\Phi(x,y,\xi) \in \mathcal{P}(a)$. Then, if we define $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$, we have:*

1. *$A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ continuously,*
2. *A is independent of the choice of mollifier,*
3. *$A^T = FIO(\Phi(y,x,\xi), a(y,x,\xi))$,*
4. *$A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ continuously.*

Proof. The proof of parts 1 and 2 are the same as the corresponding result for the Type $\mathcal{P}$ operator. Part 3 follows from the fact that the phase assumptions are symmetrical in x and y. Part 4 follows from parts 1 and 3 in the standard way. $\square$

The following Lemma will be useful when we study the composition of two generalised Type $\mathcal{P}$ operators.

Lemma 8.1.4. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$*, for any* $m_x, m_y, m_\xi \in \mathbb{R}$ *and let* $\Phi(x,y,\xi) \in \mathcal{P}(a)$*. Define* $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$*, and* $A_\epsilon u(x) := \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \exp\{i\Phi(x,y,\xi)\} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$*. Then we have*

1. $A_\epsilon : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *continuously,*

2. $A_\epsilon u(x) \to Au(x)$ *in* $\mathcal{S}(\mathbb{R}^n)$ *as* $\epsilon \to 0$.

Proof. For part 1, use the fact that that $A_\epsilon = FIO(\Phi(x,y,\xi), a(x,y,\xi)\gamma(\epsilon\xi))$ and apply part 1 of Theorem 8.1.3. For part 2, just use the operator $L_{y,\Phi(x,y,\xi)}$ (see Definition 2.5.2) to integrate by parts and apply the Lebesgue Dominated Convergence Theorem. □

Proposition 8.1.5. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$*, for any* $m_x, m_y, m_\xi \in \mathbb{R}$ *and let* $\Phi(x,y,\xi) \in \mathcal{P}(a)$*. Also, let* $\chi \in \Xi^\Delta(c)$ *where* $c \geq 2C_\Phi$*. Then, if we define* $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$ *and* $A_{Red} := FIO(\Phi(x,y,\xi), a(x,y,\xi)\chi(x,y)\chi(y,x))$*, we have*

$$A = A_{Red} \text{ modulo } \mathcal{K}.$$

Proof. This result follows from Theorem 3.2.2. □

Remark. A_{Red} will be referred to as the "reduced form " of A modulo $\mathcal{K}$.

Proposition 8.1.6. *Let* A, A_1, B, B_1 *be generalised Type* $\mathcal{P}$ *operators. Suppose* *that*

$$A = A_1 \text{ modulo } \mathcal{K} \text{ and}$$

$$B = B_1 \text{ modulo } \mathcal{K}. \tag{8.9}$$

Then we have

$$A \circ B = A_1 \circ B_1 \text{ modulo } \mathcal{K}.$$

Proof. Just follow the proof of the corresponding result for Type $\mathcal{P}$ operators. □

8.2 Technical Results

We now present several technical results which will be of use later in this chapter, starting with a Lemma about the structure of derivatives of a composition of smooth functions.

Lemma 8.2.1. *Let $a(x,y) \in C^\infty(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}, \mathbb{C})$ and $b(x,y) \in C^\infty(\mathbb{R}^{n_x} \times \mathbb{R}^{n_y}, \mathbb{R}^{n_y})$ and define $f(x,y) := a(x, b(x,y))$. Then, for any multi-index α with $|\alpha| \geq 1$,*

$$\partial_x^\alpha f = (\partial_1^\alpha a)(x, b(x,y)) +$$
$$\sum_i \left(\partial_1^{\alpha^i} \partial_2^{\beta^i} a\right)(x, b(x,y)) \prod_{j=1}^{|\beta^i|} \partial_x^{\theta^{i,j}} \left(b^{\gamma^{i,j}}\right)(x,y)$$

where the sum has finitely many terms, $\alpha^i, \beta^i, \theta^{i,j}, \gamma^{i,j}$ *are multi - indices such that*

$$|\gamma^{i,j}| = 1 \quad \forall i, j,$$
$$\alpha^i + \sum_{j=1}^{|\beta^i|} \theta^{i,j} = \alpha \quad \forall i,$$
$$\sum_{j=1}^{|\beta^i|} \gamma^{i,j} = \beta^i. \tag{8.10}$$

Remarks. The function b takes values in $\mathbb{R}^{n_y}$ and $\gamma^{i,j}$ is an n_y dimensional multi-index. In standard multi-index notation, $b^{\gamma^{i,j}} := \prod_{k=1}^{n_y} b_k^{\gamma_k^{i,j}}$, where b_k is the kth component of b and $\gamma_k^{i,j}$ is the kth component of $\gamma^{i,j}$.

Proof. By induction. Case $|\alpha| = 1$. By the standard rules for differentiation,

$$\partial_{x_i} f = (\partial_1^{e_i} a)(x, b(x,y)) + \sum_{k=1}^{n_y} (\partial_2^{e_k} a)(x, b(x,y))\, \partial_x^{e_i} (b^{e_k})(x,y),$$

where e_k is the multi - index with 1 in the kth place and zeros elsewhere. This expression has the required form.

Assume the statement is true for $|\alpha| = k$. For any $j \in \{1, \ldots, n_x\}$, by the inductive assumption, we have

$$\partial_{x_j} \partial_x^\alpha f = \partial_{x_j}[(\partial_1^\alpha a)(x, b(x,y))] +$$
$$\partial_{x_j} \left[\sum_i c_i \left(\partial_1^{\alpha^i} \partial_2^{\beta^i} a \right)(x, b(x,y)) \prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}} \right)(x,y) \right],$$

with $\alpha^i + \sum_{m=1}^{|\beta^i|} \theta^{i,m} = \alpha$ and $\sum_{j=1}^{|\beta^i|} \gamma^{i,j} = \beta^i$ for all i. Differentiating, we have,

$$\begin{aligned}\partial_{x_j}\partial_x^\alpha c = &\left(\partial_1^{\alpha+e_j} a\right)(x, b(x,y)) + \sum_{k=1}^{n_y} \left(\partial_1^\alpha \partial_2^{e_k} a\right)(x, b(x,y))\, \partial_x^{e_j}\left(b^{e_k}\right) + \\ &\sum_i c_i \partial_{x_j} \left\{ \left(\partial_1^{\alpha^i}\partial_2^{\beta^i} a\right)(x, b(x,y)) \right\} \prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y) + \\ &\sum_i c_i \left(\partial_1^{\alpha^i}\partial_2^{\beta^i} a\right)(x, b(x,y))\, \partial_{x_j} \left[\prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y) \right]. \qquad (8.11)\end{aligned}$$

The first term, $\left(\partial_1^{\alpha+e_j} a\right)(x, b(x,y))$, has the correct structure. We will check each sum separately.

Consider the first sum,

$$\sum_{k=1}^{n_y} \left(\partial_1^\alpha \partial_2^{e_k} a\right)(x, b(x,y))\, \partial_x^{e_j}\left(b^{e_k}\right).$$

Each term in the above sum has the correct structure; the number of terms in the product equals the absolute value of the order of differentiation in the second variable of a for all k and for each term, adding the orders of x derivatives gives $\alpha + e_j$ as required. Also for all k, the order of derivatives in the second variable of a matches the sum of the exponents of b. Let's consider

$$\sum_i \partial_{x_j} \left\{ \left(\partial_1^{\alpha^i}\partial_2^{\beta^i} a\right)(x, b(x,y)) \right\} \prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y). \qquad (8.12)$$

By expanding $\partial_{x_j} \left\{ \left(\partial_1^{\alpha^i}\partial_2^{\beta^i} a\right)(x, b(x,y)) \right\}$, (8.12) becomes

$$\sum_i c_i \left\{ \left[\left(\partial_1^{\alpha^i+e_j}\partial_2^{\beta^i} a\right) + \sum_{r=1}^{n} \left(\partial_1^{\alpha^i}\partial_2^{\beta^i+e_r} a\right) \partial_{x_j}\left(b^{e_r}\right) \right] \prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y) \right\}.$$

By inspection each term in this sum has the correct form. Now let's look at

$$\sum_i c_i \left(\partial_1^{\alpha^i} \partial_2^{\beta^i} a\right)(x, b(x,y))\, \partial_{x_j} \left[\prod_{m=1}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y)\right].$$

By the product rule, this is just

$$\sum_i c_i \left(\partial_1^{\alpha^i} \partial_2^{\beta^i} a\right)(x, b(x,y)) \sum_{s=1}^{|\beta^i|} \partial_x^{\theta^{i,s}+e_j} \left(b^{\gamma^{i,s}}\right) \left[\prod_{\substack{m=1 \\ m \neq s}}^{|\beta^i|} \partial_x^{\theta^{i,m}} \left(b^{\gamma^{i,m}}\right)(x,y)\right]$$

Again, by inspection, each term in this sum has the required properties. □

Lemma 8.2.2. *Assume that*

1. $a(x, y, z, \xi, \eta) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ *satisfies* $SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ *estimates on some set* $W \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}$.

2. $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ *and* $\langle \xi \rangle \sim \langle \eta \rangle$ *on* W.

Let

1. $r(x, z, \xi, \eta) \in ESG^{1,0,-1,0}_{x,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R})$

2. $s(x, z, \xi, \eta) \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R}^n)$ *with* $s(x, z, \xi, \eta)$ *satisfying* $SG^{1,0,0,0}_{x,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ *estimates on* W,

3. $F(x, y, z, \xi, \eta) := (x, \frac{y - s(x,z,\xi,\eta)}{r(x,z,\xi,\eta)}, z, \xi, \eta)$.

Suppose that we change variables to $\left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) := F(x, y, z, \xi, \eta)$. *Then the function* $f\left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) := a(\tilde{x}, \tilde{y} r(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) + s(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta})$, *satisfies the following estimates on* $F(W)$:

$$\partial^{\alpha}_{x} \partial^{\beta}_{\tilde{y}} \partial^{\gamma}_{z} \partial^{\delta}_{\xi} \partial^{\epsilon}_{\eta} f(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) \prec \langle \tilde{x} \rangle^{m_1 + m_2 - |\alpha|} \langle \tilde{z} \rangle^{m_3 - |\gamma|} \langle \tilde{\xi} \rangle^{m_4 - |\beta| - |\delta|} \langle \tilde{\eta} \rangle^{m_5 - |\epsilon|}. \tag{8.13}$$

Remarks It follows from the global inverse function theorem that F is a C^{∞} diffeomorphism from

$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}$ to $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}$. Note that $r(x, z, \xi, \eta)$ is real valued and $s(x, z, \xi, \eta)$ takes values in $\mathbb{R}^n$.

Proof. As a satisfies $SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}$ on W, it follows that on $F(W)$ we have

$$\left(\partial_1^\alpha\partial_2^\beta\partial_3^\gamma\partial_4^\delta\partial_5^\epsilon a\right)(\tilde{x},\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}),\tilde{z},\tilde{\xi},\tilde{\eta}) \prec$$

$$\langle\tilde{x}\rangle^{m_1-|\alpha|}\langle\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})\rangle^{m_2-|\beta|}\langle z\rangle^{m_3-|\gamma|}\langle\xi\rangle^{m_4-|\delta|}\langle\eta\rangle^{m_5-|\epsilon|}. \quad (8.14)$$

As $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle\xi\rangle \sim \langle\eta\rangle$ on W, it follows that

$\langle\tilde{x}\rangle \sim \langle\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})\rangle \sim \langle\tilde{z}\rangle$ and $\langle\tilde{\xi}\rangle \sim \langle\tilde{\eta}\rangle$ on $F(W)$. So, we have the following estimates on $F(W)$:

$$\left(\partial_1^\alpha\partial_2^\beta\partial_3^\gamma\partial_4^\delta\partial_5^\epsilon a\right)(\tilde{x},\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}),\tilde{z},\tilde{\xi},\tilde{\eta}) \prec$$

$$\langle\tilde{x}\rangle^{m_1+m_2-|\alpha|-|\beta|}\langle\tilde{z}\rangle^{m_3-|\gamma|}\langle\tilde{\xi}\rangle^{m_4-|\delta|}\langle\eta\rangle^{m_5-|\epsilon|}. \quad (8.15)$$

The function $r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) \in ESG^{1,0,-1,0}_{\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}}$ and $s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})$ satisfies $SG^{1,0,0,0}_{\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}}$ estimates on $F(W)$. So, since $\langle\tilde{x}\rangle \sim \langle\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})\rangle$ on $F(W)$, it follows that we have

$$|\tilde{y}| \prec \langle\tilde{\xi}\rangle \text{ on } F(W). \quad (8.16)$$

We will prove that $f\left(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}\right)$ satisfies (8.13) is three steps. We start with the case where all orders of derivatives are zero.

Case $|\alpha|=|\beta|=|\gamma|=|\delta|=|\epsilon|=0$. Just put $|\alpha|=|\beta|=|\gamma|=|\delta|=|\epsilon|=0$ in (8.15).

Case $|\beta|=0$. We apply Lemma 8.2.1 to get,

$$\partial_{\tilde{x}}^{\alpha}\partial_{\tilde{z}}^{\gamma}\partial_{\tilde{\xi}}^{\delta}\partial_{\tilde{\eta}}^{\epsilon}f = \left(\partial_1^{\alpha}\partial_3^{\gamma}\partial_4^{\delta}\partial_5^{\epsilon}a\right)\left(\tilde{x}, \tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) + s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta})\right) +$$
$$\sum_i \left(\partial_1^{\alpha^i}\partial_2^{\omega^i}\partial_3^{\gamma^i}\partial_4^{\delta^i}\partial_5^{\epsilon^i}a\right)\left(\tilde{x}, \tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) + s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta})\right) \times$$
$$\times \prod_{m=1}^{|\omega^i|} \partial_x^{\alpha^{i,m}}\partial_z^{\gamma^{i,m}}\partial_\xi^{\delta^{i,m}}\partial_\eta^{\epsilon^{i,m}}\left(\tilde{y}^{\lambda^{i,m}}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) + s^{\lambda^{i,m}}(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})\right) \tag{8.17}$$

where , $|\lambda^{i,m}| = 1 \quad \forall i, m$ and for all i in the finite sum we have

$$\alpha^i + \sum_{m=1}^{|\omega^i|} \alpha^{i,m} = \alpha,$$
$$\gamma^i + \sum_{m=1}^{|\omega^i|} \gamma^{i,m} = \gamma,$$
$$\delta^i + \sum_{m=1}^{|\omega^i|} \delta^{i,m} = \delta,$$
$$\epsilon^i + \sum_{m=1}^{|\omega^i|} \epsilon^{i,m} = \epsilon. \tag{8.18}$$

To obtain (8.17) and (8.18), just treat $(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})$ as one $2n+n_\xi+n_\eta$ dimensional variable and apply Lemma 8.2.1. Recall also that $r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})$ takes values in $\mathbb{R}$ and $s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})$ takes values in $\mathbb{R}^n$. We already have estimates of $\partial_1^{\alpha}\partial_2^{\beta}\partial_3^{\gamma}\partial_4^{\delta}\partial_5^{\epsilon}a\left(\tilde{x}, \tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) + s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta})\right)$, see (8.15).

If we can show that

$$\prod_{m=1}^{|\omega^i|} \partial_{\tilde{x}}^{\alpha^{i,m}}\partial_{\tilde{z}}^{\gamma^{i,m}}\partial_{\tilde{\xi}}^{\delta^{i,m}}\partial_{\tilde{\eta}}^{\epsilon^{i,m}}\left(\tilde{y}^{\lambda^{i,m}}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}) + s^{\lambda^{i,m}}(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})\right) \prec$$
$$\langle\tilde{x}\rangle^{|\omega^i| - \sum_{m=1}^{|\omega^i|}|\alpha^{i,m}|}\langle\tilde{z}\rangle^{-\sum_{m=1}^{|\omega^i|}|\gamma^{i,m}|}\langle\tilde{\xi}\rangle^{-\sum_{m=1}^{|\omega^i|}|\delta^{i,m}|}\langle\tilde{\eta}\rangle^{-\sum_{m=1}^{|\omega^i|}|\epsilon^{i,m}|}, \tag{8.19}$$

we'll be done by (8.18) and the estimates (8.15) .

Recall that on $F(W)$ we have

$$|\tilde{y}| \prec \langle \xi \rangle \tag{8.20}$$

By (8.20) and the fact that $r(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) \in ESG^{1,0,-1,0}_{\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}}$ and $s(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta})$ satisfies $SG^{1,0,0,0}_{\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}}$ estimates on $F(W)$, it follows that for any j we have

$$\partial^{\alpha^{i,m}}_{\tilde{x}} \partial^{\gamma^{i,m}}_{\tilde{z}} \partial^{\delta^{i,m}}_{\tilde{\xi}} \partial^{\epsilon^{i,m}}_{\tilde{\eta}} \left(\tilde{y}_j r(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) + s_j(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta})\right) \prec \langle \tilde{x} \rangle^{1-|\alpha^{i,m}|} \langle \tilde{z} \rangle^{-|\gamma^{i,m}|} \langle \tilde{\xi} \rangle^{-|\delta^{i,m}|} \langle \tilde{\eta} \rangle^{-|\epsilon^{i,m}|}.$$

on $F(W)$. Therefore, as $|\lambda^{i,m}| = 1$ for all i, m, we have

$$\prod_{m=1}^{|\omega^i|} \partial^{\alpha^{i,m}}_{x} \partial^{\gamma^{i,m}}_{z} \partial^{\delta^{i,m}}_{\xi} \partial^{\epsilon^{i,m}}_{\eta} \left(\tilde{y}^{\lambda^{i,m}} r(x, z, \xi, \eta) + s^{\lambda^{i,m}}(x, z, \xi, \eta)\right) \prec$$
$$\langle x \rangle^{|\omega^i| - \sum_{m=1}^{|\omega^i|} |\alpha^{i,m}|} \langle z \rangle^{- \sum_{m=1}^{|\omega^i|} |\gamma^{i,m}|} \langle \xi \rangle^{- \sum_{m=1}^{|\omega^i|} |\delta^{i,m}|} \langle \eta \rangle^{- \sum_{m=1}^{|\omega^i|} |\epsilon^{i,m}|}, \tag{8.21}$$

on the set $F(W)$.

Case $|\beta| > 0$. All functions are smooth so we can take the derivatives in any order. Therefore,

$$\partial^{\alpha}_{\tilde{x}} \partial^{\beta}_{\tilde{y}} \partial^{\gamma}_{\tilde{z}} \partial^{\delta}_{\tilde{\xi}} \partial^{\epsilon}_{\tilde{\eta}} f = \partial^{\alpha}_{\tilde{x}} \partial^{\gamma}_{\tilde{z}} \partial^{\delta}_{\tilde{\xi}} \partial^{\epsilon}_{\tilde{\eta}} \left[\left(\partial^{\beta}_{2} a\right) \left(\tilde{x}, \tilde{y} r(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) + s(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) r^{|\beta|} \right] \tag{8.22}$$

As $\left(\partial^{\beta}_{2} a\right)(x, y, z, \xi, \eta) \in SG^{m_1, m_2 - |\beta|, m_3, m_4, m_5}_{x,y,z,\xi,\eta}$, and the SG orders were arbitrary in the $\beta = 0$ case, we can use the earlier parts of the proof to say that we have the following estimates on the set $F(W)$:

$$\partial_{\tilde{x}}^{\alpha}\partial_{\tilde{z}}^{\gamma}\partial_{\tilde{\xi}}^{\delta}\partial_{\tilde{\eta}}^{\epsilon}\left[\left(\partial_2^{\beta}a\right)\left(\tilde{x},\tilde{y}r(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})+s(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}),\tilde{z},\tilde{\xi},\tilde{\eta})\right)\right] \prec \langle\tilde{x}\rangle^{m_1+m_2-|\alpha|-|\beta|}\langle\tilde{z}\rangle^{m_3-|\gamma|}\langle\tilde{\xi}\rangle^{m_4-|\delta|}\langle\tilde{\eta}\rangle^{m_5-|\epsilon|}. \tag{8.23}$$

As $r^{|\beta|}(\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta})$ satisfies $SG^{|\beta|,0,-|\beta|,0}_{\tilde{x},\tilde{z},\tilde{\xi},\tilde{\eta}}$ estimates on $F(W)$, the result follows from (8.22), (8.23) and the product rule.

