IMPLEMENTING ARBITRAGE-FREE MODELS for PRICING CONVERTIBLE BONDS

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ABSTRACT

This thesis deals with the challenging task of developing frameworks for pricing convertible bonds. The theoretical foundation of the work presented here is arbitrage pricing because it allows relative pricing. Based on the principle of relative pricing, parameters of the model employed can be calibrated to market prices. Following this principle, the configurations developed in this thesis aim in ensuring the correct employment of the market implied (calibrated) parameters like risk free rate curves, volatility term structures, spot FX and share prices, etc. The numerical technique employed is the trinomial tree. An approach is presented for establishing the step dates which must include the event dates (call, put, coupon dates, etc). The first implementation is a single-dimension configuration with the stochastic stock returns. This implementation is extended in order to allow capturing the effect of conditional calls and puts, and resets, features that introduce path dependency in the model. The proposed approach for dealing with these features is the calculation of the conditional probabilities on the tree. Evaluation of the implementation of the single-dimension is carried out in the form of spectrum analysis and scenario analysis. Scenario analysis involves Monte Carlo simulations of portfolios consisting of hedged convertible bond positions and calculation of the re-hedging error on the maturity date. The last part of the thesis work is devoted to the development of a two-dimensional configuration for dual currency convertible bonds with the second additional dimension involving the stochastic returns of the exchange rate. The derived configuration with non-zero correlation is not implemented because it imposes restrictions on the use of the market parameters (specifically the term structures) and the definition of the step dates; hence it is not coherent with the approach followed in the preceding work. The respective configuration which assumes zero correlation is implemented instead and results are presented in the form of spectrum analysis.

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PART A

CHAPTER 1

INTRODUCTION

A Convertible Bond is a hybrid financial instrument combining attributes of both fixed-income securities and equity. In its simplest form, the owner of such a security has a position in a bond, but he is also allowed to trade the bond for a position in equities. However, since the convertible bond market has become more sophisticated, instruments with a rich variety of embedded options and triggering conditions have been presented in this category of hybrid financial securities.

The hybrid nature of the instruments in the convertible bonds market has been the most significant contributor to the continually expanding market. The convertible bonds dual nature offers to the investors an intermediate risk/return profile between fixed income (straight bonds) and equities. The considerable gap in the risk/return profiles between these two markets has been covered by the convertible bonds. Consequently, the convertible bond market has come to meet the needs of a vast class of investors who wanted more returns than those provided by the fixed income market, but without the high risks involved in the equity market. In addition, users of hybrid securities come from several groups since investors in convertible bonds are able to participate both in the fixed-income markets and the equity markets.

Due to the tremendous complexity and diversity of hybrid instruments like convertible securities, adequate analysis tools are crucial. Quantitative tools are especially important in this market since most issues are not very liquid and it may be difficult to establish an accurate market price. The need of convertible bond analysis tools, as well as the polymorphic spectrum of computational requirements which constitute these tools challenging and complicate, is proved by the presence and popularity of specialised software packages for pricing and analysing convertible bonds. Convertible bond modelling is as much as a mathematical challenge as it is an implementation challenge. A lot of processes which each carries a lot of approximations and assumptions are involved. Correct transformation of these approximations and assumptions, as well as a good level of understanding of their effects is required in order to maintain consistency among the various processes involved and perform meaningful calculations.

The work of this thesis is concentrated on implementing pricing frameworks for convertible securities. As an underlying principle, the approach to the development of the pricing frameworks is not to limit the description and calculations on just the actual convertible values estimation, but to deal with issues like the source of the input information and their method of extraction. These issues and methods are additional factors to account for in the model derivations and implementation methods. It will also provide uniformity and consistency of the developed frameworks in this thesis with existing market standard methodologies.

Approaches that fell under the category of arbitrage pricing will be employed. Models of this category follow the main assumption that there are no arbitrage opportunities in the market. The main benefit from this assumption is that there is no requirement of establishing or adopting the risk preferences of the investor. This constitutes the models more flexible and broadly useful. Furthermore, they encourage the adaptation and absorption of market information.

Trinomial trees methodology has been the employed numerical technique in this thesis for pricing the instrument. Monte Carlo, even though it is employed for the scenario simulations in this thesis, has not been used in this work for pricing the convertible bond because of its limitations in pricing instruments with American style optionalities, like the embedded equity option in the CB. The Finite Difference methods technique is also an alternative to the trinomial trees, and is a very popular and robust approach for pricing convertible bonds. However, the tree-based methodologies were preferred because the conditional tree probabilities introduced in this thesis is an extension of the trinomial tree.

Furthermore, when contrast to binomial trees, trinomial trees provide some additional features that are fully exploited in this work. The trinomial trees allow the definition of the time step independently of the space step. It is necessary in both cases, in binomial and trinomial trees, to have a fixed space step in order to obtain a recombining tree. If the configuration is not a recombining tree, then the process can involve very few steps. Hence, the recombining feature is very important for the work presented here. A fixed space step results in a fixed time step for binomial trees, while in the case of trinomial trees, a varying time step is still allowed without distorting the recombining nature of the tree. This means that in the case of trinomial trees there is the provision of two very important features not encountered in binomial trees, the freedom of defining the step dates based on contractual conditions in the CB (coupon, call, put and reset schedules), and the allowance of employing term structures (for the rate, the stock volatility, the credit spread and the continuous dividend yield).

In the one dimensional trinomial tree framework, the stock return is the underlying stochastic process. The framework is extended to include a second underlying stochastic process, the returns of the exchange rate. Calculation of the CB price and some of the sensitivities is carried out for both frameworks. The quality of the calculated values is evaluated through spectrum analysis and scenario simulations.

The ***objective*** of this thesis it to provide unified convertible bond pricing frameworks that are consistent with market prices of other liquid securities. The applicability of these pricing frameworks is also a significant factor, and for this reason, the limitations of the introduced pricing frameworks are identified in the concluding chapter.

1.1. Thesis Overview

This thesis consists of four distinct parts. It was decided to segment the thesis into four parts because it was recognised that in this way there is significant improvement in the presentation of the various approaches, implementations and methodologies. However, the four parts are totally linked, and can not be considered separately or in isolation. In other words, there is dependency between the various parts and each part continues building based on material presented in its preceding chapters.

Part A

This is the introductory part of the thesis. Chapter 2 is devoted to the introduction of the instrument under consideration, the convertible bond, and is entirely based on material found in identified references. Terminology and contract information are presented in this chapter. The convertible market is discussed and the range of convertible securities is also outlined. The various convertible securities are distinguished based on contract conditions depicting the conversion rights of the investor and based on level and timing of cash flows (coupons and notional) during the life of the convertible bond. In the last part of chapter 2, identification of the risk factors to the issuer and investor is performed.

Chapter 3 presents the fundamental concepts of arbitrage pricing and relative pricing. A major part of this chapter is devoted to the determination and extraction of the discount function based on information from liquid instruments from both, the Treasury market and the Swaps market as described in the relevant references. It is worth noting that the methodology that will be followed in parts B and C of this thesis will assume that the discount function has been derived based on information from the swaps market and that cubic splines interpolation was employed during the curve construction or wherever it was necessary to perform interpolation in the derived pricing frameworks in the rest of the chapters. This chapter closes with a short discussion on the extraction of other implied parameters from liquid instruments, like the implied volatilities of Equity derivatives. Then, in chapter 4, the inputs to a convertible bond relative pricing frameworks are listed and related notation is introduced.

Part B

The first chapter in part B, chapter 5, is used for outlining the method for establishing the required step dates and, consequently, the values of various parameters at each step, like the forward rates and volatilities. Then, in chapter 6, the analysis of a trinomial tree implementation is performed, where the stochastic process underlies the behaviour of the stock, like in the case of American option pricing. The structure of the trinomial tree framework is presented and studied, and then extended to include additional features which will enable us to deal successfully with contractual features of the CB like, call, put and reset schedules.

The first calculation of the convertible bond price and sensitivities is carried out in chapter 7. Based on the underlying stock process introduced in chapter 6, methods are presented for accounting for conditional calls and puts, and for resets. Then, the convertible bond price is calculated at each node on the tree. The sensitivities are calculated by accounting for the value of the convertible bond at the tree nodes at the first step, instead of recalculating the whole tree with shifted stock prices, except for the case of Vega, where a shifted volatility structure is used.

The one dimensional trinomial tree framework used for pricing the CB, is evaluated in chapter 8. A spectrum analysis is carried out for the value of the CB price and the sensitivities against the stock price. Then, the performance evaluation is continued based on scenario analysis where two basic scenarios are considered. In both scenarios, a portfolio financed from a cash account consists of a position in CBs held to maturity and the hedge of short positions in stock and warrants. In one scenario, the bank account can not be affected during re-balancing, while in the other scenario it can be affected.

Part C

In chapter 9, the two-dimensional approach is presented, consisting of two underlying stochastic processes, one for the stock and the other for the exchange rate. As it is discussed in this chapter, when including the correlation between the two processes, there is the limitation that the time step has to be defined based on the relationship of the space steps of the two underlying processes. Furthermore, accounting for the correlation does not allow the use of the term structures. These significant limitations constitute the correlated two-dimensional framework inappropriate for the work of this thesis; hence we concentrate on the uncorrelated two-dimensional framework. Chapter 10 presents the Backward Induction process involved in the two-dimensional framework and then it continues with the analysis of the performance of the model introduced.

Part D

This last part of the thesis concludes based on the work presented in the preceding chapters. An overview of the results is carried out and recommendations are made for future work and extension of the work of this thesis. The thesis contribution in the area of pricing convertible bonds is summarised in this last chapter and the limitations of the proposed pricing frameworks are identified.

CHAPTER 2

INTRODUCING THE INSTRUMENT

J.J. Hill, the railroad magnate, is associated with the first convertible bond issue in 1881. As mentioned in reference [9], Hill needed to rise a secure long term financing but, even though he was shut out from the traditional debt market, he was unwilling to sell stock until his planned expansion had reaped financial rewards. He solved his problem by issuing a convertible bond. Issuing convertibles is still today one very common way for companies whose stocks are volatile to access the debt market.

By 1929, convertible debt issues made up nearly 40% of publicly issued debt. However, they were virtually absent from the market during the years of the World War II. Their presence strengthened later to become to what is a today, a continuously growing $350 billion worldwide market. There are controversial opinions on the factors affecting the popularity of convertible bonds. As argued in [9], there is some evidence that the popularity of convertibles appears to be linked to rising equity markets and high interest rates. However, past and even recent observations have driven other analysts to conclude that factors that could increase the popularity of convertibles are market uncertainty, slower growth and fears for recession (because some investors consider convertibles as suitable instruments for developing defensive strategies).

In a very expected manner, due to the convertible bond market tremendous growth over the years, convertible bonds have come to be an asset class of their own. Trading and underwriting activities, as well as product innovation, increased and are continuously expanding. A number of varieties of convertible instruments have been structured and trademarked by various investment banks. The structure of each convertible bond serves a purpose and aims in satisfying specific customer requirements. The most sophisticated convertible securities market is the US market.

Nevertheless, convertible securities with un-common and complex features can be encountered in other markets.

2.1. Terminology

Before moving to the presentation of the most popular types of existent convertible securities, it is suitable to introduce some of the basic terminology used in the convertible market.

The ***par value*** or the ***face value*** of a convertible bond is what the investor is paid on redemption at maturity. This is typically $1000 (or 1m Yen in the Japanese market) for coupon bonds and $25 or $50 for Convertible Preferred shares. The par value is also used as the basis for calculating coupon and dividend payments and yields. The ***maturity date*** of a convertible bond is the stated date on which all outstanding bonds of an issue are redeemed and the contract between the issuer and the investor expires.

The ***coupon rate*** is the stated percentage of the par value that is paid to the investors on the days the coupon is due. Typically coupons are paid semi-annually (Convertible Preferred Shares have a stated dividend instead of a coupon).

The ***issue price*** is the price at which a convertible bond is sold by the issuer to the investors on the date of issue. This price is usually equal to the par value of the bond. For zero coupon bonds the issue price is a function of the time to maturity, accretion rate and the compounding frequency.

An investor in a convertible bond can exchange the bond for a stated number of common shares of the issuing company. The predetermined number of common shares a bond converts into is called the ***Conversion Ratio***.

$$Conversion\ Ratio = \frac{Issue\ Price}{Stock\ Price\left(1+Premium\right)} \qquad \text{(Eq.1.1.1)}$$

When an investor converts a convertible bond into common stock, a certain number of shares are received in lieu of the par amount on the bond. The price per share of this number of shares that the investor effectively pays is termed the ***Conversion Price*** or the ***Strike***.

$$Conversion\ Price\ (or\ Strike) = \frac{Par\ Value}{Conversion\ Ratio} \qquad \text{(Eq.1.1.2)}$$

The market value of the number of common shares obtained by the conversion of a convertible bond is called the ***Parity Value*** of the bond. This is also known as the *Equity Parity* of the bond.

$$Parity\ Value = Market\ Price\ of\ Stock \times Conversion\ Ratio \qquad \text{(Eq.1.1.3)}$$

Because of certain advantages of the convertible bonds over common stock holdings, the convertible bonds trade at a premium over the value of an equivalent number of common stock, i.e. parity. This is called the ***Premium over Parity*** of a convertible bond.

$$Premium\ over\ Parity = \left[\frac{Bond\ Price}{Parity} - 1\right] \times 100 \qquad \text{(Eq.1.1.4)}$$

If the bond were not convertible into common stock, a convertible bond would trade similar to an equivalent corporate bond. As such, the equivalent value for the convertible bond, with no regards to the conversion feature, is known as the ***Investment Value***. A convertible bond typically never trades below this value. Also known as the ***Bond Floor***, it defines the downside protection for the investor. The difference between the price of a convertible bond and its investment value is termed as the ***Premium over Investment Value***.

Often convertible bonds have ***call features***, which provide the issuer a way to force conversion at a stipulated price or redemption of the bonds. When the conversion value of a convertible bond is higher than the call price, the issuer can issue a call notice. But for the investors it could be advantageous to convert to stock. An investor can convert a bond into stock and sell the stock immediately in the market at the market price to receive the parity value rather than let the issuer redeem the bonds at the call price, which would be lower than the parity value. Thus conversion is enforced.

Call features, if present, reduce the value of a convertible bond to the investor. In order to make a callable convertible bond more attractive to the investors, convertible bonds come often with what are termed ***Call Protection Features***. Convertible bonds with associated call schedules come with a condition that the bonds may not be called for a certain number of years. This period is called the Call Protection period. Longer call protection periods extend the life of the conversion feature and thus increases their value.

When a convertible bond is conditioned to be non-callable for a certain number of years it is known as a ***Hard Call Protection*** or Absolute Call Protection. When a convertible bond is conditioned to be callable provided that the underlying stock trades in the market at a certain level for a predetermined number of days, it is known as a Provisional or ***Soft Call Protection***. In addition, call schedules are set with a *call notice period*. Typically once the call notice is given, the investors have 30 days in which to convert before the issuer redeems the bonds at the call price.

An investor who owns a convertible bond, which has a ***put feature***, can redeem the bond with the issuer at the predetermined put price, which is usually at a premium to the par value of the bond. The investor is thus guaranteed to earn the yield to put. Thus a put feature also provides a downside protection to the bondholder.

Another important feature encountered in some convertible bonds is the possibility of a ***refix*** of the conversion price. This is also referred to as the ***reset*** of the strike and must not be confused with the *reset of the coupon*. In this thesis, the terms *refix* and *reset* will be both used for denoting the refix of the strike, unless otherwise stated. ***Convertible refix bonds*** have been around for well over a decade, but arguably gained a lot of attention with the issuance of as many as ten jumbo Japanese bank bonds between 1995 and 1998. Faced with a worsening operating environment in the mid-1990s and deteriorating capital ratios, Japanese banks were obliged to recapitalise themselves. For this reason they turned to the convertible bond market and they include the reset feature to their issues to sweeten the investors. As a result, convertible bonds with resettable conversion prices have become fairly common in the domestic Japanese convertible market and have more recently found limited application in the Asian convertible universe.

Rather than having a fixed conversion price, as is the case for conventional convertibles, these structures allow for an adjustment to that price, depending on the performance of the underlying stock. The refix, in some cases, is both downward and upward, but most commonly downward only. The effect of the downward reset to the conversion price is to boost the conversion ratio – the number of shares received per bond upon conversion, compensating for possible falls of the share price. The upward reset has the opposite effect. So, an upward reset benefits the issuer and a downward reset benefits the holder. There is always a floor to which a conversion price can reset (and a cap for the upward reset case).

2.2. Convertible Instruments

As already mentioned, a number of varieties of convertible instruments have been structured and trademarked by various investment banks. Depending upon various factors, a convertible bond behaves in different ways.

If a convertible has a small premium over equity and a low coupon yield, it will be more sensitive to the fluctuations in the equity market and will resemble the underlying stock more closely. Such a convertible is considered to be an Equity Alternative. On the other hand, a convertible with a high coupon rate and a large premium over equity value will be more sensitive to the fluctuations in the interest

Corporate Bonds	Convertible Bond			Convertible Preferred		**Common Stock**
	Zero Coupon Convertible Bonds	Original Issue Discount OID Convertible Bonds	Coupon Paying Convertible Bonds	Perpetual Convertible Preferred Stock	Mandatory Convertible Preferred Stock (PERCS & DECS)	

Figure 1.1. Convertible Bond Spectrum (from reference [B9])

rates and moves in the fixed-income markets. Such a convertible is considered to be a Fixed-income Alternative. A convertible bond with a moderate coupon rate priced at a moderate premium over equity value will be sensitive to both the equity market moves and interest rate fluctuations. Such a convertible bond can be said to be a true hybrid instrument.

Zero Coupon Convertible Bonds as the name suggests have no periodic coupon payments. However, the bonds are issued at a discount and they result to a payment equal to the par value at maturity. Unlike zero coupon bonds, *Original Issue Discount Convertible Bonds* pay a determined coupon. However, they are issued at a discount, like zero coupon bonds, and are redeemed at par at maturity. Regular *Coupon Paying Convertible Bonds* or *Debentures* have the simplest structure. They most resemble a Corporate Debt. They have a specified coupon rate and a coupon frequency. They have Call Protection periods and call notice periods associated with their call schedules. They may also have put features and sinking fund provisions. *Step-up Convertible Bonds* lie between OID Convertible Bonds and regular Coupon Paying Convertible Bonds. The difference is that after a certain pre-determined period of time in the life of the Convertible Bond, the coupon rate is adjusted to a higher rate. In the case of a *Step-down Convertible Bond*, the adjustment is a downward one.

Perpetual Convertible Preferred Stock has no maturity date. It has a pre-determined dividend rate either stated as a percentage of par or as a dollar value. It behaves more or less like coupon paying convertible bond. *Preferred Equity Redemption Cumulative Stock PERCS* has mandatory conversion to common stock. PERCS are also known as capped convertibles because there is a cap on the upside potential. They are priced at the market closing price of common stock on the date of issue and they convert into one share of the common stock upon conversion subject to stock splits and stock dividends. *Dividend Enhanced Convertible Stocks DECS* are convertible preferred shares that can be converted into common stock at any time at a premium, at the option of the investor. They include mandatory conversion at maturity. DECS are also set to give a coupon at a quarterly frequency typically on the same date when common dividend is paid. The conversion of DECS is a function of the price of the common stock on maturity date.

There are several other convertible structures in the market brought out by various investment banks. They are all designed to meet advantage of tax and accounting needs and strategies of issuers.

2.3. Example of a Vanilla Convertible Bond specifications

To better comprehend the instrument, an example of a convertible bond is presented. In this example, a Japanese convertible bond will be issued at the beginning of the second month in the following year (February of 2005). Assume that the *Face Value* of the convertible bond is 1 million Yen; the *maturity date* is the 31st of January 2015 (31/01/2015) and the convertible bond is redeemed at 100% of its face value. This 10-year convertible bond also gives a semi-annual coupon of 0.5% of its face value. The *issued price* of the convertible bond is set equal to 95% of its face value.

There are no other embedded calls or puts than the optionality provided to the investor to convert the bond to equity of the issuer company. Any further features like resets, refixs, compulsory or conditional conversion, are not included in the instrument. The case of an instrument with these specifications will be referred to in this thesis as the ***"Vanilla"*** Convertible Bond case.

The *conversion ratio* is determined based on equation (1.1.1). If the share price of the issuer company on the convertible bond issue day is equal to 4502 Yen and the company decides to include a *premium* for the optionality to convert equal to 5.5%, then the conversion ratio is found to be equal to 200 shares.

$$Conversion\ Ratio = \frac{Issue\ Price}{Stock\ Price\,(1+Premium)} = \frac{95\% \times 1mYen}{4502 \times (1+5.5\%)} = 200 \qquad \text{(Eq.1.1.5)}$$

Since the conversion ratio has been determined, the *conversion price* (or *strike*) can also be determined from equation (1.1.2) as follows.

$$Conversion\ Price\ (or\ Strike) = \frac{Par\ Value}{Conversion\ Ratio} = \frac{1mYen}{200} = 5000Yen \qquad \text{(Eq.1.1.6)}$$

All these determined parameters of the convertible bond will remain fixed up to the expiration of the convertible bond. The investor will receive a 0.5% semi-annual coupon while he holds the convertible bond, and at the same time he will have the right to convert each paper of this convertible bond issue with face value of 1m Yen to 200 shares of the issuer company. If he does convert at any point in time during the life of the convertible bond, it will be as if he is buying shares of the issuer company at the price of 5000 Yen, regardless of the actual price of the share of the issuer company. In such a case, he will no longer hold the convertible bond and he will have to forfeit the remaining coupon payments and the principal value.

At any point of time during the life of the convertible bond, the *parity* value of the convertible bond is determined by equation (1.1.3) replicated here.

$$Parity\ Value = Market\ Price\ of\ Stock \times Conversion\ Ratio \qquad \text{(Eq.1.1.7)}$$

The traded value of the convertible bond in the market will always be equal or greater than its *parity* since in any other case there would be arbitrage profit opportunities. In other words, in the case that the convertible bond is traded under its parity value, an investor may buy the convertible bond and convert it. Selling the shares in the market will provide him with a profit since their overall value will be greater than the amount he gave for buying the convertible bond.

Based on the same principle of no arbitrage opportunities, the convertible bond price can not be below its Bond Floor, also known as the Investment Value of the convertible bond. The Bond Floor is simply the sum of the discounted coupons and the discounted principal received upon redemption. If the convertible bond is traded below this floor value, then an investor may consider that he can lock into a position which guarantees future cash flows (coupons and principle) cheaper than the market conditions do determine. Again, this would provide an arbitrage opportunity, in this case, for a Fixed Income investor.

2.4. Identification of the Market Risk Factors

Based on the presentation of the features of the convertible bond market in the previous section, we are now in position to identify the risk factors associated with this instrument. In other words, we are in a position to identify the exposures of an investor (or an issuer) who has a position, long or short, in convertible bonds. These are listed as follows.

Interest Rate Exposure

Fluctuations of the interest rates in the market will have a direct effect on the value of the convertible bond due to the fixed-income nature of the instrument. The most direct effects would be on the Bond Floor value of the convertible bond since this is based on discounted cash flows. The convertible bond could also be viewed as an interest-rate derivative since the investor has the optionality to exchange (or "swap") the instrument for equity. In addition, this optionality offers to the investor a limited downside protection with respect to the interest-rate exposure.

Equity Exposure

The future stock price evolution will add or deduct value from the convertible bond. Actually, the market conception for the future probability distribution of the returns of the underlying stock will be one of the most significant driving factors of the returns on the convertible bond position, alongside with the actual realisations of the stock price. In other words, the stock price and the implied volatility, a market parameter that summarises in a way what we have referred to as the probability distribution characteristics of the returns of the stock price, are significant sources of risk and have a determinant role in pricing and hedging of a convertible bond position.

The fact that the instrument is primarily a fixed-income security offers a downside protection with respect to the stock price performance and implied volatility fluctuations. So the convertible bond can also be viewed as an equity derivative.

Credit Exposure

As in every corporate fixed-income security, the credit spread of the issuer has a significant role in the valuation of the security. Movements in the credit spread of the issuer have a direct effect on the Bond Floor value of the convertible bond.

Currency Exposure

Some convertible bonds convert into stock that is denominated into a different currency than the currency of issuance of the convertible bond. In such cases, the conversion value is subjected to currency fluctuations.

Modelling Exposure

Each of the previously identified exposures has to be taken into account when pricing a convertible bond. In addition, special features like calls and puts, conditional conversion, and reset provisions for the strike, when included in the instrument specifications, need to be accounted for in the pricing model. All these modelling requirements constitute the convertible bond a very complex instrument to price. As it was mentioned in the previous chapter, the need for adequate pricing and hedging convertible bond models has resulted in the development of specialised software products.

CHAPTER 3

RELATIVE PRICING FRAMEWORKS

3.1. Relative Pricing

The various approaches in the theory of asset pricing can be categorised under two main categories. These are *equilibrium pricing* and *arbitrage pricing*. The work presented in this report employs approaches that fell under the category of arbitrage pricing. In this section, before proceeding in the methods for extracting implied structures used in arbitrage pricing, the difference between equilibrium and arbitrage pricing is briefly discussed (material is based on references [4] and [6]).

Equilibrium pricing is directly borrowed from economic theory and is an attempt to provide *absolute pricing*. This approach is based on the effort to explain prices and returns on the financial markets by applying optimisation rules on the agents in the economy. In a model of a general equilibrium type, the demand and supply sides of the equilibrium are explicitly characterised in terms of optimised production and consumption decisions. However, this approach requires very strong assumptions about the economy, in particular, a definition of the risk preferences of the agents (investors). In other words, this approach is based on placing restrictions on the assumptions underlying the pricing method and the more restrictive these assumptions are, the more precise the pricing is.

On the other hand, *arbitrage pricing* makes the main assumption that there are no arbitrage opportunities in the market. In other words, it assumes that all the agents in the market are greedy and act in such a way as to optimise their profit. This further leads into an assumption that there is an arbitrage free equilibrium status in the market and whenever arbitrage opportunities appear these are eliminated by the agents in the market that take advantage of them in an efficient fashion.

Arbitrage pricing provides ***relative pricing***. Since in the arbitrage pricing framework there is the main assumption that all the existing securities in the market are traded in an arbitrage-free world, then non-liquid or new securities can be directly priced in terms of a portfolio of other liquid securities whose prices are known.

3.2. Determination of the Discount Function

In a *relative pricing* framework it is desired to have methods of determining the zero-coupon yield curve based on the information supplied by the market. In this section, some of the most common methods for extracting the market discount function are discussed. Even though in the associated work to this thesis, the discount factor curve employed is based on the swap market, methods for extracting the curve from the Treasury market are also presented. The methods which are based on the Treasury market are included simply for completeness and for comparison, as well as for justification of the choice for a swap market based curve.

3.2.1. Treasury Market

In reference [6], a ***direct method*** is shown for defining the zero-coupon rate curve from the coupon bond market prices. To define n distinct zero-coupon rates, a collection of n coupon (or zero-coupon) bonds is required. The usual case is to use default-free coupon bonds, like the US Treasury bonds, because they provide information about the risk-free structure of interest rates. Next, the n-dimensional vector P_t of coupon bond prices at time t (where T denotes transposition) and the $n \times n$ matrix F of cash flows (coupons and principal) corresponding to the n assets are defined.

$$P_t = \left(P_t^1, P_t^2, \ldots, P_t^j, \ldots, P_t^n\right)^T \qquad \text{(Eq.3.2.1)}$$

$$F = \left(F_{t_i}^{(j)}\right) \qquad i = 1,2,\ldots,n \ , \ j = 1,2,\ldots,n \qquad \text{(Eq.3.2.2)}$$

If the zero-coupon bond prices at time t are represented by vector B_t, then equation (3.2.4) can be assumed to hold.

$$B_t = \left(B(t,t_1), B(t,t_2), \ldots, B(t,t_j), \ldots, B(t,t_n)\right)^T \qquad \text{(Eq. 3.2.3)}$$

$$P_t = F \cdot B_t \qquad \text{(Eq. 3.2.4)}$$

Provided that the matrix F is invertible (implying that there is no linear dependence in the pay-off of the bonds), then the vector B_t can be estimated based on equation (3.2.5). Having extracted the implied zero-coupon bond values, the interest rates $R(t,t_i - t)$ can be extracted from these prices based on equation (3.2.6).

$$B_t = F^{-1} \cdot P_t \qquad \text{(Eq. 3.2.5)}$$

$$R(t,t_i - t) = -\frac{1}{t_i - t}\ln\left[B(t,t_i)\right] \qquad \text{(Eq. 3.2.6)}$$

The direct approach presented above, even though it is simple and not computationally intensive, comes with some difficulties. It requires distinct linearly independent bonds with the same coupon dates, which is something difficult to achieve. In addition, this technique is not robust with respect to the changes in the set of bonds used for extracting the implied zero-coupon rates.

Due to the drawbacks of the direct approach, ***indirect methods*** have been developed. These involved fitting the data to a pre-specified form of the zero-coupon yield curve. Like in the direct method, a choice is made of n default free bonds whose prices define the vector of equation (3.2.1) and their cash flows the matrix of equation (3.2.2). However, in the case of the indirect method, the cash flows matrix $F_s^{(j)}$ is defined for a time s where $s \geq t$.

In reference [6], the general approach followed in the indirect methods is summarised as follows. A specific form of the discount function $B(t,s) \equiv f(s-t;\beta_1)$ of the zero-coupon rates $R(t,s-t) \equiv g(s-t;\beta_2)$ needs to be postulated, where β_1 and β_2 are the vectors of parameters. In the first case, the function f is usually defined under the form of a polynomial or exponential spline functional. For the case of the function g, this is usually defined in such a way that the parameters in β_2 are easily interpretable.

Defining the theoretical prices of the securities based on equation (3.2.7) for the case of the discount function approach and based or equation (3.2.8) for the case of the zero-coupon rates approach, the set of parameters $\hat{\beta}^*$ is estimated through the optimisation program defined by equation (3.2.9) so that the estimated prices best approximate the actual observed market prices of the chosen securities.

$$P_t^j = \sum_s F_s^{(j)} f(s-t;\beta_1) \qquad \text{(Eq. 3.2.7)}$$

$$\hat{P}_t^j = \sum_s F_s^{(j)} \exp[-(s-t)g(s-t;\beta_2)] \qquad \text{(Eq. 3.2.8)}$$

$$\hat{\beta}^* = \arg\min_{\beta} \sum_{j=1}^{n} \left(P_t^j - \hat{P}_t^j \right)^2 \qquad \text{(Eq. 3.2.9)}$$

In both of references [4] and [6], it is stated that these models suffer from the risk of possible misspecification of the set of parameters of the vector β used in the definition of the zero-coupon yield curve. Even though the described approaches are mathematically simple, the practical implementation of them needs to be performed in a very delicate manner. The success of any of these approaches depends largely on the suitability of the basis function chosen to describe the behaviour of the discount function.

3.2.2. Swap Market

The above-discussed approaches were based on data obtained from the Treasury bond market. Of special interest to the work presented in this report is the derivation of the discount function from the ***swap market***. Apart from the fact that the zero-coupon yield curve based on the swap market is the reference for credit analysis, the swap-market-based curve plays a key role in the pricing and hedging of derivative products. A stochastic model for the dynamics of the zero-coupon yield curve is required when pricing and hedging interest-rate derivatives or floating rate bonds.

Information on parameters, like the volatilities and the correlation matrix of the discounting rates, used in such stochastic models are already available in the market through liquid instruments like caps and swaptions, which are derivatives on LIBOR and swap rates.

In reference [48], in addition to the recognition of the above reasons, the author extends the basis of the increased importance of the swap market to other factors and effects like decreasing liquidity and efficiency of the government debt markets, improved uniformity across the swap markets of different countries with respect to the government debt markets, swap market features like increasing liquidity, with narrow bid-ask spreads and wide spectrum of maturities, etc.

The most popular approach for extracting the zero-coupon curve involves combining and bootstrapping observed market interest rates. Commonly, the curve is divided into three term buckets, even though in some cases a fourth bucket is also included. The four possible buckets are the following:

- The ***short end*** of the term structure is derived using interbank deposit rates like LIBOR rates. This usually employs the overnight (O/N), the tomorrow/next (T/N), 1 week (1WK), 1month (1M), 2 months (2M) and 3 months (3M) rates.
- The ***middle area*** of the term structure is derived from futures (interest rate futures contracts) or FRAs (forward rate agreements). This bucket will usually cover the period up to 2 years.
- The ***long end*** of the term structure is based on swap rates that are derived from the swap market. This part of the term structure covers the period from 2 years to 10 years.
- The ***very long end*** of the term structure. It is also very common to extend the previous part of the curve to cover the period up to 30 years. Equivalently, some practitioners prefer referring to the period covering the part of the curve from 10 to 30 years (in some cases even longer) as the very long end of the curve.

The following table, named "Table T.3.1 Market Data –Currency GBP", presents a structure of market data that includes the rates corresponding to the buckets listed above. These will form the basis for the calculations presented later as an example of the construction of the discount factor swap curve.

TABLE T.3.1 Market Data - Currency GBP Close of Business 18 February 2003					
Rate Type	**Tenor Label**	**Rate**	**RateType**	**Tenor Label**	**Rate**
LIBOR	O/N	0.0328000	SWAP	2yr	0.03700
LIBOR	T/N	0.0338000	SWAP	3yr	0.03880
LIBOR	1WK	0.0353375	SWAP	4yr	0.04040
LIBOR	1M	0.0369094	SWAP	5yr	0.04170
LIBOR	2M	0.0369500	SWAP	6yr	0.04275
LIBOR	3M	0.0369375	SWAP	7yr	0.04365
FUTURE	19-Mar-03	96.325	SWAP	8yr	0.04440
FUTURE	18-Jun-03	96.515	SWAP	9yr	0.04500
FUTURE	17-Sep-03	96.545	SWAP	10yr	0.04550
FUTURE	17-Dec-03	96.485	SWAP	12yr	0.04625
FUTURE	17-Mar-04	96.360	SWAP	15yr	0.04680
FUTURE	16-Jun-04	96.215	SWAP	20yr	0.04700
FUTURE	15-Sep-04	96.070	SWAP	25yr	0.04680
			SWAP	30yr	0.04655
			SWAP	40yr	0.04585

The calculation of the continuously zero-coupon compounded rates $r_c(\tau)$ for each of the parts of the curve has its own methodology and is based on the completion of the calculations in the previous parts of the curve. The parameter τ represents the time to maturity corresponding to the rate $r_c(\tau)$ and can be quoted into number of days, in which case we use the notation τ_d, or into number of years, in which case we use the notation τ_y.

The Short End

For the *short end* of the curve the calculations are restricted into transforming the deposit (cash) rates r_d into continuously compounded rates based on the day count convention followed. The number of days in a year based on the conversion followed is denoted by $t_{d,y}$. It could be the case that the input cash rates are expressed under a conversion with $t_{d,y,cash}$ and we want to follow a different convention with $t_{d,y}$. In this way, we have two different times to maturities as number of years, $\tau_{y,cash}$ corresponding to the initial convention that the input cash rates are expressed to, and

τ_y that corresponds to the final convention $t_{d,y}$ that we want to follow in our calculations.

The terms $df_{d,y,cash}(i)$ and $df_{d,y}(i)$ are also introduced and represent the discount factors corresponding to the rates under each convention. It is also required to introduce the forward discount factors, $fdf_{d,y,cash}(i)$ and $fdf_{d,y}(i)$, and the forward rates, $fr_{d,y,cash}(i)$ and $fr_{d,y}(i)$, corresponding to each of the conventions. The forward rates and discount factors notation have been introduced because most of the cash rates are expressed as forward rates.

Equations (3.2.10) to (3.2.12) summarise the calculations for obtaining the discount factor corresponding to each of the cash rates under the original year convention of the cash rates. The parameters $SD(i)$ and $ED(i)$ correspond to the starting and ending date of each cash rate, and CD corresponds to the calculations date. The parameter $df_{CD,SD(i)}$ corresponds to the discount factor for the period extending from the calculations date to the starting date of the cash rate.

$$t_{SD,ED,cash}(i) = \frac{ED(i) - SD(i)}{t_{d,y,cash}} \qquad \text{(Eq.3.2.10)}$$

$$fdf_{d,y,cash}(i) = \frac{1}{1 + r_d(i) \times t_{SD,ED,cash}(i)} \qquad \text{(Eq.3.2.11)}$$

$$df_{d,y,cash}(i) = df_{CD,SD(i)} \times fdf_{d,y,cash}(i) \qquad \text{(Eq.3.2.12)}$$

The results from the calculations carried out when applied the three above equations to the cash rates of the example are included in table (T.3.2).