□

Lemma 8.2.3. *Let $a(x,\tilde{y},z,\xi,\eta)\in C^{\infty}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n_\xi}\times\mathbb{R}^{n_\eta},\mathbb{C})$ satisfy the following estimates on some set $V\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n_\xi}\times\mathbb{R}^{n_\eta}$:*

$$\partial_x^{\alpha}\partial_{\tilde{y}}^{\beta}\partial_z^{\gamma}\partial_{\xi}^{\delta}\partial_{\eta}^{\epsilon}a(x,\tilde{y},z,\xi,\eta)\prec\langle x\rangle^{m_1+m_2-|\alpha|}\langle z\rangle^{m_3-|\gamma|}\langle\xi\rangle^{m_4-|\beta|-|\delta|}\langle\eta\rangle^{m_5-|\epsilon|}.$$

Assume also that on V we have the following relationships between the variables: $\langle\xi\rangle\sim\langle\eta\rangle$ and $|\tilde{y}|\prec\langle\xi\rangle$. If we define an $n+n_\xi+n_\eta$ dimensional frequency variable $\theta=(\tilde{y},\xi,\eta)$, then the function

$$\tilde{a}(x,z,\theta):=a(x,\tilde{y},z,\xi,\eta),$$

satisfies $SG^{m_1+m_2,m_3,m_4+m_5}_{x,z,\theta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n+n_\xi+n_\eta})$ estimates on V.

Proof. Consider

$$\partial_x^{\alpha}\partial_z^{\beta}\partial_{\theta}^{\gamma}\tilde{a}(x,z,\theta).$$

Define $\gamma_{\tilde{y}}$ to be the first n entries of γ and define γ_ξ and γ_η similarly so that $\gamma=(\gamma_{\tilde{y}},\gamma_\xi,\gamma_\eta)$. Using this notation, we have

$$\partial_x^{\alpha}\partial_z^{\beta}\partial_{\theta}^{\gamma}\tilde{a}(x,z,\theta)=\partial_x^{\alpha}\partial_z^{\beta}\partial_{\tilde{y}}^{\gamma_{\tilde{y}}}\partial_{\xi}^{\gamma_\xi}\partial_{\eta}^{\gamma_\eta}\left[a(x,\tilde{y},z,\xi,\eta)\right].$$

So, by assumption, we have

$$\partial_x^\alpha \partial_z^\beta \partial_\theta^\gamma \tilde{a} \prec \langle x\rangle^{m_1+m_2-|\alpha|}\langle z\rangle^{m_3-|\beta|}\langle \xi\rangle^{m_4-|\gamma_{\tilde{y}}|-|\gamma_\xi|}\langle \eta\rangle^{m_5-|\gamma_\eta|}. \tag{8.24}$$

On V we assumed that $\langle \eta\rangle \sim \langle \xi\rangle$ and $|\tilde{y}| \prec \langle \xi\rangle$. It follows that $\langle \theta\rangle \sim \langle \eta\rangle \sim \langle \xi\rangle$ on V. So, by (8.24), we have the following estimates on V :

$$\partial_x^\alpha \partial_z^\beta \partial_\theta^\gamma \tilde{a} \prec \langle x\rangle^{m_1+m_2-|\alpha|}\langle z\rangle^{m_3-|\beta|}\langle \theta\rangle^{m_4+m_5-|\gamma_{\tilde{y}}|-|\gamma_\xi|-|\gamma_\eta|}.$$

Noting that $|\gamma| = |\gamma_{\tilde{y}}| + |\gamma_\xi| + |\gamma_\eta|$, the proof is complete. □

Remark. If $V = Supp(a)$, we have $\tilde{a} \in SG^{m_1+m_2,m_3,m_4+m_5}_{x,z,\theta}$.

Proposition 8.2.4. *Assume that*

1. $a(x,y,z,\xi,\eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$

   satisfies $SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ *estimates on some set* $W \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}$.

2. $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ *and* $\langle \xi\rangle \sim \langle \eta\rangle$ *on* W.

Let

1. $r(x,z,\xi,\eta) \in ESG^{1,0,-1,0}_{x,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R})$

2. $s(x,z,\xi,\eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R}^n)$ *with* $s(x,z,\xi,\eta)$ *satisfying* $SG^{1,0,0,0}_{x,z,\xi,\eta}$ *estimates on* W.

3. $F(x,y,z,\xi,\eta) := (x, \frac{y-s(x,z,\xi,\eta)}{r(x,z,\xi,\eta)}, z, \xi, \eta)$.

Suppose that we change variables $(x, y, z, \xi, \eta) \rightarrow (x, \tilde{y}, z, \xi, \eta)$ *where* $\left(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) := F(x, y, z, \xi, \eta)$. *Then if we define an* $n + n_\xi + n_\eta$ *dimensional frequency variable* $\theta = (\tilde{y}, \tilde{\xi}, \tilde{\eta})$, *the function* $c(\tilde{x}, \tilde{z}, \theta) := a(\tilde{x}, \tilde{y}r(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) + s(\tilde{x}, \tilde{z}, \tilde{\xi}, \tilde{\eta}), \tilde{z}, \tilde{\xi}, \tilde{\eta})$, *satisfies* $SG^{m_1+m_2, m_3, m_4+m_5}_{\tilde{x}, \tilde{z}, \theta}$ *estimates on* $F(W)$.

Proof. Just apply Lemmas 8.2.2 and 8.2.3.

□

Remark. If the set W in Proposition 8.2.4 is the support of a then the function $c(\tilde{x}, \tilde{z}, \theta) \in SG^{m_1+m_2, m_3, m_4+m_5}_{x,z,\theta}\left(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi + n_\eta + n}\right)$. Also, since $\langle x \rangle \sim \langle z \rangle$ on $F(W)$ it follows that $c(x, z, \theta)$ satisfies $SG^{p,q,m_4+m_5}_{x,z,\theta}$ estimates where p and q are any real numbers such that $p + q = m_1 + m_2 + m_3$.

8.3 The Generalised Type $\mathcal{P}$ class is closed under composition.

Theorem 8.3.1. *Let* $a(x,y,\xi) \in SG^{m_1,m_2,m_3}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *with* $n_\xi \geq n$ *and let* $b(x,y,\xi) \in SG^{t_1,t_2,t_2}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\eta})$ *with* $n_\eta \geq n$. *Let* $\Phi(x,y,\xi) \in \mathcal{P}(a)$, $\Psi(x,y,\xi) \in \mathcal{P}(b)$ *and define* $A := FIO(\Phi, a)$ *and* $B := FIO(\Psi, b)$.

Then the composition $C := A \circ B$ *is a generalised Type* $\mathcal{P}$ *FIO , modulo* $\mathcal{K}$. *Precisely,*

$$C = FIO(\zeta, c) \text{ modulo } \mathcal{K},$$

$$\text{where } c \in SG^{m_1+t_1+m_2+n,t_2,m_3+t_3-n}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+n_\xi+n_\eta}) \text{ and } \zeta \in \mathcal{P}(c).$$

Remark We will find that on $Supp(c)$ we have $\langle x\rangle \sim \langle z\rangle$ and so $c \in SG^{l,r,m_\xi+t_\eta-n}_{x,z,\theta}$ where l and r are any real numbers such that $l + r = m_x + m_y + t_y + t_z + n$.

Proof. The proof is in two parts. In Part 1, we show that, modulo $\mathcal{K}$, the composition has the basic structure of a FIO - we determine the amplitude c and the proposed phase ζ. In Part 2 we show that $\zeta \in \mathcal{P}(c)$.

Part 1 We introduce the following notation for the "reduced" forms of the operators modulo $\mathcal{K}$:

$$A_{Red} := FIO(\Phi(x,y,\xi), a(x,y,\xi)\chi_1(x,y)\chi_1(y,x)),$$

$$B_{Red} := FIO(\Psi(x,y,\xi), b(x,y,\xi)\chi_2(x,y)\chi_2(y,x)).$$

where $\chi_1 \in \Xi^{\Delta}(k_1)$ with $k_1 > 2C_\Phi$ and $\chi_2 \in \Xi^{\Delta}(k_2)$ with $k_2 > 2C_\Psi$.

We have $A = A_{Red}$ modulo $\mathcal{K}$ and $B = B_{Red}$ modulo $\mathcal{K}$.

As proved earlier

$$A \circ B = A_{Red} \circ B_{Red} \text{ modulo } \mathcal{K}.$$

For the sake of brevity, set $a_0(x,y,\xi) := a(x,y,\xi)\chi_1(x,y)\chi_1(y,x)$ and $b_0(x,y,\xi) = b(x,y,\xi)\chi_2(x,y)\chi_2(y,x)$. Applying standard arguments we can show that modulo $\mathcal{K}$, we have

$$\begin{aligned}(A_{Red} \circ B_{Red})\, u(x) = \lim_{\delta \to 0} \int_{\mathbb{R}^n\eta} \int_{\mathbb{R}^n\xi} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} &\exp(i\,(\Phi(x,y,\xi) + \Psi(y,z,n)))a_0(x,y,\xi)\times \\ &\times b_0(y,z,\eta)\chi(\nabla_y\Phi(x,y,\xi), -\nabla_y\Psi(y,z,\eta))\times \\ &\times \chi(\nabla_y\Psi(y,z,\eta), -\nabla_y\Phi(x,y,\xi))\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}y\mathrm{d}\xi\mathrm{d}\eta.\end{aligned} \tag{8.25}$$

where $\chi \in \Xi^{\Delta}(k)$ for any $0 < k < 1$.

Define

$$\begin{aligned}h(x,y,z,\xi,\eta) := a_0(x,y,\xi)b_0(y,z,\eta)&\chi(\nabla_y\Phi(x,y,\xi), -\nabla_y\Psi(y,z,\eta))\times \\ &\times\chi(\nabla_y\Psi(y,z,\eta), -\nabla_y\Phi(x,y,\xi))\end{aligned} \tag{8.26}$$

and recall that by definition $a_0(x,y,\xi) := a(x,y,\xi)\chi_1(x,y)\chi_1(y,x)$ and $b_0(x,y,\xi) = b(x,y,\xi)\chi_2(x,y)\chi_2(y,x)$. So, on $Supp(h)$, we have $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle\xi\rangle \sim \langle\eta\rangle$. We can therefore define the new variable $\tilde{y}$ as follows:

$$\tilde{y}r(x,z,\xi,\eta) = y,$$

where r is an arbitrary function in $ESG^{1,0,-1,0}_{x,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}, \mathbb{R})$ and apply Proposition 8.2.4.

Making this change of variable, $(A_{Red} \circ B_{Red})\, u(x)$ becomes

$$(A_{Red} \circ B_{Red})\, u(x) = \lim_{\delta \to 0} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{(i(\Phi(x,\tilde{y}r(x,z,\xi,\eta),\xi)+\Psi(\tilde{y}r(x,z,\xi,\eta),z,\eta)))} \times \tilde{h}(x,\tilde{y},z,\xi,\eta)|r^n(x,z,\xi,\eta)|\gamma(\delta\eta)u(z)\mathrm{d}\mathrm{d}z\tilde{y}\mathrm{d}\xi\mathrm{d}\eta. \tag{8.27}$$

where $\tilde{h}(x,\tilde{y},z,\xi,\eta) := h(x,\tilde{y}r(x,z,\xi,\eta),z,\xi,\eta)$ and the term $|r^n(x,z,\xi,\eta)|$ is the Jacobian of the change of variables.

Define the new $n + n_\xi + n_\eta$ dimensional frequency variable $\theta = (\tilde{y},\xi,\eta)$. By Proposition 8.2.4

$$\tilde{h}(x,\tilde{y},z,\xi,\eta) \text{ belongs to } SG^{m_1+m_2+t_1,t_2,m_3+t_3}_{x,z,\theta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+n_\xi+n_\eta}). \tag{8.28}$$

On the support of $\tilde{h}(x,\tilde{y},z,\xi,\eta)$ we have $\langle\xi\rangle \sim \langle\eta\rangle$,

$\langle x\rangle \sim \langle z\rangle \sim \langle\tilde{y}r(x,z,\xi,\eta)\rangle$ and $\tilde{y} \prec \langle\xi\rangle, \langle\eta\rangle$. Since by definition, $\theta := (\tilde{y},\xi,\eta)$, we obviously have $\langle\theta\rangle \geq \langle\xi\rangle, \langle\eta\rangle$. Given that $\tilde{y} \prec \langle\xi\rangle, \langle\eta\rangle$ we also have $\langle\theta\rangle \prec \langle\xi\rangle, \langle\eta\rangle$. So, on the support of $c(x,z,\theta)$, we have $\langle\theta\rangle \sim \langle\eta\rangle \sim \langle\xi\rangle$.

Now, $|r^n(x,z,\xi,\eta)| \in SG^{n,0,-n,0}_{x,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta})$ and as $\langle\theta\rangle \sim \langle\eta\rangle \sim \langle\xi\rangle$ on $Supp(\tilde{h})$ it follows that $|r^n(x,z,\xi,\eta)|$ satisfies $SG^{n,0,-n}_{x,z,\theta}$ estimates on $Supp(\tilde{h})$. So, if we define $c(x,z,\theta) := \tilde{h}(x,\tilde{y},z,\xi,\eta)|r^n(x,z,\xi,\eta)|$ we have

$c(x,z,\theta) \in SG^{m_1+m_2+t_1+n,t_2,m_3+t_3-n}_{x,z,\theta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+n_\xi+n_\eta})$.

Define

$$\zeta(x,z,\theta) := \Phi(x,\tilde{y}r(x,z,\xi,\eta),\xi) + \Psi(\tilde{y}r(x,z,\xi,\eta),z,\eta),$$

$$\tilde{\gamma}(\delta\theta) := \gamma(\delta\eta).$$

Employing this notation and re-writing three integrals as one integral in θ, we have

$$(A_{Red} \circ B_{Red})\,u(x) = \lim_{\delta\to 0}\int_{\mathbb{R}^{n_\eta+n_\xi+n}}\int_{\mathbb{R}^n} \exp(i\zeta(x,z,\theta))c(x,z,\theta)\tilde{\gamma}(\delta\theta)\mathrm{d}z\mathrm{d}\theta. \quad (8.29)$$

Once we have checked that $\zeta(x,z,\theta)$ satisfies the phase assumptions, we can replace $\tilde{\gamma}(\delta\theta)$ with a mollifier in θ.

On the support of $c(x,z,\theta)$, there are certain relationships between the variables $x, z, \tilde{y}, \xi, \eta$ which we now recall: $\langle x\rangle \sim \langle z\rangle, \langle\xi\rangle \sim \langle\eta\rangle, |\tilde{y}| \prec \langle\xi\rangle$.

Remark As $\langle x\rangle \sim \langle z\rangle$ on the support of c, we have $c(x,z,\theta) \in SG^{p,q,m_3+t_3-n}_{x,z,\theta}$ for any $p, q \in \mathbb{R}$ with $p+q = m_1+m_2+t_1+t_2+n$.)

This completes Part 1. It only remains to show that $\zeta \in \mathcal{P}(c)$.

Part 2 Up to now we have only assumed that the constant k in the cut-off χ is less than 1. We will choose this constant to be sufficiently small in order that $\zeta(x,z,\theta) \in \mathcal{P}(c)$. Recall that $\zeta(x,z,\theta) \in \mathcal{P}(c)$ if $\zeta(x,z,\theta)$ has the following properties on the support of $c(x,z,\eta)$:

$$\text{For all } j = 1, \ldots, n, \quad \partial_{x_j}\zeta(x,z,\theta) \text{ satisfies } \mathrm{SG}^{0,0,1}_{x,z,\theta} \text{ estimates,} \tag{8.30}$$

$$\text{For all } j = 1, \ldots, n, \quad \partial_{z_j}\zeta(x,z,\theta) \text{ satisfies } \mathrm{SG}^{0,0,1}_{x,z,\theta} \text{ estimates,} \tag{8.31}$$

$$\langle \nabla_x \zeta(x,z,\theta) \rangle \succ \langle \theta \rangle, \tag{8.32}$$

$$\langle \nabla_z \zeta(x,z,\theta) \rangle \succ \langle \theta \rangle, \tag{8.33}$$

$$\exists C_\zeta > 0 : |x - z| \geq C_\zeta \langle z \rangle \Rightarrow |\nabla_\theta \zeta(x,z,\theta)| \succ \langle x \rangle + \langle z \rangle, \tag{8.34}$$

$$\exists C_\zeta > 0 : |x - z| \geq C_\zeta \langle x \rangle \Rightarrow |\nabla_\theta \zeta(x,z,\theta)| \succ \langle x \rangle + \langle z \rangle, \tag{8.35}$$

$$\forall \gamma, \partial_\theta^\gamma \zeta(x,z,\theta) \prec (\langle x \rangle + \langle z \rangle)\langle \theta \rangle^{1-|\gamma|}. \tag{8.36}$$

Conditions 8.30 and 8.31 This is straightforward to show once we note that on the support of $h(x,y,z,\xi,\eta)$, we have:

1. $\Phi(x,y,\xi)$ satisfies $SG^{1,0,1}_{x,y,\xi}$ estimates.

2. $\Psi(y,z,\eta)$ satisfies $SG^{0,1,1}_{y,z,\eta}$ estimates.

As $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ and $\langle \xi \rangle \sim \langle \eta \rangle$ on $Supp(h)$, we can apply Proposition 8.2.4 and conclude that:

1. $\Phi(x, \tilde{y}r(x,z,\xi,\eta), \xi)$ satisfies $SG^{1,0,1}_{x,z,\theta}$ estimates on $Supp(\tilde{h})$ which contains $Supp(c)$.

2. $\Psi(\tilde{y}r(x,z,\xi,\eta), z, \eta)$ satisfies $SG^{0,1,1}_{x,z,\theta}$ estimates on $Supp(\tilde{h})$ which contains $Supp(c)$.

Conditions 8.30 and 8.31 follow immediately once we note that $\langle x\rangle \sim \langle z\rangle$ on $Supp(c)$.

Conditions 8.32 and 8.33 Differentiating ζ with respect to x_j we can write, $\partial_{x_j}\zeta = f_{1,j}(x,z,\theta) + f_{2,j}(x,z,\theta)$ where

$$f_{1,j} := \left(\partial_{1_j}\Phi\right)(x, \tilde{y}r(x,z,\xi,\eta),\xi) \quad \text{and}$$
$$f_{2,j} := \sum_{l=1}^{n}\left[\left(\partial_{2_l}\Phi\right)(x,\tilde{y}r(x,z,\xi,\eta),\xi) + \left(\partial_{1_l}\Psi\right)(\tilde{y}r(x,z,\xi,\eta),z,\eta)\right]\times$$
$$\times\,(\tilde{y})_l\,\partial_{x_j}r(x,z,\xi,\eta)$$

Let $f_1 = (f_{1,1},\ldots,f_{1,n})$ and similarly let $f_2 = (f_{2,1},\ldots,f_{2,n})$. As $\Phi(x,y,\xi)\in\mathcal{P}(a)$, we have

$$\langle f_1\rangle \succ \langle \xi\rangle. \tag{8.37}$$

on $Supp(a(x,\tilde{y}r(x,z,\xi,\eta),\xi))$ (which contains $Supp(c)$.)

Now, we show that on $Supp(c)$ we have

$$|f_2| \prec k\langle\xi\rangle \tag{8.38}$$

where k is the constant in the cut-off function χ in the definition of $c(x,z,\theta)$. If we can prove (8.38), ζ will have the correct lower bound because for any $f_1, f_2 \in \mathbb{R}^n$ we have

$$\langle f_1 + f_2\rangle \geq \langle f_1\rangle - |f_2|.$$

So, by (8.37), if we choose the constant k to be small enough, we'll have $\langle \nabla_x \zeta \rangle \succ \langle \xi \rangle$ on the support of $c(x, z, \theta)$. (Since $\langle \xi \rangle \sim \langle \theta \rangle$ on $Supp(c)$, we'll have the desired lower bound.)