TABLE T.3.2

Rate i	Rate Type	Tenor Label	Rate	Forward Rate (Yes/No)	Start Date SD	End Date ED	Time Period	fdf	df
1	LIBOR	O/N	0.0328000		18/02/2003	19/02/2003	0.002740	0.999910	0.999910
2	LIBOR	T/N	0.0338000	Forward	19/02/2003	20/02/2003	0.002740	0.999907	0.999818
3	LIBOR	1WK	0.0353375	Forward	20/02/2003	27/02/2003	0.019178	0.999323	0.999140
4	LIBOR	1M	0.0369094	Forward	20/02/2003	22/03/2003	0.082192	0.996976	0.996794
5	LIBOR	2M	0.0369500	Forward	20/02/2003	21/04/2003	0.164384	0.993963	0.993781
6	LIBOR	3M	0.0369375	Forward	20/02/2003	21/05/2003	0.246575	0.990974	0.990794

Next, the resultant discount factors must be transformed to continuous rates under the desired final year conversion. These calculations are summarised in the following equations.

$$df_{d,y,cash}(i) = df_{d,y}(i) \tag{Eq.3.2.13}$$

$$t_{CD,ED}(i) = \frac{ED(i) - CD}{t_{d,y}} \tag{Eq.3.2.14}$$

$$df_{d,y}(i) = e^{-r_c(i) \times t_{CD,ED}(i)} \tag{Eq.3.2.15}$$

The continuous rates for $\tau_y = t_{CD,ED}(i)$ where $i \in [1,6]$, can be calculated based on equation (3.2.16):

$$r_c(\tau_y) = r_c(i) = -\frac{\ln[df_{d,y}(i)]}{t_{CD,ED}(i)} \tag{Eq.3.2.16}$$

The final continuous annual rates of the example are included in table (3.3) and are obtained by applying the resultant figures of table (3.2) into equations (3.2.13) to (3.2.16). The assumption was made throughout the calculations corresponding to tables (3.2) and (3.3), that both, the input cash rates and the required final conversion to be followed, depict the use of the values $t_{d,y,cash} = t_{d,y} = 365$.

TABLE T.3.3

Rate i	Original Rate Type	Original Tenor Label	Calculations Date CD	End Date ED	Time to Maturity (ED-CD)/tdy	df	Continuous Annualised Rate rc
1	LIBOR	O/N	18/02/2003	19/02/2003	0.002740	0.999910	0.032799
2	LIBOR	T/N	18/02/2003	20/02/2003	0.005479	0.999818	0.033298
3	LIBOR	1WK	18/02/2003	27/02/2003	0.024658	0.999140	0.034875
4	LIBOR	1M	18/02/2003	22/03/2003	0.087671	0.996794	0.036631
5	LIBOR	2M	18/02/2003	21/04/2003	0.169863	0.993781	0.036724
6	LIBOR	3M	18/02/2003	21/05/2003	0.252055	0.990794	0.036695

At this point, the calculations of the continual annual rates based on the LIBOR forward rates have been completed. However, in order to proceed to the calculation of the rates of the middle area, another two rates need to be calculated, the rate for the zero tenor and the rate that corresponds to the starting date of the first future. The zero-tenor rate is needed so that we can define rates for the maturities between zero and 1 day by interpolating. The second rate is needed in order to calculate the discount factor for the starting date of the first future.

The preferred interpolation technique in the work associated with this thesis was the ***piecewise cubic spline interpolation***, which will be the default technique for interpolating on any structure, especially the risk-free rates term structures. It is desired to apply the same interpolation technique on the risk-free rate structure when running a pricing model as the technique used when the term structure was actually being developed. In this case, the term structure is developed by employing piecewise cubic spline interpolation. If it is desired to maintain consistency, then the same technique must be used later on in the pricing models when interpolation is required on the input risk-free rates term structure.

The results from applying the cubic spline interpolation technique on a set of data with n observation sets, is a group of $(n-1)$ sets of the parameters (A_i, B_i, C_i, D_i). Each set corresponds to an equation of the form shown in (3.2.17) below and represents the cubic polynomial $S_i(\tau_y)$ connecting the points of the structure $(\tau_y(i), r_c(i))$ and $(\tau_y(i+1), r_c(i+1))$. If it is required to obtain the corresponding rate r_c for a maturity τ_y, and τ_y lies in the interval $(\tau_y(i), \tau_y(i+1))$, then equation (3.2.17) will return the desired result (i.e. $r_c = S_i(\tau_y)$). Appendix II offers more details on the piecewise cubic spline interpolation technique.

$$S_i(\tau_y) = A_i + B_i(\tau_y - \tau_y(i)) + C_i(\tau_y - \tau_y(i))^2 + D_i(\tau_y - \tau_y(i))^3 \qquad \text{(Eq.3.2.17)}$$

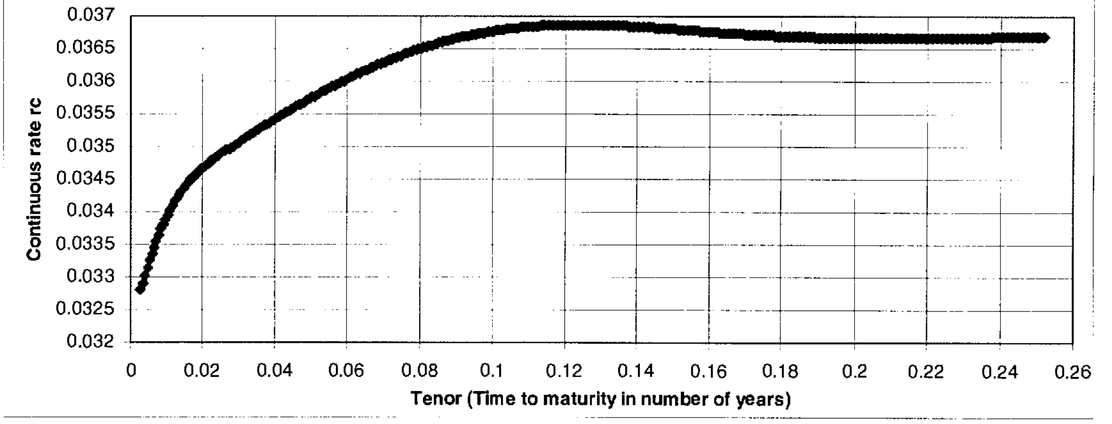
After applying cubic spline interpolation on the resultant short end of the zero-coupon risk-free rate term structure of table (3.3), the sets of parameters of the cubic polynomials corresponding to this structure were obtained. These are included in table (3.4).

TABLE T.3.4

Polynomial i	Rate i to i+1	A	B	C	D
1	1 -> 2	0.032799	0.188714	0.000000	-830.077594
2	2 -> 3	0.033298	0.170022	-6.822556	116.992286
3	3 -> 4	0.034875	0.037424	-0.091493	-0.953852
4	4 -> 5	0.036631	0.014531	-0.271810	1.323024
5	5 -> 6	0.036724	-0.003337	0.054415	-0.220684

In order to demonstrate the use of these polynomials, interpolation was carried out by employing the polynomials and the corresponding rates for 400 tenor values in the range $[0.002740, 0.251955]$ were calculated. The graph of these 400 tenor points and the resultant rates are included in graph (3.1).

Figure (F.3.1) Zero-Coupon rates for the Short End
Without the Zero-Tenor rate and based on 400 points



Even though the polynomial equations corresponding to the parameters of table (T.3.4) are supposed to be used only for interpolating, in the case of the zero tenor rate we deviate from this rule and use the first polynomial for performing extrapolation. Entering a tenor value τ_y equal to zero into the polynomial equation with $i = 1$, the returned tenor value is 0.032299. The zero-tenor rate is included into the short-end zero coupon rates as shown in table (3.5).

TABLE T.3.5

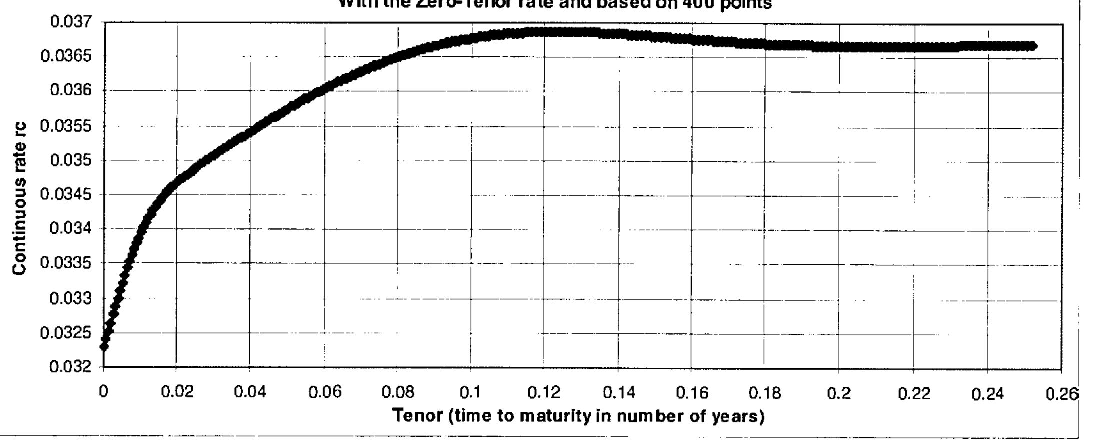
Rate i	Original Rate Type	Original Tenor Label	Calculations Date CD	End Date ED	Time to Maturity (ED-CD)/tdy	df	Continuous Annualised Rate rc
0	DUMMY	Zero Tenor	18/02/2003	18/02/2003	0.000000	1.000000	0.032299
1	LIBOR	O/N	18/02/2003	19/02/2003	0.002740	0.999910	0.032799
2	LIBOR	T/N	18/02/2003	20/02/2003	0.005479	0.999818	0.033298
3	LIBOR	1WK	18/02/2003	27/02/2003	0.024658	0.999140	0.034875
4	LIBOR	1M	18/02/2003	22/03/2003	0.087671	0.996794	0.036631
5	LIBOR	2M	18/02/2003	21/04/2003	0.169863	0.993781	0.036724
6	LIBOR	3M	18/02/2003	21/05/2003	0.252055	0.990794	0.036695

The same procedure is followed as before and a new set of polynomials is established based on the tenor values and continuous rates of table (3.5). The new set of polynomials is presented in table (3.6).

TABLE T.3.6					
Polynomial i	Rate i to i+1	A	B	C	D
0	0 -> 1	0.032299	0.180899	0.000000	211.012274
1	1 -> 2	0.032799	0.185651	1.734347	-1055.061371
2	2 -> 3	0.033298	0.171396	-6.937390	119.243966
3	3 -> 4	0.034875	0.036878	-0.076778	-1.049864
4	4 -> 5	0.036631	0.014696	-0.275246	1.340442
5	5 -> 6	0.036724	-0.003384	0.055274	-0.224168

A set of rates corresponding to 400 tenor values in the range $[0.0\,,0.251955]$ is calculated based on these polynomials and are presented in graphical form in figure (3.2). As it can be observed, the only difference with the graph of figure (3.1) is that we have included rates in the tenor range $[0.0\,,0.002740]$. In other words, with the inclusion of the dummy zero-tenor zero-coupon rate, we are able to interpolate for and calculate the continuous rates for the small additional tenor range $[0.0\,,0.002740]$.

Figure (F.3.2) Zero-Coupon rates for the Short End With the Zero-Tenor rate and based on 400 points



The middle area

The starting date of the first future which is shown in table (3.7), is the 19th of March, 2003. We will denote the time to maturity corresponding to this date as $\tau_{y,SDfutures}$ and is equal to $\tau_{y,SDfutures} = 0.079452$. This maturity period (tenor) is within the tenors corresponding to the third and fourth rates obtained for the short end of the structure. Hence, the corresponding rate to the tenor $\tau_y = 0.079452$ can be calculated by employing the third polynomial. Employing equation (3.2.17) with $i = 3$, the resultant rate is $r_c(\tau_y = 0.079452) = 0.036493$ and we denote this rate as $r_{c,SDfutures}$. From this rate and tenor values, the discount factor for the tenor which covers the period up to the starting date of the first future can be calculated and is found as shown in the following equation.

$$df_{SDfutures} = \exp(-r_{c,SDfutures} \times \tau_{y,SDfutures}) = \exp(-0.079452 \times 0.036493) = 0.997105$$

The equations governing the calculations of the continuous rates $i \in [7,13]$ corresponding to the futures prices are presented next with the resultant values shown in the last column of table (3.7). The futures prices are denoted as $V_{fut}(i)$ and correspond to the prices included in the third column of table (3.1).

$$t_{SD,ED,Future}(i) = \frac{ED(i) - SD(i)}{t_{d,y,Future}} \quad \text{(Eq.3.2.18)}$$

$$fdf_{d,y,Future}(i) = \frac{1}{1 + \left(\frac{100 - V_{fut}(i)}{100} \times t_{SD,ED,Future}(i) \right)} \quad \text{(Eq.3.2.19)}$$

$$\begin{aligned} df_{d,y,Future}(i) &= df_{SDfutures} \times fdf_{d,y,Future}(i) & i &= 7 \\ df_{d,y,Future}(i) &= df_{CD,SD}(i-1) \times fdf_{d,y,Future}(i) & i &\in [8,13] \end{aligned} \quad \text{(Eq.3.2.20)}$$

$$df_{d,y,Future}(i) = df_{d,y}(i) \quad \text{(Eq.3.2.21)}$$

$$t_{CD,ED}(i) = \frac{ED(i) - CD}{t_{d,y}} \quad \text{(Eq.3.2.22)}$$

$$df_{d,y}(i) = e^{-r_c(i) \times t_{CD,ED}(i)} \quad \text{(Eq.3.2.23)}$$

The assumption regarding the conversion on the number of days per year to be 365 has been made for the case of the futures. So, it holds that $t_{d,y,Future} = t_{d,y} = 365$. From the next equation, we can calculate the continuous rates for $\tau_y = t_{CD,ED}(i)$ where $i \in [7,13]$:

$$r_c(\tau_y) = r_c(i) = -\frac{\ln[df_{d,y}(i)]}{t_{CD,ED}(i)} \quad \text{(Eq.3.2.24)}$$

TABLE T.3.7										
Rate i	Rate Type	Tenor Lable	Future Price F(i)	Starting Date	Ending Date	Time Period	fdf	df	Time to maturity (ED - CD)/tdy	Continuous Annualised Rate rc
7	FUTURE	19-Mar-03	96.325	19/03/2003	18/06/2003	0.2493	0.990921	0.988052	0.328767	0.036561
8	FUTURE	18-Jun-03	96.515	18/06/2003	17/09/2003	0.2493	0.991386	0.979541	0.578082	0.035758
9	FUTURE	17-Sep-03	96.545	17/09/2003	17/12/2003	0.2493	0.991460	0.971176	0.827397	0.035349
10	FUTURE	17-Dec-03	96.485	17/12/2003	17/03/2004	0.2493	0.991313	0.962739	1.076712	0.035268
11	FUTURE	17-Mar-04	96.360	17/03/2004	16/06/2004	0.2493	0.991007	0.954080	1.326027	0.035450
12	FUTURE	16-Jun-04	96.215	16/06/2004	15/09/2004	0.2493	0.990652	0.945161	1.575342	0.035802
13	FUTURE	15-Sep-04	96.070	15/09/2004	15/12/2004	0.2493	0.990297	0.935990	1.824658	0.036253

It should be noted that in the above calculations an error has been introduced because the convexity adjustment required when the forward rates are calculated based on the futures rates, has not been accounted for. Interest rate futures have zero convexity since their payoff is fixed per basis point change regardless of the level of the underlying interest rates. However, the convexity nature of instruments like Forward Rate Agreements (FRAs) and swaps means that a long position in FRAs and/or swaps and a short position in futures has a net positive convexity. This positive bias in favour of the short sellers of futures contracts has to be removed from the futures rates in order to derive an unbiased estimator of the equivalent forward rates. The convexity bias is of the magnitude of one to basis points for maturities around a year and increases with term to maturity. For the purposes of this thesis, this error due to the futures bias is allowed and no additional adjustments are made to the continuous rates resulted from the futures prices.

The next step involves performing cubic spline interpolation on the sets of rates and tenors that correspond to the medium area of the term structure - the two last columns of table (3.7) - plus the last pair of values (tenor and continuous rate) of the short end. The resultant polynomials parameters based on the cubic splines technique are summarised in table (3.8).

TABLE T.3.8					
Polynomial i	Rate i to i+1	A	B	C	D
6	6 -> 7	0.036695	-0.001507	0.000000	-0.0405549
7	7 -> 8	0.036561	-0.002223	-0.009333	0.0214020
8	8 -> 9	0.035758	-0.002886	0.006674	-0.0067036
9	9 -> 10	0.035349	-0.000808	0.001660	0.0010752
10	10 -> 11	0.035268	0.000220	0.002465	-0.0016833
11	11 -> 12	0.035450	0.001135	0.001206	-0.0004000
12	12 -> 13	0.035802	0.001662	0.000906	-0.0012120

The long end and the very long end

Based on table (3.8), the continuous rate corresponding to 1 year from the calculation date is $r_c(\tau_y = 1) = 0.035265$, and is found by applying the 1 year tenor to the polynomial with index number 9. This rate will correspond to the first coupon of the 2 year swap.

The next step would be to calculate the zero-coupon rates for the long end which covers the period up to maturities of 10 years. These rates are calculated based on the bootstrapping method which is described later on. However, because these calculations are also performed for the very long end of the curve, we choose to calculate together the zero-coupon rates for the two last parts of the curve. The only difference in the calculations for the two last parts of the curve is the fact that interpolation has to be carried out on the coupon rates of the very long end in order to obtain the missing annual swap maturities.

The cubic splines technique is applied to the swap coupon-rates of table (T.3.1). The results from the cubic spline process applied on the swap rates table is shown in table (T.3.9) below. The swap rates obtained for the missing years based on the resultant polynomials from the cubic splines application of table (T.3.9) are included in table (T.3.10). The set of polynomials corresponding to the swap coupon-rates is a different and independent set to the set of polynomials derived previously for the zero-rates.

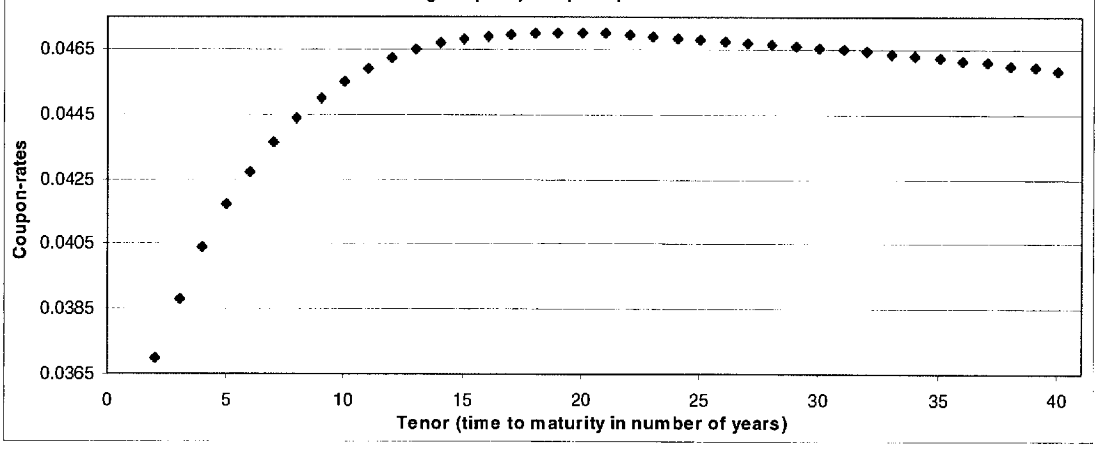
TABLE T.3.9

Coupon-Rate Index k	Rate Type	Tenor Label	Swap Rate	Polynomial m	Swap Rate k to k+1	A	B	C	D
1	SWAP	2yr	0.037000	1	1 -> 2	0.037000	0.001836	0.000000	-0.000036
2	SWAP	3yr	0.038800	2	2 -> 3	0.038800	0.001728	-0.000109	-0.000019
3	SWAP	4yr	0.040400	3	3 -> 4	0.040400	0.001454	-0.000165	0.000011
4	SWAP	5yr	0.041700	4	4 -> 5	0.041700	0.001158	-0.000131	0.000024
5	SWAP	6yr	0.042750	5	5 -> 6	0.042750	0.000966	-0.000061	-0.000005
6	SWAP	7yr	0.043650	6	6 -> 7	0.043650	0.000829	-0.000077	-0.000002
7	SWAP	8yr	0.044400	7	7 -> 8	0.044400	0.000670	-0.000082	0.000013
8	SWAP	9yr	0.045000	8	8 -> 9	0.045000	0.000543	-0.000044	0.000001
9	SWAP	10yr	0.045500	9	9 -> 10	0.045500	0.000458	-0.000040	-0.000001
10	SWAP	12yr	0.046250	10	10 -> 11	0.046250	0.000289	-0.000045	0.000003
11	SWAP	15yr	0.046800	11	11 -> 12	0.046800	0.000107	-0.000016	0.000001
12	SWAP	20yr	0.047000	12	12 -> 13	0.047000	-0.000014	-0.000008	0.000001
13	SWAP	25yr	0.046800	13	13 -> 14	0.046800	-0.000050	0.000001	0.000000
14	SWAP	30yr	0.046550	14	14 -> 15	0.046550	-0.000055	-0.000002	0.000000
15	SWAP	40yr	0.045850						

TABLE T.3.10					
Tenor Label	**Swap Coupon Rate**	**Tenor Label**	**Swap Coupon Rate**	**Tenor Label**	**Swap Coupon Rate**
11yr	0.045917	22yr	0.046943	32yr	0.046431
13yr	0.046497	23yr	0.046900	33yr	0.046366
14yr	0.046674	24yr	0.046851	34yr	0.046298
16yr	0.046891	26yr	0.046751	35yr	0.046227
17yr	0.046954	27yr	0.046703	36yr	0.046154
18yr	0.046991	28yr	0.046654	37yr	0.046080
19yr	0.047005	29yr	0.046603	38yr	0.046004
21yr	0.046978	31yr	0.046492	39yr	0.045927

The initial swap coupon-rates and the additional swap coupon-rates calculated based on the last application of the cubic splines interpolation technique are presented graphically as a complete set – covering all annual maturities from 2 years to 40 years – in figure (3.3).

Figure (F.3.3) Swap coupon-rates



For a coupon paying swap, the quoted swap rate corresponds to the annual coupon value c. This is summarised by the following equation:

$$c = quoted\ \ swap\ \ rate \times Notional \qquad \text{(Eq.3.2.25)}$$

The parameter m will be used for representing the annual coupon frequency, in other words, the number of coupons per year. The total number of remaining coupon payments is denoted by M and is given by $M = \lfloor \tau_{T,y} \times m \rfloor$, where $\tau_{T,y}$ represents the time to maturity of the swap in number of years and the operator $\lfloor \bullet \rfloor$ represents the rounding down to the first integer value. The equation relating the swap information presented here is as follows:

$$100 = \left(\sum_{j=1}^{M-1} \frac{c}{m} \times e^{-r_j \tau_j} \right) + \left(Notional + \frac{c}{m} \right) \times e^{-r_T \tau_T} \qquad \text{(Eq.3.2.26)}$$

Solving for the rate r_T, the equation will look as follows:

$$r_T = -\frac{1}{\tau_T} \times \ln \left(\frac{100 - \left(\sum_{j=1}^{M-1} \frac{c}{m} \times e^{-r_j \tau_j} \right)}{\left(Notional + \frac{c}{m} \right)} \right) \qquad \text{(Eq.3.2.27)}$$

Based on the last equation, the rate r_T can be calculated if all the rates corresponding to the coupon rates (apart the last one which coincides with the maturity) are known. This leads to the employment of the bootstrapping technique. The current calculations example can be used to demonstrate this technique. In this example, the swap rates correspond to coupon swaps with a coupon frequency equal to one and a notional of 100. This results in simplifying equation (3.2.27) in the form of equation (3.2.28) for the 2 year swap. The rate $r_{c,1yr}$ was calculated by interpolating on data of the middle area of the term structure, which was calculated based on the Futures input data. Hence, the rate $r_{c,2yr}$ can now be calculated from equation (3.2.28), where c_{2yr} corresponds to the coupon of the 2 year coupon swap.

$$r_{c,2yr} = -\frac{1}{\tau_{2yr}} \times \ln \left(\frac{100 - c_{2yr} \times e^{-r_{c,1yr} \tau_{1yr}}}{\left(100 + c_{2yr}\right)} \right) \qquad \text{(Eq.3.2.28)}$$

Having calculated the $r_{c,2yr}$ rate, the rate $r_{c,3yr}$ can now be calculated based on the $r_{c,1yr}$ and the $r_{c,2yr}$ rates, as shown by equation (3.2.29).

$$r_{c,3yr} = -\frac{1}{\tau_{3yr}} \times \ln \left(\frac{100 - c_{3yr} \times e^{-r_{c,1yr} \tau_{1yr}} - c_{3yr} \times e^{-r_{c,2yr} \tau_{2yr}}}{\left(100 + c_{3yr}\right)} \right) \qquad \text{(Eq.3.2.29)}$$

Having calculated the $r_{c,3yr}$ rate, the rate $r_{c,4yr}$ can now be calculated based on the $r_{c,1yr}$, $r_{c,2yr}$ and the $r_{c,3yr}$ rates, as shown by equation (3.2.30).

$$r_{c,4yr} = -\frac{1}{\tau_{4yr}} \times \ln\left(\frac{100 - c_{4yr} \times e^{-r_{c,1yr}\tau_{1yr}} - c_{4yr} \times e^{-r_{c,2yr}\tau_{2yr}} - c_{4yr} \times e^{-r_{c,3yr}\tau_{3yr}}}{(100 + c_{4yr})} \right) \qquad \text{(Eq.3.2.30)}$$

The same process continues until all the continuous rates are obtained up to the rate corresponding to the longest dated swap. This process is called bootstrapping. For the outlined example, the resultant rates from bootstrapping are included in table (T.3.11).

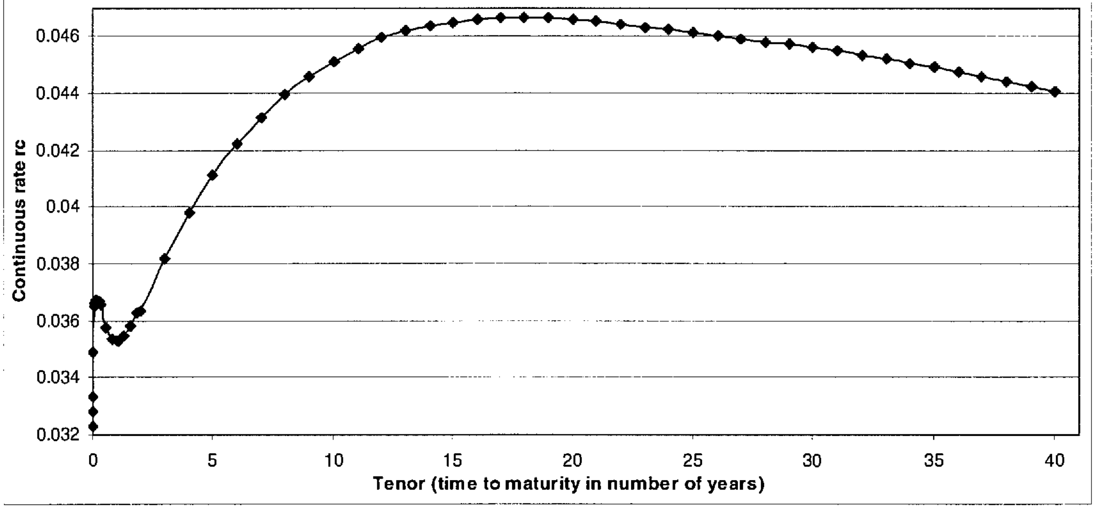
TABLE T.3.11

Tenor	Continuous annualised rate rc	Tenor	Continuous annualised rate rc	Tenor	Continuous annualised rate rc	Tenor	Continuous annualised rate rc
2yr	0.036352	12yr	0.045935	22yr	0.046434	32yr	0.045344
3yr	0.038148	13yr	0.046200	23yr	0.046336	33yr	0.045205
4yr	0.039766	14yr	0.046383	24yr	0.046230	34yr	0.045058
5yr	0.041096	15yr	0.046504	25yr	0.046123	35yr	0.044904
6yr	0.042184	16yr	0.046585	26yr	0.046019	36yr	0.044744
7yr	0.043130	17yr	0.046631	27yr	0.045917	37yr	0.044578
8yr	0.043928	18yr	0.046645	28yr	0.045814	38yr	0.044406
9yr	0.044572	19yr	0.046628	29yr	0.045708	39yr	0.044231
10yr	0.045114	20yr	0.046585	30yr	0.045595	40yr	0.044053
11yr	0.045571	21yr	0.046519	31yr	0.045474		

Final Annualised Continuous Rates Curve

Graph in figure (F.3.4) presents the final annualised continuous rates term structure which combines the short end, the middle area and the long and very-long end parts of the curve obtained in the previous calculations.

Figure (F.3.4) The zero-coupon rates curve



The resultant structure of the curve does not seem to have the shape of any of the common yield curve shapes like the following:

(i) For a *normal yield curve* the long-term rates are greater than short-term rates, so the curve has a positive slope. In other words, for a normal yield curve, the term spread, which is the difference between rates on the longer maturity and rates on the shorter maturity, is positive.

(ii) For a *flat yield curve* the yield for all the maturities is essentially the same and the term spread is roughly zero.

(iii) For an *inverted yield curve* the long-term rates are smaller than short-term rates, so the curve has a negative slope and a negative term spread.

Figure (F.3.5) The zero-coupon rates curve
Zoomed for tenors 0 to 3 years

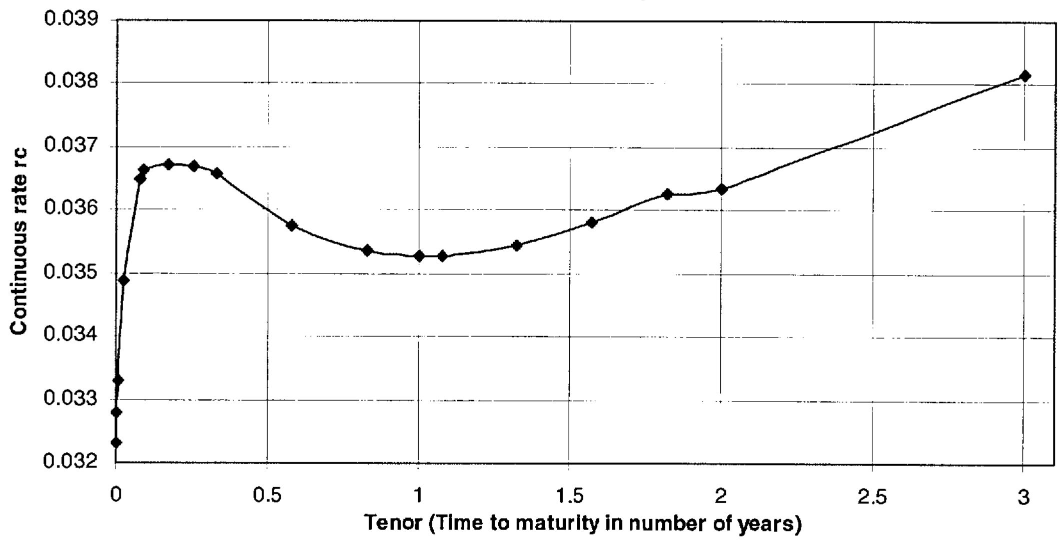


Figure (F.3.5) includes a zooming of the final curve into the lower than three years tenors. As it can be observed from both figures (F.3.4) and (F.3.5), the final curve has characteristics of two common yield curve types, the normal and the inverted types of yield curves. For the maturities up to a bit more than one year, the curve seems to be an inverted yield curve and then becomes a normal curve since it has a positive slope for the maturities beyond one year up to around 18 years. Beyond the maturities of 18 years, the curve takes once more the shape of an inverted yield curve.

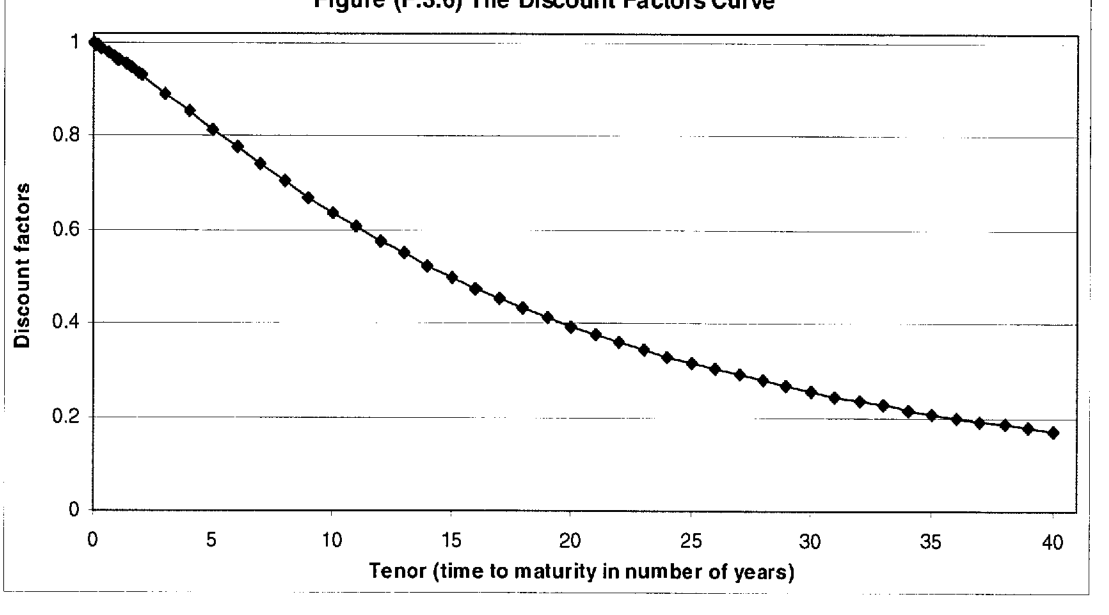
A similar shape of yield curve as the final yield curve calculated in this section is discussed in reference [50]. In that case, an inverted Treasury yield curve (negative term spread) was observed for a given period while the Swaps based curve had a positive overall term spread, presenting however features similar to the final curve presented here (inverted in the lower and middle area of the curve and with longer maturities preserving the positive term spread). The author explains that the failure of the swap curve to totally invert following the inversion of the Treasury curve might be explained by the fact that risky yield curves, like the swap curve, are more closely tied to firm behaviour and invert less often. The author extends his explanation with the argument that, in an inverted Treasury market, private firms will tend to issue more longer-term debt in the place of shorter-term debt, and the resultant supply will result in upward yield curve slope.

The Discount Factors Curve

The zero-coupon annualised continuous rates curve can equivalently be expressed as a discount factor curve. The relationship between the discount factors and the rates has already been presented previously in this section, like in equation (3.2.23). This is repeated here in equation (3.2.31). Figure (F.3.6) includes the graph of the resultant discount function for the calculations carried out in this section.

$$df_c(i) = e^{-r_c(i) \times \tau(i)} \qquad \text{(Eq.3.2.31)}$$

Figure (F.3.6) The Discount Factors Curve



Employing the Final Yield Curve in the following chapters

The graphical form of the final continuous rate curve is presented in figure (3.4) and the graphical form of the discount factors curve is included in figure (3.6) above. In table (3.12) below, the tenors and the respective annualised continuous rates (denoted as "*rc*") and discount factors (denoted as "*df*"), are set along side the indexing for the tenor points.

TABLE T.3.12

Rate Index i	Tenor (years)	Rate rc	df	Rate Index i	Tenor (years)	Rate rc	df	Rate Index i	Tenor (years)	Rate rc	df	Rate Index i	Tenor (years)	Rate rc	df
0	0.0000	0.0323	1.0000	14	1.5753	0.0358	0.9452	28	14.0000	0.0464	0.5224	42	28.0000	0.0458	0.2773
1	0.0027	0.0328	0.9999	15	1.8247	0.0363	0.9360	29	15.0000	0.0465	0.4978	43	29.0000	0.0457	0.2657
2	0.0055	0.0333	0.9998	16	2.0000	0.0364	0.9299	30	16.0000	0.0466	0.4746	44	30.0000	0.0456	0.2547
3	0.0247	0.0349	0.9991	17	3.0000	0.0381	0.8919	31	17.0000	0.0466	0.4526	45	31.0000	0.0455	0.2442
4	0.0795	0.0365	0.9971	18	4.0000	0.0398	0.8529	32	18.0000	0.0466	0.4319	46	32.0000	0.0453	0.2343
5	0.0877	0.0366	0.9968	19	5.0000	0.0411	0.8143	33	19.0000	0.0466	0.4123	47	33.0000	0.0452	0.2250
6	0.1699	0.0367	0.9938	20	6.0000	0.0422	0.7764	34	20.0000	0.0466	0.3939	48	34.0000	0.0451	0.2161
7	0.2521	0.0367	0.9908	21	7.0000	0.0431	0.7394	35	21.0000	0.0465	0.3765	49	35.0000	0.0449	0.2077
8	0.3288	0.0366	0.9881	22	8.0000	0.0439	0.7037	36	22.0000	0.0464	0.3600	50	36.0000	0.0447	0.1997
9	0.5781	0.0358	0.9795	23	9.0000	0.0446	0.6696	37	23.0000	0.0463	0.3445	51	37.0000	0.0446	0.1922
10	0.8274	0.0353	0.9712	24	10.0000	0.0451	0.6369	38	24.0000	0.0462	0.3297	52	38.0000	0.0444	0.1850
11	1.0000	0.0353	0.9653	25	11.0000	0.0456	0.6058	39	25.0000	0.0461	0.3157	53	39.0000	0.0442	0.1782
12	1.0767	0.0353	0.9627	26	12.0000	0.0459	0.5762	40	26.0000	0.0460	0.3023	54	40.0000	0.0441	0.1717
13	1.3260	0.0354	0.9541	27	13.0000	0.0462	0.5485	41	27.0000	0.0459	0.2895				

The final curve calculated in this section was used in the simulations of the convertible bond pricing frameworks presented in the chapters to be followed. This curve is referred to as the curve in the Bond currency curve or the Domestic currency and is one of the market based inputs to the pricing models.

In the calculations of the following chapter, the required tenors do not coincide with the tenor points of the final curve. Consequently, interpolation has to be carried out in order to obtain the continuous rates corresponding to the tenors encountered in the calculations. As previously discussed, in order to maintain consistency, the default interpolation technique in the pricing models is the cubic splines, the same used here for deriving the final curve. Since the same curve and the same interpolation technique are employed throughout the thesis, the resultant polynomials for the Bond currency curve used here are the polynomials used throughout the thesis for performing interpolation for the interest rate in the Bond currency. Table (3.13) includes the parameters of the resultant polynomials and the two tenor points defining the periods that each polynomial covers.

TABLE T.3.13

Polynomial i	Rate i to i+1	A	B	C	D	Polynomial i	Rate i to i+1	A	B	C	D	Polynomial i	Rate i to i+1	A	B	C	D
0	0->1	0.0323	0.1809	0.0000	211.0108	18	18->19	0.0398	0.0014	5.5E-06	-0.0001	36	36->37	0.0464	-0.0001	-7E-06	9E-07
1	1->2	0.0328	0.1857	1.7343	-1055.0538	19	19->20	0.0411	0.0012	-0.0002	4E-05	37	37->38	0.0463	-0.0001	-4E-06	1E-06
2	2->3	0.0333	0.1714	-6.9373	119.2412	20	20->21	0.0422	0.0010	-4E-05	-1E-05	38	38->39	0.0462	-0.0001	-9E-07	1E-06
3	3->4	0.0349	0.0369	-0.0769	-1.0475	21	21->22	0.0431	0.0009	-0.0001	-2E-06	39	39->40	0.0461	-0.0001	2E-06	-4E-07
4	4->5	0.0365	0.0190	-0.2491	-1.2082	22	22->23	0.0439	0.0007	-0.0001	1E-05	40	40->41	0.0460	-0.0001	1E-06	-5E-07
5	5->6	0.0366	0.0147	-0.2789	1.3873	23	23->24	0.0446	0.0006	-4E-05	1E-06	41	41->42	0.0459	-0.0001	-3E-07	-5E-07
6	6->7	0.0367	-0.0030	0.0632	-0.3703	24	24->25	0.0451	0.0005	-4E-05	-2E-06	42	42->43	0.0458	-0.0001	-2E-06	-5E-07
7	7->8	0.0367	-0.0002	-0.0281	0.0976	25	25->26	0.0456	0.0004	-5E-05	-2E-06	43	43->44	0.0457	-0.0001	-3E-06	-5E-07
8	8->9	0.0366	-0.0028	-0.0056	0.0151	26	26->27	0.0459	0.0003	-0.0001	3E-06	44	44->45	0.0456	-0.0001	-5E-06	8E-08
9	9->10	0.0358	-0.0027	0.0057	-0.0049	27	27->28	0.0462	0.0002	-4E-05	4E-06	45	45->46	0.0455	-0.0001	-5E-06	9E-08
10	10->11	0.0353	-0.0008	0.0020	0.0002	28	28->29	0.0464	0.0001	-3E-05	4E-06	46	46->47	0.0453	-0.0001	-4E-06	1E-07
11	11->12	0.0353	-0.0001	0.0021	0.0025	29	29->30	0.0465	0.0001	-2E-05	4E-07	47	47->48	0.0452	-0.0001	-4E-06	1E-07
12	12->13	0.0353	0.0002	0.0026	-0.0025	30	30->31	0.0466	0.0001	-2E-05	4E-07	48	48->49	0.0451	-0.0002	-4E-06	1E-07
13	13->14	0.0354	0.0011	0.0008	0.0022	31	31->32	0.0466	3E-05	-2E-05	5E-07	49	49->50	0.0449	-0.0002	-3E-06	1E-07
14	14->15	0.0358	0.0019	0.0024	-0.0107	32	32->33	0.0466	-2E-06	-1E-05	5E-07	50	50->51	0.0447	-0.0002	-3E-06	1E-07
15	15->16	0.0363	0.0011	-0.0056	0.0150	33	33->34	0.0466	-3E-05	-1E-05	6E-07	51	51->52	0.0446	-0.0002	-2E-06	2E-07
16	16->17	0.0364	0.0005	0.0023	-0.0010	34	34->35	0.0466	-0.0001	-1E-05	8E-07	52	52->53	0.0444	-0.0002	-2E-06	4E-08
17	17->18	0.0381	0.0021	-0.0007	0.0002	35	35->36	0.0465	-0.0001	-9E-06	9E-07	53	53->54	0.0442	-0.0002	-2E-06	6E-07

To demonstrate the performance of the interpolation approach, the rates were obtained for 1000 tenor values which were equally space in the tenor range $[0.0, 40]$ based on the resultant polynomials included in the above table. Figure (3.7) includes the graph of the resultant continuous rates and figure (3.8) includes the graph of the respective discount factors. The polynomials were also used for calculating the rates for 1000 tenor values which were equally space in the tenor range $[0.0, 3]$ and the results are presented in figure (3.9).

Figure (F.3.7) The zero-coupon rates curve (1000 points)

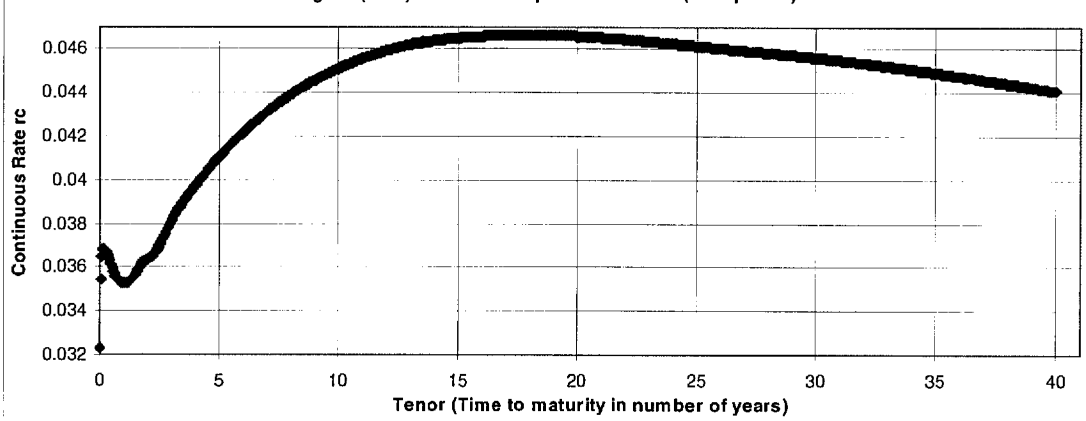
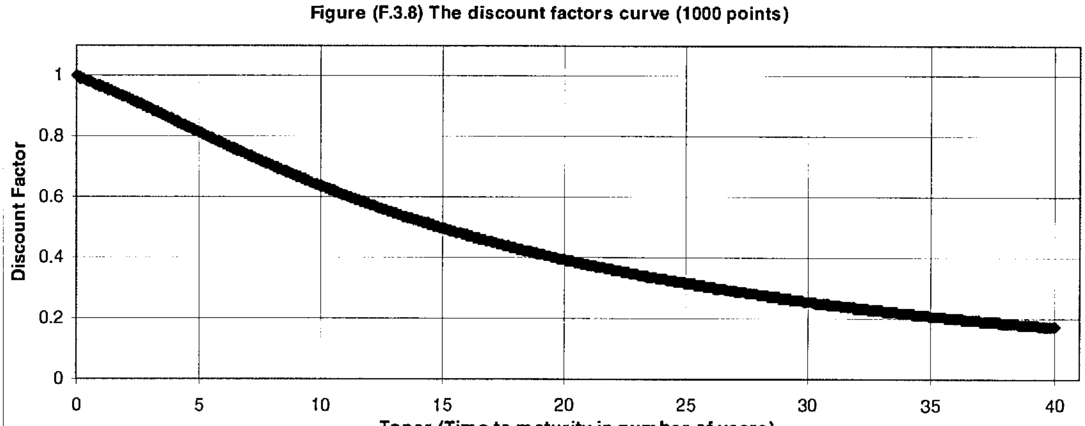
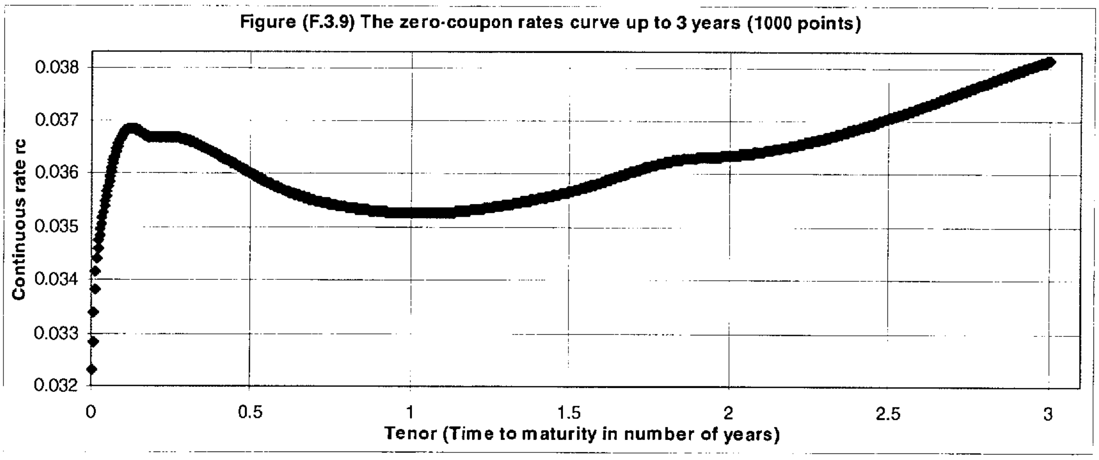


Figure (F.3.8) The discount factors curve (1000 points)





The tenor-points obtained based on the polynomials demonstrate that the cubic splines interpolation technique performs well in the case of the utilised zero-coupon rates curve and meets the requirements of the work of this thesis. As already mentioned, any tenor points required for the needs of the simulations and calculations in the following chapters are obtained based on the same polynomials.

The curve in the Equity (Foreign) currency

For the pricing of Dual currency convertibles, the zero-coupon rates curve in the Equity (Foreign) currency is required as well. Table (3.14) includes the securities used for determining the zero-coupon rates curve in the Foreign Currency.

TABLE T.3.14
Market Data - Equity (Foreign) Currency
Close of Business 18 February 2003

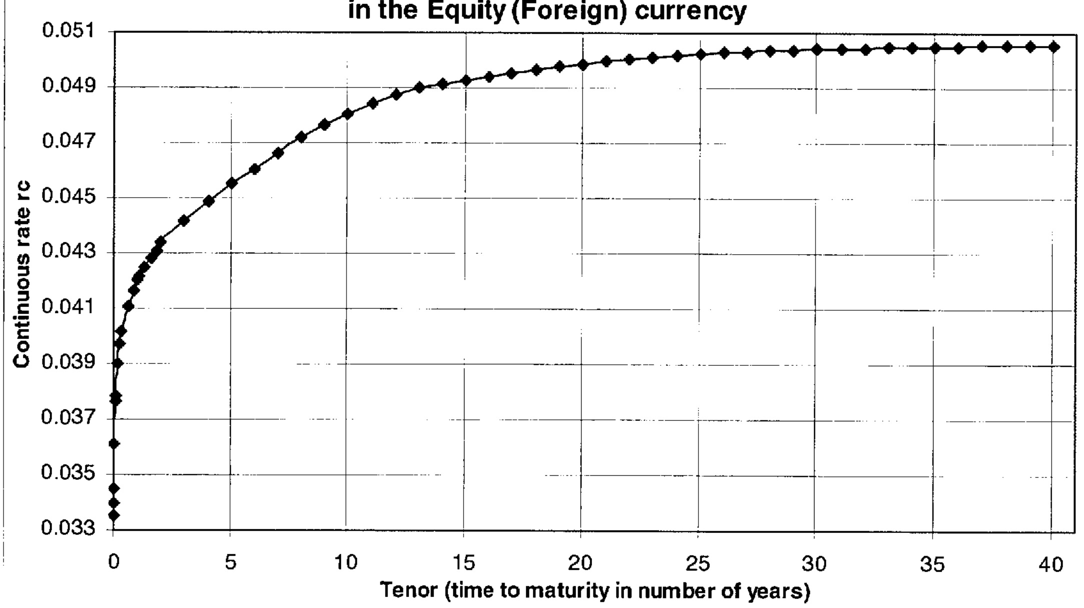
Rate Type	Tenor Label	Rate	RateType	Tenor Label	Rate
XIBOR	O/N	0.0340000	SWAP	2yr	0.04430
XIBOR	T/N	0.0350000	SWAP	3yr	0.04510
XIBOR	1WK	0.0365375	SWAP	4yr	0.04580
XIBOR	1M	0.0381094	SWAP	5yr	0.04640
XIBOR	2M	0.0392500	SWAP	6yr	0.04695
XIBOR	3M	0.0400000	SWAP	7yr	0.04750
FUTURE	19-Mar-03	95.885	SWAP	8yr	0.04800
FUTURE	18-Jun-03	95.750	SWAP	9yr	0.04845
FUTURE	17-Sep-03	95.670	SWAP	10yr	0.04880
FUTURE	17-Dec-03	95.600	SWAP	12yr	0.04940
FUTURE	17-Mar-04	95.580	SWAP	15yr	0.04990
FUTURE	16-Jun-04	95.525	SWAP	20yr	0.05040
FUTURE	15-Sep-04	95.500	SWAP	25yr	0.05070
			SWAP	30yr	0.05085
			SWAP	40yr	0.05100

The decision was made to introduce a dummy currency instead of using an actual one in order to introduce some generalisation and abstractness in the framework. For the case of the introduced currency, the cash rates are denoted as XIBOR so that the parallelism to the LIBOR rates is more indicative. In exactly the same fashion as the derivation of the continuous rates for the Bond (domestic) currency, the zero-coupon continuous rates for the Equity currency were derived and are presented below in table (3.15), as well as in the graph of figure (3.10).

TABLE T.3.15

Rate Index j	Tenor (years)	Rate rc	df	Rate Index j	Tenor (years)	Rate rc	df	Rate Index j	Tenor (years)	Rate rc	df	Rate Index j	Tenor (years)	Rate rc	df
0	0.0000	0.0335	1.0000	14	1.5753	0.0428	0.9348	28	14.0000	0.0492	0.5025	42	28.0000	0.0504	0.2442
1	0.0027	0.0340	0.9999	15	1.8247	0.0431	0.9244	29	15.0000	0.0493	0.4774	43	29.0000	0.0504	0.2320
2	0.0055	0.0345	0.9998	16	2.0000	0.0434	0.9169	30	16.0000	0.0494	0.4535	44	30.0000	0.0504	0.2204
3	0.0247	0.0361	0.9991	17	3.0000	0.0442	0.8759	31	17.0000	0.0495	0.4308	45	31.0000	0.0504	0.2095
4	0.0795	0.0377	0.9970	18	4.0000	0.0449	0.8357	32	18.0000	0.0497	0.4091	46	32.0000	0.0504	0.1991
5	0.0877	0.0378	0.9967	19	5.0000	0.0455	0.7966	33	19.0000	0.0498	0.3884	47	33.0000	0.0505	0.1892
6	0.1699	0.0390	0.9934	20	6.0000	0.0461	0.7586	34	20.0000	0.0499	0.3688	48	34.0000	0.0505	0.1797
7	0.2521	0.0397	0.9900	21	7.0000	0.0466	0.7215	35	21.0000	0.0500	0.3502	49	35.0000	0.0505	0.1708
8	0.3288	0.0401	0.9869	22	8.0000	0.0472	0.6856	36	22.0000	0.0500	0.3325	50	36.0000	0.0505	0.1623
9	0.5781	0.0411	0.9765	23	9.0000	0.0477	0.6511	37	23.0000	0.0501	0.3158	51	37.0000	0.0505	0.1542
10	0.8274	0.0417	0.9661	24	10.0000	0.0481	0.6184	38	24.0000	0.0502	0.2999	52	38.0000	0.0505	0.1466
11	1.0000	0.0420	0.9589	25	11.0000	0.0484	0.5871	39	25.0000	0.0502	0.2848	53	39.0000	0.0505	0.1393
12	1.0767	0.0422	0.9556	26	12.0000	0.0487	0.5572	40	26.0000	0.0503	0.2705	54	40.0000	0.0506	0.1323
13	1.3260	0.0425	0.9452	27	13.0000	0.0490	0.5290	41	27.0000	0.0503	0.2570				

Figure (F.3.10) The zero-coupon continuous rates curve in the Equity (Foreign) currency



The resultant curve can be characterised as a normal curve, i.e. the tenor spread is positive and the slope of the curve is positive as well. Like in the case of the Bond currency, cubic splines interpolation is performed on the resultant curve in order to obtain the values of the rates at the tenor points required in the calculations and simulations in the following chapters. The resultant polynomials when the cubic splines technique is employed on the Equity currency rates curve are included in the following table.

TABLE T.3.16

Polynomial j	Rate j to j+1	A	B	C	D	Polynomial j	Rate j to j+1	A	B	C	D	Polynomial j	Rate j to j+1	A	B	C	D
0	0->1	0.0323	0.1809	0.0000	211.0108	18	18->19	0.0398	0.0014	5.5E-06	-0.0001	36	36->37	0.0464	-0.0001	-7E-06	9E-07
1	1->2	0.0328	0.1857	1.7343	-1055.0538	19	19->20	0.0411	0.0012	-0.0002	4E-05	37	37->38	0.0463	-0.0001	-4E-06	1E-06
2	2->3	0.0333	0.1714	-6.9373	119.2412	20	20->21	0.0422	0.0010	-4E-05	-1E-05	38	38->39	0.0462	-0.0001	-9E-07	1E-06
3	3->4	0.0349	0.0369	-0.0769	-1.0475	21	21->22	0.0431	0.0009	-0.0001	-2E-06	39	39->40	0.0461	-0.0001	2E-06	-4E-07
4	4->5	0.0365	0.0190	-0.2491	-1.2082	22	22->23	0.0439	0.0007	-0.0001	1E-05	40	40->41	0.0460	-0.0001	1E-06	-5E-07
5	5->6	0.0366	0.0147	-0.2789	1.3873	23	23->24	0.0446	0.0006	-4E-05	1E-06	41	41->42	0.0459	-0.0001	-3E-07	-5E-07
6	6->7	0.0367	-0.0030	0.0632	-0.3703	24	24->25	0.0451	0.0005	-4E-05	-2E-06	42	42->43	0.0458	-0.0001	-2E-06	-5E-07
7	7->8	0.0367	-0.0002	-0.0281	0.0976	25	25->26	0.0456	0.0004	-5E-05	-2E-06	43	43->44	0.0457	-0.0001	-3E-06	-5E-07
8	8->9	0.0366	-0.0028	-0.0056	0.0151	26	26->27	0.0459	0.0003	-0.0001	3E-06	44	44->45	0.0456	-0.0001	-5E-06	8E-08
9	9->10	0.0358	-0.0027	0.0057	-0.0049	27	27->28	0.0462	0.0002	-4E-05	4E-06	45	45->46	0.0455	-0.0001	-5E-06	9E-08
10	10->11	0.0353	-0.0008	0.0020	0.0002	28	28->29	0.0464	0.0001	-3E-05	4E-06	46	46->47	0.0453	-0.0001	-4E-06	1E-07
11	11->12	0.0353	-0.0001	0.0021	0.0025	29	29->30	0.0465	0.0001	-2E-05	4E-07	47	47->48	0.0452	-0.0001	-4E-06	1E-07
12	12->13	0.0353	0.0002	0.0026	-0.0025	30	30->31	0.0466	0.0001	-2E-05	4E-07	48	48->49	0.0451	-0.0002	-4E-06	1E-07
13	13->14	0.0354	0.0011	0.0008	0.0022	31	31->32	0.0466	3E-05	-2E-05	5E-07	49	49->50	0.0449	-0.0002	-3E-06	1E-07
14	14->15	0.0358	0.0019	0.0024	-0.0107	32	32->33	0.0466	-2E-06	-1E-05	5E-07	50	50->51	0.0447	-0.0002	-3E-06	1E-07
15	15->16	0.0363	0.0011	-0.0056	0.0150	33	33->34	0.0466	-3E-05	-1E-05	6E-07	51	51->52	0.0446	-0.0002	-2E-06	2E-07
16	16->17	0.0364	0.0005	0.0023	-0.0010	34	34->35	0.0466	-0.0001	-1E-05	8E-07	52	52->53	0.0444	-0.0002	-2E-06	4E-08
17	17->18	0.0381	0.0021	-0.0007	0.0002	35	35->36	0.0465	-0.0001	-9E-06	9E-07	53	53->54	0.0442	-0.0002	-2E-06	6E-07

Concluding Remarks for Yield Curves Construction

The construction of the discount factor term structure in this section, has demonstrated in a detailed and practical manner the nature of the assumptions and approximations that usually depict such procedures. There is a considerable number of different versions for these procedures in the practitioners world, some more delicate and sophisticated, some other more simplistic and with less assumptions. For example, in reference [6], the derivation of a zero-coupon yield curve from swap prices is summarised into three main steps: turning the raw data into equivalent zero-coupon rates, writing zero-coupon rates in terms of B-spline functions, and fitting that function through an Ordinary-Least-Squares (OLS) method. The last part of this approach is considerably different from the overall approach followed in this section.

The aim of this section was not to present the best or the most popular approach to curve construction. On the contrary, the objective was to demonstrate the importance of having knowledge of the origins and the procedures involved in the extraction of market quantities necessary for the pricing models, with all the assumptions and approximations these procedures introduce. Nevertheless, a technique was chosen that returns smooth and well-fitted to market data (prices of securities used for deriving the curves) zero-coupon continuous rates curves. In addition, the polynomials used for interpolating on the Bond and Equity currency rates were also derived and their usage was demonstrated.

3.3. Implied Volatilities in Equity Derivatives

Assuming that the underlying stock follows a geometric Brownian motion process in a risk-neutral world as the process described by equation (3.4.1), then equations (3.4.2) and (3.4.3) provide the closed form solution introduced by Black and Scholes for pricing European call and put options. The options on the stock with a price S have a strike K and maturity T, and are priced for time t based on a risk free rate $r_{t,T}$ and volatility $\sigma_{t,T}$ (the standard deviation of the logarithmic returns). The resultant prices correspond to a tenor $\tau = T - t$.

$$dS_t = \left(r_{t,T} - q\right)S_t\,dt + S_t\,\sigma_{t,T}\,d_t \qquad \text{(Eq.3.4.1)}$$

$$C(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r_{t,T}(T-t)}N(d_2) \qquad \text{(Eq.3.4.2)}$$

$$P(S,t) = Ke^{-r_{t,T}(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) \qquad \text{(Eq.3.4.3)}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy \qquad \text{(Eq.3.4.4)}$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r_{t,T} - q + \frac{1}{2}\sigma_{t,T}^2\right)(T-t)}{\sigma_{t,T}\sqrt{T-t}} \qquad \text{(Eq.3.4.5)}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r_{t,T} - q - \frac{1}{2}\sigma_{t,T}^2\right)(T-t)}{\sigma_{t,T}\sqrt{T-t}} = d_1 - \sigma_{t,T}\sqrt{T-t} \qquad \text{(Eq.3.4.6)}$$

It has been observed that when the historical volatility is used in these formulas, the obtained options prices, as a rule, do not reflect the actual option prices traded in the market. This is even true for liquid options; hence there is a discrepancy between the Black-Scholes framework and the actual market. To overcome this problem, the market participants have adopted a different approach; they calculate the implied volatilities of the options based on respective traded prices. The volatility parameter in the above equations becomes another way of expressing a traded quantity, the option price, through a transformation depicted by the Black-Scholes equation.

In the options markets, the participants use the term "smile" as reference to the dependency of the implied volatility from a Black-Scholes option model, to the strike price of the option. The general form of a smile, which actually looks more like a "smirk", is presented in figure (3.11). It can be observed that the volatility used to price a low strike option is significantly higher than that used to price a high-strike-price option. In other words, the volatility decreases as the strike price increases. In addition to a volatility smile, traders use a volatility term structure when pricing options. This means that the volatility used in pricing depends on the maturity of the option. Combining the smiles with the term structures, traders create a volatility matrix for each security, which is based on the strike price and the time to maturity. In other words, the one dimension of a volatility matrix is strike price and the other is time to maturity, and the resultant graph in this case is the volatility surface.

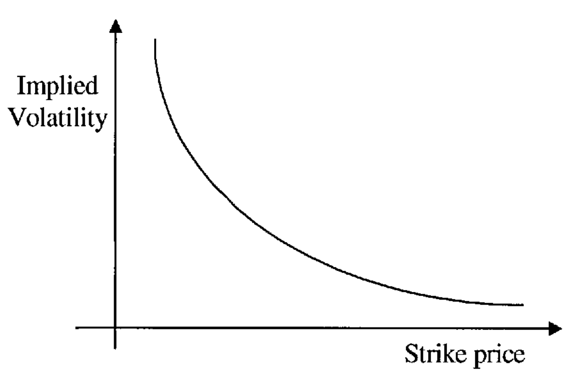


Figure 3.11 Volatility smile implied by equity options

Summarising, it is obvious that the market has its own view on the volatilities for pricing options. What is more important is that the market views on volatility are different from the actual volatilities of the securities. On the contrary, market decision on the volatilities values are more affected by factors like moneyness (how much out of the money or how much in the money an option is) and time to maturity (time horizon).

3.4. Implied Volatilities in Currency Options

Consider the following:

r_t^D The risk free rate of the domestic currency.

r_t^F The risk free rate of the foreign currency.

X_t The exchange rate for translating units in the foreign currency into units in the domestic currency.

σ_t The standard deviation of the log-normal returns of the exchange rate X_t.

Assuming that the exchange rate follows a geometric Brownian motion process similar to that assumed for the stock in the previous section, then, in a risk-neutral world this process is described by the following equation.

$$dX_t = \left(r_t^D - r_t^F\right)X_t\,dt + X_t\,\sigma_t\,dw_t \qquad \text{(Eq.3.5.1)}$$

The call and put options priced based on equations (3.5.2) and (3.5.3) are options on an exchange rate with a current value E and have a strike K and maturity T. They are priced for time t based on a domestic risk free rate $r_{t,T}^D$, a foreign risk free rate $r_{t,T}^F$ and volatility $\sigma_{t,T}$. The resultant prices correspond to a tenor $\tau = T - t$. The function $N(\bullet)$ has already been defined in equation (3.4.4).

$$C(X,t) = Xe^{-r_{t,T}^F(T-t)}N(d_1) - Ke^{-r_{t,T}^D(T-t)}N(d_2) \qquad \text{(Eq.3.5.2)}$$

$$P(X,t) = Ke^{-r_{t,T}^D(T-t)}N(-d_2) - Xe^{-r_{t,T}^F(T-t)}N(-d_1) \qquad \text{(Eq.3.5.3)}$$

$$d_1 = \frac{\log\left(\frac{X}{K}\right) + \left(r_{t,T}^D - r_{t,T}^F + \frac{1}{2}\sigma_{t,T}^2\right)(T-t)}{\sigma_{t,T}\sqrt{T-t}} \qquad \text{(Eq.3.5.4)}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r_{t,T}^D - r_{t,T}^F - \frac{1}{2}\sigma_{t,T}^2\right)(T-t)}{\sigma_{t,T}\sqrt{T-t}} = d_1 - \sigma_{t,T}\sqrt{T-t} \qquad \text{(Eq.3.5.5)}$$

If the forward exchange rate is introduced and defined as shown in equation (3.5.6), then, equations (3.5.2) to (3.5.3) can be re-written as shown in equations (3.5.7) to (3.5.10).

$$F_{t,T} = Xe^{\left(r_{t,T}^{D} - r_{t,T}^{F}\right)(T-t)} \tag{Eq.3.5.6}$$

$$C(X,t) = e^{-r_{t,T}^{D}(T-t)}\left[F_{t,T} \times N(d_1) - K \times N(d_2)\right] \tag{Eq.3.5.7}$$

$$P(X,t) = e^{-r_{t,T}^{D}(T-t)}\left[K \times N(-d_2) - F_{t,T} \times N(-d_1)\right] \tag{Eq.3.5.8}$$

$$d_1 = \frac{\log\left(\frac{F_{t,T}}{K}\right) + \frac{1}{2}\sigma_{t,T}^2(T-t)}{\sigma_{t,T}\sqrt{T-t}} \tag{Eq.3.5.9}$$

$$d_2 = \frac{\log\left(\frac{F_{t,T}}{K}\right) - \frac{1}{2}\sigma_{t,T}^2(T-t)}{\sigma_{t,T}\sqrt{T-t}} = d_1 - \sigma_{t,T}\sqrt{T-t} \tag{Eq.3.5.10}$$

Like in the case of stock options, the market convention depicts the construction and employment of volatility smiles. However, the smile in the case of currency options looks more like a "smile" as shown in the following figure, in contrast to the "smirk" more-like type of volatility structure of the stock options. Combining the smiles with volatility term structures like in the case of stock options, traders can create a volatility matrix for each exchange rate, which is based on the strike price and the time to maturity.

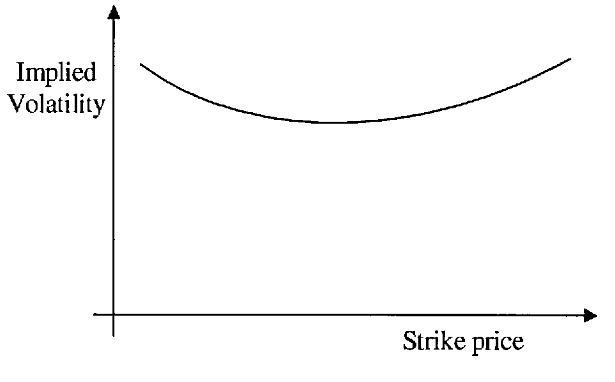


Figure 3.12 Volatility smile implied by currency options

CHAPTER 4

INPUTS TO THE CB PRICING FRAMEWORKS

In the following chapters there will be presented two mainly CB pricing frameworks, the one-dimensional and the two-dimensional configurations. However, for both cases, the required inputs are the same to some extent. So, in this chapter, we take the opportunity to define these inputs and set the respective notation that will be followed throughout the thesis. The inputs to the CB pricing framework are identified into two sources, the inputs based on the convertible bond contract description and the inputs based on market information. The two sections of this chapter are each devoted to the presentation of one source of input information.

Dates and Time parameters

Before proceeding to defining the input parameters, some general comments are included on the conversions followed in the notation of the dates and time parameters. Subscripts will always be used for indexing purposes, while superscript will always be used for adding information to the parameter, unless it is stated otherwise.

In the simulations, all actual dates are represented and input as a number of days since the end of the 19th century (Microsoft Excel format) and parameters of the form t^a are reserved for actual dates of this format. The choice was made to follow the Microsoft Excel format for representing dates, which is a number of type *double*, because all simulations were performed in DLLs called in a VBA code in Microsoft Excel (inputs and outputs were in Microsoft Excel spreadsheets). So, a value of the form $t^a = 1$ corresponds to the 1st of January, 1900, while a parameter of the form $t^a = 37622$ corresponds to the 1st of January, 2003.

The parameter t with no superscript always represents time in number of years, while the parameter t^d always represents time in number of days, both t and t^d with reference to a specified date. For the purposes of this thesis, the default referenced date is the calculations date which means that both, $t=0$ and $t^d=0$, correspond to the ***Calculations Date***, which is denoted as $t^{a,CD}$. The parameter τ is reserved for representing time to maturity in number of years, while τ^d represents time to maturity in number of days.

4.1. Inputs based on the CB Contract Description

Like every instrument, a convertible bond has some fixed contract information. The terminology around the convertible bond was presented in chapter 2. Here, the standard notation followed in this thesis for the contract information is presented, alongside with its interpretation.

The life duration of the instrument is depicted by its ***Issue Date*** $t^{a,ID}$ and its ***Expiring date*** $t^{a,ED}$, also referred to as the ***Maturity Date***. The Calculations Date must satisfy the following condition. In other words, we can only carry out the calculations for dates lying between the issue date and the expiration date of the instrument.

$$t^{ID} \le t^{CD} \le t^{ED} \qquad \text{(Eq.4.1.1)}$$

The ***Redemption Value***, denoted as P^{Rd}, and the ***Face Value***, also referred to as the ***Notional*** and denoted as P^F, are not necessarily equal, which is true for some instruments. Equation (4.1.2) must always hold, where R^{CR} denotes the ***Conversion Ratio***, and K denotes the ***Strike***.

$$P^F = R^{CR} \times K \qquad \text{(Eq.4.1.2)}$$

The ***no conversion period*** which is represented by the variable $t^{d,NoConv}$, is a parameter expressed as a number of days. This is the period prior the maturity of the convertible bond, during which no conversion to equity can take place. This is a very common and standard contract feature of convertible bonds and the following equation gives us the last date $t^{a,LastConv}$ that conversion can take place.

$$t^{a,LastConv} = t^{a,ED} - t^{d,NoConv} \qquad \text{(Eq.4.1.3)}$$

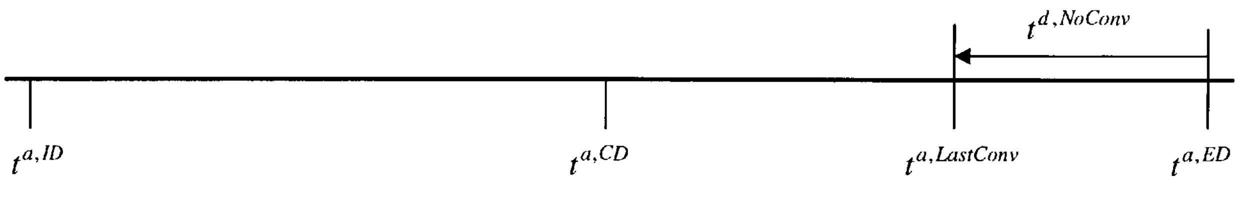


Figure (4.1)
No Conversion Period

Coupons Schedule

The coupons are always expressed in terms of an ***annual coupon rate***, denoted as r^{Cpn}, and a ***coupon frequency***, denoted as q^{Cpn}. The coupon frequency defines the number of coupon payments per year, while the coupon annual rate defines the total coupon cash flow per year as a percentage of the Notional (Face Value). In addition, the coupon dates will either be established with reference to the maturity date and with the last coupon date coinciding with the maturity date, or with respect to the ***first coupon date***, which is denoted as $t^{a,FirstCpn}$. Based on this information, the coupons structure can be established. The coupons structure with n_{Cpn} distinct coupon dates is represented by the matrix $\underline{C}^{Cpn} \in \Re^{2 \times n_{Cpn}}$ which is made of two fields, the date's field $\underline{t}^{a,Cpn} = \left\{ t^{a,Cpn}_{i=0,1,2,\ldots,n_{Cpn}-1} \right\}$ and the cash values $\underline{q}^{Cpn} = \left\{ q^{Cpn}_{i=0,1,2,\ldots,n_{Cpn}-1} \right\}$ to be received as coupons on the respective dates.