By definition

$$f_{2,j} = \sum_{l=1}^{n} \left[\left(\partial_{2_l} \Phi \right) (x, \tilde{y}r(x, z, \xi, \eta), \xi) + \left(\partial_{1_l} \Psi \right) (\tilde{y}r(x, z, \xi, \eta), z, \eta) \right] \times$$
$$\times \left(\tilde{y} \right)_l \partial_{x_j} r(x, z, \xi, \eta) \qquad (8.39)$$

We showed earlier that $(\tilde{y})_l \partial_{x_j} r(x, z, \xi, \eta) \prec 1$ on the support of c, for all l in the sum. Also, this constant is independent of k. (It does depend on k_1 though.) So,

$$|f_2| \prec |\nabla_2 \Phi(x, \tilde{y}r(x, z, \xi, \eta), \xi) + \nabla_1 \Psi(\tilde{y}r(x, z, \xi, \eta), z, \eta)|$$

on the support of c. Now, c is a product of functions, one of which is the cut-off $\chi \left(\nabla_1 \Psi(\tilde{y}r(x, z, \xi, \eta), z, \eta), -\nabla_2 \Phi(x, \tilde{y}r(x, z, \xi, \eta), \xi) \right)$. So,on the support of $c(x, z, \theta)$

$$|\nabla_2 \Phi(x, \tilde{y}r(x, z, \xi, \eta), \xi) + \nabla_1 \Psi(\tilde{y}r(x, z, \xi, \eta), z, \eta)| \prec k \langle \nabla_2 \Phi(x, \tilde{y}r(x, z, \xi, \eta)) \rangle.$$

As $\Phi \in \mathcal{P}(a)$ we have $\langle \nabla_2 \Phi(x, \tilde{y}r(x, z, \xi, \eta)) \rangle \sim \langle \xi \rangle$ on $Supp(a(x, \tilde{y}r(x, z, \xi, \eta), \xi))$ which contains $Supp(c)$. Therefore

$$f_2 \prec k \langle \xi \rangle,$$

on $Supp(c)$, which is (8.38) .

Conditions 8.34 and 8.35 We consider $\nabla_\xi \zeta$ first. Differentiating ζ with respect to ξ_j, gives

$$\partial_{\xi_j}\zeta = \left(\partial_{3_j}\Phi\right)(x, \tilde{y}r(x,z,\xi,\eta),\xi) + \sum_{l=1}^{n} \left[\left(\partial_{2_l}\Phi\right)(x,\tilde{y}r(x,z,\xi,\eta),\xi) + \left(\partial_{1_l}\Psi\right)(\tilde{y}r(x,z,\xi,\eta),z,\eta)\right] \times \times (\tilde{y})_l\, \partial_{\xi_j} r(x,z,\xi,\eta). \tag{8.40}$$

Define

$$f_{2,j} := \sum_{l=1}^{n} \left[\left(\partial_{2_l}\Phi\right)(x,\tilde{y}r(x,z,\xi,\eta),\xi) + \left(\partial_{1_l}\Psi\right)(\tilde{y}r(x,z,\xi,\eta),z,\eta)\right] \times (\tilde{y})_l\, \partial_{\xi_j} r(x,z,\xi,\eta) \tag{8.41}$$

and define $f_2 := \left(f_{2,1}, \ldots, f_{2,n_\xi}\right)$. In this notation, $\nabla_\xi \zeta = (\nabla_3\Phi)(x,\tilde{y}r(x,z,\xi,\eta),\xi) + f_2$. By the triangle inequality we have

$$|\nabla_\xi\zeta| \geq |(\nabla_3\Phi)(x,\tilde{y}r(x,z,\xi,\eta),\xi)| - |f_2|.$$

Now, we will show that $f_2 \prec k\langle x\rangle$ where k is the constant in the cut-off χ. As we argued earlier, on $Supp(c)$ we have

$$\left[\left(\partial_{2_l}\Phi\right)(x,\tilde{y}r(x,z,\xi,\eta),\xi) + \left(\partial_{1_l}\Psi\right)(\tilde{y}r(x,z,\xi,\eta),z,\eta)\right] \prec k\langle\xi\rangle.$$

Also, on $Supp(c)$ we have

$$(\tilde{y})_l\, \partial_{\xi_j} r(x,z,\xi,\eta) \prec \langle x\rangle \langle\xi\rangle^{-1},$$

with the implicit constant independent of k. It follows from the definition of f_2 that we have $f_2 \prec k\langle x\rangle$.

So, on $Supp(c)$ we have

$$|\nabla_\xi \zeta| \geq |(\nabla_3 \Phi)(x, \tilde{y}r(x,z,\xi,\eta), \xi)| - c_0 k\langle x\rangle \tag{8.42}$$

for some constant $c_0 > 0$. By similar arguments we have

$$|\nabla_\eta \zeta| \geq |(\nabla_3 \Psi)(\tilde{y}r(x,z,\xi,\eta), z, \eta)| - \tilde{c}_0 k\langle z\rangle \tag{8.43}$$

on $Supp(c)$, for some $\tilde{c} > 0$. Now, define $C_\zeta := \max\{2C_\Phi(2C_\Psi+1), 2C_\Psi(2C_\Phi+1)\}$. We remark that we have the following inequalities holding on $Supp(c)$.

$$\begin{aligned}
&\langle x\rangle \leq (2C_\Phi + 1)\langle \tilde{y}r(x,z,\xi,\eta)\rangle, \\
&\langle \tilde{y}r(x,z,\xi,\eta)\rangle \leq (2C_\Phi + 1)\langle x\rangle, \\
&\langle z\rangle \leq (2C_\Psi + 1)\langle \tilde{y}r(x,z,\xi,\eta)\rangle, \\
&\langle \tilde{y}r(x,z,\xi,\eta)\rangle \leq (2C_\Psi + 1)\langle z\rangle.
\end{aligned} \tag{8.44}$$

We will show that $|x - z| \geq C_\zeta\langle x\rangle$ implies that $\nabla_\theta \zeta \succ \langle x\rangle + \langle z\rangle$. It can be shown in the same way that Condition (8.34) is satisfied. Define

$V := \{(x, \tilde{y}, z, \xi, \eta) \in Supp(c) : |x - z| \geq C_\zeta\langle x\rangle\}$. By the triangle inequality, $|x - z| \geq C_\zeta\langle x\rangle$. implies that

$$|x - \tilde{y}r(x,z,\xi,\eta)| + |\tilde{y}r(x,z,\xi,\eta - z| \geq C_\zeta\langle x\rangle.$$

So, $|x-z| \geq C_\zeta \langle x \rangle$ implies that either

$$|x - \tilde{y}r(x, z, \xi, \eta)| \geq \frac{C_\zeta}{2} \langle x \rangle \tag{8.45}$$

or

$$|\tilde{y}r(x, z, \xi, \eta) - z| \geq \frac{C_\zeta}{2} \langle x \rangle . \tag{8.46}$$

Define $V_1 := \{(x, \tilde{y}, z, \xi, \eta) \in Supp(c) : |x - \tilde{y}r(x, z, \xi, \eta)| \geq \frac{C_\zeta}{2} \langle x \rangle\}$ and $V_2 := \{(x, \tilde{y}, z, \xi, \eta) \in Supp(c) : |\tilde{y}r(x, z, \xi, \eta) - z| \geq \frac{C_\zeta}{2} \langle x \rangle\}$.

By the above work, we have $V \subset V_1 \bigcup V_2$.

We claim that for small enough k we have

$$\nabla_\xi \zeta \succ \langle x \rangle + \langle z \rangle \text{ on } V_1. \tag{8.47}$$

$$\nabla_\eta \zeta \succ \langle x \rangle + \langle z \rangle \text{ on } V_2. \tag{8.48}$$

As $\theta := (\tilde{y}, \xi, \eta)$, we have $\nabla_\theta \zeta \succ |\nabla_\xi \zeta| + |\nabla_\eta \zeta|$. So, if we prove (8.47) and (8.48) we'll have proved that $|\nabla_\theta \zeta \succ \langle x \rangle + \langle z \rangle|$ on V because $V \subset V_1 \bigcup V_2$.

We will prove that $\nabla_\eta \zeta \succ \langle x \rangle + \langle z \rangle$ on V_2. Proving (8.47) is easier.

From the definition of V_2 and the estimates (8.44), it follows that $|(\nabla_3 \Psi)(\tilde{y}r(x, z, \xi, \eta), z, \eta)| \succ \langle z \rangle$ on V_2. So, by (8.43) we have $\nabla_\eta \zeta \succ \langle z \rangle$ on V_2 for sufficiently small k. Since $\langle x \rangle \sim \langle z \rangle$ on V_2 we have

$$\nabla_\eta \zeta \succ \langle x \rangle + \langle z \rangle \text{ on } V_2, \tag{8.49}$$

for sufficiently small k.

Similar arguments can be used to show that $|x - z| \geq C_\zeta \langle z \rangle \Rightarrow \nabla_\theta \zeta \succ \langle x \rangle + \langle z \rangle$.

Condition 8.36 As before, note that on the support of $h(x, y, z, \xi, \eta)$, we have:

1. $\Phi(x, y, \xi)$ satisfies $SG^{1,0,1}_{x,y,\xi}$ estimates.

2. $\Psi(y, z, \eta)$ satisfies $SG^{0,1,1}_{y,z,\eta}$ estimates.

As $\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ and $\langle \xi \rangle \sim \langle \eta \rangle$ on $Supp(h)$ (see (8.26)), we can apply Proposition 8.2.4 and conclude that:

1. $\Phi(x, \tilde{y}r(x, z, \xi, \eta), \xi)$ satisfies $SG^{1,0,1}_{x,z,\theta}$ estimates on $Supp(\tilde{h})$ which contains $Supp(c)$.

2. $\Psi(\tilde{y}r(x, z, \xi, \eta), z, \eta)$ satisfies $SG^{0,1,1}_{x,z,\theta}$ estimates on $Supp(\tilde{h})$ which contains $Supp(c)$. As $\zeta(x, z, \theta) := \Phi(x, \tilde{y}r(x, z, \xi, \eta), \xi) + \Psi(\tilde{y}r(x, z, \xi, \eta), z, \eta)$ we're done.

□

8.4 Generalised Type $\mathcal{Q}$ operator class.

As for the Type $\mathcal{P}$ operator, we modify the definition of the Type $\mathcal{Q}$ to allow the frequency variable to have a dimension greater than or equal to the dimension of the spatial variables.

Definition 8.4.1. *Let* $a(x, y, \xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *for* $m_1, m_2, m_3 \in \mathbb{R}$. *Let* $\Phi(x, y, \xi) = f(x, \xi) + g(y, \xi)$ *where* $f(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R})$ *and* $g(y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R})$. *We say that* $\Phi(x, y, \xi) = f(x, \xi) + g(y, \xi) \in \mathcal{Q}(a)$ *if* f *and* g *have the following properties:*

1. $f(x, \xi)$ *satisfies* $SG^{1,1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R})$ *estimates on Supp*(a),

2. $g(y, \xi)$ *satisfies* $SG^{1,1}_{y,\xi}(\mathbb{R}^n \times \mathbb{R}^{n_\xi}, \mathbb{R})$ *estimates on Supp*(a),

3. $\langle \nabla_x f(x, \xi) \rangle \sim \langle \xi \rangle$ *on Supp*(a),

4. $\langle \nabla_y g(y, \xi) \rangle \sim \langle \xi \rangle$ *on Supp*(a),

We can choose n *" prime " variables and write* $\xi = (\xi', \xi'')$ *(after re - labelling) where* $\xi' \in \mathbb{R}^n$ *and* $\xi'' \in \mathbb{R}^{n_\xi - n}$ *such that*

5. $\langle \nabla_{\xi'} g(y, \xi) \rangle \sim \langle y \rangle$ *on* $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$,

6. $\left| \det \left(\partial_{y_i} \partial_{\xi'_j} g(y, \xi) \right)^n_{i,j=1} \right| \succ 1$, *on* $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$,

7. $\partial_{x_i} \partial_{\xi'j}\, g \prec 1$, *for all* i, j, *on* $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$,

8. $|\nabla_y g(y,\xi)| \to \infty$ *as* $|\xi'| \to \infty$ *for all fixed* $y \in \mathbb{R}^n, \xi'' \in \mathbb{R}^{n_\xi - n}$,

There exists some open set $V_\Phi \subset \mathbb{R}^{n_\xi - n}$ *with* $Supp(a) \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times V_\Phi$ *such that we have:*

9. *For all* $i, j \in \{1, \ldots, n\}$,

$$\partial_{x_i} \partial_{\xi'_j} f(x,\xi) \prec 1,$$

on $\{(x,\xi) : x \in \mathbb{R}^n, \xi' \in \mathbb{R}^n, \xi'' \in V_\Phi\}$,

10.

$$\left| \det \left(\partial_{x_i} \partial_{\xi'_j} f(x,\xi) \right)_{i,j=1}^n \right| \succ 1,$$

on $\{(x,\xi) : x \in \mathbb{R}^n, \xi' \in \mathbb{R}^n, \xi'' \in V_\Phi\}$,

11. $\langle \nabla_{\xi'} f(x,\xi',\xi'') \rangle \succ \langle x \rangle$ *on* $\{(x,\xi) : x \in \mathbb{R}^n, \xi' \in \mathbb{R}^n, \xi'' \in V_\Phi\}$,

12. $|\nabla_x f(x,\xi)| \to \infty$ *as* $|\xi'| \to \infty$ *for all fixed* $x \in \mathbb{R}^n, \xi'' \in V_\Phi$,

Given $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *with* $n_\xi = n$, *then we define* $\mathcal{Q}(a) := \mathcal{Q}$.

Remark The above phase assumptions are not symmetrical in x and y. They could be made to be symmetrical but there is nothing to be gained by doing so.

Remark Let $\xi = (\xi', \xi'')$. Given a function $a(x,y,\xi)$, we may also denote this function by $a(x,y,\xi',\xi'')$.

Definition 8.4.2. *For any* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$, *with* $m_x, m_y, m_\xi \in \mathbb{R}$, $n_\xi > n$, *and* $\Phi(x,y,\xi) \in \mathcal{Q}(a)$ *we define the generalised Type* $\mathcal{Q}$ *operator* $A_{\Phi,a}$ *acting on* $\mathcal{S}(\mathbb{R}^n)$ *as follows: For* $u \in \mathcal{S}(\mathbb{R}^n)$,

$$Au(x) := \lim_{\epsilon \longrightarrow 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \exp\{i\Phi(x,y,\xi)\} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi,$$

where $\gamma(\epsilon\xi)$ *is a mollifier.*

Remark We have $\mathcal{Q}(a) \subset \mathcal{P}(a)$, so we obtain some basic facts about the generalised Type $\mathcal{Q}$ operator straightaway. Some of these are collected in the following Theorem.

Theorem 8.4.3. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$, *for any* $m_x, m_y, m_\xi \in \mathbb{R}$ *and let* $\Phi(x,y,\xi) \in \mathcal{Q}(a)$. *Then, if we define* $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$, *we have:*

1. $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *continuously,*

2. A *is independent of the choice of mollifier,*

3. $A^T = \lim_{\epsilon \to 0} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} e^{i\Phi(y,x,\xi)} a(y,x,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi$,

4. $A : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ *continuously.*

We now present the reduced form of a generalised Type $\mathcal{Q}$ operator modulo $\mathcal{K}$.

Proposition 8.4.4. *Let* $a(x,y,\xi) \in SG^{m_x,m_y,m_\xi}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$, *for any* $m_x, m_y, m_\xi \in \mathbb{R}$ *and let* $\Phi(x,y,\xi) = f(x,\xi) + g(y,\xi) \in \mathcal{Q}(a)$. *Also, let* $\chi \in \Xi^\Delta(c)$ *where* $c > 0$. *Then, if we define* $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$ *and* $A_{Red} := FIO(\Phi(x,y,\xi), a(x,y,\xi)\chi(\nabla_{\xi'} f, -\nabla_{\xi'} g)\chi(\nabla_{\xi'} g, -\nabla_{\xi'} f)$, *we have*

$$A = A_{Red} \text{ modulo } \mathcal{K}.$$

Proof. This result follows from Theorem 3.2.2. □

Remark. We will sometimes refer to a "generalised Type $\mathcal{Q}$ operator" as a "Type $\mathcal{Q}_{gen}$ operator" and similarly for the generalised Type $\mathcal{P}$ operator.

8.4.1 Some Remarks about the Generalisation of the Type $\mathcal{Q}$ operator.

As we have seen with the Type $\mathcal{P}_{gen}$ operator, it is not too hard to construct a simple SG operator class which is closed under composition. Defining an operator class which is closed under composition with the additional property that we can make "natural" changes of variable involving the phase is more difficult.

We will now discuss these "natural " changes of variable for the Type $\mathcal{Q}$ operator.

Let $A = FIO(f(x,\xi) + g(y,\xi), a(x,y,\xi))$ where $f(x,\xi) + g(y,\xi) \in \mathcal{Q}$ and $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ for any $m_1, m_2, m_3 \in \mathbb{R}$.

We assumed that $\left|\det\left(\partial_{x_i}\partial_{\xi_j} f(x,\xi)\right)^n_{i,j=1}\right| \succ 1$ and $\left|\det\left(\partial_{y_i}\partial_{\xi_j} g(y,\xi)\right)^n_{i,j=1}\right| \succ 1$. As we have seen, these “non - degeneracy ”assumptions, along with other “properness” assumptions, allow us to make the following changes of variable $(x,y,\xi) \to \left(\tilde{x},\tilde{y},\tilde{\xi}\right)$ globally on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, without destroying the SG structure of the amplitude:

$$\begin{aligned}
\left(\tilde{x},\tilde{y},\tilde{\xi}\right) &:= \left(x, \nabla_\xi g(y,\xi), \xi\right), \\
\left(\tilde{x},\tilde{y},\tilde{\xi}\right) &:= \left(x, y, \nabla_y g(y,\xi)\right). \\
\left(\tilde{x},\tilde{y},\tilde{\xi}\right) &:= \left(x, \nabla_\xi f(y,\xi), \xi\right), \\
\left(\tilde{x},\tilde{y},\tilde{\xi}\right) &:= \left(x, y, \nabla_y f(y,\xi)\right).
\end{aligned} \tag{8.50}$$

So, it seemed to me that for a Type $\mathcal{Q}_{gen}$ operator we should be able to make similar changes of variable whilst preserving the SG structure of the amplitude. For a Type $\mathcal{Q}_{gen}$ operator, the following changes of variables $(x,y,\xi) \to \left(\tilde{x},\tilde{y},\tilde{\xi}\right)$ do not destroy the SG structure of the amplitude:

$$\begin{aligned}
\left(\tilde{x}, \tilde{y}, \tilde{\xi}', \tilde{\xi}''\right) &:= \left(x, \nabla_{\xi'} g(y,\xi), \xi', \xi''\right), \\
\left(\tilde{x}, \tilde{y}, \tilde{\xi}', \tilde{\xi}''\right) &:= \left(x, y, \nabla_{y} g(y,\xi), \xi''\right). \\
\left(\tilde{x}, \tilde{y}, \tilde{\xi}', \tilde{\xi}''\right) &:= \left(x, \nabla_{\xi'} f(y,\xi), \xi', \xi''\right), \\
\left(\tilde{x}, \tilde{y}, \tilde{\xi}', \tilde{\xi}''\right) &:= \left(x, y, \nabla_{y} f(y,\xi), \xi''\right).
\end{aligned} \tag{8.51}$$

Note that the transformations involving g are globally defined but those involving f are defined on $\{(x, y, \xi', \xi'') : x, y, \xi' \in \mathbb{R}^n \text{ and } \xi'' \in V_\Phi\}$ for some open $V_\Phi \subset \mathbb{R}^{n_\xi - n}$. We now briefly discuss why this is the case.

The generalised Type $\mathcal{Q}$ operator arises naturally when we compose two type $\mathcal{Q}$ operators. We shall see later that after the composition of two type $\mathcal{Q}$ operators, we can easily retain non-degeneracy of the second phase component (corresponding to g) everywhere. It doesn't seem possible to retain non-degeneracy of the first component (corresponding to f) everywhere, although we can retain this property on an open set containing the support of the amplitude.

We could have generalised the Type $\mathcal{Q}$ operator class by making the same assumptions for g as for we did for f. Doing so would have given an operator class with the desirable property that the transpose of any operator is an operator of the same type. However, when we compose two type $\mathcal{Q}$ operators we can easily retain non-degeneracy of the second phase component globally. So, making the same assumptions about g as we do about f would mean "throwing away" some non-degeneracy of the phase.

8.5 The generalised Type $\mathcal{Q}$ operator class is closed under composition.

Theorem 8.5.1. *Let* $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ *with* $n_\xi \geq n$ *and let* $b(x,y,\xi) \in SG^{t_1,t_2,t_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\eta})$ *with* $n_\eta \geq n$*. Let* $\Phi(x,y,\xi) := f(x,\xi) + g(y,\xi) \in \mathcal{Q}(a)$*,* $\Psi(x,y,\xi) := u(x,\xi) + v(y,\xi) \in \mathcal{Q}(b)$ *and define* $A := FIO(\Phi, a)$ *and* $B := FIO(\Psi, b)$*.*

Then the composition $C := A \circ B$ *is a generalised Type* $\mathcal{Q}$ *FIO , modulo* $\mathcal{K}$*. Precisely,*

$$C = FIO(\zeta(x,z,\theta), c(x,z,\theta)) \text{ modulo } \mathcal{K},$$

where $c \in SG^{m_1+t_1+m_2+n,t_2,m_3+t_3-n}_{x,z,\theta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n+n_\xi+n_\eta})$ *with* p,q *any real numbers such that* $p+q = m_1+t_1+m_2+n+t_2$ *and* $\zeta \in \mathcal{Q}(c)$*.*

Proof. We will prove the Theorem for $n_\xi > n$ and $n_\eta > n$. The proofs of the other cases are essentially the same.

The proof is split into two parts. In part 1, we reduce the composition modulo $\mathcal{K}$, perform some changes of variable, define the θ' and the θ'' variables and finally we determine $c(x,z,\theta)$ and $\zeta(x,z,\theta)$. In part 2, we show that $\zeta \in \mathcal{Q}(c)$.

Part 1.

Given any $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi})$ with $n_\xi \geq n$, we have $\mathcal{Q}(a) \subset \mathcal{P}(a)$. So we can follow the argument in the proof that the generalised

Type $\mathcal{P}$ operator class is closed under composition. Doing so, we have that (modulo operators with Schwartz kernel applied to u),

$$(A \circ B)\, u(x) = \lim_{\delta \to 0} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\{\Phi(x,y,\xi)+\Psi(y,z,\eta)\}} h(x,y,z,\xi,\eta)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}y\mathrm{d}\xi\mathrm{d}\eta,$$

where

$$\begin{aligned} h(x,y,z,\xi,\eta) := a(x,y,\xi)b(y,z,\eta)\chi_1\left(\nabla_{\xi'} f(x,\xi), -\nabla_{\xi'} g(y,\xi)\right) \times \\ \chi_1\left(\nabla_{\xi'} g(y,\xi), -\nabla_{\xi'} f(x,\xi)\right)\chi_3\left(\nabla_{\eta'} u(y,\eta), -\nabla_{\eta'} v(z,\eta)\right) \times \\ \times \chi_3\left(\nabla_{\eta'} v(z,\eta), -\nabla_{\eta'} u(y,\eta)\right)\chi_5(\nabla_y g(y,\xi), -\nabla_y u(y,\eta)) \times \\ \times \chi_5(\nabla_y u(y,\eta), -\nabla_y g(y,\xi)), \end{aligned}$$

with $\chi_i \in \Xi^\Delta(c_i)$, where the positive constants c_i are to be fixed later. At the moment there is no restriction placed on the c_i.

We note that $h \in SG^{m_1,m_2+t_1,t_2,m_3,t_3}_{x,y,z,\xi,\eta}$ and that on the support of h, we have

$\langle x \rangle \sim \langle y \rangle \sim \langle z \rangle$ and $\langle \xi \rangle \sim \langle \eta \rangle$. We also have

$$f(x, \xi) \text{ satisfies } SG^{1,1}_{x,\xi} \text{ estimates on } Supp(h), \tag{8.52}$$

$$g(y, \xi) \text{ satisfies } SG^{1,1}_{y,\xi} \text{ estimates on } Supp(h), \tag{8.53}$$

$$u(y, \eta) \text{ satisfies } SG^{1,1}_{y,\eta} \text{ estimates on } Supp(h), \tag{8.54}$$

$$v(z, \eta) \text{ satisfies } SG^{1,1}_{z,\eta} \text{ estimates on } Supp(h), \tag{8.55}$$

$$\langle \nabla_x f(x, \xi) \rangle \sim \langle \xi \rangle \text{ on } Supp(h), \tag{8.56}$$

$$\langle \nabla_y g(y, \xi) \rangle \sim \langle \xi \rangle \text{ on } Supp(h), \tag{8.57}$$

$$\langle \nabla_y u(y, \eta) \rangle \sim \langle \eta \rangle \text{ on } Supp(h), \tag{8.58}$$

$$\langle \nabla_z v(z, \eta) \rangle \sim \langle \eta \rangle \text{ on } Supp(h). \tag{8.59}$$

Remark It is perhaps helpful at times to think of e.g. $f(x, \xi)$ as being a function of all the variables (i.e. (x, y, z, ξ, η)) rather than of x and ξ only.

We will explain how (8.52) is obtained. All the other statements follow in the same way. We assumed that $f(x, \xi)$ satisfies $SG^{1,1}_{x,\xi}$ estimates on $Supp(a)$. So, if we let $W := \{(x, y, z, \xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta} : (x, y, \xi) \in Supp(a)\}$, we have $f(x, \xi)$ satisfies $SG^{1,1}_{x,\xi}$ estimates on W. Once we note that $Supp(h) \subset W$, we have (8.52).

We now make a change in the y variable. We will perform it in two steps. We first make an SG structure preserving change $y \to s$, where s is a dummy variable. We will then make a "re-scale and shift" change $s \to \tilde{y}$. (of the type treated in

Proposition 8.2.4). We remark that the goal of these changes of variable is to have $|\tilde{y}| \prec c_1\langle\xi\rangle$ on the support of the amplitude (Recall that c_1 is the constant in the definition of the cut-off χ_1). This will be important later when we are trying to show that the proposed phase has the desired non-degeneracy in the first component.

Define $s := \nabla_{\xi'} g(y,\xi)$. Under this change the amplitude h goes over to $\tilde{h}$. The change of variables is globally defined as we now explain. For $(x,y,z,\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_\eta}$, define

$$G : \begin{pmatrix} x \\ y \\ z \\ \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} x \\ \nabla_{\xi'} g(y,\xi) \\ z \\ \xi \\ \eta \end{pmatrix}.$$

Define $\left(\tilde{x}, s, \tilde{z}, \tilde{\xi}, \tilde{\eta}\right) := G(x,y,z,\xi,\eta)$. By assumption, $|\nabla_{\xi'} g(y,\xi)|$ tends to infinity as $|y| \to \infty$ for any fixed $\xi \in \mathbb{R}^{n_\xi}$. It follows straightforwardly, that G is a proper map from $\mathbb{R}^{3n+n_\xi+n_\eta}$ to $\mathbb{R}^{3n+n_\xi+n_\eta}$. By assumption, $|\det\left(\partial_{y_i}\partial_{\xi'_j} g(y,\xi)\right)_{i,j=1}^{n}| \succ 1$. So G is a proper, smooth function with non-zero Jacobian everywhere. Therefore, G is a C^∞ diffeomorphism from $\mathbb{R}^{3n+n_\xi+n_\eta}$ to $\mathbb{R}^{3n+n_\xi+n_\eta}$.

Making this change of variables, $(A \circ B)\, u(x)$ (modulo Ku for $K \in \mathcal{K}$) becomes

$$\lim_{\delta \to 0} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i([f(x,\xi)+g\left(\left(\nabla_\xi g\right)^{-1}(s,\xi),\xi\right)+u\left(\left(\nabla_\xi g\right)^{-1}(s,\xi),\eta\right)+v(z,\eta)]} \times$$

$$\times h\left(x, \left(\nabla_{\xi'} g\right)^{-1}(s,\xi), z, \xi, \eta\right) |\det \partial_s \left(\nabla_{\xi'} g\right)^{-1}(s,\xi)| \gamma(\delta\eta) \times$$

$$u(z) \mathrm{d}z \mathrm{d}s \mathrm{d}\xi \mathrm{d}\eta, \qquad (8.60)$$

where $\left(\nabla_{\xi'} g\right)^{-1}(s,\xi)$ denotes the inverse of the ξ section of $\left(\nabla_{\xi'} g\right)$ and $|\det \partial_s \left(\nabla_{\xi'} g\right)^{-1}(s,\xi)|$ is the Jacobian of the change. Note also that $\langle \left(\nabla_{\xi'} g\right)^{-1}(s,\xi)\rangle \sim \langle s\rangle$ on $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$. This follows from the fact that $\langle \nabla_{\xi'} g(y,\xi)\rangle \sim \langle y\rangle$ on $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$ by assumption. Also, by Proposition 6.2.3 , the function $\left(\nabla_{\xi'} g\right)^{-1}(s,\xi)$ satisfies $SG^{1,0}_{s,\xi}$ estimates on $Supp(h\left(x, \left(\nabla_{\xi'} g\right)^{-1}(s,\xi), z, \xi, \eta\right))$.

For convenience, we have dropped the tildes over the x, z, ξ, η variables.

Define

$$\tilde{h}(x,s,z,\xi,\eta) := h\left(x, \left(\nabla_{\xi'} g\right)^{-1}(s,\xi), z, \xi, \eta\right) |\det \partial_s \left(\nabla_{\xi'} g\right)^{-1}(s,\xi)|.$$

By Proposition 6.2.4 we have

$$\tilde{h}(x,s,z,\xi,\eta) \in SG^{m_1,m_2+t_1,t_2,m_3,t_3}_{x,s,z,\xi,\eta}. \qquad (8.61)$$

By Proposition (6.2.4) we also have:

$$f(x,\xi) \text{ satisfies } SG^{1,1}_{x,\xi} \text{ estimates on } Supp(\tilde{h}), \tag{8.62}$$

$$g((\nabla_{\xi'} g)^{-1}(s,\xi),\xi) \text{ satisfies } SG^{1,1}_{s,\xi} \text{ estimates on } Supp(\tilde{h}), \tag{8.63}$$

$$u((\nabla_{\xi'} g)^{-1}(s,\xi),\eta) \text{ satisfies } SG^{1,0,1}_{s,\xi,\eta} \text{ estimates on } Supp(\tilde{h}), \tag{8.64}$$

$$v(z,\eta) \text{ satisfies } SG^{1,1}_{z,\eta} \text{ estimates on } Supp(\tilde{h}). \tag{8.65}$$

It follows from (??) to (8.59), and the properties of G that:

$$\langle \nabla_1 f(x,\xi)\rangle \sim \langle \xi \rangle \text{ on } Supp(\tilde{h}), \tag{8.66}$$

$$\langle \nabla_1 g\left((\nabla_{\xi'} g)^{-1}(s,\xi),\xi\right)\rangle \sim \langle \xi \rangle \text{ on } Supp(\tilde{h}) \tag{8.67}$$

$$\langle \nabla_1 u\left((\nabla_{\xi'} g)^{-1}(s,\xi),\eta\right)\rangle \sim \langle \eta \rangle \text{ on } Supp(\tilde{h}) \tag{8.68}$$

$$\langle \nabla_1 v(z,\eta)\rangle \sim \langle \eta \rangle \text{ on } Supp(\tilde{h}) \tag{8.69}$$

We now make the "re-scale and shift " change in the s variable. Let $r(x,\xi)$ be an arbitrary function in $ESG^{1,-1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^{n_\xi})$ and define

$$\tilde{y}r(x,\xi) := s + \nabla_{\xi'} f(x,\xi).$$

Each component of $\nabla_{\xi'} f(x,\xi)$ satisfies $SG^{1,0}_{x,\xi}$ estimates on $Supp(\tilde{h})$, and $\langle x\rangle \sim \langle s\rangle \sim \langle z\rangle$ and $\langle \xi\rangle \sim \langle \eta\rangle$ on $Supp(\tilde{h})$. So, we are in a position to use Proposition 8.2.4.

Define $H(x,s,z,\xi,\eta) := (x, \frac{s+\nabla_{\xi'} f(x,\xi)}{r(x,\xi)}, z, \xi, \eta)$ and we remark that H is a C^∞ diffeomorphism from $\mathbb{R}^{3n+n_\xi+n_\eta}$ to $\mathbb{R}^{3n+n_\xi+n_\eta}$.

Making this change of variables, we have the following equality (modulo Ku where $K \in \mathcal{K}$),

$$
\begin{aligned}
&A \circ Bu(x) = \\
&\lim_{\delta \to 0} \int_{\mathbb{R}^{n_\eta}} \int_{\mathbb{R}^{n_\xi}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \exp\left[i\left(f(x,\xi) + g\left((\nabla_{\xi'} g)^{-1}\left[\tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right], \xi\right)\right.\right. \\
&\qquad \left.\left. + u\left((\nabla_{\xi'} g)^{-1}\left[\tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right], \eta\right) + v(z,\eta)\right)\right] \times \\
&\qquad \times \tilde{h}\left(x, \tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta\right) \gamma(\delta\eta) \times \\
&\qquad |r(x,\xi)^n| u(z) \mathrm{d}z \mathrm{d}\tilde{y} \mathrm{d}\xi \mathrm{d}\eta.
\end{aligned}
\tag{8.70}
$$

In an effort to make things easier to read, the variables in $(\nabla_{\xi'} g)^{-1}$ have been delimited with square brackets.

Now define the $n + n_\xi + n_\eta$ dimensional frequency variable $\theta := (\tilde{y}, \xi, \eta)$. We choose $\theta' := \eta'$ and $\theta'' := (\tilde{y}, \xi, \eta'')$. Also define

$$
\begin{aligned}
c(x,z,\theta) :=& \tilde{h}\left(x, \tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta)\right) |r(x,\xi)^n|, \\
w(x,\theta) :=& f(x,\xi) + g((\nabla_{\xi'} g)^{-1}\left(\tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right), \xi) + \\
&+ u((\nabla_{\xi'} g)^{-1}\left(\tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right), \eta) \\
\zeta(x,z,\theta) :=& w(x,\theta) + v(z,\eta)
\end{aligned}
$$

We'll now show that $c(x,z,\theta) \in SG^{p,q,m_3+t_3-n}_{x,z,\theta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi+n_\eta+n})$ where p, q are real numbers with $p + q = m_1 + m_2 + t_1 + t_2 + n$. By Proposition 8.2.4, we have

$$
\tilde{h}\left(x, \tilde{y} r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta\right) \in SG^{m_1+m_2+t_1, t_2, m_3+t_3}_{x,z,\theta}. \tag{8.71}
$$

We have the following relationships between the variables on

$Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$:

$$\langle x \rangle \sim \langle \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi) \rangle \sim \langle z \rangle, \tag{8.72}$$

$$\langle \xi \rangle \sim \langle \eta \rangle, \tag{8.73}$$

$$|\tilde{y}| \prec c_1 \langle \xi \rangle, \tag{8.74}$$

$$|\nabla_1 u((\nabla_{\xi'} g)^{-1} [\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta), -$$

$$\nabla_1 g((\nabla_{\xi'} g)^{-1} [\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \xi)| \prec c_5 \langle \xi \rangle, \tag{8.75}$$

$$\langle \theta \rangle \sim \langle \xi \rangle \sim \langle \eta \rangle. \tag{8.76}$$

Statements (8.72) and (8.73) are obvious since we had $\langle x \rangle \sim \langle s \rangle \sim \langle z \rangle$ and $\langle \xi \rangle \sim \langle \eta \rangle$ on $Supp(\tilde{h}(x, s, z, \xi, \eta))$. Statement (8.75) follows immediately from (??). For (8.74) recall that we had $|\nabla_{\xi'} f(x,\xi), -s| \leq c_1 \langle s \rangle$ on $Supp(\tilde{h}(x, s, z, \xi, \eta))$. So on $Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$ we have

$$\tilde{y}r(x,\xi) \leq c_1 \langle \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi) \rangle.$$

Since $\langle x \rangle \sim \langle \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi) \rangle$ on $Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$

$$\tilde{y}r(x,\xi) \prec c_1 \langle x \rangle,$$

on $Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$. Since $r(x,\xi) \in ESG^{1,-1}_{x,\xi}$, it follows that

$$|\tilde{y}| \prec c_1 \langle \xi \rangle,$$

on $Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$, which is (8.74). Statement (8.76) follows from (8.74) and (8.73).

Since $r(x,\xi) \in ESG^{1,-1}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^{n_\xi})$ and $\langle \xi \rangle \sim \langle \theta \rangle$ on $Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))$ it follows that

$$|r(x,\xi)^n| \text{ satisfies } SG^{n,-n}_{x,\theta} \text{ estimates on } Supp(\tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta)). \tag{8.77}$$

By definition,

$$c(x,z,\theta) := \tilde{h}\,(x, \tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), z, \xi, \eta))|r(x,\xi)^n|,$$

so by (8.71) and (8.77),we have $c(x,z,\theta) \in SG^{m_1+m_2+t_1+n, t_2, m_3+t_3-n}_{x,z,\theta}$, by the basic facts about SG functions. Since $\langle x \rangle \sim \langle z \rangle$ on $Supp(c)$ we have $c(x,z,\theta) \in SG^{p,q,m_3+t_3-n}_{x,z,\theta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n_\xi+n_\eta+n})$ where p, q are any real numbers with $p + q = m_1 + m_2 + t_1 + t_2 + n$.

Returning to (8.70), by writing the $\tilde{y}, \xi, \eta$ integrals as one $n_\xi+n_\eta+n$ dimensional θ integral and using our definitions of c and ζ we have

$$(A \circ B)\,u(x) = \lim_{\delta \longrightarrow 0} \int_{\mathbb{R}^{n_\xi+n_\eta+n}} \int_{\mathbb{R}^n} e^{i\zeta(x,z,\theta)} c(x,z,\theta)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}\theta.$$

The mollifier $\gamma(\delta\eta)$ can easily be replaced by a mollifer in θ once we have checked that $\zeta \in \mathcal{Q}(c)$. In the above, equality is modulo Ku for $K \in \mathcal{K}$. This completes part 1.

Part 2

We start by presenting a few facts.

It follows from Proposition 8.2.4 and (8.62) ... (8.65) that on $Supp(c)$ we have:

$$f(x,\xi) \text{ satisfies } SG^{1,1}_{x,\theta} \text{ estimates,} \tag{8.78}$$

$$g((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\xi) \text{ satisfies } SG^{1,1}_{x,\theta} \text{ estimates,} \tag{8.79}$$

$$u((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\eta) \text{ satisfies } SG^{1,1}_{x,\theta} \text{ estimates,} \tag{8.80}$$

$$v(z,\eta) \text{ satisfies } SG^{1,1}_{z,\theta} \text{ estimates,} \tag{8.81}$$

From (8.66) to (8.68) , we also have the following estimates on $Supp(c)$:

$$\langle \nabla_1 f(x,\xi)\rangle \sim \langle \xi \rangle \tag{8.82}$$

$$\langle \nabla_1 g\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\xi\right)\rangle \sim \langle \xi \rangle \tag{8.83}$$

$$\langle \nabla_1 u\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\eta\right)\rangle \sim \langle \eta \rangle \tag{8.84}$$

$$\sim \langle (\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi]\rangle \sim \langle \tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi)\rangle \tag{8.85}$$

We remark that $(\nabla_{\xi'}g)_l^{-1}[\tilde{y}r(x,\xi) - (\nabla_{\xi'}f)_r(x,\xi),\xi])$ satisfies $SG^{1,0}_{x,\theta}$ estimates on $Supp(c)$. To see this note that $(\nabla_{\xi'}g)_l^{-1}(s,\xi)$ satisfies $SG^{1,0}_{s,\xi}$ estimates on $Supp(\tilde{h}(x,s,z,\xi,\eta))$ and apply Proposition 8.2.4

We will postpone the definition of V_ζ until later and we'll start checking that $\zeta \in \mathcal{Q}(c)$.