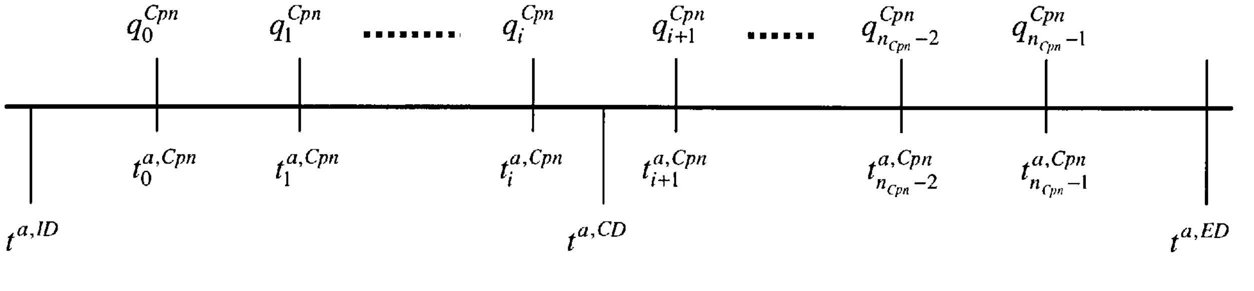


Figure (4.2)
The Coupon Schedule

For the purposes of the work presented in this thesis, it was decided to define the coupon dates and cash flows based on the first coupon date. So, the first coupon date $t_0^{a,Cpn}$ is set equal to the respective contract based input parameter if this input parameter satisfies the condition that it is equal or greater than the issue date of the convertible bond. Then, the rest of the coupon dates, starting with the second coupon date $t_1^{a,Cpn}$, are calculated by adding to the previous date the time interval between the coupon dates (this is equal to the number of days in one year divided by the coupon frequency). This action is repeated until we have reached the expiration date of the convertible bond, since no coupon date can be greater than the expiration date. The coupon cash flows are then calculated based on the face amount. The calculations are summarized in the following three equations. Division by 100 in equation (4.1.6) is required only in the case that the coupon rate r^{Cpn} is quoted as a percentage.

$$t_0^{a,Cpn} = t^{a,FirstCpn} \geq t^{a,ID} \qquad \text{(Eq.4.1.4)}$$

$$\begin{aligned} & t_i^{a,Cpn} = t_{i-1}^{a,Cpn} + \frac{NumDaysPerYear}{q^{Cpn}} \quad , \ 1 \leq i < n_{Cpn} \\ & t_i^{a,Cpn} \leq t^{a,ED} \end{aligned} \qquad \text{(Eq.4.1.5)}$$

$$q_i^{Cpn} = \frac{r^{Cpn}}{100 \times q^{Cpn}} \times P^F \quad , \ 1 \leq i < n_{Cpn} \qquad \text{(Eq.4.1.6)}$$

For each coupon date greater than the calculations date, the time parameters are calculated based on equations (4.1.7) to (4.1.10).

$$t_i^{d,Cpn} = t_i^{a,Cpn} - t^{a,CD} \quad , \ t_i^{a,Cpn} \geq t^{a,CD}, \ 0 \leq i < n_{Cpn} \qquad \text{(Eq.4.1.7)}$$

$$t_i^{Cpn} = \frac{t_i^{d,Cpn}}{NumDaysPerYear} \quad , \ t_i^{a,Cpn} \geq t^{a,CD}, \ 0 \leq i < n_{Cpn} \qquad \text{(Eq.4.1.8)}$$

$$\tau_i^{d,Cpn} = t^{a,ED} - t_i^{a,Cpn} \quad , \ t_i^{a,Cpn} \geq t^{a,CD}, \ 0 \leq i < n_{Cpn} \qquad \text{(Eq.4.1.9)}$$

$$\tau_i^{Cpn} = \frac{\tau_i^{d,Cpn}}{NumDaysPerYear} \quad , \ t_i^{a,Cpn} \geq t^{a,CD}, \ 0 \leq i < n_{Cpn} \qquad \text{(Eq.4.1.10)}$$

Call Schedule

A call schedule with n_{CS} distinct call periods is represented by the matrix $\underline{C}^{CS} \in \Re^{6 \times n_{CS}}$ which is made of six fields (columns):

(1) The call values field $\underline{v}^{CS} = \left\{v_{i=0,1,2,\ldots,n_{CS}-1}^{CS}\right\}$.

(2) The starting dates field $\underline{t}^{a,SD,CS} = \left\{t_{i=0,1,2,\ldots,n_{CS}-1}^{a,SD,CS}\right\}$.

(3) The ending dates field $\underline{t}^{a,ED,CS} = \left\{t_{i=0,1,2,\ldots,n_{CS}-1}^{a,ED,CS}\right\}$.

(4) The triggers (conditions) field $\underline{c}^{CS} = \left\{c_{i=0,1,2,\ldots,n_{CS}-1}^{CS}\right\}$.

(5) The grace periods (in number of days) field $\underline{w}^{d,CS} = \left\{w_{i=0,1,2,\ldots,n_{CS}-1}^{d,CS}\right\}$.

(6) The field with the additional flags for the conditions $\underline{y}^{CS} = \left\{y_{i=0,1,2,\ldots,n_{CS}-1}^{CS}\right\}$.

For the purposes of this thesis, the default type of calls is the American type since European type of calls are a more simplistic case and can be easily dealt within a framework already supporting American type of calls. An American type of call with a call value v_i^{CS} is consider to be active (valid) during the period starting from the date $t_i^{a,CS,SD}$ to the date $t_i^{a,CS,ED}$, which in most cases is just the day before the next call date $t_{i+1}^{a,CS}$. In other words, a call value v_i^{CS} is consider to be active for the period $t_i^{a,CS,SD} \leq t^a < t_{i+1}^{a,CS,SD}$. Usually, the last call is active for the period $t_{n_{CS}-1}^{a,CS,SD} \leq t^a < t^{a,ED}$, and, in this way, its active period ends just before the maturity of the instrument, i.e. $t_{n_{CS}-1}^{a,CS,ED} = t^{a,ED} - 1$.

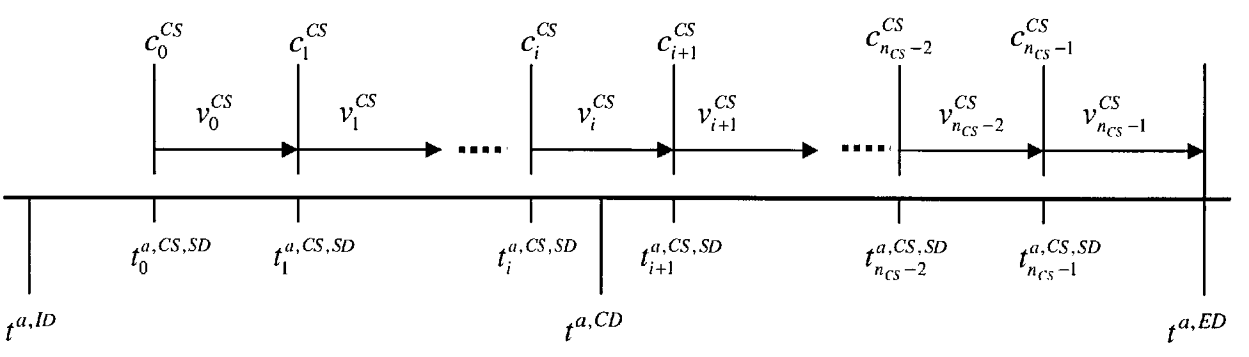


Figure (4.3)
The Conditional Call Schedule

The following relationships hold.

$$t_0^{a,CS,SD} > t^{a,ID}, \quad t_{n_{CS}-1}^{a,CS,ED} < t^{a,ED} \tag{Eq.4.1.11}$$

$$t_i^{a,CS,SD} < t_i^{a,CS,ED} \quad ,0 \le i < n_{CS} \tag{Eq.4.1.12}$$

$$t_i^{a,CS,SD} > t_{i-1}^{a,CS,ED} \quad ,1 \le i < n_{CS} \tag{Eq.4.1.13}$$

For each starting and ending call date greater than the calculations date, i.e. $t_i^{a,CS,SD} \ge t^{a,CD}$ and $t_i^{a,CS,ED} \ge t^{a,CD}$ respectively, the time parameters are calculated based on the following equations.

$$t_i^{CS,SD} = \frac{t_i^{d,CS,SD}}{NumDaysPerYear} = \frac{t_i^{a,CS,SD} - t^{a,CD}}{NumDaysPerYear} \tag{Eq.4.1.14}$$

$$\tau_i^{CS,SD} = \frac{\tau_i^{d,CS,SD}}{NumDaysPerYear} = \frac{t^{a,ED} - t_i^{a,CS,SD}}{NumDaysPerYear} \tag{Eq.4.1.15}$$

$$t_i^{CS,ED} = \frac{t_i^{d,CS,ED}}{NumDaysPerYear} = \frac{t_i^{a,CS,ED} - t^{a,CD}}{NumDaysPerYear} \tag{Eq.4.1.16}$$

$$\tau_i^{CS,ED} = \frac{\tau_i^{d,CS,ED}}{NumDaysPerYear} = \frac{t^{a,ED} - t_i^{a,CS,ED}}{NumDaysPerYear} \tag{Eq.4.1.17}$$

A call event can either take place conditionally or unconditionally. In the case of an unconditional call, the issuer will have the right to accelerate the redemption of the bond within the specified call period and at the specific call value. However, in the case of a conditional call, the issuer's right for early redemption exist only if a condition is satisfied. For the case of convertible bonds, a condition on a call would be that the price of the underlying share exceeds a level for a number of consequent days; this number of days is commonly referred to as the *grace period.*

The share price level which is used in the condition for determining whether the issuer can call the bond is defined as a percentage of the strike. Consider the example where there is a call value v_i^{CS} equal to 102, a condition value c_i^{CS} equal to 115 and a grace period $w_i^{d,CS}$ equal to 20. For this example, the convertible bond can be called at the price of 102% of its face value, given that the underlying share price has exceeded the level of 115% of the strike for 20 consequent days.

The last column of the call schedule contains a flag which is useful for the cases of convertible bonds with a reset schedule. Because in these cases the strike can change during the life of the convertible bond, it needs to be specified whether the level for determining the level of the trigger must be applied on the prevailing strike or on the initial strike (as it was defined on the issue date of the convertible bond).

Put Schedule

The Put Schedule has been defined in the same fashion as the call schedule. A put schedule with n_{PS} distinct put periods is represented by the matrix $\underline{C}^{PS} \in \Re^{6 \times n_{PS}}$ which is made of six fields (columns):

(1) The put values field $\underline{v}^{PS} = \left\{ v_{i=0,1,2,\ldots,n_{PS}-1}^{PS} \right\}$.

(2) The starting dates field $\underline{t}^{a,SD,PS} = \left\{ t_{i=0,1,2,\ldots,n_{PS}-1}^{a,SD,PS} \right\}$.

(3) The ending dates field $\underline{t}^{a,ED,PS} = \left\{ t_{i=0,1,2,\ldots,n_{PS}-1}^{a,ED,PS} \right\}$.

(4) The triggers (conditions) field $\underline{c}^{PS} = \left\{ c_{i=0,1,2,\ldots,n_{PS}-1}^{PS} \right\}$.

(5) The grace periods (in number of days) field $\underline{w}^{d,PS} = \left\{ w_{i=0,1,2,\ldots,n_{PS}-1}^{d,PS} \right\}$.

(6) The field with the additional flags for the conditions $\underline{y}^{PS} = \left\{ y_{i=0,1,2,\ldots,n_{PS}-1}^{PS} \right\}$.

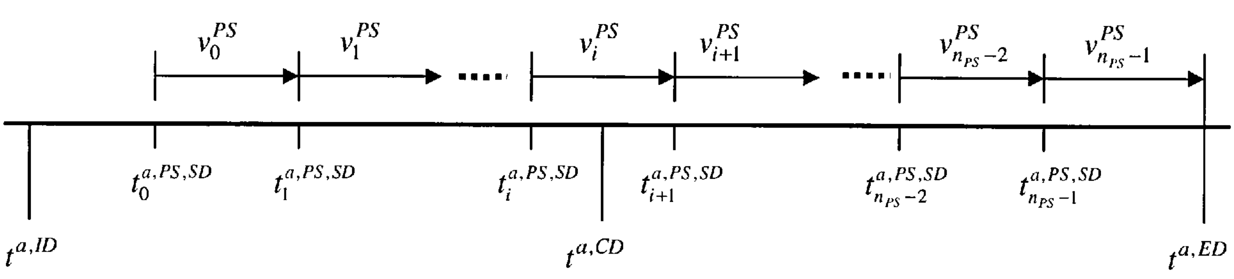


Figure (4.4)
The Put Schedule

The periods for the puts have been defined in exactly the same fashion as for the call schedule, and these are calculated based on equations (14.1.18) to (14.1.24).

$$t_0^{a,PS,SD} > t^{a,ID}, \quad t_{n_{PS}-1}^{a,PS,ED} < t^{a,ED} \qquad \text{(Eq.4.1.18)}$$

$$t_i^{a,PS,SD} < t_i^{a,PS,ED} \quad ,0 \le i < n_{PS} \qquad \text{(Eq.4.1.19)}$$

$$t_i^{a,PS,SD} > t_{i-1}^{a,PS,ED} \quad ,1 \le i < n_{PS} \qquad \text{(Eq.4.1.20)}$$

The following equations hold for each starting and ending put date greater than the calculations date, i.e. $t_i^{a,PS,SD} \ge t^{a,CD}$ and $t_i^{a,PS,ED} \ge t^{a,CD}$ respectively.

$$t_i^{PS,SD} = \frac{t_i^{d,PS,SD}}{NumDaysPerYear} = \frac{t_i^{a,PS,SD} - t^{a,CD}}{NumDaysPerYear} \qquad \text{(Eq.4.1.21)}$$

$$\tau_i^{PS,SD} = \frac{\tau_i^{d,PS,SD}}{NumDaysPerYear} = \frac{t^{a,ED} - t_i^{a,PS,SD}}{NumDaysPerYear} \qquad \text{(Eq.4.1.22)}$$

$$t_i^{PS,ED} = \frac{t_i^{d,PS,ED}}{NumDaysPerYear} = \frac{t_i^{a,PS,ED} - t^{a,CD}}{NumDaysPerYear} \qquad \text{(Eq.4.1.23)}$$

$$\tau_i^{PS,ED} = \frac{\tau_i^{d,PS,ED}}{NumDaysPerYear} = \frac{t^{a,ED} - t_i^{a,PS,ED}}{NumDaysPerYear} \qquad \text{(Eq.4.1.24)}$$

Even though the conditionality feature encountered in call schedules, no conditionality is encountered in put schedules. However, we have included, or actually introduced, a form of conditionality in the put schedule used in this thesis. This conditionality almost mimics the conditionality included in the call schedule. The only difference is in the triggering condition which has been reversed. A condition on a put depicts that the price of the underlying share is lower (instead of higher like in the case of a conditional call) than a level for the grace period. So, both conditional and unconditional puts are allowed in the put schedule.

Resets Schedule

Another structure that could be included in the contract information of a convertible bond is the ***Reset Schedule***. A reset schedule with n_{RS} distinct reset dates is represented by the matrix $\underline{C}^{RS} \in \Re^{5xn}$ which is made of five fields.

(1) The reset dates field $\underline{t}^{a,RS} = \left\{t_{i=0,1,2,\ldots,n_{RS}-1}^{a,RS}\right\}$.

(2) The lower reset limits field $\underline{v}^{lower,RS} = \left\{v_{i=0,1,2,\ldots,n_{RS}-1}^{lower,RS}\right\}$.

(3) The upper reset limits field $\underline{v}^{upper,RS} = \left\{v_{i=0,1,2,\ldots,n_{RS}-1}^{upper,RS}\right\}$.

(4) The number of days for averaging field $\underline{w}^{d,RS} = \left\{ w^{d,RS}_{i=0,1,2,\ldots,n_{RS}-1} \right\}$.

(5) The field with the additional flags for the conditions $\underline{y}^{RS} = \left\{ y^{RS}_{i=0,1,2,\ldots,n_{RS}-1} \right\}$.

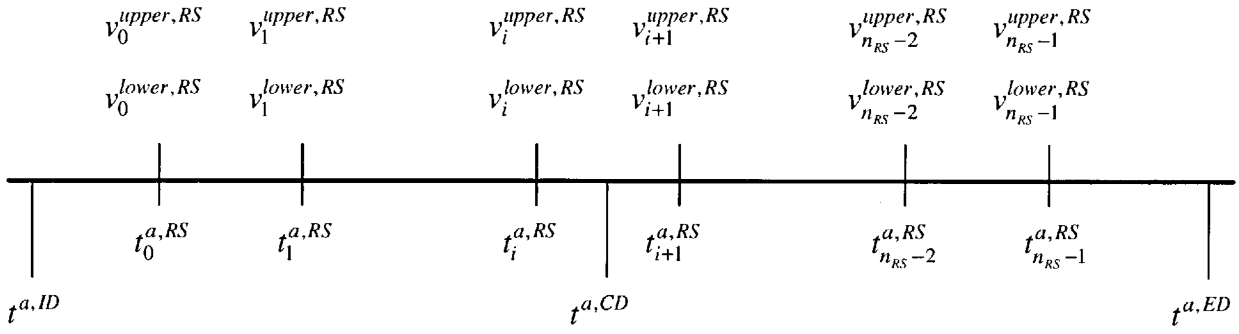


Figure (4.5)
The Reset Schedule

The following equations include the relationships that must hold in order to include a reset schedule in the calculations.

$$t_0^{a,RS,SD} > t^{a,ID}, \quad t_{n_{RS}-1}^{a,RS} < t^{a,ED} \qquad \text{(Eq.4.1.25)}$$

$$t_i^{a,RS} > t_{i-1}^{a,RS} \quad , 1 \le i < n_{RS} \qquad \text{(Eq.4.1.26)}$$

$$v_i^{upper,RS} > v_i^{lower,RS} \quad , \ 0 \le i < n_{RS} \qquad \text{(Eq.4.1.27)}$$

The following time parameters are calculated for each reset date greater than the calculations date, i.e. $t_i^{a,RS} \ge t^{a,CD}$.

$$t_i^{RS} = \frac{t_i^{d,RS}}{NumDaysPerYear} = \frac{t_i^{a,RS} - t^{a,CD}}{NumDaysPerYear} \qquad \text{(Eq.4.1.28)}$$

$$\tau_i^{RS} = \frac{\tau_i^{d,RS}}{NumDaysPerYear} = \frac{t^{a,ED} - t_i^{a,RS}}{NumDaysPerYear} \qquad \text{(Eq.4.1.29)}$$

A reset with the set of parameters $\left\{ t_i^{a,RS}, v_i^{lower,RS}, v_i^{upper,RS}, w_i^{d,RS}, y_i^{RS} \right\}$ can take place only the date $t_i^{a,RS}$. The flag y_i^{RS} determines whether the initial strike (defined on the issued date) or the prevailing strike will be used as reference for determining the new strike on the date $t_i^{a,RS}$. Whichever is the case, we refer to the chosen strike as K'. The strike resets to a new value K'' based on the average share price over the last

$w_i^{d,RS}$ number of days and this average is denoted as $\bar{S}_{t_i^{a,RS}}$. The rules for defining the new strike K'' are summarised by the following equations.

$$\begin{array}{ll}
K'' = \bar{S}_{t_i^{a,RS}} & \dfrac{v_i^{lower,RS}}{100} \times K' \le \bar{S}_{t_i^{a,RS}} \le \dfrac{v_i^{upper,RS}}{100} \times K' \\
K'' = \dfrac{v_i^{upper,RS}}{100} \times K' & \bar{S}_{t_i^{a,RS}} > \dfrac{v_i^{upper,RS}}{100} \times K' \\
K'' = \dfrac{v_i^{lower,RS}}{100} \times K' & \bar{S}_{t_i^{a,RS}} < \dfrac{v_i^{lower,RS}}{100} \times K'
\end{array} \qquad \text{(Eq.4.1.30)}$$

4.2. Inputs based on Market Information

The rest of the inputs to the pricing framework of a convertible bond are market based information. The ***share price on the calculations date*** which is denoted as S^{CD} and the ***continuous dividend yield*** which is denoted as q^{Div} and expressed as an annual percentage of the stock price, are both two standard inputs to the convertible bond pricing framework. As an alternative, we also allow for a continuous dividend yield term structure to be input in the framework. A continuous dividend yield term structure with n_{CDS} distinct sets of parameters is represented by the matrix $\underline{C}^{CDS} \in \Re^{2 \times n_{CDS}}$ which is made of two fields, the dates field $\underline{t}^{a,CDS} = \left\{t_{i=0,1,2,\ldots,n_{CDS}-1}^{a,CDS}\right\}$ and the continuous dividend yield values field $\underline{q}^{CDS} = \left\{q_{i=0,1,2,\ldots,n_{CDS}-1}^{CDS}\right\}$.

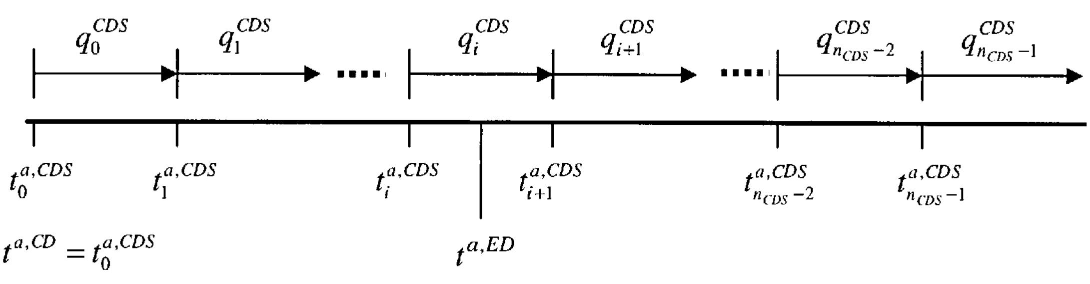


Figure (4.6) The Continuous Dividend Yield Term Structure

A continuous dividend yield q_i^{CDS} corresponds to the time period $\left[t_i^{a,CDS}, t_{i+1}^{a,CDS}\right)$. i.e. from the date and including the date $t_i^{a,CDS}$ up to the date and excluding the date $t_{i+1}^{a,CDS}$.

In the case of the last continuous dividend yield $q^{CDS}_{n_{CDS}-1}$ in the term structure, this yield is valid indefinitely from the date and including the date $t^{a,CDS}_{n_{CDS}-1}$. In addition, the relationships shown in equations (4.2.1) and (4.2.2) must hold and the respective time parameters are calculated for the corresponding dates based on equation (4.2.3).

$$t_0^{a,DDS} = t^{a,CD} \qquad \text{(Eq.4.2.1)}$$

$$t_i^{a,CDS} > t_{i-1}^{a,CDS} \qquad , \ 1 \le i < n_{CDS} \qquad \text{(Eq.4.2.2)}$$

$$t_i^{CDS} = \frac{t_i^{d,CDS}}{NumDaysPerYear} = \frac{t_i^{a,CDS} - t^{a,CD}}{NumDaysPerYear} \qquad \text{(Eq.4.2.3)}$$

However, in some cases, instead of the continuous dividend yield, a ***discrete dividends structure*** is used. The discrete dividends structure with n_{DDS} distinct dividend dates is represented by the matrix $\underline{C}^{DDS} \in \Re^{2 \times n_{DDS}}$ which is made of two fields, the dates field $\underline{t}^{a,DDS} = \{t^{a,DDS}_{i=0,1,2,\dots,n_{DDS}-1}\}$ and the cash values $\underline{q}^{DDS} = \{v^{DDS}_{i=0,1,2,\dots,n_{DDS}-1}\}$ to be received as dividends on the respective dates.

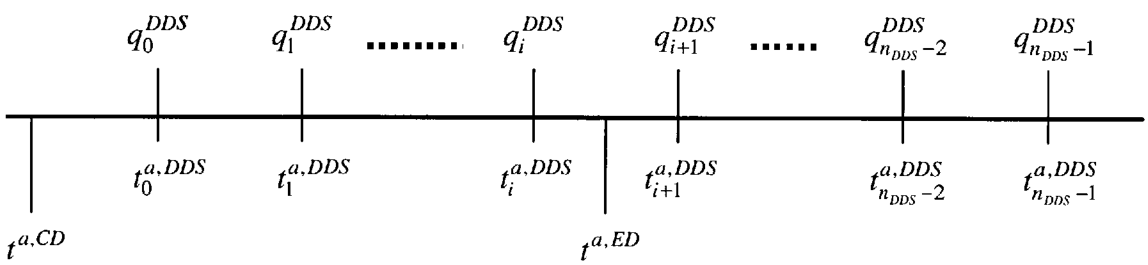


Figure (4.7)
The Discrete Dividends Schedule (Term Structure)

In the case of a dual currency convertible bond, the ***exchange rate*** between the bond currency and the equity currency on the calculations date is an additional and required input and is denoted as $X^{b/e}_{t=t^{CD}}$. In general, the exchange rate for translating units from the bond currency to the equity currency is denoted as $X_t^{b/e}$, while the exchange rate for translating units from the equity currency to the bond currency is denoted as $X_t^{e/b}$. For any point in time, the following equation holds.

$$X_t^{b/e} \times X_t^{e/b} = 1 \qquad \text{(Eq.4.2.4)}$$

Volatility Term Structure

The ***volatility of the underlying stock***, which will be used for referring to the standard deviation of the logarithmic returns of the underlying stock, will be denoted as σ^S. This is a significant input parameter to the pricing framework since, as it was pointed out in the previous section, it summarises the view of the market on the future probability distribution of the returns of the underlying stock. Instead of a single volatility parameter, a ***volatility term structure*** can also be used. A volatility term structure with n_{VS} distinct sets of parameters is represented by the matrix $\underline{C}^{VS} \in \Re^{2\times n_{VS}}$ which is made of two fields, the dates field $\underline{t}^{a,VS} = \{t^{a,VS}_{i=0,1,2,\dots,n_{VS}-1}\}$ and the volatility values field $\underline{\sigma}^{VS} = \{\sigma^{VS}_{i=0,1,2,\dots,n_{VS}-1}\}$. The volatilities are expressed as percentages. An input volatility value equal to 40 corresponds to 40%.

A volatility value σ_i^{VS} corresponds to the implied volatility of an option on the underlying stock expiring on the date $t_i^{a,VS}$. In addition, the relationships shown in equations (4.2.5) and (4.2.6) must hold and the respective time parameters are calculated for the corresponding dates based on equation (4.2.7).

$$t_0^{a,VS} = t^{a,CD} \qquad \text{(Eq.4.2.5)}$$

$$t_i^{a,VS} > t_{i-1}^{a,VS} \quad , \; 1 \le i < n_{VS} \qquad \text{(Eq.4.2.6)}$$

$$t_i^{VS} = \frac{t_i^{d,VS}}{NumDaysPerYear} = \frac{t_i^{a,VS} - t^{a,CD}}{NumDaysPerYear} \qquad \text{(Eq.4.2.7)}$$

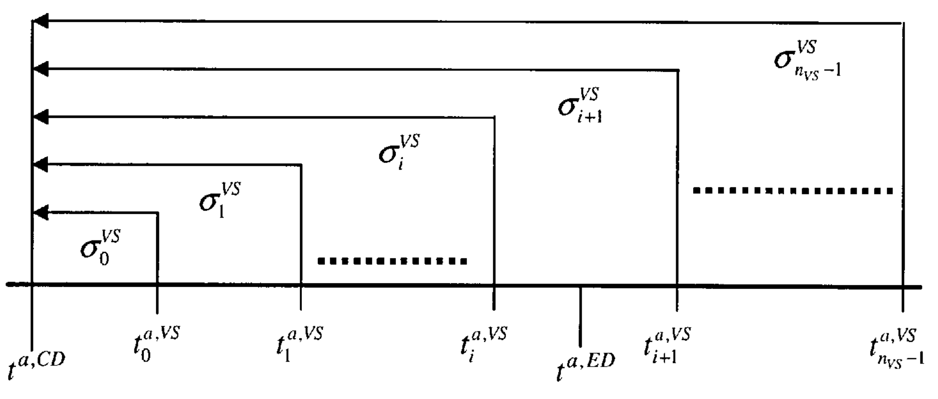


Figure (4.8) : The Volatility Term Structure

Discount Factors Curves (General)

It was preferred to input the discount factors instead of inputting the corresponding interest rates because, in this way, it was avoided the need for defining which market convention for discounting has been used in the derivation of the rates. The approach here is to input the discount factors and calculate the corresponding annual continuous rates.

Three discount factors curves were employed in the calculations:

(i) The Risk-Free Discount Factors curve in the Bond (Domestic) currency which is denoted as $\underline{C}^{RFb} \in \Re^{2 \times n_{RFb}}$. This is actually the discount factors curve that was calculated in the previous chapter based on the swap market prices of the GBP currency. We will refer to this curve as the risk-free curve in the domestic currency, even though there is embedded risk in the swap market.

(ii) The Risk-Free Discount Factors curve in the Equity (Foreign) currency which is denoted as $\underline{C}^{RFe} \in \Re^{2 \times n_{RFe}}$. This discount factors curve was calculated based on the derived zero-coupon rates curve of the dummy currency introduced in the previous chapter as the Equity (Foreign) currency.

(iii) The Risky Discount Factors curve in the Bond (Domestic) currency which is denoted as $\underline{C}^{RiskyDF} \in \Re^{2 \times n_{RiskyDF}}$. This is an abstract curve introduced as the discount factors curve used for discounting cash flows of the corporate that issued the abstract convertible bond employed in the calculations presented in the following chapters. The corporate curve is considered more risky than the swap market based curve, hence we denote the corporate curve as the Risky Discount Factors curve.

Before proceeding in defining each of the required discount factors curves, some definitions and conditions which are accountable for all the employed discount factors curves, are outlined in order to avoid repetition.

Let us consider a general discount factor curve that is similar in structure to the three discount factors curves used in the calculations. This curve has n_{DF} distinct sets of parameters and is represented by the matrix $\underline{C}^{DF} \in \Re^{2 \times n_{DF}}$ which is made of two fields, the dates field $\underline{t}^{a,DF} = \left\{ t^{a,DF}_{i=0,1,2,\ldots,n_{DF}-1} \right\}$ and the discount factors field $\underline{d}^{DF} = \left\{ d^{DF}_{i=0,1,2,\ldots,n_{DF}-1} \right\}$. A discount factor d_i^{DF} is used for discounting cash flows from the date $t_i^{a,DF}$ to the calculations date $t^{a,CD}$.

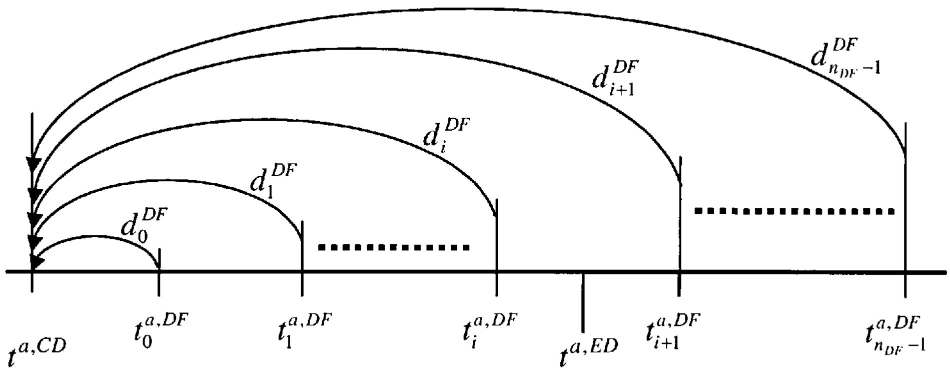


Figure (4.9): The Risk Free Discount Factors Curve

The following relationships must hold for a given discount factors curve:

(i) The first date is greater than the calculations date. This means that the zero-tenor rates and discount factors introduced in the previous chapter for the two calculated curves, which are the Bond and Equity curves, are not included in the structures of the two curves as presented here.

$$t_0^{a,DF} > t^{a,CD} \qquad \text{(Eq.4.2.7)}$$

(ii) The first discount factor is a positive quantity that is equal to or smaller than 1 and the last included discount factor is greater than zero.

$$d_0^{DF} \leq 1 \qquad \text{(Eq.4.2.8)}$$

$$d_{n_{DF}-1}^{DF} > 0 \qquad \text{(Eq.4.2.9)}$$

(iii) All the dates must be in ascending order.

$$t_i^{a,DF} > t_{i-1}^{a,DF} \quad , \ 1 \leq i < n_{DF} \qquad \text{(Eq.4.2.10)}$$

(iv) All the discount factors must be in descending order.

$$d_i^{DF} \leq d_{i-1}^{DF} \quad , \ 1 \leq i < n_{DF} \qquad \text{(Eq.4.2.11)}$$

The following quantities are also calculated based on an input discount factor curve:

$$t_i^{DF} = \frac{t_i^{d,DF}}{NumDaysPerYear} = \frac{t_i^{a,DF} - t^{a,DF}}{NumDaysPerYear} \qquad \text{(Eq.4.2.11)}$$

$$r_i^{DF} = -\frac{\ln\left(d_i^{DF}\right)}{t_i^{DF}} \quad , \ 0 \le i < n_{DF} \qquad \text{(Eq.4.2.12)}$$

At this point, we can introduce the notation specifically to each of the discount factors curve and the correspondence of the specific parameters of each curve to the general case presented above.

Risk-Free Discount Factors Curve for the Bond (Domestic) Currency

A risk-free discount factors curve of the Bond Currency with n_{RFb} distinct sets of parameters is represented by the matrix $\underline{C}^{RFb} \in \Re^{2 \times n_{RFb}}$ which is made of two fields, the dates field $\underline{t}^{a,RFb} = \left\{t_{i=0,1,2,\ldots,n_{RFb}-1}^{a,RFb}\right\}$ and the discount factors field $\underline{d}^{RFb} = \left\{d_{i=0,1,2,\ldots,n_{RFb}-1}^{RFb}\right\}$. The two sets of time parameters $\underline{t}^{d,RFb} = \left\{t_{i=0,1,2,\ldots,n_{RFb}-1}^{d,RFb}\right\}$ and $\underline{t}^{RFb} = \left\{t_{i=0,1,2,\ldots,n_{RFb}-1}^{RFb}\right\}$ are calculated based on equation (4.2.11) and the set of the rate parameters $\underline{r}^{RFb} = \left\{r_{i=0,1,2,\ldots,n_{RFb}-1}^{RFb}\right\}$ is calculated based on equation (4.2.12).

Risk-Free Discount Factors Curve for the Equity (Foreign) Currency

A risk-free discount factors curve of the Equity Currency with n_{RFe} distinct sets of parameters is represented by the matrix $\underline{C}^{RFe} \in \Re^{2 \times n_{RFe}}$ which is made of two fields, the dates field $\underline{t}^{a,RFe} = \left\{t_{i=0,1,2,\ldots,n_{RFe}-1}^{a,RFe}\right\}$ and the discount factors field $\underline{d}^{RFe} = \left\{d_{i=0,1,2,\ldots,n_{RFe}-1}^{RFe}\right\}$. The two sets of time parameters $\underline{t}^{d,RFe} = \left\{t_{i=0,1,2,\ldots,n_{RFe}-1}^{d,RFe}\right\}$ and $\underline{t}^{RFe} = \left\{t_{i=0,1,2,\ldots,n_{RFe}-1}^{RFe}\right\}$ are calculated based on equation (4.2.11) and the set of the rate parameters $\underline{r}^{RFe} = \left\{r_{i=0,1,2,\ldots,n_{RFe}-1}^{RFe}\right\}$ is calculated based on equation (4.2.12).

Risky Discount Factors Curve for the Bond (Domestic) Currency

A risky discount factors curve of the Bond Currency with $n_{RiskyDF}$ distinct sets of parameters is represented by the matrix $\underline{C}^{RiskyDF} \in \Re^{2 \times n_{RiskyDF}}$ which is made of two fields, the dates field $\underline{t}^{a,RiskyDF} = \left\{t^{a,RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$ and the discount factors field $\underline{d}^{RiskyDF} = \left\{d^{RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$. The two sets of time parameters $\underline{t}^{d,RiskyDF} = \left\{t^{d,RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$ and $\underline{t}^{RiskyDF} = \left\{t^{RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$ are calculated based on equation (4.2.11) and the set of the rate parameters $\underline{r}^{RiskyDF} = \left\{r^{RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$ is calculated based on equation (4.2.12).

The set of risky rates $\underline{r}^{RiskyDF,Interp} = \left\{r^{RiskyDF,Interp}_{i=0,1,2,\ldots,n_{RFb}-1}\right\}$ corresponding to the dates $\underline{t}^{a,RFb} = \left\{t^{a,RFb}_{i=0,1,2,\ldots,n_{RFb}-1}\right\}$ of the risk-free discount factors curve can be established by interpolating on the risky rates $\underline{r}^{RiskyDF} = \left\{r^{RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$ which correspond to the dates $\underline{t}^{RiskyDF} = \left\{t^{RiskyDF}_{i=0,1,2,\ldots,n_{RiskyDF}-1}\right\}$. The default interpolation technique is the cubic splines method. Based on the last set of risky rates $\underline{r}^{RiskyDF,Interp} = \left\{r^{RiskyDF,Interp}_{i=0,1,2,\ldots,n_{RFb}-1}\right\}$ and the set of risk-free rates $\underline{r}^{RFb} = \left\{r^{RFb}_{i=0,1,2,\ldots,n_{RFb}-1}\right\}$, the set of spreads $\underline{s}^{RiskyDF} = \left\{s^{RiskyDF}_{i=0,1,2,\ldots,n_{RFb}-1}\right\}$ can be calculated as shown by the following equation. This set of spreads is the spread structure of the corporate that have issued the convertible bond used in the calculations and is over the zero-coupon rates of GBP swap market which was calculated in the previous section.

$$s_i^{RiskyDF} = r_i^{RiskyDF,Interp} - r_i^{RFb} \qquad i = 0,1,2,\ldots,n_{RFb} - 1 \qquad \text{(Eq.4.2.13)}$$

Imported Credit Spread Structure

Information regarding the ***credit*** standards of the issuer is required for pricing purposes. The credit of the issuer will affect the discounting process of the cash flows and instrument values. There are three ways that this information can be input in the pricing frameworks presented in this thesis. Based on the first way, the credit spread can be simply considered as a ***constant spread over the risk-free rate*** and represented by the variable s^b. The second way which has already presented above when the discount factors curves were defined, involves employing the ***risky discount factors curve*** and calculating the corresponding risky rates. Then, based on the risky rates and the risk-free rates, the credit spread structure can be calculated based on equation (4.2.13).