Condition 1: $w(x,\theta)$ **satisfies** $SG^{1,1}_{x,\theta}$ **estimates on** $Supp(c)$. This is true by (8.78),(8.79),(8.80) and the basic facts about SG functions.

Condition 2: $v(z,\theta)$ **satisfies** $SG^{1,1}_{x,\theta}$ **estimates on** $Supp(c)$. This has been proved already. See (8.81)

Condition 3: $\langle \nabla_x w(x,\theta)\rangle \sim \langle\theta\rangle$ **on** $Supp(c)$. For convenience, define

$$\tilde{g}(x,\tilde{y},\xi) := g((\nabla_{\xi'} g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi),\xi),\xi)$$

$$\tilde{u}(x,\tilde{y},\xi,\eta) := u((\nabla_{\xi'} g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi),\xi),\eta)$$

Using this notation, we have

$$w(x,\theta) = f(x,\xi) + \tilde{g}(x,\tilde{y},\xi) + \tilde{u}(x,\tilde{y},\xi,\eta).$$

By assumption, $\nabla_x f(x,\xi) \succ \langle\xi\rangle$ when ξ'' so the inequality holds on $Supp(c)$. As $\langle\theta\rangle \sim \langle\xi\rangle$ on $Supp(c)$, we'll be done if we can show that

$$\nabla_x \left[\tilde{g}(x,\tilde{y},\xi) + \tilde{u}(x,\tilde{y},\xi,\eta)\right] \prec c_i\langle\xi\rangle \tag{8.86}$$

on $Supp(c)$ for some c_i. Recall that c_i is the constant in the cut-off χ_i.

Differentiating $[\tilde{g}(x,\tilde{y},\xi) + \tilde{u}(x,\tilde{y},\xi,\eta)]$ we have

$$\begin{aligned} &\partial_{x_j}\left[\tilde{g}(x,\tilde{y},\xi) + \tilde{u}(x,\tilde{y},\xi,\eta)\right] = \\ &\sum_{l=1}^{n}\Big[(\partial_{1_l} g)\left((\nabla_{\xi'} g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi),\xi),\xi\right) + \\ &+ (\partial_{1_l} u)\left((\nabla_{\xi'} g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi),\xi),\eta\right)\Big]\times \\ &\times \partial_{x_j}(\nabla_{\xi'} g)^{-1}_l(\tilde{y}r(x,\xi) - (\nabla_{\xi'} f)_r(x,\xi)). \end{aligned} \tag{8.87}$$

The function $c(x, z, \theta)$ is a product of functions, one of which is

$$\chi_5\left(\nabla_1 u\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r - \nabla_{\xi'}f, \xi], \eta\right), \nabla_1 g\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r - \nabla_{\xi'}f, \xi], \xi\right)\right).$$

So

$$\begin{aligned} &|\nabla_1 g\left((\nabla_{\xi'}g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi),\xi\right) + \\ &\nabla_1 u\left((\nabla_{\xi'}g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi),\eta\right)| \\ &\prec c_5\langle\nabla_1 g\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\xi\right)\rangle, \end{aligned} \tag{8.88}$$

on $Supp(c)$. Also

$$\langle\nabla_1 g\left((\nabla_{\xi'}g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi],\xi\right)\rangle \sim \langle\xi\rangle$$

globally. So, by (8.88) we have

$$\begin{aligned} &|\nabla_1 g\left((\nabla_{\xi'}g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi),\xi\right) + \\ &\nabla_1 u\left((\nabla_{\xi'}g)^{-1}(\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi),\eta\right)| \prec c_5\langle\xi\rangle, \end{aligned} \tag{8.89}$$

on $Supp(c)$. Also, we showed earlier $(\nabla_{\xi'}g)_l^{-1}(\tilde{y}r(x,\xi) - (\nabla_{\xi'}f)_r(x,\xi))$ satisfies $SG^{1,0}_{x,\theta}$ estimates on $Supp(c)$. So, on $Supp(c)$, we have

$$\partial_{x_j}(\nabla_{\xi'}g)_l^{-1}(\tilde{y}r(x,\xi) - (\nabla_{\xi'}f)_r(x,\xi)) \prec 1, \tag{8.90}$$

for $l = 1, \ldots, n$. Therefore, by (8.87), (8.89) and (8.90) we have $\nabla_x[\tilde{g} + \tilde{u}] \prec c_5\langle\xi\rangle$, which is (8.86).

Condition 4: $\langle \nabla_z v(z,\theta)\rangle \sim \langle\theta\rangle$ on $Supp(c)$.

We have already established that $\langle \nabla_z v(z,\eta)\rangle \sim \langle\eta\rangle$ on $Supp(c)$. Noting that $\langle\eta\rangle \sim \langle\theta\rangle$ on $Supp(c)$ shows that Condition 4 is satisfied.

The phase component v is unaltered by the changes of variable. So the following conditions involving v are trivially satisfied.

Condition 5: $\langle \nabla_{\theta'} v\rangle \sim \langle z\rangle$.

Condition 6: $\left|\det\left(\partial_{z_i}\partial_{\eta'_j} v(z,\theta)\right)_{j=1}^{n}\right| \succ 1$

Condition 7: $\partial_{z_i}\partial_{\theta'_j} v \prec 1$ for all i,j

Condition 8: $|\nabla_z v(z,\theta)| \to \infty$ as $|\theta'| \to \infty$.

We will now discuss our choice of V_ζ. Given any $\epsilon > 0$, define

$$V_{\zeta,\epsilon} := \{(\tilde{y},\xi',\xi'',\eta'') \text{ such that } \tilde{y},\xi' \in \mathbb{R}^n, \xi'' \in V_\Phi, \eta'' \in V_\Psi \text{ and } |\tilde{y}| < \epsilon\langle\xi\rangle\}.$$

We have $|\tilde{y}| \prec c_1\langle\xi\rangle$ on $Supp(c)$ (as well as $Supp(c(x,z,\eta',\tilde{y},\xi',\xi'',\eta'')) \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times V_\Phi \times V_\eta$.) So, for any $\epsilon > 0$, we can choose c_1 small enough so that

$$Supp(c(x,z,\theta',\theta'')) \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}. \tag{8.91}$$

We will prove that for ϵ small enough we have conditions 9 to 12 holding with $V_\zeta = V_{\zeta,\epsilon}$

Condition 9 $\partial_{x_i}\partial_{\theta_j} w(x,\theta',\theta'') \prec 1$ for all $i,j=1,\dots,n$.

We will show that this condition is satisfied on $\{(x,\theta) : x,\theta' \in \mathbb{R}^n \theta'' \in V_{\zeta,\epsilon}\}$ for any ϵ. We have

$$\partial_{x_i}\partial_{\eta'_j} w(x,\theta) = \sum_{r=1}^{n} \left(\partial_{1_r}\partial_{2'_j} u\right) \left((\nabla_{\xi'} g)^{-1} [\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta\right) \times$$

$$\times \partial_{x_i}\big[\, (\nabla_{\xi'} g)_r^{-1} [\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta)\big]. \quad (8.92)$$

By assumption,

$$\partial_{1_r}\partial_{2'} u\,(y,\eta) \prec 1 \text{ on } \{(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^{n_\eta} : \eta'' \in V_\Psi\}. \quad (8.93)$$

Also:

1. For any $i,r \in \{1,\dots,n\}$ we have $\partial_1 \left(\nabla_{\xi'} g\right)_r^{-1} \prec 1$ on $\mathbb{R}^n \times \mathbb{R}^n$ by assumptions 5, 6, 7 and 8,
2. By assumption we have $\partial_{x_i}\partial_{\xi'_j} f(x,\xi) \prec 1$ on $\{x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, \xi'' \in V_\Phi\}$ for any $i,j \in \{1,\dots,n\}$,
3. Straight from the definitions of $V_{\zeta,\epsilon}$ and $r(x,\xi)$ we have $\partial_{x_j}\tilde{y}r(x,\xi) \prec \epsilon$ for $\theta'' \in V_{\zeta,\epsilon}$.

It follows from 1, 2 and 3 above that for any fixed ϵ, we have

$$\partial_{x_i} \left(\nabla_{\xi'} g\right)_r^{-1} [\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta) \prec 1, \quad (8.94)$$

on $\{x, \xi', \eta' \in \mathbb{R}^n, \eta'' \in V_\Psi, \xi'' \in V_\Phi\}$.

We're done by (8.92), (8.93) and (8.94).

Condition 10 $\det\left(\partial_{x_i}\partial_{\theta'_j}w(x,\theta)\right)_{i,j=1}^{n} \succ 1$ on $\mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}$ for sufficiently small ϵ.

We have

$$\partial_{\eta'_j}w = \left(\partial_{2'_j}u\right)\left((\nabla_{\xi'}g)^{-1}\left[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi\right],\eta\right). \tag{8.95}$$

Differentiating with respect to x_i gives

$$\sum_{\rho=1}^{n}\left(\partial_{1_\rho}\partial_{2'_j}u\right)\left((\nabla_{\xi'}g)^{-1}\left[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi\right],\eta\right)\times$$
$$\times\partial_{x_i}\left\{(\nabla_{\xi'}g)_\rho^{-1}\left[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi\right]\right\}. \tag{8.96}$$

Now, $|\det\left(\left(\partial_{1_\rho}\partial_{2'_j}u\right)(y,\eta)\right)_{\rho,j=1}^{n}| \succ 1$ on $\{(y,\eta) : \eta'' \in V_\Psi\}$. So we have

$$|\det\left(\partial_{1_\rho}\partial_{2'_j}u\right)\left((\nabla_{\xi'}g)^{-1}\left[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi\right],\eta\right)| \succ 1 \tag{8.97}$$

for $(x,\theta',\theta'') \in \mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}$, for any $\epsilon > 0$. Indeed the implicit constant does not depend on ϵ. We have only used the fact that on $V_{\zeta,\epsilon}$ we have $\eta'' \in V_\Psi$.

Let W be the matrix with (i,j) entry

$$\partial_{x_i}\left\{(\nabla_{\xi'}g)_j^{-1}\left[\tilde{y}r(x,\xi) - \nabla_{\xi'}f(x,\xi),\xi\right]\right\}. \tag{8.98}$$

So, by (8.96) and (8.97), if we can show that $\det W \succ 1$ on $\{(x,\theta) : \theta'' \in V_{\zeta,\epsilon}\}$ for some ϵ we'll be done. Suppose that $W = M + N$ where the matrices M and N have the following properties:

1. $\det M \succ 1$ on $\{(x,\theta) : x,\theta' \in \mathbb{R}^n, \theta'' \in V_{\zeta,\epsilon}\}$ for any ϵ, (with implicit constant independent of ϵ)

2. $M_{i,j} \prec 1$ on $\{(x,\theta) : x, \theta' \in \mathbb{R}^n, \theta'' \in V_{\zeta,\epsilon}\}$ for any ϵ, (with implicit constant independent of ϵ)

3. $N_{i,j} \prec \epsilon$ on $\{(x,\theta) : x, \theta' \in \mathbb{R}^n, \theta'' \in V_{\zeta,\epsilon}\}$ for any ϵ.

Then we can write

$$\det W = \det M + \sum_j h_j$$

where the sum is finite and each h_j is a product elements of M and N, where at least one of the elements comes from N. Clearly, we will then be able to choose $\epsilon = \epsilon_0$ small enough so that $\det W \succ 1$ on $\{(x,\theta) : x, \theta' \in \mathbb{R}^n, \theta'' \in V_{\zeta,\epsilon_0}\}$.

We will now show that W can indeed be written as a sum of matrices M and N which have the properties 1, 2 and 3 above. Expanding (8.98), we obtain

$$W_{i,j} = \sum_{k=1}^{n} \partial_{1_k} \left(\nabla_{\xi'} g\right)_j^{-1} \left(\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right) \left[\partial_{x_i}(\tilde{y}_k r(x,\xi)) + \partial_{x_i}\partial_{\xi'_k} f(x,\xi)\right].$$

Define

$$\begin{aligned} M &:= \sum_{k=1}^{n} \partial_{1_k} \left(\nabla_{\xi'} g\right)_j^{-1} \left[\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right] \partial_{x_i}\partial_{\xi'_k} f(x,\xi) \\ N &:= \sum_{k=1}^{n} \partial_{1_k} \left(\nabla_{\xi'} g\right)_j^{-1} \left[\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi\right] \partial_{x_i} \left(\tilde{y}_k r(x,\xi)\right) \end{aligned} \tag{8.99}$$

M **has properties 1 and 2.** We have $\det \left(\partial_{1_k} \left(\nabla_{\xi'} g\right)_j^{-1}\right)_{k,j=1}^{n} (s,\xi) \succ 1$ on $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$. We have $\det \left(\partial_{x_i}\partial_{\xi'_k} f(x,\xi)\right)_{i,k=1}^{n} \succ 1$ on

$\{(x,\xi) : x, \xi' \in \mathbb{R}^n \text{ and } \xi'' \in V_\Phi\}$ So, we have $\det M \succ 1$ on $\{(x,\xi) : x, \xi' \in \mathbb{R}^n \text{ and } \xi'' \in V_{\zeta,\epsilon}\}$ for all ϵ which is property 1. (and the implicit constant does not depend on ϵ)

We also have $M_{i,j} \prec 1$ on because $\partial_{1_k} (\nabla_{\xi'} g)_j^{-1} (y, \xi) \prec 1$ on $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$ and $\partial_{x_i} \partial_{\xi'_k} f(x, \xi) \prec 1$ on $\{(x, \xi) : x, \xi' \in \mathbb{R}^n \text{ and } \xi'' \in V_\Phi\}$ by assumption. Again the implicit constant does not depend on ϵ.

N has property 3. For $N_{i,j}$ note that $\partial_{1_k} (\nabla_{\xi'} g)_j^{-1} \prec 1$, and $\partial_{x_i} \tilde{y}_k r(x, \xi) \prec \epsilon$ on $\{(x, \theta) : x, \theta' \in \mathbb{R}^n \text{ and } \theta'' \in V_{\zeta,\epsilon}\}$. So, it is clear that $N_{i,j} \prec \epsilon$ on $\{(x, \theta) : \theta'' \in V_{\zeta,\epsilon}\}$.

Condition 11 $\nabla_{\theta'} w \succ \langle x \rangle$ on $\mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}$. We have

$$\nabla_{\theta'} w(x, \theta) = (\nabla_{2'} u) \left((\nabla_{\xi'} g)^{-1} (\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi), \eta \right).$$

On the set $\{(y, \eta) : y, \eta' \in \mathbb{R}^n \text{ and } \eta'' \in V_\Psi$ we have $(\nabla_{2'} u(y, \eta)) \succ \langle y \rangle$. So, on $\{(x, \theta', \theta'') \in \mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}\}$ we have

$$(\nabla_{2'} u) \left((\nabla_{\xi'} g)^{-1} (\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi), \eta \right) \succ \langle (\nabla_{\xi'} g)^{-1} (\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi) \rangle$$

We have $\langle (\nabla_{\xi'} g)^{-1} (s, \xi) \rangle \sim \langle s \rangle$ on $\mathbb{R}^n \times \mathbb{R}^{n_\xi}$. So it follows that

$$\langle \nabla_{\theta'} w(x, \theta) \rangle \succ \langle \tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi) \rangle,$$

on the set $\{(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi + n_\eta + n} : \theta'' \in V_{\zeta,\epsilon}\}$. By the triangle inequality,

$$\langle \tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi) \rangle \geq \langle \nabla_{\xi'} f(x, \xi) \rangle - |\tilde{y} r(x, \xi)|$$

By assumption, $\xi'' \in V_\Phi$ implies that $\langle \nabla_{\xi'} f(x, \xi) \rangle \succ \langle x \rangle$ and on $\{(x, \theta) \in \mathbb{R}^n \times \mathbb{R}^{n_\xi + n_\eta + n} : \theta'' \in V_{\zeta,\epsilon}\}$ we have

$$|\tilde{y} r(x, \xi)| \prec \epsilon \langle x \rangle.$$

So, for small enough ϵ, we'll have $\nabla_{\theta'} w \succ \langle x \rangle$ on $\{(x, \theta', \theta'') \in \mathbb{R}^n \times \mathbb{R}^n \times V_{\zeta,\epsilon}\}$.

Condition 12. For fixed $x \in \mathbb{R}^n$ and $\theta'' \in V_{\zeta,\epsilon}$ we have $|\nabla_x w(x, \theta)| \to \infty$ and $|\theta'| \to \infty$.

We are interested in the behaviour of $\nabla_x w$ as $|\eta'| \to \infty$ for fixed values of the other variables. Recall that $\theta' = \eta'$. So we only need to consider the limiting behaviour of the x gradient of the parts of w which depend on η'. That is, the behaviour of

$$\nabla_x \left[u \left((\nabla_{\xi'} g)^{-1} [\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi], \eta \right) \right].$$

We have

$$\begin{aligned} \partial_{x_j} u \left((\nabla_{\xi'} g)^{-1} [\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi], \eta \right) = \\ \sum_{s=1}^{n} (\partial_{1_s} u) \left((\nabla_{\xi'} g)^{-1} [\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi], \eta \right) \times \\ \times \partial_{x_j} \left[(\nabla_{\xi'} g)_s^{-1} [\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi] \right]. \end{aligned} \tag{8.100}$$

As argued earlier (see condition 10), we can choose ϵ small enough so that for any fixed $x \in \mathbb{R}^n, \theta'' \in V_{\zeta,\epsilon}$ the matrix

$$\left(\partial_{x_j} \left[(\nabla_{\xi'} g)_s^{-1} [\tilde{y} r(x, \xi) - \nabla_{\xi'} f(x, \xi), \xi] \right] \right)_{s,j=1}^{n}$$

has bounded inverse. Then, by (8.100), we can conclude that for fixed $x \in \mathbb{R}^n, \theta'' \in$

$V_{\zeta,\epsilon}$ we have

$$\nabla_x \left[u\left((\nabla_{\xi'} g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta\right)\right] \succ$$

$$|\,(\nabla_1 u)\left((\nabla_{\xi'} g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta\right)|. \tag{8.101}$$

For $\theta'' \in V_{\zeta,\epsilon}$, and $x \in \mathbb{R}^n$, we have

$$|\,(\nabla_1 u)\left((\nabla_{\xi'} g)^{-1}[\tilde{y}r(x,\xi) - \nabla_{\xi'} f(x,\xi), \xi], \eta\right)| \to \infty$$

as $|\eta'| \to \infty$ because u is a phase component and $\theta'' \in V_{\zeta,\epsilon}$ implies that $\eta'' \in V_\Psi$. □

Chapter 9

Type $\mathcal{R}$ Fourier Integral Operator

9.1 Introduction

In the definition of the Type $\mathcal{Q}$ operator class we assumed that the mixed spatial derivatives of the phase were zero everywhere - i.e. that $\partial_{x_i}\partial_{y_j}\Phi(x,y,\xi) \equiv 0$ for all $i,j = 1,\ldots,n$. The reason for this assumption was purely technical. I wanted the Type $\mathcal{Q}$ operator class to have the property that the composition of a type $\mathcal{Q}$ operator and its adjoint was pseudodifferential and I could not make the necessary (structure preserving) change of variables without this mixed derivative assumption. The Type $\mathcal{R}$ class was born out of the desire to define an SG FIO operator class with the property that $A \circ A^\star$ and $A^\star \circ A$ are pseudodifferential without assuming that $\partial_{x_i}\partial_{y_j}\Phi(x,y,\xi) \equiv 0$ for all $i,j = 1,\ldots,n$.

9.2 Operator Definition.

Notation For $\Phi(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ we will write $\partial_1 \partial_2 \Phi(x, y, \xi)$ to mean the $n \times n$ matrix with i, j entry $\partial_{1_j} \partial_{2_i} \Phi(x, y, \xi)$ with $\partial_1 \partial_3 \Phi(x, y, \xi)$ etc. similarly defined.

The definition of the Type $\mathcal{R}$ phase is on the next page.

Definition 9.2.1. *Let $\Phi(x,y,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. We say that $\Phi(x,y,\xi) \in \mathcal{R}$ if it has the following properties:*

$$\text{For } j = 1, \ldots, n, \text{ we have } \partial_{x_j}\Phi \in SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n), \tag{9.1}$$

$$\text{For } j = 1, \ldots, n, \text{ we have } \partial_{y_j}\Phi \in SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n), \tag{9.2}$$

$$\langle \nabla_x \Phi \rangle \succ \langle \xi \rangle, \tag{9.3}$$

$$\langle \nabla_y \Phi \rangle \succ \langle \xi \rangle, \tag{9.4}$$

$$\text{For all multi-indices } \gamma \text{ we have } \partial_\xi^\gamma \Phi \prec (\langle x \rangle + \langle y \rangle) \langle \xi \rangle^{1-|\gamma|}, \tag{9.5}$$

$$\text{For } i,j = 1, \ldots, n, \text{ we have } (\langle x \rangle + \langle y \rangle)\, \partial_{x_i}\partial_{y_j}\Phi \in SG^{0,0,1}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n), \tag{9.6}$$

$$\det\left(\partial_{x_i}\partial_{\xi_j}\Phi\right)_{i,j=1}^n \succ 1, \tag{9.7}$$

$$\det\left(\partial_{y_i}\partial_{\xi_j}\Phi\right)_{i,j=1}^n \succ 1, \tag{9.8}$$

There exists some function $g(x,\xi) \in SG^{1,0}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ with $\langle g \rangle \succ \langle x \rangle$ and

$$\det\left(\partial_{x_i} g_j\right)_{i,j=1}^n \succ 1, \text{ such that } |\nabla_\xi \Phi(x,y,\xi)| \sim |y - g(x,\xi)|, \tag{9.9}$$

$$\det \begin{pmatrix} \partial_1\partial_2\Phi(x,y,\xi) & \partial_3\partial_2\Phi(x,y,\xi) \\ \partial_1\partial_3\Phi(x,y,\xi) & \partial_3\partial_3\Phi(x,y,\xi) \end{pmatrix} \succ 1, \tag{9.10}$$

$$\det\big[-\partial_1\partial_2\Phi(x,g(x,\xi),\xi)\,(\partial_1\partial_3\Phi)^{-1}(x,g(x,\xi),\xi)\partial_3\partial_3\Phi(x,g(x,\xi),\xi) + \partial_3\partial_2\Phi(x,g(x,\xi),\xi)\big] \succ 1, \tag{9.11}$$

$$\det\big[-\partial_2\partial_1\Phi(x,g(x,\xi),\xi)\,(\partial_2\partial_3\Phi)^{-1}(x,g(x,\xi),\xi)\partial_3\partial_3\Phi(x,g(x,\xi),\xi) + \partial_3\partial_1\Phi(x,g(x,\xi),\xi)\big] \succ 1 \tag{9.12}$$

Remarks Define

$$M := \begin{pmatrix} \partial_1\partial_2\Phi(x,y,\xi) & \partial_3\partial_2\Phi(x,y,\xi) \\ \partial_1\partial_3\Phi(x,y,\xi) & \partial_3\partial_3\Phi(x,y,\xi) \end{pmatrix}$$

Assumptions (9.1),(9.2),(9.5) and (9.6) mean that $\det(M)$ belongs to $SG^{0,0,0}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. So assumption (9.10) is an ellipticity assumption on $\det M$.

We shall call the set $\{x,y,\xi \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \nabla_\xi\Phi(x,y,\xi) = 0\}$ the "zero set" of $\nabla_\xi\Phi(x,y,\xi)$. Assumption (9.9) means the zero set has a specific global parameterisation.

Our phase assumptions mean that we have two "main" global charts on the zero set of $\nabla_\xi\Phi(x,y,\xi)$. Using our parameterisation $y = g(x,\xi)$ of the zero set, the coordinates are $\big(x, \nabla_1\Phi(x,g(x,\xi),\xi)\big)$ and $\big(g(x,\xi), -\nabla_2\Phi(x,g(x,\xi),\xi)$. The assumptions (9.11) and (9.12) (along with some others) mean that the coordinate change is a global SG diffeomorphism. So, the zero set of $\nabla_\xi\Phi(x,y,\xi)$ is an SG manifold.

We define the Type $\mathcal{R}$ operator in the standard way.

Definition 9.2.2. *Let $\Phi(x,y,\xi) \in \mathcal{R}$, $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ with m_1, m_2, m_3 arbitrary real numbers and let $\gamma(\epsilon\xi)$ be a mollifier. Define the Type $\mathcal{R}$ operator $A_{\Phi,a}$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$ as follows:*

$$A_{\Phi,a}u(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\Phi(x,y,\xi)} a(x,y,\xi)\gamma(\epsilon\xi)u(y)\mathrm{d}y\mathrm{d}\xi. \tag{9.13}$$

Given that Type $\mathcal{R}$ operators are also Type $\mathcal{P}$ operators we obtain the following result straightaway.

Proposition 9.2.3. *Let $\Phi(x,y,\xi) \in \mathcal{R}$ let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ for any $m_1, m_2, m_3 \in \mathbb{R}$ and let $A = FIO\left(\Phi(x,y,\xi), a(x,y,\xi)\right)$. Then*

1. *$A : \mathcal{S}\left(\mathbb{R}^n\right) \to \mathcal{S}\left(\mathbb{R}^n\right)$ continuously,*

2. *A is independent of the choice of mollifier,*

3. *$A^T = FIO(\Phi(y,x,\xi), a(y,x,\xi))$,*

4. *A extends to a continuous operator from $\mathcal{S}'\left(\mathbb{R}^n\right)$ to $\mathcal{S}'\left(\mathbb{R}^n\right)$.*

5. *For $\chi \in \Xi^{\Delta}(c)$ with $c < 1$, we have*

$$A = FIO(\Phi, a(x,y,\xi)\chi_1(y, g(x,\xi))\chi_1(g(x,\xi), y)) \text{ modulo } \mathcal{K}.$$

Proof. If $\Phi(x,y,\xi) \in \mathcal{R}$, then $\Phi(x,y,\xi) \in \mathcal{P}$. So the first 4 statements follow from the corresponding results for Type $\mathcal{P}$ operators. Statement 5 follows from Theorem 3.2.2. □

Before proving that the composition of a Type $\mathcal{R}$ with its adjoint is pseudodifferential we present some sufficient conditions for a real-valued smooth function Φ to have the zero-set parameterisation structure described in (9.9).

Theorem 9.2.4. *Suppose that a real valued function $\Phi(x, y, \xi)$ has the following properties:*

$$\Phi(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}), \tag{9.14}$$

$$\det \partial_{x_i}\partial_{\xi_j}\Phi \succ 1, \tag{9.15}$$

$$\det \partial_{y_i}\partial_{\xi_j}\Phi \succ 1, \tag{9.16}$$

$$\partial_{x_j}\Phi \in SG^{0,0,1}_{x,y,\xi}, \tag{9.17}$$

$$\partial_{y_j}\Phi \in SG^{0,0,1}_{x,y,\xi}, \tag{9.18}$$

$$\partial^\gamma_\xi\Phi \prec (\langle x\rangle + \langle y\rangle)\langle\xi\rangle^{1-|\gamma|}, \tag{9.19}$$

$$\text{For all fixed } x, \xi \in \mathbb{R}^n \quad \nabla_\xi\Phi(x, y, \xi) \to \infty \text{ as } |y| \to \infty, \tag{9.20}$$

$$\text{For all fixed } y, \xi \in \mathbb{R}^n \quad \nabla_\xi\Phi(x, y, \xi) \to \infty \text{ as } |x| \to \infty, \tag{9.21}$$

Then we have $\nabla_\xi\Phi(x, y, \xi) \sim |y - g(x, \xi)|$, where the vector valued function $g(x, \xi)$ has the following properties:

$$g(x, \xi) \in SG^{1,0}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n), \tag{9.22}$$

$$\langle g(x, \xi)\rangle \succ \langle x\rangle, \tag{9.23}$$

$$\det\,(\partial_{x_i}g_j)^n_{i,j=1} \succ 1. \tag{9.24}$$

Remark In [3], Coriasco calls functions $g(x, \xi)$ satisfying (9.22) , (9.23) and (9.24) "SG diffeomorphisms in x with parameter ξ. "

Proof. **Existence of** $g(x, \xi)$. Define $F := (x, \nabla_\xi\Phi(x, y, \xi), \xi)$, and define $w :=$

(x, y, ξ). We claim that the function $F \in Diffeo(\mathbb{R}^{3n}, \mathbb{R}^{3n})$. By assumption (9.20), we have $|F(w)| \to \infty$ as $|w| \to \infty$. Also, the Jacobian of F is $\det \left(\partial_{y_i}\partial_{\xi_j}\Phi\right)_{i,j=1}^{n}$ which is globally bounded below by some real constant. So by the Global Inverse Function Theorem , $F \in Diffeo(\mathbb{R}^{3n}, \mathbb{R}^{3n})$. As F is a bijection, for all $x, \xi \in \mathbb{R}^n$, there is a unique $y \in \mathbb{R}^n$ such that $\nabla_\xi \Phi(x, y, \xi) = 0$. Let $g(x, \xi)$ denote this y. So $\nabla_\xi \Phi(x, y, \xi) = 0 \iff y = g(x, \xi)$. It follows from the local Implicit Function Theorem that $g(x, \xi)$ is smooth on $\mathbb{R}^n \times \mathbb{R}^n$.

Similarly define $G := (\nabla_\xi \Phi(x, y, \xi), y, \xi)$. Arguing as for F, we have $G \in Diffeo(\mathbb{R}^{3n}, \mathbb{R}^{3n})$. So there exists a function $h(y, \xi)$ such that $\nabla_\xi \Phi(x, y, \xi) = 0 \iff x = h(y, \xi)$.

It follows that $y = g(h(y, \xi), \xi)$ for $y, \xi \in \mathbb{R}^n$. So, for all fixed $\xi \in \mathbb{R}^n$, $g(x, \xi)$ is globally invertible in x.

Determinant Condition.

Now, we have $\nabla_3 \Phi(x, g(x, \xi), \xi) = 0$ for all $x, \xi \in \mathbb{R}^n$. Differentiating the *ith* component with respect to x_j and rearranging gives

$$\partial_{1_j}\partial_{3_i}\Phi(x, g(x, \xi), \xi) = -\sum_{r=1}^{n} \partial_{2_r}\partial_{3_i}\Phi(x, g(x, \xi), \xi)\partial_{x_j} g_r(x, \xi). \tag{9.25}$$

Now, consider the matrix M with i, r entry $\partial_{2_r}\partial_{3_i}\Phi(x, g(x, \xi), \xi)$. By assumption $-\partial_{2_r}\partial_{\xi_i}\Phi(x, y, \xi) \in SG^{0,0,0}_{x,y,\xi}$ for all $i, r = 1, \dots, n$. It follows that $\det M \prec 1$ for all $x, \xi \in \mathbb{R}^n$. It follows from assumption (9.16) that $\det M(x, \xi) \succ 1$. So $\det M^{-1} = \frac{1}{\det M} \sim 1$. Let N be the matrix with i, j entry $\left(\partial_{1_j}\partial_{3_i}\Phi(x, g(x, \xi), \xi)\right)$. It follows from assumption (9.15) that $\det N \succ 1$ for $x, \xi \in \mathbb{R}^n$. (We actually have $\det N \sim 1$). It follows from (9.25) that the matrix with i, r entry $\partial_{x_i} g_r(x,\xi) = M^{-1}N$. So, $(\partial_x g(x,\xi))_{i,j=1}^n$ a product of two matrices with determinants which are globally bounded below. So, $\det (\partial_x g(x,\xi))_{i,j=1}^n \succ 1$.

Condition: $\nabla_\xi \Phi(x,y,\xi) \sim y - g(x,\xi)$

Consider $\nabla_\xi \Phi(x,y,\xi) - \nabla_\xi \Phi(x, g(x,\xi), \xi)$. It follows easily from the Mean Value Theorem and assumption (9.18) that $\nabla_\xi \Phi(x,y,\xi) \prec y - g(x,\xi)$. Now, for all fixed $x, \xi \in \mathbb{R}^n$, $\nabla_\xi \Phi(x,y,\xi)$ is globally invertible as a function of y. So, setting $y = (\nabla_\xi \Phi)^{-1}(x,v,\xi)$ and recalling that $g(x,\xi) = (\nabla_\xi)^{-1}(x,0,\xi)$, we have

$$y - g(x,\xi) = (\nabla_\xi \Phi)^{-1}(x,v,\xi) - (\nabla_\xi \Phi)^{-1}(x,0,\xi). \tag{9.26}$$

It follows from assumptions (9.16) and (9.18) , that $\nabla_v (\nabla_\xi \Phi)^{-1}(x,v,\xi) \prec 1$. (Just differentiate the identity $\nabla_\xi \Phi(x, (\nabla_\xi \Phi)^{-1}(x,v,\xi), \xi) = v$ for all $x, v, \xi \in \mathbb{R}^n$.) Using this fact and the the Mean Value Theorem, it follows from (9.26) that we have

$$y - g(x,\xi) \prec v.$$

Since $v = \nabla_\xi \Phi(x,y,\xi)$, we have

$$y - g(x,\xi) \prec \nabla_\xi \Phi(x,y,\xi). \tag{9.27}$$

SG properties of $g(x,\xi)$

To start with, note that by assumption 9.19 we have

$$\nabla_\xi \Phi(x,0,\xi) \prec \langle x \rangle. \tag{9.28}$$

Since $\nabla_\xi \Phi(x, y, \xi) \sim y - g(x, \xi)$, we have $\nabla_\xi \Phi(x, 0, \xi) \sim g(x, \xi)$. So it follows from (9.28) that

$$\langle g(x, \xi) \rangle \prec \langle x \rangle.$$

By considering $\nabla_\xi \Phi(0, x, \xi)$ and arguing similarly we can show that $g^{-1}(x, \xi) \prec \langle x \rangle$ which implies that

$$\langle g(x, \xi) \rangle \succ \langle x \rangle.$$

It follows from differentiation of the identity $\nabla_\xi \Phi(x, g(x, \xi), \xi) = 0$ and the assumptions (9.14), ..., (9.19) that $g(x, \xi) \in SG^{1,0}_{x,\xi}$. We also use the fact that $\langle g(x, \xi) \rangle \sim \langle x \rangle$. □

9.3 Composition Theorems

9.3.1 Composition with a Pseudodifferential Operator

Since $\mathcal{R} \subset \mathcal{P}$, we obtain the following result by Theorem 5.0.2.

Theorem 9.3.1. *Let $\Phi(x,y,\xi) \in \mathcal{R}$, $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}$ and $p(x,y,\xi) \in SG^{t_1,t_2,t_3}_{x,y,\xi}$. Define $A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$ and $P := Op(p(x,y,\xi))$. Then the compositions $P \circ A$ and $A \circ P$ are Type $\mathcal{R}$ FIOs with amplitudes in the expected SG classes. We also obtain the expected asymptotic expansions for the amplitudes.*

9.3.2 Composition with the Adjoint Operator

Before proving that the composition of a Type $\mathcal{R}$ operator with its adjoint is a ψdo, we collect a few Lemmas which we'll need.

Lemma 9.3.2. *Let $V_c := \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^n : |x-z| \leq c\langle x\rangle\}$ for $c > 0$ and let $\Phi \in \mathcal{R}$. Then, for sufficiently small c we have*

1. *$\int_0^1 \nabla_2\Phi(y, x+t(z-x), \eta)\mathrm{d}t$ satisfies $SG^{0,0,0,1}_{x,y,z,\eta}$ estimates on $\{(x,z,y,\eta) \in uR4n : (x,z) \in V_c\}$,*

2. *$\langle \int_0^1 \nabla_2\Phi(y, x+t(z-x), \eta)\mathrm{d}t\rangle \sim \langle\eta\rangle$, on $\{(x,z,y,\eta) \in uR4n : (x,z) \in V_c\}$,*

3. *$\partial_{y_j} \int_0^1 \nabla_2\Phi(y, x+t(z-x), \eta)\mathrm{d}t \prec \frac{\langle\eta\rangle}{\langle x\rangle+\langle y\rangle}$, on $\{(x,z,y,\eta) \in uR4n : (x,z) \in V_c\}$,*

4. $\partial_{x_i}\partial_{y_j}\int_0^1 \nabla_2\Phi(y, x+t(z-x), \eta)\mathrm{d}t \prec \frac{\langle\eta\rangle}{(\langle x\rangle+\langle y\rangle)\langle x\rangle}$, *on* $\{(x, z, y, \eta) \in uR4n : (x, z) \in V_c\}$,

5. *The implicit constants in 1,2, 3 and 4 above are independent of c.*

Proof. We claim that if $c < \frac{1}{2}$, we have $\langle x+t(z-x)\rangle \sim \langle x\rangle \sim \langle z\rangle$ for $(x, z) \in V_c$ and $t \in [0, 1]$ with the implicit constants are independent of c. Once we prove the claim the statements 1 and 2 can be proved by following the proof of 7.2.2. Statement 3 follows easily from the claim and phase assumption 9.6. Statement 4 follows from the claim and assumptions (9.2) and (9.7), as we now explain. By assumption (9.7), we have $(\langle x\rangle + \langle y\rangle)\, \partial_{x_i}\partial_{y_j}\Phi(x, y, \xi) \in SG^{0,0,1}_{x,y,\xi}$, for all i, j. Statement 4 follows from differentiating $(\langle x\rangle + \langle y\rangle)\, \partial_{x_i}\partial_{y_j}\Phi(x, y, \xi)$ and using assumption (9.2).

We now prove the claim. Since $|x-z| \leq c\langle x\rangle$ on V_c and $|z-x| \geq \langle z\rangle - \langle x\rangle$ everywhere, we have $\langle z\rangle \leq (c+1)\langle x\rangle$. By similar arguments, we have $(1-c)\langle x\rangle \leq \langle z\rangle$ on V_c. Consider $\langle x+t(z-x)\rangle$. It's obvious that $\langle x+t(z-x)\rangle \leq (c+1)\langle x\rangle$ for $(x, z) \in V_c$ and $t \in [0, 1]$ (by the triangle inequality). Now, $\langle x+t(z-x)\rangle \geq \langle x\rangle - t|z-x|$ everywhere, so $\langle x+t(z-x)\rangle \geq (1-c)\langle x\rangle$ for $(x, z) \in V_c$ and $t \in [0, 1]$. This is enough to prove the claim.