Finally, based on the third way, a function for the credit spread in the form of a curve structure can be imported directly and used to calculate the risky rates. A ***credit spread structure*** with n distinct sets of parameters is represented by the matrix $\underline{C}^{spread} \in \Re^{2\times n}$ which is made of two fields, the dates field $\underline{t}^{a,spread} = \left\{t^{a,spread}_{i=0,1,2,\dots,n-1}\right\}$ and the credit spread values field $\underline{s}^{spread} = \left\{s^{spread}_{i=0,1,2,\dots,n-1}\right\}$. The credit spread values are expressed as percentages. An input credit spread value equal to 1.2 corresponds to 1.2%. A credit spread value s_i^{spread} is used in discounting cash flows from the date $t_i^{a,spread}$ to the calculations date $t^{a,CD}$.

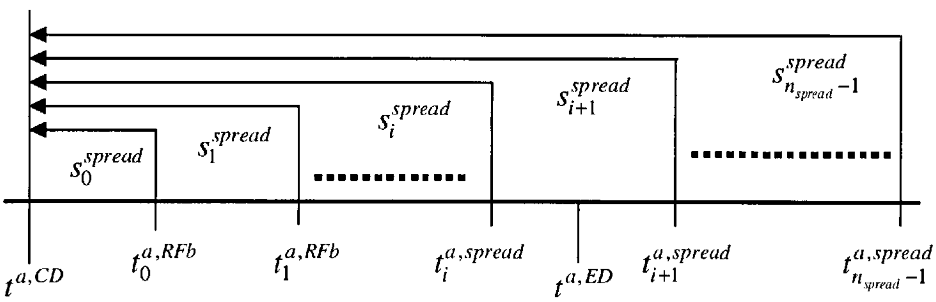


Figure (B.11) : The Credit Spread Structure

The default method for defining the credit spread structure in the following chapters will be based on the risky discount factors curve and the corresponding risky rates, as shown in equation (4.2.13).

CHAPTER 5

STEP DATES

The choice of the dates corresponding to the steps of the trees used in the pricing frameworks of this thesis was of significant importance for the methodology developed. First of all, the chosen step dates are a determining factor for the sampling quality of the trees. In addition, the presence of many "events" in the pricing of the convertible bonds pre-determines a number of dates that need to be included in the tree. "Events" is a term used in this thesis for referring to coupon dates, call dates, etc and is further defined later on in this chapter.

Even though the number of steps to be used in the calculations is an input parameter and is denoted as N_{input}, this is only an initial estimate of the number of steps to be used, or, stated more correct, this is just the origin in the process for determining the final number of steps N and the respective final step dates. In this chapter, it is demonstrated how this number N_{input} is redefined and the processes it goes through in order to get to its final value N.

The remaining part of this chapter is devoted to the presentation of the calculations for obtaining the values of various parameters at the tree nodes, like the forward rates, forward volatilities, the bond floor, etc.

Overview of the process for defining N

The first two sections of this chapter are devoted to this process. In the first section, using the input parameter N_{input} as an origin, an initial estimate of the number of steps denoted as N_{equity} is defined through a process that aims in improving the sampling quality resulting from the chosen step dates. In the second section, an additional number of dates are included in order to account for the event dates. At the end of the process in the second section, there is an additional short procedure for adding some more steps which we will refer to as the intermediate steps. This short procedure aims in improving the sampling quality when the initial number of steps N_{input} exceeded the number of days to expiration. The result of the overall process of the two sections is the final number of steps N.

5.1. Basic Step Dates

First of all, the number of days $n_{daysToED}$ to the expiration date is established.

$$n_{daysToED} = t^{a,ED} - t^{a,CD} \qquad \text{(Eq.5.1.1)}$$

The number of steps N^*_{equity} represents an initial estimate of the final number of steps N_{equity} of this section – not the final number of steps N - and is calculated as follows:

$$\begin{aligned} N^*_{Equity} &= 2\,, \quad N_{input} \leq 1 \\ N^*_{Equity} &= N_{input}\,, \quad 1 < N_{input} \leq n_{daysToED} \\ N^*_{Equity} &= n_{daysToED}\,, \quad n_{daysToED} < N_{input} \end{aligned} \qquad \text{(Eq.5.1.2)}$$

Then the time step $\Delta t^{d,equity}$ between two steps on the tree is calculated. The result $\Delta t^{d,equity}$ of equation (5.1.3) can only be an integer number of days since the floor value of the division is used. In addition, the value $\Delta t^{d,equity}$ can not be smaller than one because in equation (5.1.2) the value N^*_{equity} was set equal to $n_{daysToED}$ if N^*_{equity} was smaller than $n_{daysToED}$. The value $\Delta t^{d,Equity,*}_{last}$ is set as shown in equation (5.1.4). This value is actually the reminder of the division in equation (5.1.3) and is used to determine the date of the last step in such a way as to coincide with the expiration date. The *floor* function rounds down to the closest integer value.

$$\Delta t^{d,Equity} = floor\left(\frac{n_{daysToED}}{N^*_{Equity}}\right) \qquad \text{(Eq.5.1.3)}$$

$$\Delta t^{d,Equity,*}_{last} = n_{daysToED} - N^*_{Equity} \times \Delta t^{d,Equity} \qquad \text{(Eq.5.1.4)}$$

The actual new estimate of the steps on the tree as determined up to this point is denoted as N^{**}_{equity} and is defined based on the following equations.

$$\begin{aligned} N^{**}_{Equity} &= N^{*}_{Equity}\ , \quad \Delta t^{d,Equity,*}_{last} = 0 \\ N^{**}_{Equity} &= N^{*}_{Equity} + 1\ , \quad \Delta t^{d,Equity,*}_{last} > 0 \end{aligned} \qquad \text{(Eq.5.1.5)}$$

However, for many cases, the process up to this point results in step dates that are very unevenly distributed over the time line starting from the calculation date and ending with the expiration date. Let us consider an example where the number of days to expiration $n_{daysToED}$ is equal to 705 and the original number of steps N_{input} is equal to 300. Based on equation (5.1.2), the value N^{*}_{equity} will also be equal to 300. The calculated time step $\Delta t^{d,equity}$ is found to be equal to 2 based on equation (5.1.3) and the time step $\Delta t^{d,Equity,*}_{last}$ is found to be equal to 105 based on equation (5.1.4). This means that the first 300 steps will be evenly spaced over the first 600 days of the life of the security and for the last 105 days of the life of the security their will not be included any steps in the calculations apart from the last step corresponding to the maturity.

To overcome this problem, the procedures corresponding to the following equations have been introduced. We are also introducing the new time steps $\Delta t^{d,Equity}_{1st}$, $\Delta t^{d,Equity}_{2nd}$ and $\Delta t^{d,Equity}_{last}$, the factor M and the number of steps $N_{equity,1st}$ and $N_{equity,2nd}$. There are two possible paths in the calculations at this point, distinguished into cases A and B. For the cases where the last time step is greater than the rest of the time steps by more than two times, the calculations are carried based on case B, otherwise the calculations are carried out based on case A.

$$\begin{aligned} &Case\ A: \quad \Delta t^{d,Equity,*}_{last} \le 2 \times \Delta t^{d,equity} \\ &\Delta t^{d,Equity}_{1st} = \Delta t^{d,equity} \\ &\Delta t^{d,Equity}_{2nd} = 0 \\ &\Delta t^{d,Equity}_{last} = \Delta t^{d,Equity,*}_{last} \\ &N_{equity,1st} = N^{*}_{Equity} \\ &N_{equity,2nd} = 0 \end{aligned} \qquad \text{(Eq.5.1.6)}$$

The *ceil* function rounds up to the closest integer value.

$$
\begin{aligned}
&Case\ B: \quad \Delta t_{last}^{d,Equity,*} > 2 \times \Delta t^{d,equity} \\
&M = 1 + \Delta t^{d,equity} \\
&\Delta t_{1st}^{d,Equity} = \Delta t^{d,equity} \\
&\Delta t_{2nd}^{d,Equity} = M \times \Delta t_{1st}^{d,Equity} \\
&N_{equity,1st} = ceil\left(\frac{\left(N_{Equity}^{*} \times \Delta t_{2nd}^{d,Equity}\right) - n_{daysToED}}{(M-1) \times \Delta t^{d,equity}} \right) \\
&N_{equity,2nd} = N_{Equity}^{*} - N_{equity,1st} \\
&\Delta t_{last}^{d,Equity} = n_{daysToED} - \left[\left(N_{equity,1st} \times \Delta t_{1st}^{d,Equity}\right) + \left(N_{equity,2nd} \times \Delta t_{2nd}^{d,Equity}\right)\right]
\end{aligned}
\qquad \text{(Eq.5.1.7)}
$$

The total number of steps N_{equity} (including the maturity) up to this point – this means without the dates that will be added in the next section - is calculated as follows.

$$
\begin{aligned}
N_{equity} &= \left(N_{equity,1st} \times \Delta t_{1st}^{d,Equity}\right) + \left(N_{equity,2nd} \times \Delta t_{2nd}^{d,Equity}\right), \quad \Delta t_{last}^{d,Equity} = 0 \\
N_{equity} &= \left(N_{equity,1st} \times \Delta t_{1st}^{d,Equity}\right) + \left(N_{equity,2nd} \times \Delta t_{2nd}^{d,Equity}\right) + 1, \quad \Delta t_{last}^{d,Equity} > 0
\end{aligned}
\qquad \text{(Eq.5.1.8)}
$$

For the example with the number of days to expiration $n_{daysToED}$ equal to 705 and the original number of steps N_{input} equal to 300, the case B would be applicable. The resultant number of steps and time steps would be as follows: $\Delta t_{1st}^{d,Equity} = 2$, $\Delta t_{2nd}^{d,Equity} = 6$, $\Delta t_{last}^{d,Equity} = 1$, $N_{equity,1st} = 274$ and $N_{equity,2nd} = 26$. This means that for 26 steps, the time step between them has been increased from 2 days to 6 days and as a result the last time step has been decreased from 105 days to 1 day. This example has demonstrated the benefit of including the procedures corresponding to case B above.

Depending on whether the parameter $N_{equity,2nd}$ is equal to zero or not, the respective step dates $\underline{t}^{a,EquitySteps} = \left\{ t_{i=0,1,2,\ldots,N_{Equity}}^{a,EquitySteps} \right\}$ are calculated based on one of cases *A* and *B*.

Case A: $N_{equity,2nd} = 0$

$$
t_0^{a,EquitySteps} = t^{a,CD} \qquad \text{(Eq.5.1.9)}
$$

$$
t_i^{a,EquitySteps} = t_{i-1}^{a,EquitySteps} + \Delta t_{1st}^{d,Equity} \quad , \ 1 \le i < N_{Equity} - 1 \qquad \text{(Eq.5.1.10)}
$$

$$t_{N_{Equity}}^{a,EquitySteps} = t_{N_{Equity-1}}^{a,EquitySteps} + \Delta t_{1st}^{d,Equity} \quad , if \;\; \Delta t_{last}^{d,Equity} = 0$$
$$t_{N_{Equity}}^{a,EquitySteps} = t_{N_{Equity-1}}^{a,EquitySteps} + \Delta t_{last}^{d,Equity} \quad , if \;\; \Delta t_{last}^{d,Equity} > 0 \qquad \text{(Eq.5.1.11)}$$

Case B: $\qquad N_{equity,2nd} > 0$

$$t_0^{a,EquitySteps} = t^{a,CD} \qquad \text{(Eq.5.1.12)}$$

$$t_i^{a,EquitySteps} = t_{i-1}^{a,EquitySteps} + \Delta t_{1st}^{d,Equity} \quad , \; 1 \le i \le N_{Equity,1st} \qquad \text{(Eq.5.1.13)}$$

$$t_i^{a,EquitySteps} = t_{i-1}^{a,EquitySteps} + \Delta t_{2nd}^{d,Equity} \quad , \; 1 \le i < N_{Equity} - 1 \qquad \text{(Eq.5.1.14)}$$

$$t_{N_{Equity}}^{a,EquitySteps} = t_{N_{Equity-1}}^{a,EquitySteps} + \Delta t_{2nd}^{d,Equity} \quad , if \;\; \Delta t_{last}^{d,Equity} = 0$$
$$t_{N_{Equity}}^{a,EquitySteps} = t_{N_{Equity-1}}^{a,EquitySteps} + \Delta t_{last}^{d,Equity} \quad , if \;\; \Delta t_{last}^{d,Equity} > 0 \qquad \text{(Eq.5.1.15)}$$

The above calculations, in both cases A and B, take into account the case where the last time step is not equal to the time step used in the rest of the steps, but it is equal to $\Delta t_{last}^{a,Equity}$. So, care has been taken for ensuring that the procedures up to this point will result in a last step date that is equal to the expiration date ($t_{N_{Equity}-1}^{a,EquitySteps} = t^{ED}$). The equity dates vector $\underline{t}^{a,EquitySteps} = \left\{ t_{i=0,1,2,\dots,N_{Equity}}^{a,EquitySteps} \right\}$ is also shown diagrammatically in the following figure in the case that $N_{equity,2nd}$ is equal to zero. If it is not equal to zero, it means that some of the time steps are not equal to $\Delta t_{1st}^{d,Equity}$, but are equal to $\Delta t_{2nd}^{d,Equity}$ instead.

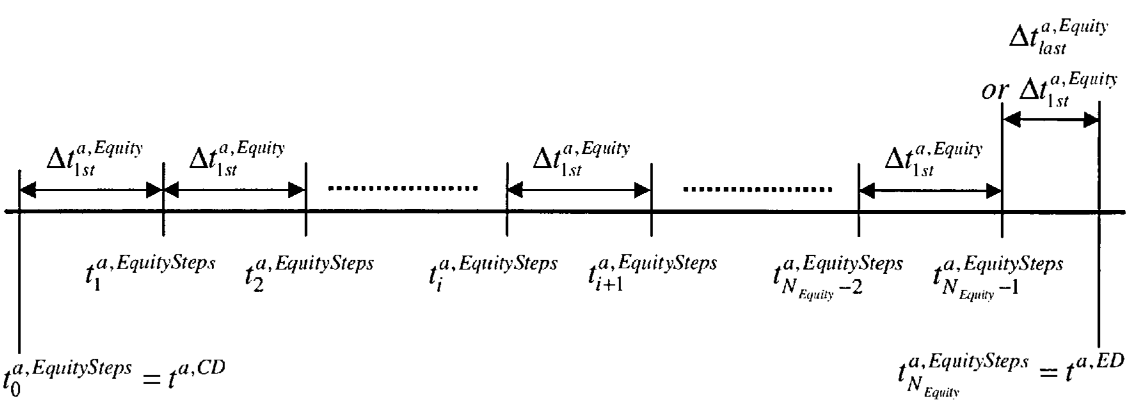


Figure (5.1) : The Equity Steps Dates

5.2. Additional Step Dates

At this point, a number of steps dates have been specified based on the input required number of steps. This forms the minimum number of steps to include in the tree. However, there is still the possibility that some event dates have not been included in the steps of the tree. It is desirable to have steps on the event dates since most of the types of events, like discrete dividends and coupons, introduce discontinuities in the pricing process. Since one of the reasons that the trinomial tree implementation was chosen over the binomial tree implementation was the benefit of having a variable time step, it is reasonable to introduce additional steps to include any event dates that are not already included in the equity steps.

The following dates qualify as event dates:

(1) Coupon Dates.

(2) Discrete Dividends Dates.

(3) Call Dates (Starting and Ending Dates).

(4) Dates of which the stock price is included in the grace period for conditional calls.

(5) Put Dates (Starting and Ending Dates).

(6) Dates of which the stock price is included in the grace period for conditional puts.

(7) Reset Dates.

(8) Dates of which the stock price is included in the averaging process for the resets.

(9) The last conversion date $t^{a,LastConv}$.

The algorithm goes through all the event dates and adds a step date for each event date not already included in the steps dates. The final number of steps dates is denoted as N^* and the final steps dates are denoted as $\underline{t}^{a,Steps} = \left\{t_{i=0,1,2,\ldots,N^*}^{a,Steps}\right\}$.

In figure (5.2), there is a diagrammatically description of an example where there are N_{Equity} steps defined based on the input required number of steps N_{Input} and by following the procedures of section (5.1). However, there is a number of event dates, hence more steps dates have been included in the final steps dates as depicted by the procedures presented in this section. The first line of the figure includes the initial step dates as resulted for this example based on the procedures of section (5.1), the second line includes the events dates and the third includes the final step dates.

For the event dates that were coinciding with any of the initial steps dates $\underline{t}^{a,EquitySteps} = \left\{t_{i=0,1,2,\dots,N_{Equity}}^{a,EquitySteps}\right\}$, there was no need to include extra steps. In more detail, the final steps $\underline{t}^{a,Steps} = \left\{t_{i=0,2,6,8,10,12,14,15}^{a,Steps}\right\}$ correspond to the initial step dates $\underline{t}^{a,EquitySteps} = \left\{t_{i=0,1,2,\dots,N_{Equity}}^{a,EquitySteps}\right\}$. The final dates $\underline{t}^{a,Steps} = \left\{t_{i=1,3,5,7,8,11,13}^{a,Steps}\right\}$ where included because of the event dates, like, for example, step date $t_3^{a,Steps}$ was added because of the event date $t_0^{a,CS,SD}$ which is a staring call date. No extra dates were required for the cases of the event dates $t_0^{a,Cpn}$, $t_0^{a,RS}$ and $t_2^{a,DDS}$ since corresponding dates are already included in the initial steps dates. For example, $t_0^{a,Cpn}$ is equal to the initial step date $t_2^{a,EquitySteps}$, so there is no need to add another step date.

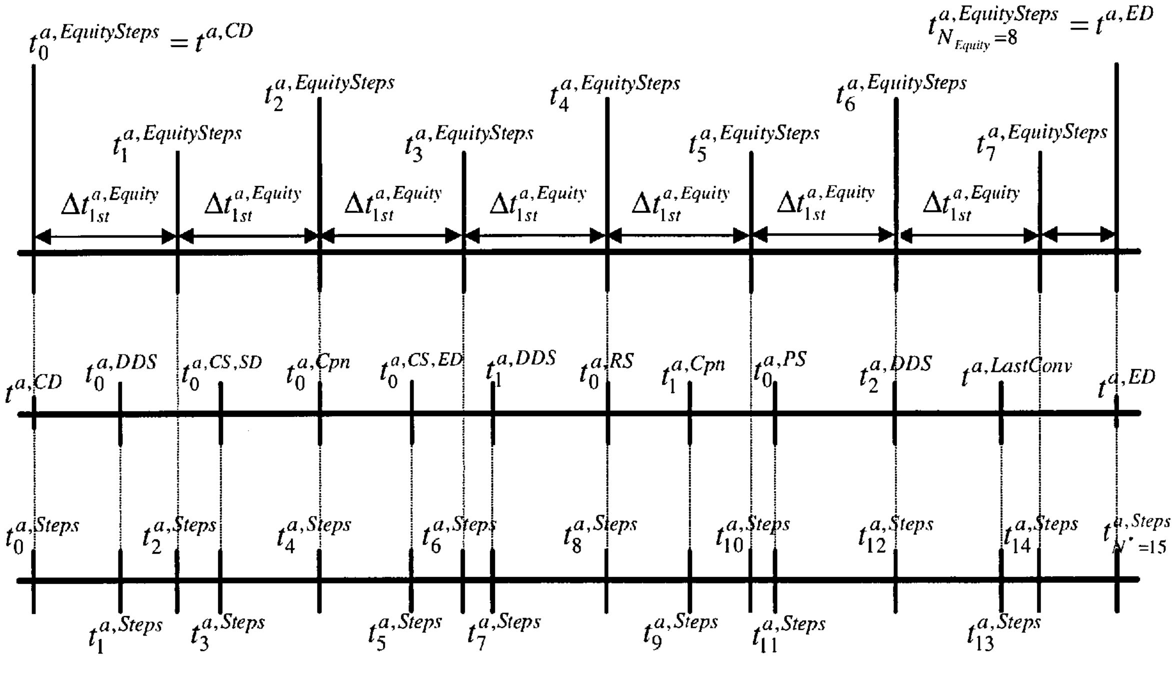


Figure (5.2) : Establishing the final steps dates

Intermediate Steps

Up to this point, steps have been included by permitting time steps with size only an integer number of days greater than or equal to zero. In other words, no allowance was made for time steps smaller than one day length. In this thesis, the term *intermediate steps* will be used for the additional steps added between the dates $t^{a,CD}$ and $t^{a,CD}+1$. These additional steps are included only in two cases as described next and the time step between them is smaller than one.

In the first case, intermediate steps are added because the final number of steps N^* is not smaller than the initially required number of steps N_{input}. This would most probably be the case where $n_{daysToED} < N_{input}$. In this case, the temporary number of additional steps is referred to as N_{1st}^{Interm} and is simply calculated as follows.

$$\begin{aligned} N_{1st}^{Interm} &= 0, & N^* &\geq N_{input} \\ N_{1st}^{Interm} &= N_{input} - N^*, & N^* &< N_{input} \end{aligned} \qquad \text{(Eq.5.1.16)}$$

One of the inputs to the pricing frameworks is the minimum number of steps $N_{MinFirstEvent}$ before the first event date. For example, if there is an event date just 2 days after the calculations date, then in the best case scenario, only two steps have added up to that point. This means that there will be only 5 nodes at the first event date, meaning that the sampling at that node will be poor. If a number of intermediate nodes is added between the dates $t^{a,CD}$ and $t^{a,CD}+1$, then the sampling for the first event date will be improved.

So, in the case that $N_{MinFirstEvent} > N_{FirstEvent}$ where $N_{FirstEvent}$ is the number of steps to the first event date, the number of additional steps to be included is equal to $N_{MinFirstEvent} - N_{FirstEvent}$. If this number is positive, then, before calculating the required number of intermediate steps, a check is made if there are any dates between the calculations date and the first event date that have not been included in the step dates. If there are any available dates, then these are included in the steps dates. The number of the added step dates because of the required number of steps to the first event date is denoted as $N_{AddToFirstEvent}$. Then, if it is the case that the number of step dates to the first event is still smaller to the required number of dates (to the first event), then the number of intermediate numbers to be added N_{2nd}^{Interm} is calculated as follows.

$$\begin{aligned} &N_{2nd}^{Interm} = 0, && N_{AddToFirstEvent} + N_{FirstEvent} \geq N_{MinFirstEvent} \\ &N_{2nd}^{Interm} = N_{MinFirstEvent} - \left(N_{AddToFirstEvent} + N_{FirstEvent}\right), && otherwise \end{aligned} \quad \text{(Eq.5.1.17)}$$

The final number of intermediate steps to be included is denoted as N^{Interm} and is calculated based on N_{1st}^{Interm} and N_{2nd}^{Interm}.

$$\begin{aligned} &N^{Interm} = N_{1st}^{Interm}, && N_{1st}^{Interm} \geq N_{2nd}^{Interm} \\ &N^{Interm} = N_{2nd}^{Interm}, && N_{1st}^{Interm} < N_{2nd}^{Interm} \end{aligned} \quad \text{(Eq.5.1.18)}$$

<u>The Final Step Dates</u>

The final set of step dates is denoted as $\underline{t}^{a,Steps} = \left\{t_{i=0,1,2,\dots,N}^{a,Steps}\right\}$ and includes the intermediate step dates as well. The number of the final step dates is denoted as N and is calculated as follows.

$$N = N^{*} + N^{Interm} + N_{AddToFirstEvent} \quad \text{(Eq.5.1.19)}$$

Having specified the final steps dates, the following parameters can also be calculated.

$$\begin{aligned} &\Delta t_0^{d,Steps} = 0 \\ &\Delta t_i^{d,Steps} = t_i^{a,Steps} - t_{i-1}^{a,Steps} \qquad 1 \leq i \leq N \end{aligned} \quad \text{(Eq.5.1.20)}$$

$$\Delta t_i^{Steps} = \frac{\Delta t_i^{d,Steps}}{NumDaysPerYear} \qquad 1 \leq i \leq N \quad \text{(Eq.5.1.21)}$$

$$t_i^{d,Steps} = t_i^{a,Steps} - t^{a,CD} \qquad 0 \leq i \leq N \quad \text{(Eq.5.1.22)}$$

$$t_i^{Steps} = \frac{t_i^{d,Steps}}{NumDaysPerYear} \qquad 0 \leq i \leq N \quad \text{(Eq.5.1.23)}$$

$$\tau_i^{d,Steps} = t^{a,ED} - t_i^{a,Steps} \qquad 0 \leq i \leq N \quad \text{(Eq.5.1.24)}$$

$$\tau_i^{Steps} = \frac{\tau_i^{d,Steps}}{NumDaysPerYear} \qquad 0 \leq i \leq N \quad \text{(Eq.5.1.25)}$$

5.3. Forward Values

Having established the number of steps and the respective dates, we continue with the calculations of the remaining information at the steps.

Risk-free and Risky rates

Initially, the polynomials presented in chapter 3 as the result of the cubic splines technique applied on the risk free rates curve in the bond currency, are used for interpolating and obtaining the values of this quantity at the step dates. The resultant bond currency risk free rates are denoted as $\underline{r}^{RFb,steps} = \{r_{i=0,1,2,\dots,N}^{RFb,steps}\}$. Based on these rates, the discount factors $\underline{d}^{RFb,steps} = \{d_{i=0,1,2,\dots,N}^{RFb,steps}\}$, the forward discount factors $\underline{fd}^{RFb,steps} = \{fd_{i=1,2,\dots,N}^{RFb,steps}\}$ and the forward rates $\underline{fr}^{RFb,steps} = \{fr_{i=1,2,\dots,N}^{RFb,steps}\}$ can also be calculated.

$$d_i^{RFb,steps} = e^{-r_i^{RFb,steps} \times t_i^{Steps}} = \exp\left(-r_i^{RFb,steps} \times t_i^{Steps}\right) \qquad 0 \le i \le N \qquad \text{(Eq.5.3.1)}$$

$$fd_i^{RFb,steps} = \frac{d_i^{RFb,steps}}{d_{i-1}^{RFb,steps}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.2)}$$

$$fr_i^{RFb,steps} = -\frac{\ln\left(fd_i^{RFb,steps}\right)}{\Delta t_i^{Steps}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.3)}$$

In the previous chapter, three possible ways were presented for obtaining the credit spread structure based on the inputs to the pricing framework. At this point, cubic splines interpolation is applied on the resultant credit spread structure in order to obtain the credit spread value at the each step and the resultant spread values are denoted as $\underline{s}^{steps} = \{s_{i=0,1,2,\dots,N}^{steps}\}$. Based on the risk free rate in the bond currency and the credit spread value at each step, the risky rates $\underline{r}^{Risky,steps} = \{r_{i=0,1,2,\dots,N}^{Risky,steps}\}$ can now be calculated, and based on these the risky discount factors $\underline{d}^{Risky,steps} = \{d_{i=0,1,2,\dots,N}^{Risky,steps}\}$ and the risky forward discount factors $\underline{fd}^{Risky,steps} = \{fd_{i=1,1,2,\dots,N}^{Risky,steps}\}$.

$$r_i^{Risky,steps} = r_i^{RFb,steps} + s_i^{steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.3.4)}$$

$$d_i^{Risky,steps} = e^{-r_i^{Risky,steps} \times t_i^{Steps}} = \exp\left(-r_i^{Risky,steps} \times t_i^{Steps}\right) \qquad 0 \le i \le N \qquad \text{(Eq.5.3.5)}$$

$$fd_i^{Risky,steps} = \frac{d_i^{Risky,steps}}{d_{i-1}^{Risky,steps}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.6)}$$

Dual Currency Convertible Bonds

If the convertible bond is a dual currency one, then the same procedures need to be carried out for obtaining the risk free rates in the equity currency at the steps, $\underline{r}^{RFe,steps} = \left\{ r_{i=0,1,2,\ldots,N}^{RFe,steps} \right\}$, and the respective discount factors $\underline{d}^{RFe,steps} = \left\{ d_{i=0,1,2,\ldots,N}^{RFe,steps} \right\}$ and forward discount factors $\underline{fd}^{RFe,steps} = \left\{ fd_{i=1,2,\ldots,N}^{RFe,steps} \right\}$. The risk free rates are obtained by interpolating on the input equity currency risk free curve while the discount factors are obtained based on equations (5.3.7) and (5.3.8). In addition, for dual currency convertible bonds, the forward exchange rates are also required at the steps, and these are calculated based on equations (5.3.9) to (5.3.11). The exchange rate for translating units from the bond currency to the equity currency at step i is denoted as $X_i^{b/e}$, while the exchange rate for translating units from the equity currency to the bond currency is denoted as $X_i^{e/b}$. The calculations of the exchange rates are initialized by setting the exchange rate for translating from the bond currency to the equity currency at step 0 equal to the input parameter $X_{t=t^{CD}}^{b/e}$, as shown in equation (5.3.9).

$$d_i^{RFe,steps} = e^{-r_i^{RFe,steps} \times t_i^{Steps}} = \exp\left(-r_i^{RFe,steps} \times t_i^{Steps}\right) \qquad 0 \le i \le N \qquad \text{(Eq.5.3.7)}$$

$$fd_i^{RFe,steps} = \frac{d_i^{RFe,steps}}{d_{i-1}^{RFe,steps}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.8)}$$

$$X_{i=0}^{b/e} = X_{t=t^{CD}}^{b/e} \qquad \text{(Eq.5.3.9)}$$

$$\left.\begin{array}{l} X_i^{b/e} = X_0^{b/e} \dfrac{e^{r_i^{RFb,steps} \times t_i^{Steps}}}{e^{r_i^{RFe,steps} \times t_i^{Steps}}} = X_0^{b/e} \dfrac{e^{-r_i^{RFe,steps} \times t_i^{Steps}}}{e^{-r_i^{RFb,steps} \times t_i^{Steps}}} \\ X_i^{b/e} = X_0^{b/e} \dfrac{d_i^{RFe,steps}}{d_i^{RFb,steps}} \end{array}\right\} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.10)}$$

$$X_i^{b/e} \times X_i^{e/b} = 1 \Rightarrow X_i^{e/b} = \frac{1}{X_i^{b/e}} \qquad 0 \le i \le N \qquad \text{(Eq.5.3.11)}$$

Continuous Dividend Yield

We allow for a different continuous dividend yield to be defined at each step and the respective vector is denoted as $\underline{q} = \{q_{i=1,1,2,\ldots,N}\}$. If the continuous dividend yield is not employed, then the data of the vector $\underline{q} = \{q_{i=1,1,2,\ldots,N}\}$ are set equal to either the input value q^{Div} or to zero if the case is that no continuous dividend yield will be employed. However, if it is the case that the continuous dividend yield term structure $\underline{q}^{CDS}$ will be employed, then the data of the vector $\underline{q} = \{q_{i=1,1,2,\ldots,N}\}$ are obtained by interpolating on the term structure. For the continuous dividend yields the interpolation is performed based on the step function method as described by the following equations.

$$\begin{array}{lll} q_i = q_k^{CDS} & t_k^{a,CDS} \le t_i^{a,Steps} < t_{k+1}^{a,CDS} & k = 0,1,2,\ldots,n_{CDS}-2 \\ q_i = q_k^{CDS} & t_k^{a,CDS} \le t_i^{a,Steps} & k = n_{CDS}-1 \end{array} \qquad \text{(Eq.5.3.12)}$$

Volatility

By performing cubic splines interpolation on the input volatility structure $\underline{C}^{VS} \in \Re^{2\times n_{VS}}$, the volatility is obtained at each step. The new values are denoted as $\underline{\sigma}^{steps} = \{\sigma_{i=0,1,2,\ldots,N}^{steps}\}$. Because the calculation of Vega sensitivity is based on shifting the volatility term structure by 1%, the new set of volatilities $\underline{\sigma}^{steps} = \{\sigma_{i=0,1,2,\ldots,N}^{steps}\}$ is shifted by 1% and the shifted volatilities are denoted as $\underline{\sigma}^{steps,1\%} = \{\sigma_{i=0,1,2,\ldots,N}^{steps,1\%}\}$.

$$\sigma_i^{steps,1\%} = 1.01 \times \sigma_i^{steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.3.13)}$$

The set of forward volatilities $\underline{f\sigma}^{steps} = \{f\sigma_{i=1,2,\ldots,N}^{steps}\}$ and $\underline{f\sigma}^{steps,1\%} = \{f\sigma_{i=1,2,\ldots,N}^{steps,1\%}\}$ are established for each set of volatilities based on the following equations.

$$f\sigma_i^{steps} = \sqrt{\frac{\left(\sigma_i^{steps}\right)^2 t_i^{Steps} - \left(\sigma_{i-1}^{steps}\right)^2 t_{i-1}^{Steps}}{\Delta t_i^{Steps}}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.14)}$$

$$f\sigma_i^{steps,1\%} = \sqrt{\frac{\left(\sigma_i^{steps,1\%}\right)^2 t_i^{Steps} - \left(\sigma_{i-1}^{steps,1\%}\right)^2 t_{i-1}^{Steps}}{\Delta t_i^{Steps}}} \qquad 1 \le i \le N \qquad \text{(Eq.5.3.15)}$$

5.4. The Bond Floor

The Bond Floor $\underline{V}_b^{straight} = \left\{V_{b,i=0,1,2,\ldots,N}^{straight}\right\}$ and the Accrued Interest $\underline{AI} = \left\{AI_{i=0,1,2,\ldots,N}\right\}$ are calculated at each step. The Bond Floor is simply the value of the straight bond, the convertible bond without the optionality to convert to Equity. Since the coupon dates are accounted for as event dates, it is the case that the coupon dates are included in the step dates.

The coupon cash flows $\underline{q}^{Cpn,steps} = \left\{q_{i=0,1,2,\ldots,N}^{Cpn,steps}\right\}$ are established at each step. For the steps that correspond to a date in the coupons schedule $\underline{C}^{Cpn} \in \Re^{2 \times n_{Cpn}}$, the value is non-zero as shown in equation (5.3.19), while for the rest of the steps the coupon cash flows are simply equal to zero.

$$\left.\begin{array}{ll} q_i^{Cpn,steps} = q_k^{Cpn} & if\ t_i^{a,Steps} = t_k^{a,Cpn} \\ q_i^{Cpn,steps} = 0 & otherwise \end{array}\right\} \quad 0 \le i \le N,\ 0 \le k < n_{Cpn} \qquad \text{(Eq.5.4.1)}$$

The set of discounted coupons $\underline{q}^{DiscCpn,steps} = \left\{q_{i=0,1,2,\ldots,N}^{DiscCpn,steps}\right\}$ is calculated by discounting the coupon cash flows back to the calculations date as shown in equation (5.3.20).

$$q_i^{DiscCpn,steps} = df_i^{Risky,steps} \times q_i^{Cpn,steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.4.2)}$$

Finally, the sums of the discounted coupon cash flows $\underline{S}^{Cpn,steps} = \left\{S_{i=0,1,2,\ldots,N}^{Cpn,steps}\right\}$ at each step are calculated based on equation (5.4.3). For each step, the sum of all the discounted coupon cash flows with greater or equal step index is compounded in order to obtain the value of the sum of the coupon cash flows on the step date.

$$S_i^{Cpn,steps} = \frac{1}{df_i^{Risky,steps}} \sum_{k=i}^{N} v_k^{DiscCpn,steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.4.3)}$$

The Bond Floor value at each step is calculated by adding the discounted notional and the sum of the discounted coupon cash flows.

$$V_{b,i}^{straight} = \frac{df_N^{Risky,steps}}{df_i^{Risky,steps}} \times P^{Rd} + S_i^{Cpn,steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.4.4)}$$

The Accrued Interest is the part of the next coupon payment that has accrued. This is calculated at each step by multiplying the next coupon payment by the proportion of time between the next and the previous coupon date that has elapsed.

$$\left.\begin{array}{ll} AI_{i=0} = 0 & if\ \ t_{i=0}^{a,Steps} = t^{a,ID} \\ AI_i = \dfrac{t_0^{a,Cpn} - t_i^{a,Steps}}{t_0^{a,Cpn} - t^{a,ID}} \times q_0^{Cpn} & if\ \ t^{a,ID} < t_i^{a,Steps} \le t_0^{a,Cpn} \\ AI_i = \dfrac{t_k^{a,Cpn} - t_i^{a,Steps}}{t_k^{a,Cpn} - t_{k-1}^{a,Cpn}} \times q_k^{Cpn} & if\ \ t_{k-1}^{a,Cpn} < t_i^{a,Steps} \le t_k^{a,Cpn},\ 1 \le k < n_{Cpn} \end{array}\right\} 0 \le i \le N$$

(Eq.5.4.5)

5.5. Sum of Discounted Discrete Dividends

In the case that a Discrete Dividends structure is employed, we need to establish the sum of the discounted discrete dividends at each step on the tree. These values are denoted as $\underline{S}^{Divs,steps} = \left\{S_{i=0,1,2,\ldots,N}^{Divs,steps}\right\}$. The sum of discounted discrete dividends at each step is established in exactly the same way as the Bond Floor value. The main difference is that the risk-free discount factors are used instead of the risky discount factors.

Since the discrete dividends dates are accounted for as event dates, it is the case that the discrete dividend dates are included in the step dates. The first action involves establishing the dividend cash flows $\underline{v}^{Divs,steps} = \left\{v_{i=0,1,2,\ldots,N}^{Divs,steps}\right\}$ at each step. For the steps that correspond to a date in the input discrete dividends structure $\underline{C}^{DDS} \in \Re^{2\times n_{DDS}}$, the value is non-zero as shown in equation (5.5.1), while for the rest of the steps the dividend cash flows are equal to zero.

$$\left.\begin{array}{ll} v_i^{Divs,steps} = q_k^{DDS} & if\ \ t_i^{a,Steps} = t_k^{a,DDS} \\ v_i^{Divs,steps} = 0 & otherwise \end{array}\right\} \quad 0 \le i \le N,\ 0 \le k < n_{DDS} \qquad \text{(Eq.5.5.1)}$$

Then, the set of discounted discrete dividends $\underline{v}^{DiscDivs,steps} = \{v_{i=0,1,2,\dots,N}^{DiscDivs,steps}\}$ is calculated by discounting the discrete dividend cash flows back to the calculations date as shown in equation (5.5.2).

$$v_i^{DiscDivs,steps} = df_i^{RFb,steps} \times v_i^{Divs,steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.5.2)}$$

Finally, the sums of the discounted discrete dividends at each step are calculated based on equation (5.5.3). For each step, the sum of all the discounted discrete dividends with greater or equal step index is compounded. The resultant value is the sum of the discrete dividends as it would be on the date corresponding to the step.

$$S_i^{Divs,steps} = \frac{1}{df_i^{RFb,steps}} \sum_{k=i}^{N} v_k^{DiscDivs,steps} \qquad 0 \le i \le N \qquad \text{(Eq.5.5.3)}$$

CHAPTER 6

TRINOMIAL TREE FOR THE STOCK PROCESS

The most volatile factor involved in the pricing of a convertible bond is the price of the underlying equity. Interest rate fluctuations and spread fluctuations play a significant role as well, but their magnitude and their effect on the convertible bond price is not of the same level as the equity. Only in the case of very out of the money convertible bonds can be stated that the effect of the interest rates and spreads is greater since the convertible bonds are traded as straight bonds. Another significant factor for dual currency convertible bond prices is the exchange rate, which is accounted for in a two-dimensional tree approach in the next part of the thesis.

The stochastic process for the underlying equity was implemented in the form of a recombining trinomial tree. The important benefits arising from implementing a trinomial tree like variable time step and employment of term structures for interest rates, spreads, volatilities and continuous dividend yields, have already been discussed.

A trinomial tree implementation involves two basic processes, the forward induction and the backward induction. During the forward induction, the transition probabilities from one node to the nodes of the next step are calculated, as well as the realisations of the parameter of the stochastic process, in our case this would be the equity stock price. The pricing of the instrument, in our case this would be the convertible bond, is performed during the second process, the backward induction. In this chapter, we present the various calculations involved in the forward induction part.

In the first section, the general approach to trinomial trees implementation is presented. Then, in section (6.2), the equations involved in the calculation of the transition probabilities and the stock prices are outlined. Section (6.3) is devoted in establishing the conditions under which the calculations result in valid probabilities, because the structure of trinomial trees does not guarantee valid probabilities, in contrast to the binomial trees and the implicit finite difference methods.

In section (6.4) we are introducing the conditional probabilities. The conditional probabilities were introduced through the work of this thesis in order to enable accounting for the conditional calls and puts, as well as the resets. The calculations during the forward induction which involved the conditional probabilities are presented in the last section of this chapter.

6.1. The General Approach

The general methodology followed in the implementation of the trinomial tree Equity model here is based on the description of a basic implementation of the same model included in chapter 3 of reference [8], where the authors also recognise the advantages of variable time step and employing term structures for the interest rate and the volatility, offered by this configuration. The first extension to their model as it was presented in the reference is fitting the model to structured data, while a more significant extension is the introduction of the conditional probabilities.

The stochastic differential equation for the risk-neutral geometric Brownian motion (GBM) model of an asset price paying a continuous dividend yield δ is given by equation (6.1.1). This is the Black-Scholes based stochastic differential equation, which ensures arbitrage-free conditions for the implementation.

$$dS = (r_t - \delta_t)S\,dt + \sigma_t\,S\,dz \qquad \text{(Eq.6.1.1)}$$

By setting:

$$x = \ln(S) \qquad \text{(Eq.6.1.2)}$$

the stochastic differential equation is changed as follows:

$$dx = \mu_t dt + \sigma_t\,dz \qquad \text{(Eq.6.1.3)}$$

$$\mu_t = r_t - \delta_t - \frac{\sigma_t^2}{2} \qquad \text{(Eq.6.1.4)}$$

Up to this point, the equations governing the behaviour of the stochastic process have been in continuous form. Replacing the continuous form time step dt with the discrete form time step Δt and the space step will result in changing the equations form into discrete form.

$$\Delta x = \mu_i \Delta t + \sigma_i \Delta z \qquad \text{(Eq.6.1.5)}$$

$$\mu_i = r_i - \delta_i - \frac{\sigma_i^2}{2} \qquad \text{(Eq.6.1.6)}$$

In figure (6.1) a representative branching configuration of a node in a trinomial tree with the evolution of x as defined by equation (6.1.5), is presented. We have allowed for a variable time step configuration and the employment of structured data. The relationship between the parameters of the processes depicted by equations (6.1.5) and (6.1.6), and the parameters of the trinomial process as defined in figure (6.1) is obtained by equating the mean and the variance over the time interval Δt and requiring that the three probabilities on the tree sum to one. These are summarized by equations (6.1.7) to (6.1.9).

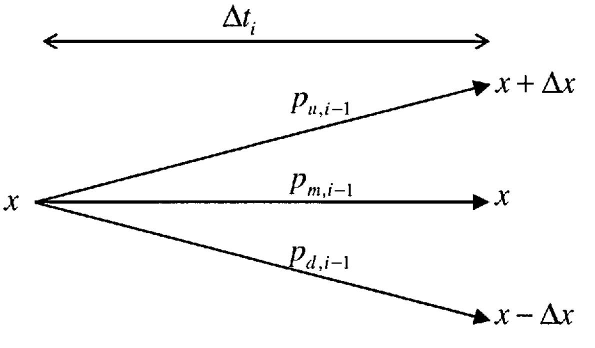


Figure (6.1) Representative Node Configuration (General Case)

$$E[\Delta x] = p_{u,i-1}(\Delta x) + p_{m,i-1}(0) + p_{d,i-1}(-\Delta x) = v_i \Delta t_i \qquad \text{(Eq.6.1.7)}$$

$$E[\Delta x^2] = p_{u,i-1}(\Delta x^2) + p_{m,i-1}(0) + p_{d,i-1}(\Delta x^2) = \sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2 \qquad \text{(Eq.6.1.8)}$$

$$p_{u,i-1} + p_{m,i-1} + p_{d,i-1} = 1 \qquad \text{(Eq.6.1.9)}$$

The equations above will result in the probabilities given by equations (6.1.10) to (6.1.12).

$$p_{u,i-1} = \frac{1}{2}\left(\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} + \frac{\mu_i \Delta t_i}{\Delta x}\right) \quad \text{(Eq.6.1.10)}$$

$$p_{m,i-1} = 1 - \frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} \quad \text{(Eq.6.1.11)}$$

$$p_{d,i-1} = \frac{1}{2}\left(\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} - \frac{\mu_i \Delta t_i}{\Delta x}\right) \quad \text{(Eq.6.1.12)}$$

As a final comment, in reference [8] it is proposed that the space step Δx is set according to equation (6.1.13).

$$\Delta x = \sigma\sqrt{3\Delta t} \quad \text{(Eq.6.1.13)}$$

6.2. Forward Induction

Having presented the general framework, we continue with the specifics of the implementations of this thesis. First of all, we need to specify a referencing system for the nodes of the tree. It was desired to create and maintain a nodes reference system of the form (i, j) where the letter *i* is used for the indexing of the steps and the letter *i* is used for the indexing of the nodes at each step. The step index *i* can take the values $i = \{0,1,\ldots,N\}$, while at the i^{th} step, the nodes index j can take the values $j = \{-i,-i+1,\ldots,-1,0,1,\ldots,i-1,i\}$. This is also demonstrated in figure (6.2) included in the next page.

At each step there is a set of parameters involved in the calculations. Calculation of most of these parameters has already been presented in the previous chapters. The remaining parameters to calculate are the drift μ_i and the probabilities $p_{u,i-1}$, $p_{d,i-1}$ and $p_{d,i-1}$. These are calculated based on the following equations which are based on the general equations presented above but adjusted to notation of this thesis.

$$\mu_i = fr_i^{RFb,steps} - \delta_i - \frac{\left(f\sigma_i^{steps}\right)^2}{2} \quad \text{(Eq.6.2.1)}$$

$$p_{u,i-1} = \frac{1}{2}\left(\frac{\left(f\sigma_i^{steps}\right)^2 \Delta t_i^{steps} + \mu_i^2 \left(\Delta t_i^{steps}\right)^2}{\Delta x^2} + \frac{\mu_i \Delta t_i^{steps}}{\Delta x}\right) \quad \text{(Eq.6.2.2)}$$

$$p_{m,i-1} = 1 - \frac{\left(f\sigma_i^{steps}\right)^2 \Delta t_i^{steps} + \mu_i^2 \left(\Delta t_i^{steps}\right)^2}{\Delta x^2} \quad \text{(Eq.6.2.3)}$$

$$p_{d,i-1} = \frac{1}{2}\left(\frac{\left(f\sigma_i^{steps}\right)^2 \Delta t_i^{steps} + \mu_i^2 \left(\Delta t_i^{steps}\right)^2}{\Delta x^2} - \frac{\mu_i \Delta t_i^{steps}}{\Delta x}\right) \quad \text{(Eq.6.2.4)}$$

If it is the case that the convertible bond is a dual currency one, then equation (6.2.1) is replaced by equation (6.2.5).

$$\mu_i = fr_i^{RFe,steps} - \delta_i - \frac{\left(f\sigma_i^{steps}\right)^2}{2} \quad \text{(Eq.6.2.5)}$$

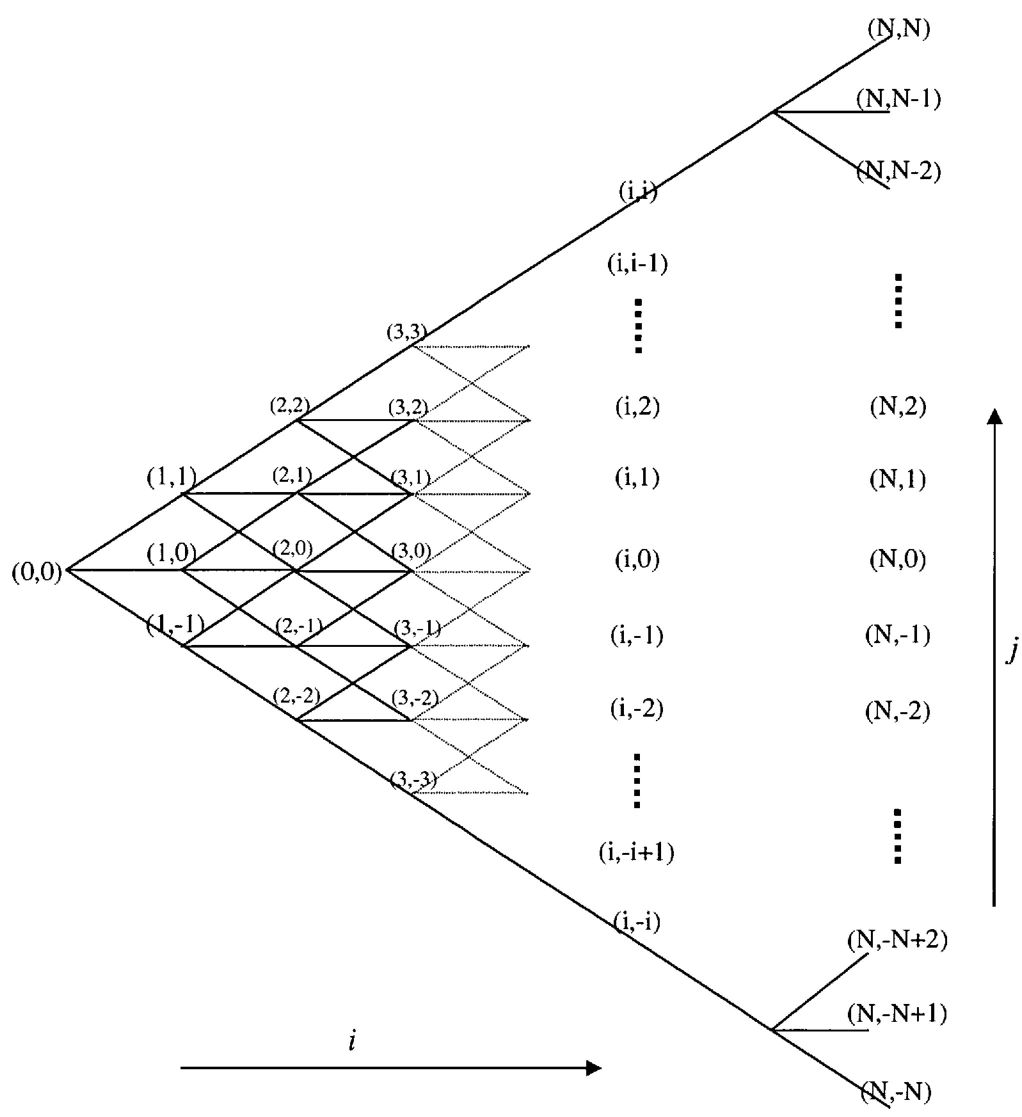


Figure (6.2) Indexing On The TrinomialTree

In figure (6.3) there is a diagrammatical presentation of the calculations involved in the transition from a node $(i-1, j)$ to the three possible nodes in the next step.

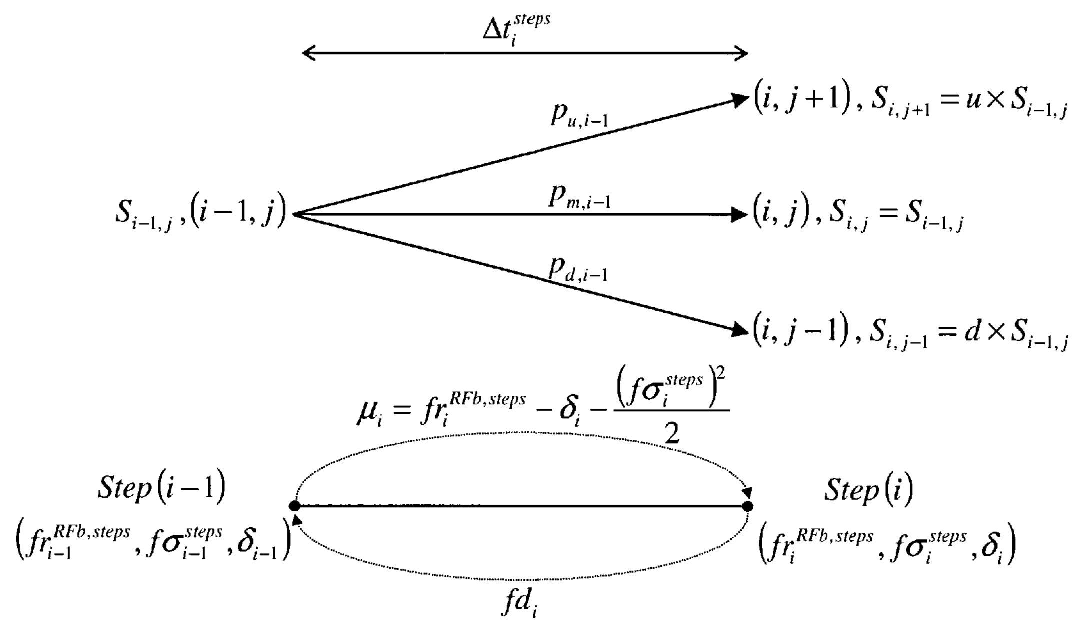


Figure (6.3) Representative Node Configuration

In the above figure, a multiplication factor u has been used for calculating the stock price $S_{i+1,j+1}$ at node $(i+1, j+1)$ and a multiplication factor d has been used for calculating the stock price $S_{i-1,j-1}$ at node $(i-1, j-1)$. These multiplication factors are fixed across the tree since they are calculated based on the fixed space step, as illustrated by the following equations.

$$u = e^{\Delta x} = \exp(\Delta x) \qquad \text{(Eq.6.2.6)}$$

$$d = e^{-\Delta x} = \exp(-\Delta x) \qquad \text{(Eq.6.2.7)}$$

$$u \times d = 1 \qquad \text{(Eq.6.2.8)}$$

Taking advantage of the recombining structure of the trinomial tree, there is no need for establishing the stock values at all the time steps. A vector $\underline{S} \in \Re^{1\times(2N+1)}$ is initialized for containing the values of the stock at the respective nodes at the time step N of the tree. For the rest of the time steps $0 \rightarrow (N-1)$, any stock values can be mapped to respective stock values of the last time step because the following relationship holds.

$$S_{0,j} = S_{1,j} = S_{2,j} = \cdots = S_{i,j} = \cdots = S_{N-1,j} = S_{N,j} \quad ,\forall j \in [-N,N] \qquad \text{(Eq.6.2.9)}$$

Of course, for each $S_{i,j}$ there is the limitation that the nodes index reference j for the time step i is limited between $-i$ to $+i$. References outside this range are not permitted in the implementation, and the mapping of $S_{i,j}$ on the vector $\underline{S} \in \Re^{1\times(2N+1)}$ limited in the permitted range of j at the specific step. This is also illustrated by the following equation where the stock price S_j is an element of the vector $\underline{S} \in \Re^{1\times(2N+1)}$ and the stock price $S_{i,j}$ is an element of the vector $\underline{S}_i \in \Re^{1\times(2i+1)}$.

$$S_{i,j} = S_j \qquad \forall j \in [-i,i] \qquad \text{(Eq.6.2.10)}$$

The elements of vector $\underline{S} \in \Re^{1\times(2N+1)}$ are calculated based on the following equation.

$$S_j = S_0 e^{jdx} \qquad \forall j \in [-N,N] \qquad \text{(Eq.6.2.11)}$$

Employing the constants $u = d^{-1} = e^{dx}$ and $u^{-1} = d = e^{-dx}$, the above equation can be simplified as follows:

$$\left.\begin{aligned} S_j &= S_0\left(e^{\Delta x}\right)^j \\ S_j &= S_0 u^j \end{aligned}\right\} \quad \forall j \in [-N,N] \qquad \text{(Eq.6.2.12)}$$

Alternatively, this can be rewritten as follows.

$$S_j = u \times S_{j-1} \qquad \forall j \in [-N,N] \qquad \text{(Eq.6.2.13)}$$

The respective equations utilizing the down multiplication factor are as follows:

$$\left.\begin{aligned} S_{-j} &= S_0\left(e^{-\Delta x}\right)^j \\ S_{-j} &= S_0 d^j \end{aligned}\right\} \quad \forall j \in [-N,N] \qquad \text{(Eq.6.2.14)}$$

$$S_{-j} = d \times S_{-j+1} \qquad \forall j \in [-N,N] \qquad \text{(Eq.6.2.15)}$$

In the actual computations, the stock value S_0 is set equal to the current share price (with the exclusion of the sum of the discounted discrete dividends on that date), and then, recursively, the rest of the stock prices are established in vector $\underline{S} \in \Re^{1\times(2N+1)}$, as demonstrated by the following equations. The parameter $S^{CD,NoDivs}$ denotes the actual share price on the calculations date after the discrete dividends have been subtracted.

$$S_0 = S^{CD,NoDivs} = S^{CD} - S_0^{Divs,steps} \qquad \text{(Eq.6.2.16)}$$

$$S_j = u \times S_{j-1} \qquad j = 1 \to N \qquad \text{(Eq.6.2.17)}$$

$$S_j = d \times S_{j+1} \qquad j = -1 \to -N \qquad \text{(Eq.6.2.18)}$$

6.3. Valid Tree Probabilities

We are introducing the derivation of the conditions that must be satisfied in order to have transition probabilities that are strictly positive and smaller than one. One drawback, perhaps the only significant one, of the trinomial tree configuration is that it does not guarantee by construction greater than zero (strictly positive) and smaller than one transition probabilities. Negative and greater than one transition probabilities arise in a trinomial tree for certain combinations of market based input data like rates and volatilities. It has been proved that these combinations of data actually arise when there are arbitrage conditions in the market, or at least during of the derivation of the input data, arbitrage-free conditions were not maintained.

The transition probabilities at any node on the tree are given by the following three equations.

$$p_{u,i-1} = \frac{1}{2}\left(\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} + \frac{\mu_i \Delta t_i}{\Delta x}\right) \qquad \text{(Eq.6.3.1)}$$

$$p_{m,i-1} = 1 - \frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} \qquad \text{(Eq.6.3.2)}$$

$$p_{d,i-1} = \frac{1}{2}\left(\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} - \frac{\mu_i \Delta t_i}{\Delta x}\right) \qquad \text{(Eq.6.3.3)}$$

The parameter μ_i is given by equation (6.3.4).

$$\mu_i = r_i - \delta_i - \frac{\sigma_i^2}{2} \qquad \text{(Eq.6.3.4)}$$

The space step Δx must of course satisfy the basic requirement depicted by the following relationship.

$$\Delta x > 0 \qquad \text{(Eq.6.3.5)}$$

Starting with $p_{m,i-1}$,

$$p_{m,i-1} < 1 \quad \Rightarrow 1 - \frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} < 1$$

$$\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} > 0 \qquad \text{(Eq.6.3.6)}$$

All the parameters in equation (6.3.6) are squared, so they are definitely positive. The time step Δt_i is by definition positive. So, the condition depicted by equation (6.3.6) is always satisfied.

Based on the condition that the probability $p_{m,i}$ must be strictly positive:

$$p_{m,i-1} > 0 \quad \Rightarrow \quad 1 - \frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} > 0 \quad \Rightarrow \quad \frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} < 1$$

$$\Delta x > \sqrt{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2} \qquad \text{(Eq.6.3.7)}$$

Next, we consider the conditions for the probability $p_{u,i-1}$. In the case that $\mu_i = 0$, the probability $p_{u,i-1}$ is simplified as below.

$$p_{u,i-1} = \frac{\sigma_i^2\,\Delta t_i}{2\Delta x^2} \qquad \text{(Eq.6.3.8)}$$

The requirement $p_{u,i-1} < 1$ results in the following.

$$p_{u,i-1} < 1 \quad \Rightarrow \quad \frac{\sigma_i^2\,\Delta t_i}{2\Delta x^2} < 1$$

$$\Delta x > \sqrt{\frac{\sigma_i^2\,\Delta t_i}{2}} \qquad \text{(Eq.6.3.9)}$$

In the case that $\mu_i \neq 0$, for the condition that $p_{u,i-1} < 1$:

$$p_{u,i-1} < 1 \quad \Rightarrow \quad \frac{1}{2}\left(\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} + \frac{\mu_i \Delta t_i}{\Delta x}\right) < 1$$

$$2\Delta x^2 - \mu_i \Delta t_i \Delta x - \left(\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2\right) > 0 \qquad \text{(Eq.6.3.10)}$$

The roots of the equation are given by:

$$\Delta x_{1,2} = \frac{\mu_i \Delta t_i \pm \sqrt{\mu_i^2 \Delta t_i^2 + 4 \times 2 \times \left(\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2\right)}}{4}$$

$$\Delta x_{1,2} = \frac{\mu_i \Delta t_i \pm \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.11)}$$

So the resultant ranges are given by:

$$\Delta x < \Delta x_1 = \frac{\mu_i \Delta t_i - \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.12)}$$

$$\Delta x > \Delta x_2 = \frac{\mu_i \Delta t_i + \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.13)}$$

Now, we consider the condition that the probability $p_{u,i-1}$ must be strictly positive. In the case that $\mu_i = 0$, this requirement results in the following.

$$\frac{\sigma_i^2\,\Delta t_i}{2\Delta x^2} > 0 \qquad \text{(Eq.6.3.14)}$$

It is obvious that the requirement $p_{u,i-1} > 0$ is always satisfied if $\mu_i = 0$.

In the case that $\mu_i \neq 0$, the requirement $p_{u,i-1} > 0$ results in the following.

$$p_{u,i-1} > 0 \quad \Rightarrow \quad \frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} + \frac{\mu_i \Delta t_i}{\Delta x} > 0$$

$$\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} > -\frac{\mu_i \Delta t_i}{\Delta x} \qquad \text{(Eq.6.3.15)}$$

In the case that $\mu_i > 0$, the requirement depicted by the relationship in equation (6.3.15) is always true since the right-hand side is always negative and the left-hand side is positive.

In the case that $\mu_i < 0$, the resultant relationship is derived as following.

$$\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} - \frac{|\mu_i|\Delta t_i}{\Delta x} > 0$$

$$\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} > \frac{|\mu_i|\Delta t_i}{\Delta x}$$

$$\Delta x < \frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{|\mu_i|\Delta t_i} \qquad \text{(Eq.6.3.16)}$$

Next, we consider the conditions for the probability $p_{d,i-1}$. In the case that $\mu_i = 0$, the probability $p_{d,i-1}$ is as shown by equation (6.3.17). This is the same with equation (6.3.9) for $p_{u,i-1}$. If we consider μ_i being the actual drift after the application of the branching process, then it should be expected that $p_{u,i-1} = p_{d,i-1}$ when $\mu_i = 0$.

$$p_{d,i-1} = \frac{\sigma_i^2\,\Delta t_i}{2\Delta x^2} \qquad \text{(Eq.6.3.17)}$$

In the case that $\mu_i \neq 0$, the requirement $p_{d,i-1} < 1$ results in the following.

$$p_{d,i-1} < 1 \quad \Rightarrow \quad \frac{1}{2}\left(\frac{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}{\Delta x^2} - \frac{\mu_i \Delta t_i}{\Delta x}\right) < 1$$

$$2\Delta x^2 + \mu_i \Delta t_i \Delta x - \left(\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2\right) > 0 \qquad \text{(Eq.6.3.18)}$$

The roots of the equation are given by:

$$\Delta x_{3,4} = \frac{-\mu_i \Delta t_i \pm \sqrt{\mu_i^2 \Delta t_i^2 + 4\times 2\times\left(\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2\right)}}{4}$$

$$\Delta x_{3,4} = \frac{-\mu_i \Delta t_i \pm \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.19)}$$

So the resultant ranges are given by:

$$\Delta x < \Delta x_3 = \frac{-\mu_i \Delta t_i - \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.20)}$$

$$\Delta x > \Delta x_4 = \frac{-\mu_i \Delta t_i + \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4} \qquad \text{(Eq.6.3.21)}$$

In the case that $\mu_i \neq 0$, the requirement $p_{d,i-1} > 0$ results in the following.

$$p_{d,i-1} > 0 \quad \Rightarrow \quad \frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} - \frac{\mu_i \Delta t_i}{\Delta x} > 0$$

$$\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} > \frac{\mu_i \Delta t_i}{\Delta x} \qquad \text{(Eq.6.3.22)}$$

In the case that $\mu_i < 0$, the requirement depicted by the relationship in equation (6.3.22) is always true since the right-hand side is always negative and the left-hand side is positive.

In the case that $\mu_i > 0$, the resultant relationship is derived as following.

$$\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} - \frac{|\mu_i| \Delta t_i}{\Delta x} > 0$$

$$\frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{\Delta x^2} > \frac{|\mu_i| \Delta t_i}{\Delta x}$$

$$\Delta x < \frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{|\mu_i| \Delta t_i} \qquad \text{(Eq.6.3.23)}$$

Summarizing, the required conditions are as follows:

$$\Delta x > 0 \qquad \text{(Eq.6.3.5)}$$

$$\Delta x > \sqrt{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2} \qquad \text{(Eq.6.3.7)}$$

$$\Delta x > \sqrt{\frac{\sigma_i^2 \Delta t_i}{2}}, \quad \textit{if } \mu_i = 0 \qquad \text{(Eq.6.3.9)}$$

$$\Delta x < \frac{\sigma_i^2 \Delta t_i + \mu_i^2 \Delta t_i^2}{|\mu_i| \Delta t_i}, \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.16)}$$

$$\left.\begin{aligned} &\Delta x < \Delta x_1 = \frac{\mu_i \Delta t_i - \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2 \Delta t_i}}{4} \\ &or\ \Delta x > \Delta x_2 = \frac{\mu_i \Delta t_i + \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2 \Delta t_i}}{4} \end{aligned}\right\} \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.12)/ (Eq.6.3.13)}$$

$$\left.\begin{aligned} &\Delta x < \Delta x_3 = \frac{-\mu_i \Delta t_i - \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2 \Delta t_i}}{4} \\ &or\ \Delta x > \Delta x_4 = \frac{-\mu_i \Delta t_i + \sqrt{9\mu_i^2 \Delta t_i^2 + 8\sigma_i^2 \Delta t_i}}{4} \end{aligned}\right\} \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.20)/ (Eq.6.3.21)}$$

The effort is now concentrated on excluding conditions from the above list. If it is proved that all the roots $\Delta x_{\{1,2,3,4\}}$ are smaller than $\sqrt{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}$, then because of condition in equation (6.3.7), the conditions of the two sets of equations, (6.3.12)/(6.3.13) and (6.3.20)/(6.3.21), can be removed from the required conditions. In other words, if the smallest permissible value by the condition of equation (6.3.7) satisfies the conditions of the two sets of equations (6.3.12)/(6.3.13) and (6.3.20)/(6.3.21), then we can remove the two sets of equations from the conditions.

To prove that all the roots $\Delta x_{\{1,2,3,4\}}$ are smaller than $\sqrt{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2}$, we need to prove the following:

$$\sqrt{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2} > \frac{|\mu_i|\Delta t_i + \sqrt{9\mu_i^2\Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4}, \quad if\ \mu_i \neq 0 \qquad \text{(Eq.6.3.24)}$$

By squaring both sides of the equation (both sides are positive):

$$\sqrt{\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2} > \left(\frac{|\mu_i|\Delta t_i + \sqrt{9\mu_i^2\Delta t_i^2 + 8\sigma_i^2\,\Delta t_i}}{4}\right)^2$$

$$\left(\sigma_i^2\,\Delta t_i + \mu_i^2\,\Delta t_i^2\right)\times 16 > \mu_i^2\Delta t_i^2 + 2\times|\mu_i|\Delta t_i \times \sqrt{8\sigma_i^2\,\Delta t_i + 9\mu_i^2\Delta t_i^2} + 8\sigma_i^2\,\Delta t_i + 9\mu_i^2\Delta t_i^2$$

$$8\sigma_i^2\,\Delta t_i + 6\,\mu_i^2\,\Delta t_i^2 > 2\times|\mu_i|\Delta t_i \times \sqrt{8\sigma_i^2\,\Delta t_i + 9\,\mu_i^2\,\Delta t_i^2}$$

We can further square both sides as follows:

$$\left(8\sigma_i^2\,\Delta t_i + 6\,\mu_i^2\,\Delta t_i^2\right)^2 > \left(2\times|\mu_i|\Delta t_i \times \sqrt{8\sigma_i^2\,\Delta t_i + 9\,\mu_i^2\,\Delta t_i^2}\right)^2$$

$$64\sigma_i^4\,\Delta t_i^2 + 96\sigma_i^2\,\Delta t_i\,\mu_i^2\Delta t_i^2 + 36\,\mu_i^4\,\Delta t_i^4 > 4\,\mu_i^2\Delta t_i^2\left(8\sigma_i^2\,\Delta t_i + 9\,\mu_i^2\,\Delta t_i^2\right)$$

$$64\sigma_i^4\,\Delta t_i^2 + 96\sigma_i^2\,\mu_i^2\Delta t_i^3 + 36\,\mu_i^4\,\Delta t_i^4 > 32\sigma_i^2\mu_i^2\Delta t_i^3 + 36\,\mu_i^4\,\Delta t_i^4$$

$$64\sigma_i^4\,\Delta t_i^2 + 64\sigma_i^2\,\mu_i^2\Delta t_i^3 > 0$$

$$64\sigma_i^2\,\Delta t_i^2\left(\sigma_i^2 + \mu_i^2\Delta t_i\right) > 0 \qquad \text{(Eq.6.3.25)}$$

Proving that relationship of equation (6.3.24) is valid comes down to proving that (6.3.25) holds. And (6.3.25) holds because the right-hand side will always be a positive non-zero value.

For $\mu_i = 0$, equation (6.3.7) becomes $\Delta x > \sqrt{\sigma_i^2 \, \Delta t_i}$ which is greater than $\sqrt{\dfrac{\sigma_i^2 \, \Delta t_i}{2}}$, the minimum value depicted by equation (6.3.9) for $\mu_i = 0$. So, the condition of equation (6.3.9) is covered by equation (6.3.7). Putting together equations (6.3.7) and (6.3.16):

$$\begin{aligned} &\sqrt{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2} < \Delta x < \frac{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2}{|\mu_i| \Delta t_i}, \quad \textit{if } \mu_i \neq 0 \\ &\Delta x > \sqrt{\sigma_i^2 \, \Delta t_i}, \quad \textit{if } \mu_i = 0 \end{aligned} \qquad \text{(Eq.6.3.26)}$$

The conditions of equation (6.3.26) also cover the requirement that $\Delta x > 0$, since both of the bounds of the conditions in equation (6.3.26) are positive numbers.

It must also be pointed out that if the conditions of equation (6.3.26) reverse, then we will not have a viable model. In other words, if quantity A_i could become greater than quantity B_i, as defined in equations (6.3.27) and (6.3.28), then the system collapses. If condition shown in equation (6.3.29) holds, then there is no such case.

$$A_i = \sqrt{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2} \qquad \text{(Eq.6.3.27)}$$

$$B_i = \frac{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2}{|\mu_i| \Delta t_i}, \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.28)}$$

$$A_i < B_i, \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.29)}$$

or,

$$\sqrt{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2} < \frac{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2}{|\mu_i| \Delta t_i}, \quad \textit{if } \mu_i \neq 0 \qquad \text{(Eq.6.3.29)}$$

From equation (6.3.29) we have the following:

$$\begin{aligned} &1 < \frac{\sqrt{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2}}{|\mu_i| \Delta t_i} \quad \Rightarrow \quad |\mu_i| \Delta t_i < \sqrt{\sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2} \\ &\mu_i^2 \Delta t_i^2 < \sigma_i^2 \, \Delta t_i + \mu_i^2 \, \Delta t_i^2 \quad \Rightarrow \quad \Delta t_i > 0 \end{aligned} \qquad \text{(Eq.6.3.30)}$$

Since it has been set from the beginning that the time step is positive non-zero quantity, this means that equation (6.3.29) always holds. However, this is proved only to hold between the quantities A_i and B_i of a specific step. Due to the fixed value of the space step Δx across all the steps, we redefine equations (6.3.26) and (6.3.29) as follows.

$$A_{\max} < \Delta x < B_{\min} \qquad \text{(Eq.6.3.31)}$$

$$A_{\max} < B_{\min} \qquad \text{(Eq.6.3.32)}$$

If the last equation does not hold, then there is no way that a space step Δx value can be calculated that will satisfy the conditions for valid probabilities.

6.4. The Conditional Tree Probabilities

In this section, the conditional tree probabilities are introduced as a concept and defined. For this task, the following notation is introduced and employed:

- The probability $\pi_{i,j}$ is defined as the probability of being at a tree node (i,j) given that we have started from the origin which is node $(0,0)$.
- The probability $\psi^u_{i,j}$ is defined as the probability of getting to the tree node (i,j) by first getting to node $(i-1, j-1)$ and then branching upwards.
- The probability $\psi^m_{i,j}$ is defined as the probability of getting to the tree node (i,j) by first getting to node $(i-1, j)$ and then branching to the middle.
- The probability $\psi^d_{i,j}$ is defined as the probability of getting to the tree node (i,j) by first getting to node $(i-1, j+1)$ and then branching downwards.
- The probability $\lambda^u_{i,j}$ is defined as the probability of branching upwards from the tree node $(i-1, j-1)$ to the tree node (i,j), given the fact that we are at the tree node $(i-1, j-1)$.
- The probability $\lambda^m_{i,j}$ is defined as the probability of branching to the middle from the tree node $(i-1, j)$ to the tree node (i,j), given the fact that we are at the tree node $(i-1, j)$.

- The probability $\lambda_{i,j}^{d}$ is defined as the probability of branching downwards from the tree node $(i-1, j+1)$ to the tree node (i, j), given the fact that we are at the tree node $(i-1, j+1)$.

We will refer to the probabilities $\lambda_{i,j}^{u}$, $\lambda_{i,j}^{m}$ and $\lambda_{i,j}^{d}$ as the conditional probabilities. In order to introduce the conditional probabilities, we consider the more general case than the case of the trinomial tree presented here with respect to the defined transition probabilities. The transition probabilities $p_{u,i}$, $p_{m,i}$ and $p_{d,i}$, are the same for all the nodes $j \in [-i, i]$ at step i for the purposes of the work of this thesis. Since we want to account for the more general case, we assume that the transition probabilities are different for each node at each step, hence, we follow the notation $p_{i,j}^{u}$, $p_{i,j}^{m}$ and $p_{i,j}^{d}$, for introducing the conditional probabilities.