□

Lemma 9.3.3. *Let* $c = (c_1, c_2)$ *where* $c_1, c_2 > 0$, *define* $W_c := \{(x, y, z, \xi, \eta) \in \mathbb{R}^{5n} : |x-z| < 2c_1\langle x\rangle$ *and* $|\xi| < 2c_2\}$ *and let* $\Phi \in \mathcal{R}$. *Then, for* c_1 *and* c_2 *sufficiently small we have*

1. $\eta + \xi\langle \int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t\rangle$ *satisfies* $SG^{0,0,0,0,1}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ *estimates on* W_c,

2. $\langle \eta + \xi\langle \int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t\rangle\rangle \sim \langle\eta\rangle$ *on* W_c,

3. $\langle \eta + s\xi\langle \int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t\rangle\rangle \sim \langle\eta\rangle$ *for* $s \in [0, 1]$ *and* $(x, y, z, \xi, \eta) \in W_c$,

4. *If* $h(x, y, z, \omega, \eta) \in SG^{m_1,m_2,m_3,m_4,m_5}_{x,y,z,\omega,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$, *then*

$$h(x, y, z, \eta + \xi\langle \int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t\rangle, \eta)$$

satisfies $SG^{m_1,m_2,m_3,0,m_4+m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ *on* W_c,

5. *The implicit constants in 1,2 ,3 and 4 above are independent of* c_1, c_2.

Proof. Define $r(x, y, z, \eta) := \int_0^1 \nabla_2\Phi(y, x+t(z-x), \eta)\mathrm{d}t$. By Lemma 9.3.2 $r(x, y, z, \eta)$ satisfies $SG^{0,0,0,1}_{x,y,z,\eta}$ estimates on W_c and $\langle r(x, y, z, \eta)\rangle \sim \langle\eta\rangle$ with implicit constants independent of c when c_1 is small enough. The statements 1,2,3,4 and 5 follow easily by the now standard arguments. □

Lemma 9.3.4. *Let* $\Phi \in \mathcal{R}$ *and let* $c = (c_1, c_2)$ *where* $c_1, c_2 > 0$. *Define the map*

$F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ *as follows:*

$$F : \begin{pmatrix} x \\ y \\ z \\ \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} x \\ \int_0^1 \nabla_3\Phi(y, x, \eta + s\xi\langle\int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t\rangle)\mathrm{d}s + x \\ z \\ \xi \\ \int_0^1 \nabla_2\Phi(y, x + t(z - x), \eta)\mathrm{d}t \end{pmatrix} := \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix},$$

and define $W_c := \{(x, y, z, \xi, \eta) \in \mathbb{R}^{5n} : |x - z| < 2c_1\langle x\rangle$ *and* $|\xi| < 2c_2\}$. *For sufficiently small* c_1 *and* c_2 *the map* F *is a smooth diffeomorphism from* W_c *to itself.*

Proof. The fact that $F(W_c) \subset W_c$ is obvious from the definition of W_c and the fact that F leaves x, z and ξ unchanged. Define $V_c := \{(x, z, \xi) \in \mathbb{R}^{3n} : |x - z| < 2c_1\langle x\rangle$ and $|\xi| < 2c_2\}$. We will prove two facts.

1. We can choose c_1, c_2 to be sufficiently small to ensure that for any $(x, z, \xi) \in V_c$, the (x, z, ξ) section of F is a proper map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^n$,

2. We can choose c_1, c_2 to be sufficiently small to ensure that the Jacobian of $F \succ 1$ on W_c.

If 1 and 2 hold, we're done. To see this, note that because of the form of F, the Jacobian of F is equal to the Jacobian of the x, z, ξ section of F. So, statements 1 and 2 mean that the x, z, ξ section of F is a bijection from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}^n$

(by the Global Inverse Function Theorem.) This implies that F is a bijection from W_c to W_c. Smoothness of the inverse of F follows from statement 2, the Local Inverse Function Theorem and the fact that F is smooth.

Statement 1. By Lemma 9.3.2 for sufficiently small c_1 we have $\langle F_5\rangle \sim \langle\eta\rangle$ on W_c. So, we only need to check that c can be taken small enough so that for fixed any $x, z, \xi \in V_c$ and η bounded we have $|F_2| \to \infty$ as $|y| \to \infty$. Consider

$$F_2 = \int_0^1 \nabla_3\Phi(y, x, \eta + t\xi\langle F_5(x, y, z, \eta)\rangle)\mathrm{d}t$$

Adding and subtracting $\int_0^1 \nabla_3\Phi(y, x, \eta)\mathrm{d}t$ we have

$$\int_0^1 \nabla_3\Phi(y, x, \eta + t\xi\langle F_5(x, y, z, \eta)\rangle)\mathrm{d}t = H(x, y, z, \xi, \eta) + G(x, y, z, \xi, \eta)$$

where $H := \int_0^1 \left[\nabla_3\Phi(y, x, \eta + t\xi\langle F_5(x, y, z, \eta)\rangle) - \nabla_3\Phi(y, x, \eta)\right]\mathrm{d}t$ and $G := \int_0^1 \nabla_3\Phi(y, x, \eta)\mathrm{d}t$. It follows from the Mean Value Theorem, Lemma 9.3.3 part 3, Lemma 9.3.2 and the phase assumptions that we have

$$H \prec c_2(\langle y\rangle + \langle x\rangle). \tag{9.29}$$

on W_c for c_1 and c_2 small enough. We are interested in the behavior of F_2 as $|y| \to \infty$, for x, z, ξ fixed in V_c and η bounded. So we can assume that $|y| > k\langle x\rangle$ for some $k > 1$. Doing so, we have

$$H \prec c_2\langle y\rangle, \tag{9.30}$$

on $W_c \bigcap\{(x, y, z, \xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |y| > k\langle x\rangle\}$. By assumption, $G \succ y - g(x, \eta)$ where $g(x, \eta) \in SG^{1,0}_{x,\eta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\langle g(x, \xi)\rangle \sim \langle x\rangle$. By the triangle inequality, $|G| \succ \frac{1}{2}\langle y\rangle + \frac{1}{2}\langle y\rangle - \langle g(x,\xi)\rangle$. If we choose k so that $\frac{1}{2}\langle y\rangle - \langle g(x,\xi)\rangle > 0$ we have

$$G \succ \langle y\rangle, \tag{9.31}$$

on $W_c \bigcap \{(x,y,z,\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |y| > k\langle x\rangle\}$. Since $F_2 = H(x,y,z,\xi,\eta) + G(x,y,z,\xi,\eta)$ we're done by (9.30) and (9.31).

Statement 2 We now show that, for sufficiently small c_1 and c_2, the Jacobian of $F \succ 1$ on W_c. We will denote the Jacobian matrix of F by N and we will show that $\det N = \det(R+S)$ where $\det R \succ 1$ on W_c, $R_{i,j} \prec 1$ on W_c, and $S_{i,j} \prec f_{i,j}(c_1,c_2)$ where the real valued functions $f_{i,j}$ tend to zero as $c_1, c_2 \to 0$. This is enough, as we've argued previously. Define

$$M(y,x,\eta) := \begin{pmatrix} \partial_1\partial_3\Phi(y,x,\eta) & \partial_3\partial_3\Phi(y,x,\eta) \\ \partial_1\partial_2\Phi(y,x,\eta) & \partial_3\partial_2\Phi(y,x,\eta). \end{pmatrix} \tag{9.32}$$

where $\partial_1\partial_3\Phi(y,x,\eta)$ is the $n \times n$ matrix with i,j entry $\partial_{1_j}\partial_{3_i}\Phi(y,x,\eta)$ and the other blocks are defined similarly. By assumption $\det M \succ 1$. We obviously have

$$N = M + (N - M).$$

Consider $N - M$. By the Mean Value Theorem, we have

$$N - M = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

where

$$\begin{aligned}
\tilde{A}_{i,j} &= \sum_{r=1}^{n} \int_0^1 \int_0^1 \partial_{1_j}\partial_{3_i}\partial_{3_r}\Phi(y,x,n+tt'\xi\langle F_5\rangle)(t\xi_r\langle F_5\rangle)\mathrm{d}t\mathrm{d}t' + \\
&\quad + \sum_{r=1}^{n} \int_0^1 \partial_{3_r}\partial_{3_i}\Phi(y,x,n+t\xi\langle F_5\rangle)t\xi_r\partial_{y_j}\langle F_5\rangle \mathrm{d}t \\
\tilde{B}_{i,j} &= \sum_{r=1}^{n} \int_0^1 \int_0^1 \partial_{3_j}\partial_{3_i}\partial_{3_r}\Phi(y,x,n+tt'\xi\langle F_5\rangle)[t\xi_r\langle F_5\rangle]\mathrm{d}t\mathrm{d}t' + \\
&\quad + \sum_{r=1}^{n} \int_0^1 \partial_{3_r}\partial_{3_i}\Phi(y,x,n+t\xi\langle F_5\rangle)[t\xi_r\partial_{\eta_j}\langle F_5\rangle]\mathrm{d}t \\
\tilde{C}_{i,j} &= \int_0^1 \partial_{1_j}\partial_{2_j}\partial_{2_r}\Phi(y,x+t(z-x),\eta)(z_r-x_r)\mathrm{d}t \\
\tilde{D}_{i,j} &= \int_0^1 \partial_{3_j}\partial_{2_j}\partial_{2_r}\Phi(y,x+t(z-x),\eta)(z_r-x_r)\mathrm{d}t
\end{aligned} \tag{9.33}$$

It follows from the above, Lemma 9.3.2 (in particular parts 3 and 4) , Lemma 9.3.3 and the phase assumptions that

$$\begin{aligned}
\tilde{A}_{i,j} &\prec c_2 \\
\tilde{B}_{i,j} &\prec c_2 \frac{(\langle y\rangle + \langle x\rangle)}{\langle\eta\rangle} \\
\tilde{C}_{i,j} &\prec c_1 \frac{\langle\eta\rangle}{\langle y\rangle + \langle x\rangle} \\
\tilde{D}_{i,j} &\prec c_1.
\end{aligned} \tag{9.34}$$

Consider $M+(N-M)$. If we multiply the first n rows of $M+(N-M)$ by $\frac{\langle\eta\rangle}{\langle y\rangle+\langle x\rangle}$ and then multiply the first n columns by $\frac{(\langle y\rangle+\langle x\rangle)}{\langle\eta\rangle}$ we obtain a new matrix $R+S$. (R and S are the matrices obtained by performing the forementioned operations on M and $N - M$ respectively.) Because of cancellation, the determinant of $\det[M + (N - M)] = \det[R + S]$. Obviously $\det[M + (N - M)] = \det[N]$. So we have

$$\det[N] = \det[R + S].$$

Recall that N denotes the Jacobian matrix of F. Also, because of cancellation, $\det[R] = \det[M] \succ 1$. It follows from the phase assumptions that $R_{i,j} \prec 1$. By our estimates 9.34 we have $S_{i,j} \prec f_{i,j}(c_1, c_2)$ for $i, j = 1, \ldots, n$ where the real valued functions $f_{i,j}$ tend to zero as $c_1, c_2 \to 0$. It follows that for sufficiently small c_1 and c_2 we have $\det N \succ 1$ on W_c.

□

Lemma 9.3.5. *Let $b(x, y, z, \xi, \eta) \in SG^{m_1,m_2,m_3,0,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ be such that $|\xi| < \epsilon < 1$ on Supp(b). Then if we define an new variable $\omega := \eta + \xi\langle\eta\rangle$ then the function*

$$f(x, y, z, \omega, \eta) := b(x, y, z, \frac{\omega - \eta}{\langle\eta\rangle}, \eta)$$

belongs to $SG^{m_1,m_2,m_3,0,m_5}_{x,y,z,\omega,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$.

Proof. First note that on *Supp(f)* we have $|\omega - \eta| < \epsilon\langle\eta\rangle$. Since $\epsilon < 1$ this implies that $\langle\omega\rangle \sim \langle\eta\rangle$ on *Supp(f)*. We only need to consider derivatives with respect to ω and η. As $b(x, y, z, \xi, \eta) \in SG^{m_1,m_2,m_3,0,m_5}_{x,y,z,\xi,\eta}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ the other estimates are obvious, once we obtain the required estimates for ω and η derivatives. Now, $\partial^\delta_\omega f = \langle\eta\rangle^{-|\delta|}(\partial^\delta_4 b)(x, y, z, \frac{\omega-\eta}{\langle\eta\rangle}, \eta)$ and $(\partial^\delta_4 b)(x, y, z, \xi, \eta) \in$

$SG^{m_1,m_2,m_3,0,m_5}_{x,y,z,\xi,\eta}$. Given that $\langle\eta\rangle \sim \langle\omega\rangle$ on $Supp(f)$, and $\langle\eta\rangle^{-|\delta|} \in SG^{-|\delta|}_{\eta}$ we'll be done if we can show that

$$\partial^{\epsilon}_{\eta}\left[(\partial^{\delta}_{4}b)(x,y,z,\frac{\omega-\eta}{\langle\eta\rangle},\eta)\right] \prec \langle x\rangle^{m_1}\langle y\rangle^{m_2}\langle z\rangle^{m_3}\langle\eta\rangle^{m_5-|\epsilon|}.$$

This follows from Lemma 8.2.1 on the structure of derivatives of compositions of smooth functions and the fact that $\partial^{\theta}_{\eta}\left(\frac{\omega-\eta}{\langle\eta\rangle}\right) \prec \langle\eta\rangle^{-|\theta|}$ on $Supp(f)$.

□

Theorem 9.3.6. *Let $\Phi(x,y,\xi) \in \mathcal{R}$ and let $a(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$. Define*

$A := FIO(\Phi(x,y,\xi), a(x,y,\xi))$ and let $A^\star$ denote the adjoint of A. Then the compositions $A \circ A^\star$ and $A^\star \circ A$ are pseudodifferential operators and we obtain asymptotic expansions for their respective symbols.

Proof. We will prove the theorem for $A^\star \circ A$. Once we prove the theorem for $A^\star \circ A$ the result for $A \circ A^\star$ follows. (We can just define $B := A^\star$ and consider $B^\star \circ B = A^\star \circ A$.) In this proof $\int$ means $\int_{\mathbb{R}^n}$. As ever, when considering this composition modulo $\mathcal{K}$, we only need to study the composition of the reduced forms of the operators. By composing the reduced forms, and following the now standard arguments we have

$$(A^\star \circ A)\, u(x) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \iiiint \exp\{i(\Phi(y,z,\eta) - \Phi(y,x,\xi))\}\times$$
$$\chi_1(x, g(y,\xi))\chi_1(g(y,\xi),x)\chi_2(z,g(y,\eta))\chi_2(g(y,\eta),z)\times$$
$$a(y,z,\eta)\overline{a}(y,x,\xi)\gamma(\delta\eta)\gamma(\delta\xi)u(z)\mathrm{d}z\mathrm{d}y\mathrm{d}\xi\mathrm{d}\eta.$$

where $\chi_i \in \Xi^\Delta(c_i)$ and we can choose c_1, c_2 freely in $(0, \frac{1}{2})$. In fact we will take $c_2 = c_1$. Now, we introduce a cut-off $\chi_3\left(\nabla_1\Phi(y, g(y,\xi),\xi), \nabla_1\Phi(y, g(y,\eta),\eta)\right)$ with $\chi_i \in \Xi^\Delta(c_i)$. It's easy to check that for any c_3, we can choose c_1 to be small enough so that on the support of

$$\chi_1(x, g(y,\xi))\chi_1(g(y,\xi),x)\chi_2(z,g(y,\eta))\chi_2(g(y,\eta),z)\times$$
$$\times\,(1-\chi_3)\left(\nabla_1\Phi(y, g(y,\xi),\xi), \nabla_1\Phi(y, g(y,\eta),\eta)\right)$$

we have $\nabla_y\Phi(y,z,\eta) - \nabla_y\Phi(y,x,\xi) \succ \langle\eta\rangle + \langle\xi\rangle$. So, by the now standard arguments, we have

$$(A^\star \circ A)\, u(x) = \lim_{\epsilon\to 0}\lim_{\delta\to 0} \iiiint \exp\{i(\Phi(y,z,\eta) - \Phi(y,x,\xi))\}\times$$
$$\chi_1(x, g(y,\xi))\chi_1(g(y,\xi),x)\chi_2(z,g(y,\eta))\chi_2(g(y,\eta),z)\times$$
$$\chi_3\left(\nabla_1\Phi(y, g(y,\xi),\xi), \nabla_1\Phi(y, g(y,\eta),\eta)\right) a(y,z,\eta)\overline{a}(y,x,\xi)\gamma(\delta\eta)\gamma(\delta\xi)u(z)\mathrm{d}z\mathrm{d}y\mathrm{d}\xi\mathrm{d}\eta.$$

modulo Ku for $K \in \mathcal{K}$.

For convenience define

$$h(x,y,z,\xi,\eta) := a(y,z,\eta)\overline{a}(y,x,\xi)\chi_1(x,g(y,\xi))\chi_1(g(y,\xi),x)\times$$

$$\chi_2(z,g(y,\eta))\chi_2(g(y,\eta),z)\chi_3\left(\nabla_1\Phi(y,g(y,\xi),\xi),\nabla_1\Phi(y,g(y,\eta),\eta)\right). \tag{9.35}$$

Note that we choose c_1 small after choosing c_3. This is not a restriction as we can still take each constant c_i to be as small as we want.

On $Supp(h)$ we have $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle\xi\rangle \sim \langle\eta\rangle$ with the implicit constants independent of the c_i. Further, we have $\eta-\xi \prec c_3\langle\eta\rangle$ on $Supp(h)$. To see this, define $\tilde{\nabla_1}\Phi(y,\xi) := \nabla_1\Phi(y,g(y,\xi),\xi)$. It follows from the phase assumptions that the y section of $\tilde{\nabla_1}\Phi(y,\xi)$ is invertible with $\left(\tilde{\nabla_1}\Phi\right)^{-1}(y,\xi) \in SG^{0,1}_{y,\xi}$. This fact implies that $\xi-\eta \prec \nabla_1\Phi(y,g(y,\xi),\xi)-\nabla_1\Phi(y,g(y,\eta),\eta)$. Since $\langle\nabla_1\Phi(y,g(y,\eta),\eta)\rangle \sim \langle\eta\rangle$ by assumption, it's clear that $\eta-\xi \prec c_3\langle\eta\rangle$ on $Supp(h)$.

We have shown that $\eta-\xi \prec c_3\langle\eta\rangle$ on the support of $\chi_3\left(\nabla_1\Phi(y,g(y,\xi),\xi),\nabla_1\Phi(y,g(y,\eta),\eta)\right)$. Since h is a product of functions, some of which are $\chi_2(z,g(y,\eta))$ and $\chi_1(x,g(y,\xi))$, it follows that for any $c_4>0$ we can have c_3 small enough so that $Supp(h)$ is contained in $\{(x,y,z,\xi,\eta)\in\mathbb{R}^{5n} : |x-z| < 2c_4\langle z\rangle\}$.

To summarize, after reduction, modulo operators with Schwartz kernel applied to u we have

$$(A^\star \circ A)\, u(x) = \lim_{\delta \to 0} \iiiint \exp\{i(\Phi(y,z,\eta) - \Phi(y,x,\xi))\} \times$$
$$h(x,y,z,\xi,\eta)\gamma_\epsilon(\xi)\gamma(\delta\eta)u(z)\mathrm{d}z\mathrm{d}y\mathrm{d}\xi\mathrm{d}\eta.$$

where

1. $h(x,y,z,\xi,\eta) \in SG^{m_2,2m_1,m_2,m_3,m_3}_{x,y,z,\xi,\eta}$,

2. $\langle x\rangle \sim \langle y\rangle \sim \langle z\rangle$ and $\langle\xi\rangle \sim \langle\eta\rangle$ on $Supp(h)$ with the implicit constants independent of the c_i,

3. for any $c_4, c_5 > 0$ we can choose c_1 and c_3 sufficiently small so that $Supp(h)$ is contained in the set $\{(x,y,z,\xi,\eta) \in \mathbb{R}^{5n} : |x - z| < 2c_4\langle z\rangle \text{and } |\xi - \eta| < 2c_5\langle\eta\rangle\}$.