We consider that the transition probabilities are available and known quantities, calculated based on the previous material presented in this thesis. The following equations determine the values of the introduced conditional probabilities.

$$\psi_{i,j}^{u} = p_{i-1,j-1}^{u} \cdot \pi_{i-1,j-1} \qquad \text{(Eq.6.4.1)}$$

$$\psi_{i,j}^{m} = p_{i-1,j}^{m} \cdot \pi_{i-1,j} \qquad \text{(Eq.6.4.2)}$$

$$\psi_{i,j}^{d} = p_{i-1,j+1}^{d} \cdot \pi_{i-1,j+1} \qquad \text{(Eq.6.4.3)}$$

$$\pi_{i,j} = \psi_{i,j}^{u} + \psi_{i,j}^{m} + \psi_{i,j}^{d} \qquad \text{(Eq.6.4.4)}$$

$$\lambda_{i,j}^{u} = \frac{\psi_{i,j}^{u}}{\pi_{i,j}} \qquad \text{(Eq.6.4.5)}$$

$$\lambda_{i,j}^{m} = \frac{\psi_{i,j}^{m}}{\pi_{i,j}} \qquad \text{(Eq.6.4.6)}$$

$$\lambda_{i,j}^{d} = \frac{\psi_{i,j}^{d}}{\pi_{i,j}} \qquad \text{(Eq.6.4.7)}$$

The probability $\pi_{0,0}$ of the origin node is equal to one.

$$\pi_{0,0} = 1 \qquad \text{(Eq.6.4.8)}$$

It is important to specify that any conditional probabilities with node reference (i, j) not in the permissible range, are equal to zero. This is also true for the probabilities $\pi_{i,j}$.

$$\lambda_{i,j}^{u} = 0 \quad if \;\; i \notin [0, N] \;\; or \;\; j \notin [-i, i] \tag{Eq.6.4.9}$$

$$\lambda_{i,j}^{m} = 0 \quad if \;\; i \notin [0, N] \;\; or \;\; j \notin [-i, i] \tag{Eq.6.4.10}$$

$$\lambda_{i,j}^{d} = 0 \quad if \;\; i \notin [0, N] \;\; or \;\; j \notin [-i, i] \tag{Eq.6.4.11}$$

$$\pi_{i,j} = 0 \quad if \;\; i \notin [0, N] \;\; or \;\; j \notin [-i, i] \tag{Eq.6.4.12}$$

The sum of the conditional probabilities for a given node (i, j) is equal to one by construction, as shown next.

$$\lambda_{i,j}^{u} + \lambda_{i,j}^{m} + \lambda_{i,j}^{d} = \frac{\psi_{i,j}^{u}}{\pi_{i,j}} + \frac{\psi_{i,j}^{m}}{\pi_{i,j}} + \frac{\psi_{i,j}^{d}}{\pi_{i,j}} = \frac{\psi_{i,j}^{u} + \psi_{i,j}^{m} + \psi_{i,j}^{d}}{\pi_{i,j}} = \frac{\pi_{i,j}}{\pi_{i,j}} = 1 \tag{Eq.6.4.13}$$

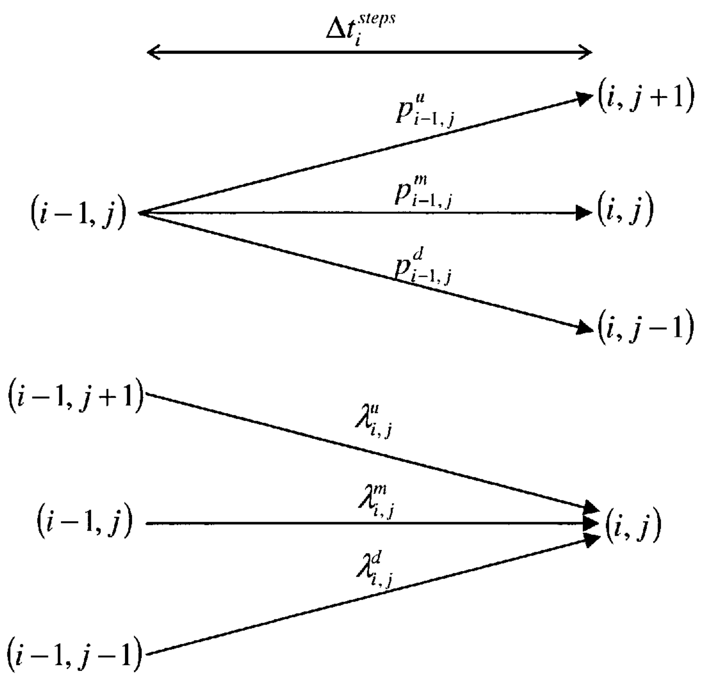


Figure (6.4) Representative Node Configurations for the transition probabilities and the conditional probabilities

6.5. Accounting for Schedules in Forward Induction

Having established the conditional probabilities, we are now in position for developing a methodology for dealing with challenging features encountered in convertible bonds. To be more specific, in this section, we present the part of the introduced methodology for dealing with conditional calls, conditional puts and resets, that takes place during the forward induction process.

Conditional Calls

For a conditional call, we need to specify whether the call will be active at each of the nodes of the tree falling in the time period of the call (between the starting and ending dates of the call). Because of the inclusion of the trigger, this becomes a path-dependent issue which we overcome by proposing a probability-weighted approach.

Let us consider a conditional call with the set of parameters $\{v_k^{CS}, t_k^{a,SD,CS}, t_k^{a,ED,CS}, c_k^{CS}, w_k^{d,CS}, y_k^{CS}\}$ in the call schedule. This set means that this call covers the period from $t_k^{a,SD,CS}$ to $t_k^{a,ED,CS}$, it has a grace period $w_k^{d,CS}$ and a trigger c_k^{CS} which is applicable on the prevailing strike $K_{i,j}$ at each node if the flag y_k^{CS} is equal to 1 or on the original contract-defined strike K if the flag y_k^{CS} is equal to 0. We are assuming that all the call period extended to include the grace period is within the period used for pricing. In other words, we are assuming that, for this example, $\left(t_k^{a,SD,CS} - w_k^{d,CS} + 1\right) > t^{a,CD}$, where $t^{a,CD}$ is the calculations date.

The first action would be to define the flag $\xi_{i,j}^{Trigger,Call}$ for each node in the tree that lies in the period $\left(t_k^{a,SD,CS} - w_k^{d,CS} + 1\right)$ to $t_k^{a,ED,CS}$. The flag is set equal to one if the stock price at the node plus the sum of the discounted discrete dividends at that step is greater than the trigger times the appropriate strike (either the prevailing or the original). The strike used is denoted as $K_{i,j}^{*}$.

$$K_{i,j}^{*} = K_{i,j} \quad if \;\; y_k^{CS} = 1$$
$$K_{i,j}^{*} = K \quad if \;\; y_k^{CS} = 0 \qquad \text{(Eq.6.5.1)}$$

$$\xi_{i,j}^{Trigger,Call} = 1 \qquad if \;\; S_{i,j} + S_i^{Divs,steps} \geq \frac{c_k^{CS}}{100} \times K_{i,j}^{*}$$
$$\xi_{i,j}^{Trigger,Call} = 0 \qquad if \;\; S_{i,j} + S_i^{Divs,steps} < \frac{c_k^{CS}}{100} \times K_{i,j}^{*} \qquad \text{(Eq.6.5.2)}$$

The next action would be to define the probability $\pi_{i,j}^{Trigger,Call}$ for each node in the tree that lies in the period $t_k^{a,SD,CS}$ to $t_k^{a,ED,CS}$. The following process is repeated for each node in the defined range. Let us consider the node (m,n) which lies in this range. To define the value of $\pi_{m,n}^{Trigger,Call}$ of this node we need to include the flags $\xi_{i,j}^{Trigger,Call}$ of all the nodes in the time period $\left(t_m^{a,Steps} - w_k^{d,CS} + 1\right)$ to $t_m^{a,Steps}$. The next figure includes all the nodes involved in defining $\pi_{m,n}^{Trigger,Call}$ if the grace period $w_k^{d,CS}$ was equal to 3.

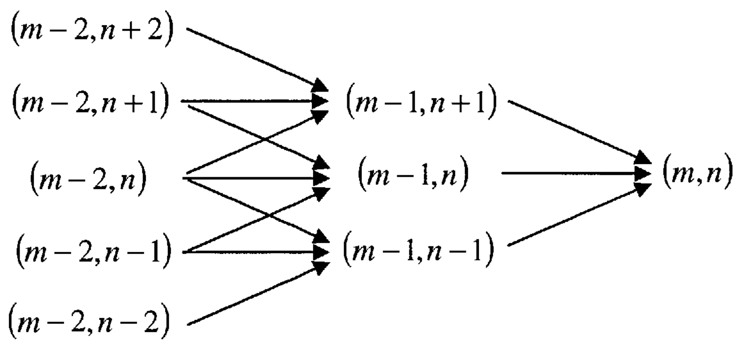


Figure (6.5) Nodes Included in calculating $\pi_{m,n}^{Trigger,Call}$

For the specific case, first we calculate the parameter $\pi_{i,j}^{Trigger,Call,*}$ for each of the nodes $(m-1,n-1)$, $(m-1,n)$ and $(m-1,n+1)$.

$$\pi_{m-1,n-1}^{Trigger,Call,*} = \xi_{m-1,n-1}^{Trigger,Call} \times \left[\left(\lambda_{m-1,n-1}^{d} \cdot \xi_{m-2,n-2}^{Trigger,Call} \right) + \left(\lambda_{m-1,n-1}^{m} \cdot \xi_{m-2,n-1}^{Trigger,Call} \right) + \left(\lambda_{m-1,n-1}^{u} \cdot \xi_{m-2,n}^{Trigger,Call} \right) \right]$$
(Eq.6.5.3)

$$\pi_{m-1,n}^{Trigger,Call,*} = \xi_{m-1,n}^{Trigger,Call} \times \left[\left(\lambda_{m-1,n}^{d} \cdot \xi_{m-2,n-1}^{Trigger,Call} \right) + \left(\lambda_{m-1,n}^{m} \cdot \xi_{m-2,n}^{Trigger,Call} \right) + \left(\lambda_{m-1,n}^{u} \cdot \xi_{m-2,n+1}^{Trigger,Call} \right) \right]$$
(Eq.6.5.4)

$$\pi_{m-1,n+1}^{Trigger,Call,*} = \xi_{m-1,n+1}^{Trigger,Call} \times \left[\left(\lambda_{m-1,n+1}^{d} \cdot \xi_{m-2,n}^{Trigger,Call}\right) + \left(\lambda_{m-1,n+1}^{m} \cdot \xi_{m-2,n+1}^{Trigger,Call}\right) + \left(\lambda_{m-1,n+1}^{u} \cdot \xi_{m-2,n+2}^{Trigger,Call}\right)\right]$$

(Eq.6.5.5)

The probability $\pi_{m,n}^{Trigger,Call}$ can now be calculated as follows.

$$\pi_{m,n}^{Trigger,Call} = \xi_{m,n}^{Trigger,Call} \times \left[\left(\lambda_{m,n}^{d} \cdot \pi_{m-1,n-1}^{Trigger,Call,*}\right) + \left(\lambda_{m,n}^{m} \cdot \pi_{m-1,n}^{Trigger,Call,*}\right) + \left(\lambda_{m,n}^{u} \cdot \pi_{m-1,n+1}^{Trigger,Call,*}\right)\right]$$

(Eq.6.5.6)

As already pointed out, the probability $\pi_{i,j}^{Trigger,Call}$ is calculated for each node in the tree that lies in the period $t_k^{a,SD,CS}$ to $t_k^{a,ED,CS}$. Calculating these probabilities means that the process for accounting for this conditional call during the forward induction has completed. For any additional conditional calls, the same procedure is repeated.

Conditional Puts

The same actions are involved for accounting for conditional puts during forward induction. The only difference is in the calculation of the flag $\xi_{i,j}^{Trigger,Put}$.

Let us consider a conditional put with the set of parameters $\left\{v_k^{PS}, t_k^{a,SD,PS}, t_k^{a,ED,PS}, c_k^{PS}, w_k^{d,PS}, y_k^{PS}\right\}$ in the put schedule. This set means that this put covers the period from $t_k^{a,SD,PS}$ to $t_k^{a,ED,PS}$, it has a grace period $w_k^{d,PS}$ and a trigger c_k^{PS} which is applicable on the prevailing strike $K_{i,j}$ at each node if the flag y_k^{PS} is equal to 1 or on the original contract-defined strike K if the flag y_k^{PS} is equal to 0. We are assuming that all the put period extended to include the grace period is within the period used for pricing. In other words, we are assuming that, for this example, $\left(t_k^{a,SD,PS} - w_k^{d,PS} + 1\right) > t^{a,CD}$, where $t^{a,CD}$ is the calculations date.

The first action would be to define the flag $\xi_{i,j}^{Trigger,Put}$ for each node in the tree that lies in the period $\left(t_k^{a,SD,PS} - w_k^{d,PS} + 1\right)$ to $t_k^{a,ED,PS}$. The flag is set equal to one if the stock price at the node plus the sum of the discounted discrete dividends at that step is greater than the trigger times the appropriate strike (either the prevailing or the original). The strike used is denoted as $K_{i,j}^{*}$.

$$\begin{aligned} K_{i,j}^{*} &= K_{i,j} \quad if \;\; y_k^{PS} = 1 \\ K_{i,j}^{*} &= K \quad\;\; if \;\; y_k^{PS} = 0 \end{aligned} \qquad \text{(Eq.6.5.7)}$$

$$\begin{aligned} \xi_{i,j}^{Trigger,Put} &= 1 \qquad if \;\; S_{i,j} + S_i^{Divs,steps} \leq \frac{c_k^{PS}}{100} \times K_{i,j}^{*} \\ \xi_{i,j}^{Trigger,Put} &= 0 \qquad if \;\; S_{i,j} + S_i^{Divs,steps} > \frac{c_k^{PS}}{100} \times K_{i,j}^{*} \end{aligned} \qquad \text{(Eq.6.5.8)}$$

The next action would be to define the probability $\pi_{i,j}^{Trigger,Put}$ for each node in the tree that lies in the period $t_k^{a,SD,PS}$ to $t_k^{a,ED,PS}$. Let us consider the node (m,n) which lies in this range. To define the value of $\pi_{m,n}^{Trigger,Put}$ of this node we need to include the flags $\xi_{i,j}^{Trigger,Put}$ of all the nodes in the time period $\left(t_m^{a,Steps} - w_k^{d,PS} + 1\right)$ to $t_m^{a,Steps}$. The configuration of figure (6.5) includes all the nodes involved in defining $\pi_{m,n}^{Trigger,Put}$ if the grace period $w_k^{d,PS}$ was equal to 3.

For the specific case, first we calculate the parameter $\pi_{i,j}^{Trigger,Put,*}$ for each of the nodes $(m-1,n-1)$, $(m-1,n)$ and $(m-1,n+1)$.

$$\pi_{m-1,n-1}^{Trigger,Put,*} = \xi_{m-1,n-1}^{Trigger,Put} \times \left[\left(\lambda_{m-1,n-1}^{d} \cdot \xi_{m-2,n-2}^{Trigger,Put}\right) + \left(\lambda_{m-1,n-1}^{m} \cdot \xi_{m-2,n-1}^{Trigger,Put}\right) + \left(\lambda_{m-1,n-1}^{u} \cdot \xi_{m-2,n}^{Trigger,Put}\right)\right]$$

(Eq.6.5.9)

$$\pi_{m-1,n}^{Trigger,Put,*} = \xi_{m-1,n}^{Trigger,Put} \times \left[\left(\lambda_{m-1,n}^{d} \cdot \xi_{m-2,n-1}^{Trigger,Put}\right) + \left(\lambda_{m-1,n}^{m} \cdot \xi_{m-2,n}^{Trigger,Put}\right) + \left(\lambda_{m-1,n}^{u} \cdot \xi_{m-2,n+1}^{Trigger,Put}\right)\right]$$

(Eq.6.5.10)

$$\pi_{m-1,n+1}^{Trigger,Put,*} = \xi_{m-1,n+1}^{Trigger,Put} \times \left[\left(\lambda_{m-1,n+1}^{d} \cdot \xi_{m-2,n}^{Trigger,Put}\right) + \left(\lambda_{m-1,n+1}^{m} \cdot \xi_{m-2,n+1}^{Trigger,Put}\right) + \left(\lambda_{m-1,n+1}^{u} \cdot \xi_{m-2,n+2}^{Trigger,Put}\right)\right]$$

(Eq.6.5.11)

The probability $\pi_{m,n}^{Trigger,Put}$ can now be calculated as follows.

$$\pi_{m,n}^{Trigger,Put} = \xi_{m,n}^{Trigger,Put} \times \left[\left(\lambda_{m,n}^{d} \cdot \pi_{m-1,n-1}^{Trigger,Put,*}\right) + \left(\lambda_{m,n}^{m} \cdot \pi_{m-1,n}^{Trigger,Put,*}\right) + \left(\lambda_{m,n}^{u} \cdot \pi_{m-1,n+1}^{Trigger,Put,*}\right)\right]$$

(Eq.6.5.12)

Resets

Whenever there is even a single reset in the convertible bond contract information, the pricing framework of this thesis includes the parameter $K_{i,j}$ which is the prevailing strike at each node (i, j). The prevailing strike was already included above in the calculations for accounting for conditional calls and puts. The strike $K_{0,0}$ is set equal to the original – contract – strike, or, in the case that there has been a reset date before the calculations date, the strike $K_{0,0}$ is set equal to the prevailing strike on the calculations date, which is a parameter that has to be included in the inputs of the pricing framework in this case. For the rest of the steps, the prevailing strike is defined as shown in the next equation, with the exception of the reset dates.

$$K_{i,j} = \left(\lambda^{d}_{m,n} \cdot K_{i-1,j-1}\right) + \left(\lambda^{m}_{m,n} \cdot K_{i-1,j}\right) + \left(\lambda^{u}_{m,n} \cdot K_{i-1,j+1}\right) \qquad \text{(Eq.6.5.13)}$$

Let us consider a reset with the set of parameters $\left\{t^{a,RS}_{k}, v^{lower,RS}_{k}, v^{upper,RS}_{k}, w^{d,RS}_{k}, y^{RS}_{k}\right\}$ in the call schedule. This set means that this reset will take place on the $t^{a,RS}_{k}$, it has an averaging period $w^{d,RS}_{k}$ and a lower and upper reset levels of $v^{lower,RS}_{k}$ and $v^{upper,RS}_{k}$ which are applicable on the prevailing strike $K_{i,j}$ at each node if the flag y^{RS}_{k} is equal to 1 or on the original contract-defined strike K if the flag y^{RS}_{k} is equal to 0. We are also assuming that, for this example, $\left(t^{a,RS}_{k} - w^{d,RS}_{k} + 1\right) > t^{a,CD}$, where $t^{a,CD}$ is the calculations date.

The first action would be to calculate the average stock price $S^{Average}_{i=m,j}$ for each node in the step m where $t^{a,Steps}_{m} = t^{a,RS}_{k}$, i.e. the reset k takes place at step m. Let us consider the node (m, n) which is one of the nodes at step m. To define the value $S^{Average}_{i=m,j}$ of this node we need to include the stock price of all the nodes in the time period $\left(t^{a,Steps}_{m} - w^{d,CS}_{k} + 1\right)$ to $t^{a,Steps}_{m}$ that are part of the paths passing through node (m, n). The next figure includes all the nodes involved in defining $S^{Average}_{i=m,j}$ if the averaging period $w^{d,CS}_{k}$ was equal to 3.

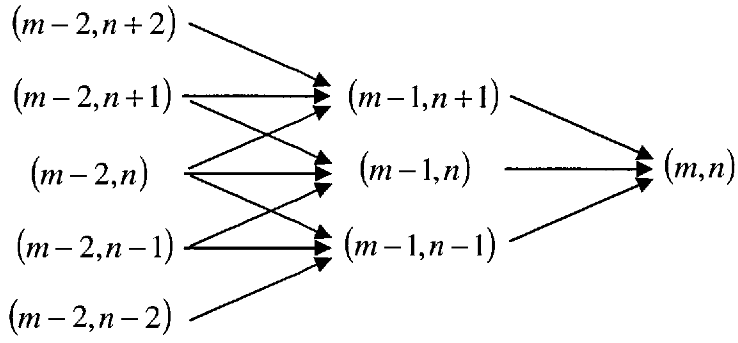


Figure (6.6) Nodes Included in calculating $S_{m,n}^{Average}$

For the specific case, first, we define the values of the parameter $S_{i,j}^{Average,*}$ for each of the nodes $(m-2,n-2)$, $(m-2,n-1)$, $(m-2,n)$, $(m-2,n+1)$ and $(m-2,n+2)$.

$$S_{m-2,n-2}^{Average,*} = S_{m-2,n-2} + S_{m-2}^{Divs,Steps} \qquad \text{(Eq.6.5.14)}$$

$$S_{m-2,n-1}^{Average,*} = S_{m-2,n-1} + S_{m-2}^{Divs,Steps} \qquad \text{(Eq.6.5.15)}$$

$$S_{m-2,n}^{Average,*} = S_{m-2,n} + S_{m-2}^{Divs,Steps} \qquad \text{(Eq.6.5.16)}$$

$$S_{m-2,n+1}^{Average,*} = S_{m-2,n+1} + S_{m-2}^{Divs,Steps} \qquad \text{(Eq.6.5.17)}$$

$$S_{m-2,n+2}^{Average,*} = S_{m-2,n+2} + S_{m-2}^{Divs,Steps} \qquad \text{(Eq.6.5.18)}$$

Then, we calculate the parameter $S_{i,j}^{Average,*}$ for each of the nodes $(m-1,n-1)$, $(m-1,n)$ and $(m-1,n+1)$.

$$\begin{aligned} S_{m-1,n-1}^{Average,*} = S_{m-1,n-1} + S_{m-1}^{Divs,Steps} &+ \left(\lambda_{m-1,n-1}^{d} \cdot S_{m-2,n-2}^{Average,*}\right) \\ &+ \left(\lambda_{m-1,n-1}^{m} \cdot S_{m-2,n-1}^{Average,*}\right) + \left(\lambda_{m-1,n-1}^{u} \cdot S_{m-2,n}^{Average,*}\right) \end{aligned} \qquad \text{(Eq.6.5.19)}$$

$$\begin{aligned} S_{m-1,n}^{Average,*} = S_{m-1,n} + S_{m-1}^{Divs,Steps} &+ \left(\lambda_{m-1,n}^{d} \cdot S_{m-2,n-1}^{Average,*}\right) \\ &+ \left(\lambda_{m-1,n}^{m} \cdot S_{m-2,n}^{Average,*}\right) + \left(\lambda_{m-1,n}^{u} \cdot S_{m-2,n+1}^{Average,*}\right) \end{aligned} \qquad \text{(Eq.6.5.20)}$$

$$\begin{aligned} S_{m-1,n+1}^{Average,*} = S_{m-1,n+1} + S_{m-1}^{Divs,Steps} &+ \left(\lambda_{m-1,n+1}^{d} \cdot S_{m-2,n}^{Average,*}\right) \\ &+ \left(\lambda_{m-1,n+1}^{m} \cdot S_{m-2,n+1}^{Average,*}\right) + \left(\lambda_{m-1,n+1}^{u} \cdot S_{m-2,n+2}^{Average,*}\right) \end{aligned} \qquad \text{(Eq.6.5.21)}$$

The parameter $S_{i,j}^{Average,*}$ for node (m,n) can now be calculated.

$$S_{m,n}^{Average,*} = S_{m,n} + S_{m}^{Divs,Steps} + \left(\lambda_{m,n}^{d} \cdot S_{m-1,n-1}^{Average,*}\right) + \left(\lambda_{m,n}^{m} \cdot S_{m-1,n}^{Average,*}\right) + \left(\lambda_{m,n}^{u} \cdot S_{m-1,n+1}^{Average,*}\right) \quad \text{(Eq.6.5.22)}$$

Finally, the average stock price $S_{i=m,n}^{Average}$ is calculated as follows.

$$S_{m,n}^{Average} = \frac{S_{m,n}^{Average,*}}{w_{k}^{d,CS}} \quad \text{(Eq.6.5.23)}$$

Once the averaging process has been completed and all the average stock prices $S_{m,j=[-m.m]}^{Average}$ have been calculated at the reset date, we are in position for resetting the strike at each of these nodes. The strike is set based on the lower and upper levels which are found by multiplying the values $v_{k}^{lower,RS}$ and $v_{k}^{upper,RS}$ by the appropriate strike (either the prevailing or the original). The strike used is denoted as $K_{i,j}^{*}$.

$$\begin{aligned} K_{i,j}^{*} &= K_{i,j} \quad if \ y_{k}^{RS} = 1 \\ K_{i,j}^{*} &= K \quad if \ y_{k}^{RS} = 0 \end{aligned} \quad \text{(Eq.6.5.24)}$$

$$\begin{aligned} K_{i,j} &= S_{m,j}^{Average} & \frac{v_{k}^{lower,RS}}{100} \times K_{i,j}^{*} &\le S_{m,j}^{Average} \le \frac{v_{k}^{upper,RS}}{100} \times K_{i,j}^{*} \\ K_{i,j} &= \frac{v_{k}^{upper,RS}}{100} \times K_{i,j}^{*} & S_{m,j}^{Average} &> \frac{v_{k}^{upper,RS}}{100} \times K_{i,j}^{*} \\ K_{i,j} &= \frac{v_{k}^{lower,RS}}{100} \times K_{i,j}^{*} & S_{m,j}^{Average} &< \frac{v_{k}^{lower,RS}}{100} \times K_{i,j}^{*} \end{aligned} \quad \text{(Eq.6.5.25)}$$

CHAPTER 7

CALCULATING THE CB PRICE AND THE SENSITIVITIES

The Backward Induction process and the calculation of the sensitivities are the subjects of this chapter. In the previous chapter, the Forward Induction process was presented, while, in its proceeding chapters the input data structures were listed. The outcomes of the Forward Induction process are the stock prices on the trinomial tree with their associated probabilities. For the case of conditional calls and puts there is the additional outcome which is the probability of triggering the conditional call/put at each node, while for the case of resets the additional outcome is the average probability-weighted stock price at the nodes of the step on the reset date. So, at this point, we are in position to price the convertible bond and calculate it sensitivities by carrying out Backward Induction.

7.1. Backward Induction

The first action involved in the Backward Induction process is establishing the convertible bond price at the nodes of the tree on the expiration date (last step on the tree). Then, by working backwards on the tree, the price of the convertible bond is calculated at the nodes of the remaining steps until the convertible bond price for the calculations date (step with index zero) has been established.

First, some notation is introduced. The indexing of all the parameters introduced here follows that of the equity trinomial tree, as shown in figure (6.2). The convertible bond price at a node on a tree is denoted as $V_{i,j}^{b}$, while the holding value of the convertible bond is denoted as $V_{i,j}^{h}$ and the converting value of the convertible bond is denoted as

$V_{i,j}^{conv}$. The holding value $V_{i,j}^{h}$ at a node represents the value of the convertible bond to the investor at node (i,j) which is one of the possible realisations at time $t_i^{a,Steps}$, if the investor ***does not convert*** at node (i,j), holds the convertible bond, and either converts later on in time or holds the convertible bond to maturity. The converting value $V_{i,j}^{conv}$ at a node represents the value of the convertible bond to the investor at node (i,j) which is one of the possible realisations at time $t_i^{a,Steps}$, if the investor ***does convert*** at node (i,j). The final value of the convertible bond at node (i,j) is the value $V_{i,j}^{b}$, which is the ***optimal value*** to the investor between the holding value $V_{i,j}^{h}$ and the converting value $V_{i,j}^{conv}$.

In the cases where the convertible bond is callable, then the call value at any node (i,j) is denoted as $V_{i,j}^{c}$, while in the cases that the convertible bond is puttable, then the put value at any node (i,j) is denoted as $V_{i,j}^{p}$. In this implementation, a convertible bond is allowed to be callable and puttable at the same time and at the same node. However, no calls or puts can be active on the expiration date since the value of the convertible is defined by its redemption value. The put value at any node (i,j) is calculated based on the following equation.

$$\left.\begin{array}{ll} V_{i,j}^{p} = v_k^{PS} & t_k^{a,PS,SD} \le t_i^{a,steps} \le t_{k+1}^{a,PS,ED},\ 0 \le k \le (n_{PS}-1) \\ V_{i,j}^{p} = 0 & otherwise\ (no\ put\ is\ activated) \end{array}\right\} , \forall i \in [0, N-1], \forall j \in [-i, i]$$

(Eq.7.1.1)

In the case of the call price $V_{i,j}^{c}$, the accrued interest has to be taken into account as denoted in the following equation.

$$\left.\begin{array}{ll} V_{i,j}^{c} = v_k^{CS} + AI_i & t_k^{a,CS,SD} \le t_i^{a,steps} \le t_{k+1}^{a,CS,ED},\ 0 \le k \le (n_{CS}-1) \\ V_{i,j}^{c} = 0 & otherwise\ (no\ call\ is\ activated) \end{array}\right\} , \forall i \in [0, N-1], \forall j \in [-i, i]$$

(Eq.7.1.2)

If it is the case that there is a call and put active at the same time, then, usually, in these cases the call value is higher than the put value, locking in this way the price of a straight bond (without optionality to convert) into the range between the two values, the put and the call value. However, in the case of the convertible bonds, the range is violated on the upward side because the conversion value comes into play since there is the optionality to convert to Equity.

The ***exchange rate parameter*** $X_i^{e/b}$ will appear in the following equations without making any distinguish between dual currency and single currency convertible bonds. To simplify the calculations and description here, the exchange rates are set equal to one across all the steps in the case of a single currency convertible bond. In the case of a dual currency convertible bond, the parameter $X_i^{e/b}$ represents the forward exchange rate as was described in a previous chapter.

Another important note to make regards the distinguish between "pure" stock values $S_{i,j}$ and "actual" stock values $S_{i,j}^*$. The pure stock values are the stock values that are included on the equity trinomial tree. As you can recall, those stock prices are the diffused stock prices based on an initial stock price that did not include the sum of the discounted discrete dividends (the sum of the discounted discrete dividends was subtracted from the initial stock price used in the diffusion). Adding the sum of the discounted discrete dividends at any step to the pure stock prices will result in the actual stock prices.

$$S_{i,j}^* = S_{i,j} + S_i^{Divs,steps} \qquad \forall i \in [0,N], \forall j \in [-i,i] \qquad \text{(Eq.7.1.3)}$$

At all the nodes of all the steps, the conversion flag $\xi_{i,j}^{ConvAllowed}$ is simply defined based on the following equation. Conversion is not allowed during the no-conversion period at the end of the convertible bond life which includes at least the maturity date. Conversion is also not allowed at the intermediate steps, the steps added between the calculations date and the calculations date plus one. As a reminder, these steps were added for improving the sampling performance of the tree and they have time step sizes smaller than one day.

$$\left.\begin{array}{ll} \textit{Conversion Allowed at } i: & \xi_{i,j}^{ConvAllowed} = 1 \\ \textit{Conversion Not Allowed at } i: & \xi_{i,j}^{ConvAllowed} = 0 \end{array}\right\} \forall i \in [0,N], \forall j \in [-i,i] \qquad \text{(Eq.7.1.4)}$$

Probability Weighted Discounting

For the cash-flows discounting, we follow the *probability weighted discounting* approach, which is an important notion already followed in the industry when there are risk-free and risky cash flows involved in a pricing framework. In the case of the convertible bond, the risky cash flows are a result of the bond-like features of the convertible bond, which are the coupon cash flows and the redemption (maturity) cash flow. The risk-free cash flows are a result of the equity-like features of the convertible bond, which is the conversion value – the stock price and the dividends.

At each node, based on which is more optimal, the convertible bond price can either reflect the conversion value of the convertible bond or the holding value of the convertible bond. If the convertible bond value reflects the conversion value, then the forward risk-free rate – the forward risk-free discount factor actually - should be used for discounting it. In the case that the convertible bond reflects the holding value, things become more complicated because the holding value reflects all the possible future outcomes and these include cases where the conversion takes place at some point in the future and cases where no conversion takes place and the convertible bond is held to maturity. So, it is not clear which discounting factor to use, the risk-free or the risky one.

To overcome this challenging situation, we are introducing another two parameters for each node on the tree, the flag $\xi_{i,j}^{Converted}$ and the probability $\pi_{i,j}^{Converted}$. During the backward induction, it is decided at each node whether to convert or not. If at a node it is decided to convert, then the flag $\xi_{i,j}^{Converted}$ is set equal to one, otherwise is set equal to zero. Then the probability of conversion $\pi_{i,j}^{Converted}$ is set as shown in the following equations. The initialisation of this process starts with setting both the flag and the probability for all the nodes at the maturity step equal to zero, since no conversion is allowed ever on the maturity date.

$$\left.\begin{aligned} \xi_{i=N,j}^{Converted} &= 0 \\ \pi_{i=N,j}^{Converted} &= 0 \end{aligned}\right\} \forall j \in [-N, N] \qquad \text{(Eq.7.1.5)}$$

$$\left.\begin{array}{ll} \xi_{i,j}^{Converted} = 0 & \textit{if decided to hold} \\ \xi_{i,j}^{Converted} = 1 & \textit{if decided to convert} \end{array}\right\} \forall i \in [0, N-1], \forall j \in [-i, i] \qquad \text{(Eq.7.1.6)}$$

$$\pi_{i,j}^{Converted} = \xi_{i,j}^{Converted} \times \left[\left(p_{i,u} \cdot \pi_{i+1,j+1}^{Converted}\right) + \left(p_{i,m} \cdot \pi_{i+1,j}^{Converted}\right) + \left(p_{i,d} \cdot \pi_{i+1,j-1}^{Converted}\right)\right] \quad \forall i \in [0, N-1], \forall j \in [-i, i] \qquad \text{(Eq.7.1.7)}$$

The probabilities $\pi_{i,j}^{Converted}$ are employed in the discounting process, and this is demonstrated in the descriptions to follow for the rest of the calculations.

<u>Calculations at Maturity:</u> $(i = N)$

Backward Induction begins with the calculation of the convertible bond price at the last step, which is the step that corresponds to the maturity date. The holding value of the convertible bond is calculated as the sum of the redemption value and any possible coupon cash flows on that day.

$$V_{i,j}^{h} = P^{Rd} + q_i^{Cpn,steps} \qquad i = N, \forall j \in [-N, N] \qquad \text{(Eq.7.1.8)}$$

The final convertible bond value is simply calculated based on the following equation:

$$V_{i,j}^{b} = V_{i,j}^{h} \qquad i = N, \forall j \in [-N, N] \qquad \text{(Eq.7.1.9)}$$

<u>Calculations at the rest of the steps:</u> $(i = (N-1) \to 0)$

Starting from step $i = N-1$ and working backwards on the tree until step $i = 0$ (inclusively), the following calculations are repeated at each step.

First, the conversion value is calculated as depicted by equation (7.1.4) above and re

$$\left.\begin{array}{ll} V_{i,j}^{conv} = R^{CR} \times X_i^{e/b} \times S_{i,j}^{*} & \xi_{i,j}^{ConvAllowed} = 1 \\ V_{i,j}^{conv} = 0 & \xi_{i,j}^{ConvAllowed} = 0 \end{array}\right\} \forall j \in [-i, i] \qquad \text{(Eq.7.1.10)}$$

The holding value is calculated as the probability weighted sum of the connected nodes in the following in time step, plus any coupon values. Three intermediate values

$V_{i,j}^{h,u}$, $V_{i,j}^{h,m}$ and $V_{i,j}^{h,d}$, are calculated before the final value $V_{i,j}^{h}$ is established. The probability of converting is also taken into account.

$$V_{i,j}^{h,u} = \left[\left(\pi_{i+1,j+1}^{Converted} \times fd_{i+1}^{RFb,steps}\right) + \left(\left(1-\pi_{i+1,j+1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{u,i} \times V_{i+1,j+1}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.11)

$$V_{i,j}^{h,m} = \left[\left(\pi_{i+1,j}^{Converted} \times fd_{i+1}^{RFb,steps}\right) + \left(\left(1-\pi_{i+1,j}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{m,i} \times V_{i+1,j}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.12)

$$V_{i,j}^{h,d} = \left[\left(\pi_{i+1,j-1}^{Converted} \times fd_{i+1}^{RFb,steps}\right) + \left(\left(1-\pi_{i+1,j-1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{d,i} \times V_{i+1,j-1}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.13)

$$V_{i,j}^{h} = q_i^{Cpn,steps} + V_{i,j}^{h,u} + V_{i,j}^{h,m} + V_{i,j}^{h,d} \qquad \forall j \in [-i,i] \qquad \text{(Eq.7.1.14)}$$

Then, according to the presence of calls and puts, the final convertible bond value at each node is calculated based on equation (Eq.7.1.15). The convertible bond is initially set equal to the minimum of the holding value and the call value (if the convertible bond is callable at that step). Then, the result is compared to the converting value and the put value (if the convertible bond is puttable at that node), and the maximum of the three values is used as the final convertible bond value at that node. In the absence of any calls at a node (i, j), then the parameter $V_{i,j}^{c}$ is simply not included in the equation. In the same manner, in the absence of any puts at a node (i, j), then the parameter $V_{i,j}^{p}$ is simply not included in the equation. Finally, in the cases where no conversion is allowed at a node (i, j), then the parameter $V_{i,j}^{conv}$ is simply not included in the equation. Actually, $V_{i,j}^{conv}$ is equal to zero in the cases where no conversion is allowed, so the result of the following equation is not affected anyway.

$$V_{i,j}^{b} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \forall j \in [-i,i] \qquad \text{(Eq.7.1.15)}$$

Once the above calculations have been performed at all the steps, the value of the convertible bond on the calculation date, V^{CD}, is set equal to the value $V_{0,0}^{b}$ which is the value of the convertible bond at the zero step of the tree.

$$V^{CD} = V_{0,0}^{b} \qquad \text{(Eq.7.1.16)}$$

Dual Currency Convertible Bonds

As it has already been pointed out, for the case of dual currency convertible bonds the forward exchange rate $X_i^{e/b}$ at each node is not equal to one like in the case of single currency convertible bonds. The calculation of the forward exchange rates for dual currency convertible bonds has already been presented in the previous chapter.