Define the new variable $\tilde{\xi}$ implicitly by the equation

$$\tilde{\xi}\langle \int_0^1 \nabla_2\Phi(y, x + t(z-x), \eta)\mathrm{d}t\rangle = \xi - \eta.$$

This change of variables is globally defined. Making this change, $h(x,y,z,\xi,\eta)$ goes over to the function $m(x,y,z,\tilde{\xi},\eta) := h(x,y,z,\eta+\tilde{\xi}\langle\int_0^1 \nabla_2\Phi(y,x+t(z-x),\eta)\mathrm{d}t\rangle,\eta)$ with the support of m contained in

$W_c := \{(x,y,z,\xi,\eta) \in \mathbb{R}^{5n} : |x-z| < 2c_4\langle z\rangle \text{and } |\tilde{\xi}| \prec c_5\}$. By Lemma 9.3.3 part 4, for c_4 and c_5 sufficiently small, the function m satisfies $SG^{m_2,2m_1,m_2,0,2m_3}_{x,y,z,\tilde{\xi},\eta}$ estimates on W_c which contains its support.

(The choice of zero as the SG order of $\tilde{\xi}$ is arbitrary. Since $|\tilde{\xi}|$ is bounded on $Supp(m)$ we could have chosen any real number.)

Defining

$$P(x, y, z, \tilde{\xi}, \eta) := \Phi(y, z, \eta) - \Phi(y, x, \eta + \tilde{\xi}\langle \int_0^1 \nabla_2 \Phi(y, x + t(z - x), \eta) \mathrm{d}t \rangle),$$

and writing out the integrals we have

$$(A^\star \circ A)\, u(x) = \lim_{\delta \to 0} \iiiint \exp\{iP(x, y, z, \tilde{\xi}, \eta)\} \times$$
$$m(x, y, z, \tilde{\xi}, \eta) \langle \int_0^1 \nabla_2 \Phi(y, x + t(z - x), \eta) \mathrm{d}t \rangle^n \gamma(\delta\eta) u(z) \mathrm{d}z \mathrm{d}y \mathrm{d}\tilde{\xi} \mathrm{d}\eta.$$

For convenience, we'll drop the tilde on the ξ variable. By adding and subtracting $\Phi(y, x, \eta)$ we can re-write P in the following way:

$$P(x, y, z, \xi, \eta) = \Phi(y, z, \eta) - \Phi(y, x, \eta) + \Phi(y, x, \eta) -$$
$$-\Phi(y, x, \eta + \xi \langle \int_0^1 \nabla_2 \Phi(y, x + t(z - x), \eta) \mathrm{d}t \rangle). \qquad (9.36)$$

By the mean value theorem we have

$$P = \sum_{r=1}^{n} \Big[\int_0^1 \partial_{2_r} \Phi(y, x + t(z - x), \eta)(z_r - x_r) \mathrm{d}t -$$
$$\int_0^1 \partial_{3_r} \Phi(y, x, \eta + s\xi \langle \int_0^1 \nabla_2 \Phi(y, x + t(z - x), \eta) \mathrm{d}t \rangle) \times$$
$$\times \xi_r \langle \int_0^1 \nabla_2 \Phi(y, x + t(z - x), \eta) \mathrm{d}t \rangle \mathrm{d}s \Big].$$

Define the map $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$F : \begin{pmatrix} x \\ y \\ z \\ \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} x \\ \int_0^1 \nabla_3\Phi(y, x, \eta + s\xi\langle\int_0^1 \nabla_2\Phi(y, x + t(z-x), \eta)\mathrm{d}t\rangle)\mathrm{d}s + x \\ z \\ \xi \\ \int_0^1 \nabla_2\Phi(y, x + t(z-x), \eta)\mathrm{d}t \end{pmatrix} := \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{pmatrix}$$

We will define new variables as follows

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} := \begin{pmatrix} F_1(x, y, z, \xi, \eta) \\ F_2(x, y, z, \xi, \eta) \\ F_3(x, y, z, \xi, \eta) \\ F_4(x, y, z, \xi, \eta) \\ F_5(x, y, z, \xi, \eta) \end{pmatrix}. \tag{9.37}$$

Note that $F_2 = \int_0^1 \nabla_3\Phi(y, x, \eta + t\xi\langle F_5\rangle)) + x$. Define

$$W_c := \{(x, y, z, \xi, \eta) \in \mathbb{R}^{5n} : |x - z| < 2c_4\langle x\rangle \text{ and } |\xi| < 2c_5\}.$$

By Lemma 9.3.4 F is a smooth diffeomorphism from W_c to itself. (For sufficiently small c_4, c_5.)

As we explain below, we also have

$$\begin{aligned}
&F_1(x,y,z,\xi,\eta) \text{ satisfies } SG^{1,0,0,0,0}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)\\
&\qquad\qquad\qquad\qquad\text{estimates on } Supp(m),\\
&F_2(x,y,z,\xi,\eta) \text{ satisfies } SG^{0,1,0,0,0}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)\\
&\qquad\qquad\qquad\qquad\text{estimates on } Supp(m),\\
&F_3(x,y,z,\xi,\eta) \text{ satisfies } SG^{0,0,1,0,0}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)\\
&\qquad\qquad\qquad\qquad\text{estimates on } Supp(m),\\
&F_4(x,y,z,\xi,\eta) \text{ satisfies } SG^{0,0,0,1,0}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)\\
&\qquad\qquad\qquad\qquad\text{estimates on } Supp(m),\\
&F_5(x,y,z,\xi,\eta) \text{ satisfies } SG^{0,0,0,0,1}_{x,y,z,\xi,\eta}(\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)\\
&\qquad\qquad\qquad\qquad\text{estimates on } Supp(m).
\end{aligned} \tag{9.38}$$

The above statement about F_5 follows from Lemma 9.3.2. The above statement about F_2 follows from Lemma 9.3.3 part 3, the phase assumptions and the fact that $\langle x\rangle \sim \langle y\rangle$ on $Supp(m)$. The rest are obvious. It follows from (9.38) that

$$\text{The Jacobian of } F \text{ satisfies } SG^{0,0,0,0,0}_{x,y,z,\xi,\eta} \text{ estimates on } Supp(m) \tag{9.39}$$

To obtain (9.39), just multiply rows and columns of the Jacobian matrix by SG functions to reduce it to a matrix of $SG^{0,0,0,0,0}_{x,y,z,\xi,\eta}$ functions as we did in Chapter 6. Everything we multiply by cancels.

In the proof of Lemma 9.3.4 we showed that for sufficiently small c_4, c_5

$$\text{the Jacobian of } F \succ 1 \text{ on } W_c. \tag{9.40}$$

By standard arguments, if $c_1 = c_2$ and c_3 are small enough we have

$$\begin{aligned}
\langle F_1 \rangle &\sim \langle x \rangle \text{ on } Supp(m) \\
\langle F_2 \rangle &\sim \langle y \rangle \text{ on } Supp(m) \\
\langle F_3 \rangle &\sim \langle z \rangle \text{ on } Supp(m) \\
\langle F_4 \rangle &\sim \langle \xi \rangle \text{ on } Supp(m) \\
\langle F_5 \rangle &\sim \langle \eta \rangle \text{ on } Supp(m)
\end{aligned} \tag{9.41}$$

We can use (9.38), (9.39), (9.40) and (9.41) to complete the proof by our SG change of variables results. Define new variables as follows:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} := \begin{pmatrix} F_1(x,y,z,\xi,\eta) \\ F_2(x,y,z,\xi,\eta) \\ F_3(x,y,z,\xi,\eta) \\ F_4(x,y,z,\xi,\eta) \\ F_5(x,y,z,\xi,\eta) \end{pmatrix}. \tag{9.42}$$

By (9.38), (9.39), (9.40) and (9.41) the conditions of Proposition 6.2.5 are satisifed taking $V = W_c$ therein. Making the change of variables, we obtain

$$(A^\star \circ A)\, u(\tilde{x}) = \lim_{\delta\to 0} \iiiint \exp\{i(\tilde{x}-\tilde{z})\cdot\tilde{\eta} + (\tilde{y}-\tilde{x})\cdot\tilde{\xi}\langle\tilde{\eta}\rangle\}\times$$
$$b(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta})\gamma(\delta(F)_5^{-1}(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}))\langle\tilde{\eta}\rangle^n u(\tilde{z})\mathrm{d}\tilde{z}\mathrm{d}\tilde{y}\mathrm{d}\tilde{\xi}\mathrm{d}\tilde{\eta}.$$

where $b \in SG^{m_2,2m_1,m_2,0,2m_3}_{\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}}$ by Proposition 6.2.5. (The function b is the transformed version of the function m times the absolute value of the Jacobian of F^{-1}.) We also have $\langle\tilde{x}\rangle \sim \langle\tilde{y}\rangle \sim \langle\tilde{z}\rangle$ and $\tilde{\xi} < 2c_5$ on $Supp(b)$.

The function $\gamma(\delta(F)_5^{-1}(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}))$ can be replaced by the mollifier $\gamma(\delta\tilde{\eta})$ by integration by parts.

It is clear from (9.43) that the proof will be complete if we can show that

$$h(\tilde{x},\tilde{z},\tilde{\eta}) := \iint \exp\{i(\tilde{y}-\tilde{x})\cdot\tilde{\xi}\langle\tilde{\eta}\rangle\} b(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta})\langle\tilde{\eta}\rangle^n \mathrm{d}\tilde{y}\mathrm{d}\tilde{\xi}. \tag{9.43}$$

belongs to the appropriate SG class. By changing variables $\tilde{\xi} \to \omega$ where $\omega := \tilde{\xi}\langle\tilde{\eta}\rangle + \tilde{\eta}$ we obtain

$$h(\tilde{x},\tilde{z},\tilde{\eta}) := \iint \exp\{i(\tilde{y}-\tilde{x})\cdot(\omega-\tilde{\eta})\} b(\tilde{x},\tilde{y},\tilde{z},\frac{\omega-\tilde{\eta}}{\langle\tilde{\eta}\rangle},\tilde{\eta})\mathrm{d}\tilde{y}\mathrm{d}\omega. \tag{9.44}$$

Define

$$f(\tilde{x},\tilde{y},\tilde{z},\omega,\tilde{\eta}) := b(\tilde{x},\tilde{y},\tilde{z},\frac{\omega-\tilde{\eta}}{\langle\tilde{\eta}\rangle},\tilde{\eta}).$$

Since $\tilde{\xi} < 2c_5$ on $Supp(b(\tilde{x},\tilde{y},\tilde{z},\tilde{\xi},\tilde{\eta}))$, it follows from Lemma 9.3.5 that $b(\tilde{x},\tilde{y},\tilde{z},\frac{\omega-\tilde{\eta}}{\langle\tilde{\eta}\rangle},\tilde{\eta})$ belongs to $SG^{m_2,2m_1,m_2,0,2m_3}_{\tilde{x},\tilde{y},\tilde{z},\omega,\tilde{\eta}}$, provided $c_5 < \frac{1}{2}$.

We have $\langle \tilde{x} \rangle \sim \langle \tilde{y} \rangle \sim \langle \tilde{z} \rangle$ and $\langle \omega \rangle \sim \langle \tilde{\eta} \rangle$ on $Supp(f)$. So we can apply Proposition 4.0.4 to see that $h(\tilde{x}, \tilde{z}, \tilde{\eta}) \in SG^{p,q,2m_3}_{\tilde{x},\tilde{z},\tilde{\eta}}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ where p and q are any real numbers with $p + q = 2(m_1 + m_2)$.

□

9.4 Further Work

In addition to the study of the applications to Hyperbolic PDEs described in the introduction, it is of interest to know when two type $\mathcal{R}$ phases Φ and Ψ are equivalent. We say that two phases Φ, Ψ are equivalent if for any $a \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with m_1, m_2, m_3 arbitrary there exists an amplitude $b \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with m_1, m_2, m_3 such that

$$FIO(\Phi, a) = FIO(\Psi, b) \quad \text{modulo } \mathcal{K}.$$

In this section we prove some sufficient conditions under which a type $\mathcal{R}$ phase is equivalent to the pseudodifferential phase $(x - y) \cdot \xi$. We show that if $\Phi \in \mathcal{R}$ is such that $g(x, \xi) = x$ and $\nabla_1 \Phi = -\nabla_2 \Phi$ on $y = x$ then Φ is equivalent to $(x - y) \cdot \xi$.

In order to prove this result we need the following Lemma.

Lemma 9.4.1. *Let $\Phi(x, y, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ be such that $\nabla_\xi \Phi(x, x, \xi) = 0$ for all $x, \xi \in \mathbb{R}^n$ and $\nabla_1 \Phi(x, x, \xi) = -\nabla_2 \Phi(x, x, \xi)$ for all $x, \xi \in \mathbb{R}^n$. Then*

$$\exp\{i\Phi(x, x, \xi)\} \in SG^{0,0}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n). \tag{9.45}$$

Proof. All derivatives of $\exp\{i\Phi(x,x,\xi)\}$ of non-zero order are identically zero. □

Theorem 9.4.2. *Let $\Phi \in \mathcal{R}$ be such that $\nabla_\xi \Phi(x,x,\xi) \equiv 0$ and $\nabla_1 \Phi(x,x,\xi) = -\nabla_2\Phi(x,x,\xi)$ for all $x, \xi \in \mathbb{R}^n$. Then, for any $a \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ with m_1, m_2, m_3 arbitrary, there exists $b \in SG^{m_1,m_2,m_3}_{x,y,\xi}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$FIO(\Phi(x,y,\xi), a(x,y,\xi)) = Op(b(x,y,\xi))$$

modulo $\mathcal{K}$.

Proof. Let $\chi(y,x) \in \Xi^{\Delta}(c)$ where $c \in (0,1)$. We have

$$FIO(\Phi(x,y,\xi), a(x,y,\xi)) = FIO(\Phi(x,y,\xi), a(x,y,\xi)\chi(y,x))$$

modulo $\mathcal{K}$. Writing out the integrals, we have

$$Au(x) = \lim_{\delta\to 0} \iint \exp\{i\Phi(x,y,\xi)\} a(x,y,\xi)\chi(y,x)\gamma(\delta\xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

We now multiply the integrand by $\exp\{-i\Phi(x,x,\xi)\}\exp\{i\Phi(x,x,\xi)\}$ to get

$$Au(x) = \lim_{\delta\to 0} \iint \exp\{i\left[\Phi(x,y,\xi) - \Phi(x,x,\xi)\right]\}\tilde{a}(x,y,\xi)\gamma(\delta\xi)u(y)\mathrm{d}y\mathrm{d}\xi.$$

where we have defined $\tilde{a} := \exp\{i\Phi(x,x,\xi)\}a(x,y,\xi)\chi(y,x)$. By Lemma 9.4.1 $\exp\{i\Phi(x,x,\xi)\} \in SG^{0,0}_{x,\xi}(\mathbb{R}^n \times \mathbb{R}^n)$ and so by the basic facts about SG functions

$$\tilde{a}(x,y,\xi) \in SG^{m_1,m_2,m_3}_{x,y,\xi}.$$

Define $h(x,y,\xi) := -\int_0^1 \nabla_2\Phi(x, x+t(y-x),\xi)$ and set $\eta := h(x,y,\xi)$. We can apply the standard change of variables arguments to see that for sufficiently small

c we have

$$Au(x) = \lim_{\delta\to 0}\iint \exp\{i(x-y)\cdot\eta\}\tilde{a}(x,y,h^{-1}(x,y,\eta))|\det\partial_\eta h^{-1}(x,y,\eta)|\gamma(\delta\eta)u(y)\mathrm{d}y\mathrm{d}\eta.$$

where

1. h^{-1} means the inverse of the x, y section of h, which is well defined on $W_c := \{(x,y,\eta)\in\mathbb{R}^n : |y-x| < 2c\langle x\rangle\}$,
2. $\partial_\eta h^{-1}(x,y,\eta)$ is shorthand for the $n\times n$ matrix with i,j entry $\partial_{\eta_j (h^{-1})_j}(x,y,\xi)$,
3. $\tilde{a}(x,y,h^{-1}(x,y,\eta))|\det\partial_\eta h^{-1}(x,y,\eta)| := 0$ outside W_c,
4. $\tilde{a}(x,y,h^{-1}(x,y,\eta))|\det\partial_\eta h^{-1}(x,y,\eta)| \in SG^{m_1,m_2,m_3}_{x,y,\xi}$.

So A is equivalent to a pseudodifferential operator. □

Finally, it should be fairly easy to extend the definition of the Type $\mathcal{R}$ operator (to allow the dimension of the frequency variable ξ to be $\geq n$) and obtain closedness of this generalised class under composition.

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Bibliography

[1] M.S. Berger: *Nonlinearity and Functional Analysis* Academic Press (1977)

[2] H.O. Cordes: *The Technique of Pseudodifferential Operators*, London Mathematical Society Lecture Note Series. 202, Cambridge University Press (1995)

[3] S. Coriasco: *Fourier Integral Operators in SG Classes*, Rend. Sem. Mat. Univ. Pol. Torino, Volume 57, 4 (1999)

[4] S. Coriasco: *Fourier Integral Operators in SG Classes II: Application to SG Hyperbolic Cauchy Problems*, Ann. Univ. Ferrara - Sez. VII - Sc.Mat., Volume XLIV, 81-122 (1998)

[5] S. Coriasco, P. Panarese: *Fourier Integral Operators Defined by Classical Symbols with Exit Behaviour*, Math. Nachr., Volume 242, 61 - 78 (2002)

[6] S. Coriasco, L. Maniccia: *Wave Front Set at Infinity and Hyperbolic Operators with Multiple Characteristics*, Annals of Global Analysis and Geometry, Volume 24, 375 - 400 (2003)

[7] S. Coriasco, L. Rodino: *Cauchy problem for SG hyperbolic equations with constant multiplicities* Richerche Mat. 48 (1999)

[8] J.J.. Duistermaat: *Fourier Integral Operators*, Birkhauser(1996)

[9] Y.V. Egorov, M.A. Shubin: *Foundations of the Classical Theory of Partial Differential Equations*, Springer (1998)

[10] L.C. Evans: *Partial Differential Equations*, Graduate Studies in Mathematics, Volume 19 American Mathematical Society (1998)

[11] G.B. Folland: *Real Analysis: Modern Techniques and Their Applications*, Second Edition, Wiley-Interscience (1999)

[12] W.B. Gordon: *On the Diffeomorphisms of Euclidean Space* American Mathematical Monthly, Volume 79, No. 7, pp. 755-759 (1972)

[13] L. Hormander: *The Analysis of Linear Partial Differential Operators I*, Reprint of Second Edition, Springer (1990)

[14] L. Hörmander: *The Analysis of Linear Partial Differential Operators 2*, Reprint of Second Edition, Springer (1990)

[15] L. Hörmander: *The Analysis of Linear Partial Differential Operators 3*, Reprint of Second Edition, Springer (1990)

[16] L. Hörmander: *The Analysis of Linear Partial Differential Operators 4*, Reprint of Second Edition, Springer (1990)

[17] L. Hörmander: *Fourier Integral Operators 1*,

[18] L. Hörmander: *Pseudodifferential Operators* Comm. Pure Appl. Math. 18, 1965, 501 – 517

[19] C.J. Isham: *Modern Differential Geometry for Physicists* Second Edition World Scientific Lecture Notes in Physics - Vol. 61 (1999)

[20] I. Kamotski, M. Ruzhansky: *Regularity Properties, Representation of Solutions and Spectral Asymptotics of Systems with Multiplicities.*, Comm. Partial Differential Equations, Volume 32, 1 - 35 (2007)

[21] S.G. Krantz and H.R. Parks *The Implicit Function Theorem, history, theory and applications*, Birkhauser (2002)

[22] H. Kumano - go: *Pseudodifferential Operators*, MIT Press (1981)

[23] W. Rudin: *Real and Complex Analysis* Third edition, McGraw Hill International Edition (1987)

[24] A. Seeger, C.D. Sogge, E.M. Stein *Regularity Properties of Fourier Integral Operators* The Annals of Mathematics Vol. 134, No.2, pp. 231-251

[25] M.E. Taylor *Partial Differential Equations II; Qualitative Studies of Linear Equations* Springer (1996)

[26] F. Treves: *Introduction to Pseudodifferential and Fourier Integral Operators,* Volume Two, Plenum Press (1980)