Another difference in the approach for the pricing of the dual currency convertible bonds is in the discounting process. The risk-free cash flows for dual currency convertible bonds are discounted based on the equity (foreign) currency. The risky cash flows are still being discounted based on the risky discount factors which correspond to the bond (domestic) currency. Equations (7.1.11) to (7.1.13) are replaced by the following three equations when pricing dual currency CBs.

$$V_{i,j}^{h,u} = \left[\left(\pi_{i+1,j+1}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j+1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{u,i} \times V_{i+1,j+1}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.17)

$$V_{i,j}^{h,m} = \left[\left(\pi_{i+1,j}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{m,i} \times V_{i+1,j}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.18)

$$V_{i,j}^{h,d} = \left[\left(\pi_{i+1,j-1}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j-1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{d,i} \times V_{i+1,j-1}^{b} \quad \forall j \in [-i,i]$$

(Eq.7.1.19)

7.2. Backward Induction for Resets and Conditional Calls/Puts

Conditional Calls

When conditional calls are present, then there are some modifications in the calculations. In the previous chapter, it was described how the conditional calls are accounted for. Essentially, the probability of triggering a conditional call $\pi_{i,j}^{Trigger,Call}$ was calculated for all the nodes (i,j) that fell within the period of a conditional call.

To complete the calculations for accounting for a conditional call, we need, for each node in the conditional call period, to calculate the value of the convertible bond in the case that the call is activated – triggered – and in the case that is not activated. Then, the final convertible bond price at that node will be the probability weighted sum of the two values, based on the probabilities $\pi_{i,j}^{Trigger,Call}$ and $\left(1-\pi_{i,j}^{Trigger,Call}\right)$.

In more detail, let us consider the general case of a node (i,j) which falls within the conditional call period. Let us also denote the convertible bond price at that node with the call activated as $V_{i,j}^{b,CallTriggered}$ and the price without the call activated as $V_{i,j}^{b,CallNotTriggered}$. Then, equation (7.1.15) is used as described in the calculations above, once with a call value $V_{i,j}^{c}$ which is calculated by adding the conditional call's call value and the accrued interest, and once without the call value. This is also shown by the following equations. Like before, the conversion value is included in the equations if conversion is allowed and the put value is included in the calculations if a put is activated.

$$V_{i,j}^{b,CallTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.7.2.1)}$$

$$V_{i,j}^{b,CallNotTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, V_{i,j}^{h}\right) \qquad \text{(Eq.7.2.2)}$$

The final value of the convertible bond at a node within the period covered by a conditional call is calculated as follows.

$$V_{i,j}^{b} = \pi_{i,j}^{Trigger,Call} \cdot V_{i,j}^{b,CallTriggered} + \left(1-\pi_{i,j}^{Trigger,Call}\right) \cdot V_{i,j}^{b,CallNotTriggered} \qquad \text{(Eq.7.2.3)}$$

Conditional Puts

During the Backward Induction process, the treatment of the conditional puts is in exactly the same fashion as that for the conditional calls. To complete the calculations for accounting for a conditional put, we need, for each node (i,j) in the conditional put period, to calculate the value of the convertible bond in the case that the put is triggered and in the case that is not. Then, the final convertible bond price at that node will be the probability weighted sum of the two values, based on the probabilities $\pi_{i,j}^{Trigger,Put}$ and $\left(1-\pi_{i,j}^{Trigger,Put}\right)$.

Let us consider the general case of a node (i,j) which falls within the conditional call period. The convertible bond price at that node with the put activated is denoted as $V_{i,j}^{b,PutTriggered}$ and the price without the put activated is denoted as $V_{i,j}^{b,PutNotTriggered}$. Then, equation (7.1.15) is used as described in the calculations above, once with a put value $V_{i,j}^{p}$ and once without the put value. This is also shown by the following equations. Like before, the conversion value is included in the equations if conversion is allowed and the call value is included in the calculations if a call is activated.

$$V_{i,j}^{b,PutTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.7.2.4)}$$

$$V_{i,j}^{b,PutNotTriggered} = \max\left(V_{i,j}^{conv}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.7.2.5)}$$

The final value of the convertible bond at a node within the period covered by a conditional put is calculated as follows.

$$V_{i,j}^{b} = \pi_{i,j}^{Trigger,Put} \cdot V_{i,j}^{b,PutTriggered} + \left(1 - \pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j}^{b,PutNotTriggered} \qquad \text{(Eq.7.2.6)}$$

Combining Conditional Puts and Calls

To cover the extreme, but plausible, case where the period of a conditional call and the period of a conditional put overlap, we consider a general node (i,j) which falls in this overlapping period. We calculate four possible values: The convertible bond price $V_{i,j}^{b,CallPutTriggered}$ at that node with the conditional put and the conditional call both triggered, the convertible bond price $V_{i,j}^{b,CallTriggered}$ at that node with the call activated and the put deactivated, the convertible bond price $V_{i,j}^{b,PutTriggered}$ at that node with the put activated and the call deactivated, and, the convertible bond price $V_{i,j}^{b,NoneTriggered}$ at that node with both the put and the call deactivated.

$$V_{i,j}^{b,CallPutTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.7.2.7)}$$

$$V_{i,j}^{b,CallTriggered} = \max\left(V_{i,j}^{conv}, \min\left(V_{i,j}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.7.2.8)}$$

$$V_{i,j}^{b,PutTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{p}, V_{i,j}^{h}\right) \qquad \text{(Eq.7.2.9)}$$

$$V_{i,j}^{b,NoneTriggered} = \max\left(V_{i,j}^{conv}, V_{i,j}^{h}\right) \qquad \text{(Eq.7.2.10)}$$

The final value of the convertible bond at a node within the period covered by both a conditional call and a conditional put is calculated as shown next.

$$\begin{aligned} V_{i,j}^{b} = & \left(\pi_{i,j}^{Trigger,Call} \cdot \pi_{i,j}^{Trigger,Put} \cdot V_{i,j}^{b,CallPutTriggered}\right) \\ & + \left(\pi_{i,j}^{Trigger,Call} \cdot \left(1-\pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j}^{b,CallTriggered}\right) \\ & + \left(\left(1-\pi_{i,j}^{Trigger,Call}\right) \cdot \pi_{i,j}^{Trigger,Put} \cdot V_{i,j}^{b,PutTriggered}\right) \\ & + \left(\left(1-\pi_{i,j}^{Trigger,Call}\right) \cdot \left(1-\pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j}^{b,NoneTriggered}\right) \end{aligned} \qquad \text{(Eq.7.2.11)}$$

Resets

Resets have been fully accounted for in the Forward Induction process. The only comment to make for the calculations during the Backward Induction process regarding the resets, is on the use of a different conversion ratio at each node instead of a fixed strike like in equation (7.1.10). It was shown in the previous chapter how the strike $K_{i,j}$ is calculated for each node (i,j) on the tree when reset dates are present. This means, that for each node (i,j), a conversion ratio $R_{i,j}^{CR}$ must be calculated based on equation (7.2.12), where P^F is the Notional (Face Value). Finally, equation (7.1.10) is changed into the form of equation (7.2.13) when resets are present.

$$R_{i,j}^{CR} = \frac{P^F}{K_{i,j}} \qquad \text{(Eq.7.2.12)}$$

$$\left.\begin{aligned} V_{i,j}^{conv} &= R_{i,j}^{CR} \times X_i^{e/b} \times S_{i,j}^{*} & \xi_{i,j}^{ConvAllowed} &= 1 \\ V_{i,j}^{conv} &= 0 & \xi_{i,j}^{ConvAllowed} &= 0 \end{aligned}\right\} \forall j \in [-i,i] \qquad \text{(Eq.7.2.13)}$$

7.3. Sensitivities

At the end of the previous section, a convertible bond price was established based on the current input information from the contract and the market. Even though the contract information are fixed and do not vary for an instrument, market information are subjected to changes and variations. It is of the greatest importance to investors, issuers and analysts to be able to quantify these variations, or the sensitivity of the instrument to the various market quantities that affect the instrument's price, and, as an extension, to use these sensitivities for creating positions for hedging most of the factors that the price of the instrument is exposed to.

This section can be considered as a step further to the calculation of the convertible bond price. The sensitivities of the convertible bond to the share price and the volatility of the share price are studied, as well as the sensitivity to the decaying of the time to maturity.

Delta and Gamma

The delta is defined as the sensitivity of the convertible bond price to fluctuations in the underlying share price. The gamma is defined as the rate of change of the delta with respect to share price fluctuations. The delta and gamma sensitivities are calculated in this pricing framework based on two methods.

Numerical Differentiation

We use the term numerical differentiation to refer to the calculation of the sensitivities when the pricing framework is run more than one times, with the additional runs carried out with shifted parameters. In the case of delta, after the convertible bond price V^{CD} has been established based on the current market information, it is recalculated again twice based on the same information but with a shifted share price. We are considering a shift of 0.1% applied to the pure share price on the calculations date as shown in equation (7.3.1). The total shift in the equity value of the convertible bond is shown in equation (7.3.2), and this is simply the share price shift multiplied by the conversion ratio. The convertible bond price is recalculated based on the two shifted share prices and the two new calculated convertible bond prices are denoted as $V^{CD,up}$ and $V^{CD,down}$. The delta is calculated as demonstrated in equation (7.3.5).

$$\Delta S = 0.001 \times S^{CD,NoDivs} \qquad \text{(Eq.7.3.1)}$$

$$\Delta S_{Total} = R^{CR} \times \Delta S \qquad \text{(Eq.7.3.2)}$$

$$S^{CD,up} = S^{CD,NoDivs} + \Delta S \qquad \text{(Eq.7.3.3)}$$

$$S^{CD,down} = S^{CD,NoDivs} - \Delta S \qquad \text{(Eq.7.3.4)}$$

$$\Delta_{ND} = \frac{V^{CD,up} - V^{CD,down}}{2 \times \Delta S_{Total}} \qquad \text{(Eq.7.3.5)}$$

The gamma equation is derived by starting from the definition of equation (7.3.8) where gamma is expressed as the rate of change of the delta with respect to share price fluctuations. The definitions of equations (7.3.6) and (7.3.7) are pre-requested for the process. The parameters are also presented diagrammatically in figure (7.1).

$$\Delta_{ND}^{up} = \frac{V^{CD,up} - V^{CD}}{\Delta S_{Total}} \qquad \text{(Eq.7.3.6)}$$

$$\Delta_{ND}^{down} = \frac{V^{CD} - V^{CD,down}}{\Delta S_{Total}} \qquad \text{(Eq.7.3.7)}$$

$$\Gamma = \frac{\Delta_{ND}^{up} - \Delta_{ND}^{down}}{\Delta S_{Total}} \qquad \text{(Eq.7.3.8)}$$

$$\Gamma = \frac{\Delta_{ND}^{up} - \Delta_{ND}^{down}}{\Delta S_{Total}} = \frac{\dfrac{V^{CD,up} - V^{CD}}{\Delta S_{Total}} - \dfrac{V^{CD} - V^{CD,down}}{\Delta S_{Total}}}{\Delta S_{Total}}$$

$$= \frac{V^{CD,up} - V^{CD} - \left(V^{CD} - V^{CD,down}\right)}{\left(\Delta S_{Total}\right)^2} = \frac{V^{CD,up} - 2 \times V^{CD} + V^{CD,down}}{\left(\Delta S_{Total}\right)^2}$$

$$\Gamma = \frac{V^{CD,up} - 2 \times V^{CD} + V^{CD,down}}{\left(\Delta S_{Total}\right)^2} \qquad \text{(Eq.7.3.9)}$$

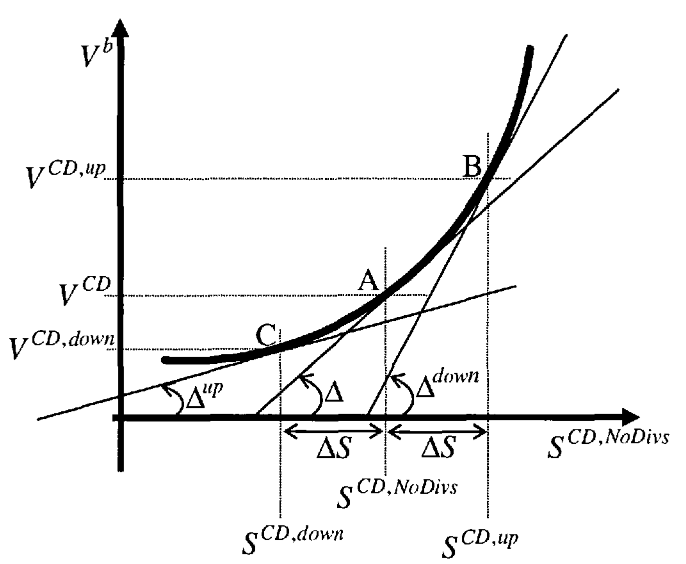


Figure (7.1) Delta and Gamma Sensitivities

Tree Embedded Sensitivities

Based on this approach, instead of re-calculating the convertible bond price by re-running the whole pricing framework for shifted share prices, we use the calculated convertible bond prices at the first step after the calculation date. Following this method, the delta sensitivity is calculated based on the following equation.

$$\Delta = \frac{V_{1,1}^{b} - V_{1,-1}^{b}}{\left(R_{1,1}^{CR} \times X_{1}^{e/b} \times S_{1,1}^{*}\right) - \left(R_{1,-1}^{CR} \times X_{1}^{e/b} \times S_{1,-1}^{*}\right)} \qquad \text{(Eq.7.3.10)}$$

As it can be observed, the conversion value has been used in the above equation. This is the general case of the equation and covers the cases of both single currency CBs where the exchange rate is simply $X_{1}^{e/b} = 1$ and dual currency CBs where the forward exchange rate $X_{1}^{e/b}$ is calculated based on the interest rate differentials of the two currencies. In addition, it covers the case where there are resets and the conversion ratio could be different across the nodes, as well as the cases where there are discrete dividends, since it uses the actual stock prices which include the sum of the discounted discrete dividends in their value.

Since we have identified that the conversion value of the convertible is used in the calculation of the delta, equation (7.3.10) is re-written as follows.

$$\Delta = \frac{V_{1,1}^{b} - V_{1,-1}^{b}}{V_{1,1}^{conv} - V_{1,-1}^{conv}} \qquad \text{(Eq.7.3.11)}$$

For the gamma calculation, the necessary equations are derived as follows.

$$\Delta^{up} = \frac{V_{1,1}^{b} - V_{1,0}^{b}}{V_{1,1}^{conv} - V_{1,0}^{conv}} \qquad \text{(Eq.7.3.12)}$$

$$\Delta^{down} = \frac{V_{1,0}^{b} - V_{1,-1}^{b}}{V_{1,0}^{conv} - V_{1,-1}^{conv}} \qquad \text{(Eq.7.3.13)}$$

$$\Gamma = \frac{\Delta^{up} - \Delta^{down}}{\left(V_{1,0}^{conv} + \frac{V_{1,1}^{conv} - V_{1,0}^{conv}}{2}\right) - \left(V_{1,0}^{conv} - \frac{V_{1,0}^{conv} - V_{1,-1}^{conv}}{2}\right)} = \frac{\Delta^{up} - \Delta^{down}}{\left(\frac{V_{1,1}^{conv} - V_{1,-1}^{conv}}{2}\right)} \qquad \text{(Eq.7.3.14)}$$

$$\Gamma = \frac{\Delta^{up} - \Delta^{down}}{\left(\dfrac{V_{1,1}^{conv} - V_{1,-1}^{conv}}{2}\right)} = \frac{\dfrac{V_{1,1}^{b} - V_{1,0}^{b}}{V_{1,1}^{conv} - V_{1,0}^{conv}} - \dfrac{V_{1,0}^{b} - V_{1,-1}^{b}}{V_{1,0}^{conv} - V_{1,-1}^{conv}}}{\left(\dfrac{V_{1,1}^{conv} - V_{1,-1}^{conv}}{2}\right)} \qquad \text{(Eq.7.3.15)}$$

The tree embedded sensitivities is preferred over the numerical differentiation method because of the following two main reasons (also outlined in reference [35]):

- Numerical differentiation involves running the trinomial tree for an additional time in order to obtain the delta sensitivity and for an additional extra time, two additional times in total, in order to calculate the gamma sensitivity. This means that the numerical differentiation approach is significantly more computationally demanding than the tree embedded sensitivities approach. Actually, for the latter, the additional computations involved are neglible compare to the overall computations of the tree.
- In the case of the numerical differentiation, the resultant delta sensitivity is a ladder-like function of the underlying value, in our case, the conversion value. This means that the delta function of the conversion value is not differentiable at the kinks of the ladder; hence we can not actually calculate a gamma value.

Both drawbacks of the numerical differentiation are significant. For these reasons it was decided that the default calculation of the sensitivities was based on the tree embedded sensitivities approach. Finally, it should be pointed out that because the convertible bond prices and the conversion values used in the calculations of the delta and gamma sensitivities are located at the first step after the calculations date step, this means that the calculated sensitivities correspond to that point of time which is $t_{i=1}^{a,Steps}$ and not to the calculations date. In other words, an approximation is involved when following the preferred approach, since we are approximating the sensitivities on the calculation date by using the sensitivities of another point in time. The smaller the first time step $\Delta t_{i=1}^{Steps}$ is, the smaller the approximation error becomes.

Theta

The rate of change of the convertible bond price with respect to the passage of time is referred to as the ***theta*** sensitivity, sometimes also referred to as the time decay of the convertible bond price. By construction, the theta sensitivity is calculated based on the tree embedded sensitivity approach. The time shifted value of the convertible bond $V_{1,0}^{b}$ is there ready to be used.

$$\Theta = \frac{V_{1,0}^{b} - V_{0,0}^{b}}{\Delta t_{i=1}^{steps}} \qquad \text{(Eq.7.3.16)}$$

Vega

Vega is defined as the sensitivity of the convertible bond price to changes in the volatility of the underlying stock price. In the implementation of the pricing framework of this thesis, Vega is calculated as the change in the value of the convertible bond for a parallel upward shift to the volatility structure of 1%. A new convertible price $V^{CD,vega1\%}$ is calculated after the actual volatility structure $\underline{\sigma}^{steps}$ is replaced by its shifted version $\underline{\sigma}^{steps,1\%}$ and the pricing framework is run for one more time.

$$Vega = V^{CD,vega1\%} - V^{CD} \qquad \text{(Eq.7.3.17)}$$

CHAPTER 8

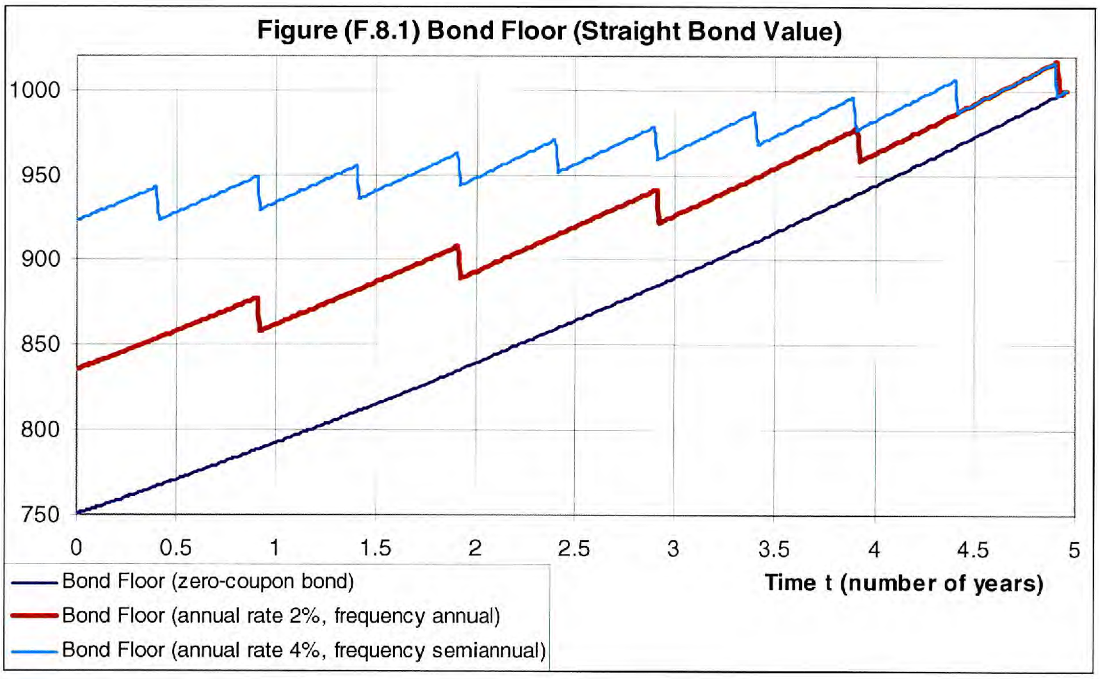
EVALUATING THE PRFORMANCE OF THE MODEL

Having established the pricing framework of the convertible bond in the previous chapter, we are now in position to evaluate the resultant model and study various aspects of the convertible bond. For this task, we follow two approaches, the spectrum analysis and the scenario analysis and simulation, each approach offering different insights to the developed model's assumptions and output parameters.

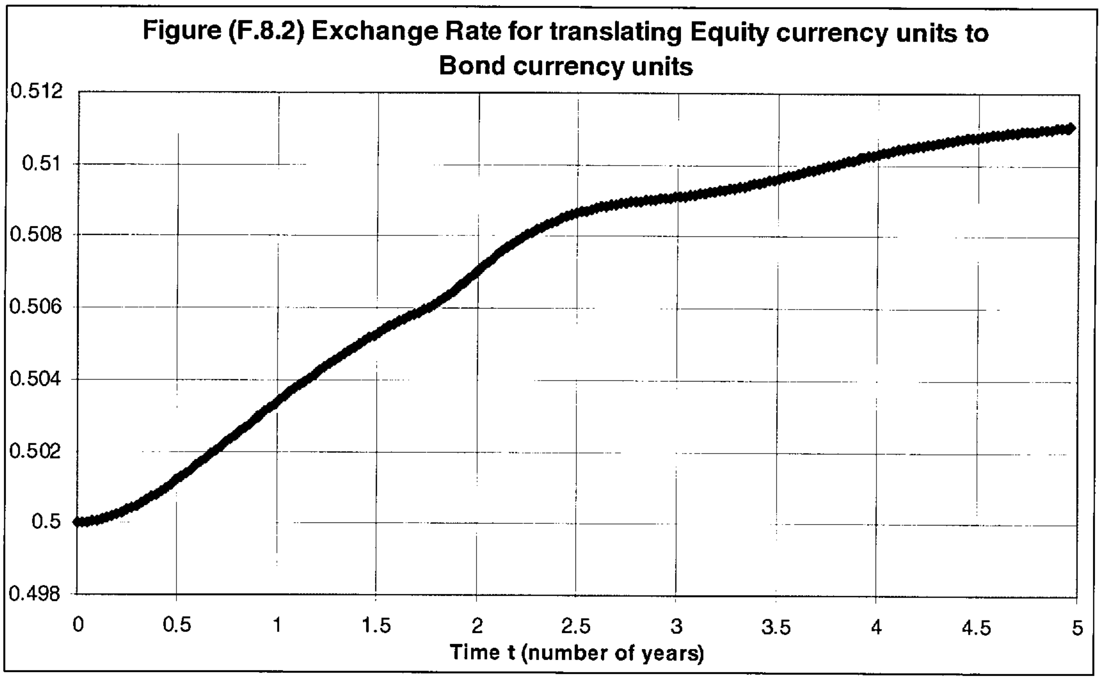
The characteristics of a dummy convertible security were defined and used throughout the simulations. This security is a GBP (British bound) denominated 5-year convertible bond with a face value and redemption value both equal to 1000. The issue date of the security is the 16/01/2003 and the expiration date is the 30/01/2008. The initial conversion ratio is equal to 20 resulting in an initial strike equal to 50, and the no conversion period is equal to 5 days. The valuation date is the 18/02/2003, and the share price of the underlying of the CB on that date is assumed to be equal to 40.2, meaning that the CB is slightly out of the money. The volatility term structure and the continuous term structure of the underlying security are included in the next table.

TABLE T.8.1			
Share Price Volatility Structure		**Continuous Dividend Yield Structure**	
Date	Volatility (%)	Date	Continuous Dividend Yield (%)
19/02/2003	20	19/02/2003	1
12/03/2003	21	12/03/2003	1.1
10/03/2004	22	10/03/2004	1.2
14/01/2005	23	14/01/2005	1.3
16/04/2006	23.5	16/04/2006	1.35
19/04/2007	24	19/04/2007	1.4
17/10/2008	24.5	17/10/2008	1.45
18/05/2009	24.75	18/05/2009	1.475
20/03/2010	25	20/03/2010	1.5
19/07/2011	25.15	19/07/2011	1.515
20/01/2031	25.25	20/01/2031	1.525

The characteristics of the security presented up to this point were kept fixed throughout the simulations. However, there is a significant range of characteristics that are outlined next which are included in the definition of the security only if it is stated so. For example, the default case is that the security is a zero-coupon security. For the simulations that it is stated that the security is a coupon-paying security, a coupon schedule is employed which is defined based on a first coupon date set as the 16/01/2004, a coupon frequency of once per year (annual) and an overall annual coupon rate of 2%. The inclusion of coupon payments increases the bond floor level and this increase is proportional to the coupon rate. This is demonstrated in the following graph.



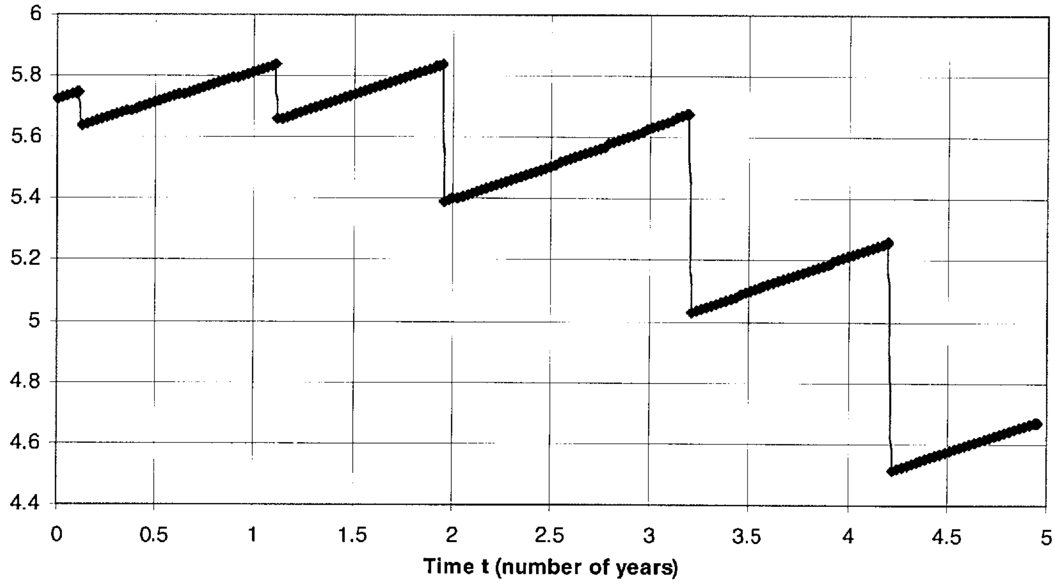
It is also the default case that the convertible bond is a single-currency security. In the cases that it is stated that the security is a dual currency one, then the currency of the underlying security is assumed to be the dummy currency whose interest rate and discount factors curves were presented and calculated in chapter 3. The exchange rate for translating the equity currency (dummy currency) units to the bond currency units (GBP) is assumed to be equal to 0.5 on the calculations date. Of course, the exchange rate on the dates corresponding to the step dates of the trees are calculated based on the interest rate differentials of the two currencies, as it was shown in the previous chapters. The following graph presents the exchange rate evolution with time when the initial exchange rate is equal to 0.5 and the interest rate differentials are defined based on the two interest rate curves presented in chapter 3.



Discrete dividends schedule can be included in the implemented pricing framework. Nevertheless, the default security does not include a discrete dividends schedule and the discrete dividends schedule presented in the next table, table (T.8.2), is only included in the pricing only in the cases that is specified so. The evolution of the sum of the discounted discrete dividends is presented in figure (F.8.3).

TABLE (T.8.2)	
Discrete Dividends Structure	
Date	Dividends (Value)
30/03/2003	0.11
26/03/2004	0.18
28/01/2005	0.45
28/04/2006	0.65
29/04/2007	0.75
25/10/2008	0.92
24/05/2009	0.98
24/03/2010	1.04
21/07/2011	1.11
20/01/2012	1.2

Figure (F.8.3) The evolution of the Sum of Discounted Discrete Dividends



The default security is not callable, puttable or resetable. However, in the cases that it is stated otherwise, the call schedule, the put schedules and the reset schedule included in the following tables, are employed. In the schedules, calls and puts without a trigger are hard calls, while in the case of resets there may or may not be an averaging period.

TABLE T.8.3

Call Schedule - Hard Calls & Soft (Conditional) Calls

Call Value	Starting Date	Ending Date	Trigger (%)	Grace Period (days)	Prevailing Strike (1) or Initial Strike (0)
1040	16/01/2004	15/01/2005	115	12	1
1030	16/01/2005	16/01/2006			
1020	17/01/2006	17/01/2007			
1020	18/01/2007	18/01/2008			

TABLE T.8.4

Put Schedule - Hard Puts & Soft (Conditional) Puts

Put Value	Starting Date	Ending Date	Trigger (%)	Grace Period (days)	Prevailing Strike (1) or Initial Strike (0)
980	19/07/2005	19/07/2006	90	10	1
990	20/07/2006	20/07/2007			
1000	21/07/2007	17/01/2008			

TABLE T.8.5

Reset Schedule

Date	Lower Limit Reset Range (%)	Upper Limit Reset Range (%)	Number of Days for Averaging	Prevailing Strike (1) or Initial Strike (0)
15/04/2005	75	100	25	1
12/10/2006	85	115		

8.1. Spectrum Analysis

In this thesis, spectrum analysis for a quantity is performed by calculating the value of the quantity for a given range of share price values. Each of these share price values is input to the developed pricing framework as the share price on the calculation date and the returned value or output is one of the realisations of the quantity which can be the convertible bond price or any of the sensitivities.

Three inputs are requested in order to specify which stock prices we need to calculate the respective realisations of the quantity for. The starting stock price of the spectrum is denoted as $S^{CD,SSP}$ and the ending stock price of the spectrum is denoted as $S^{CD,ESP}$, while the number of samples is denoted as M. It worths reminding that the superscript "*CD*" denotes that the value corresponds to the calculation date defined in the pricing framework. The floor (minimum) value for the starting stock price is denoted as $S^{CD,SSP,\min} = 0.0001$ and the final starting stock price $S^{*,CD,SSP}$ employed is defined based on equation (8.1.1). However, in the cases where a discrete dividends schedule is included, this minimum value is offset by the sum of the discounted discrete dividends.

$$\begin{aligned} S^{*,CD,SSP} &= S^{CD,SSP} \qquad if\ S^{CD,SSP} \geq S^{CD,SSP,\min} \\ S^{*,CD,SSP} &= S^{CD,SSP,\min} \qquad otherwise \end{aligned} \tag{Eq.8.1.1}$$

In order to carry out the spectrum calculations successfully, it must holds that the ending stock price is greater than the starting stock price and that the number of samples is greater than 1. These conditions are represented by the following two equations.

$$S^{CD,ESP} > S^{*,CD,SSP} \tag{Eq.8.1.2}$$

$$M > 1 \tag{Eq.8.1.3}$$

Based on these three input values, the spectrum space step $\Delta S^{spectrum}$ and, consequently the stock prices employed in the spectrum are established. These stock prices are denoted as $\underline{S}^{CD,spectrum} = \left\{S^{CD,spectrum}_{m=0,1,2,\ldots,M-1}\right\}$.

$$\Delta S^{spectrum} = \frac{S^{CD,ESP} - S^{*,CD,SSP}}{M-1} \qquad \text{(Eq.8.1.4)}$$

$$\begin{aligned} S_0^{CD,spectrum} &= S^{*,CD,SSP} \\ S_m^{CD,spectrum} &= S_{m-1}^{CD,spectrum} + \Delta S^{spectrum} \qquad m \in [1, M-1] \end{aligned} \qquad \text{(Eq.8.1.5)}$$

The starting stock price of the spectrum and the ending stock price of the spectrum were set equal to 0.01 and 100.01 respectively, while the number of samples was set equal to 100. The resultant spectrums of the convertible bond price, the option only value, the delta, the gamma and the theta, for the zero-coupon convertible coupon (the default case) are presented in figures (F.8.4) to (F.8.8). The respective spectrums for the coupon-paying convertible bond, the convertible bond with a discrete dividends term structure, and the dual currency convertible bond are included in these figures as well.

Coupon Schedule

The main effect of the inclusion of a coupon schedule in the features of a convertible bond is on the level on the bond floor, as it was also pointed out earlier. It also results in reducing the absolute level of the sensitivities delta, gamma and theta, of the convertible bond.

Discrete Dividends term structure

Based on the resultant spectrums, it can be concluded that the inclusion of a discrete dividends affects the valuation of the convertible bond. This was expected since the inclusion of discrete dividends results in jumps in the share price at the nodes corresponding to the discrete dividends dates.

Dual currency

Since for the single currency convertible bond the conversion ratio was 20, setting the exchange rate equal to 0.5 and the conversion ratio for the dual currency convertible bond equal to 40, the parity value of the convertible bond when translated to the bond currency remains unchanged. However, this is true only for the calculation date since for the future dates the exchange rate does not remain fixed at 0.5 until maturity of the convertible bond. Actually, the exchange rate moves based on the interest rate differentials as it was demonstrated in figure (F.8.2) for the two yield curves employed in this thesis. This has an effect on the convertible bond pricing as it is shown in the following five figures and it should be accounted for.

Figure (F.8.4) Convertible Bond Price Spectrum

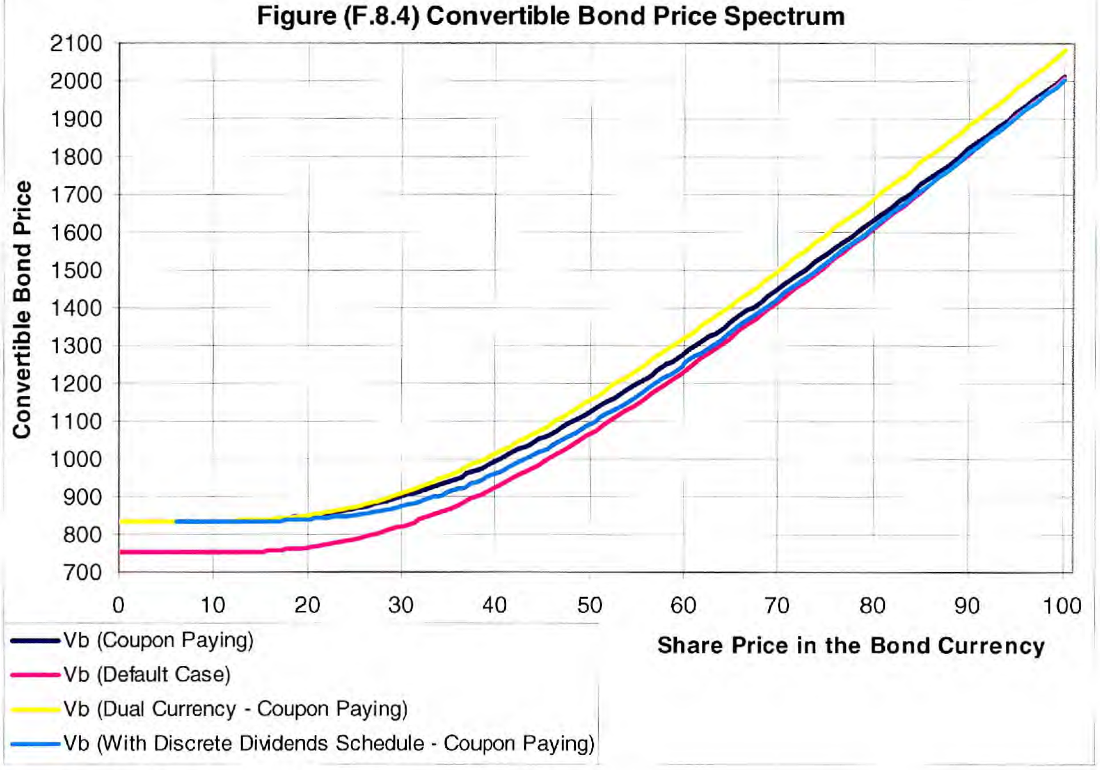


Figure (F.8.5) Option Only Value Spectrum

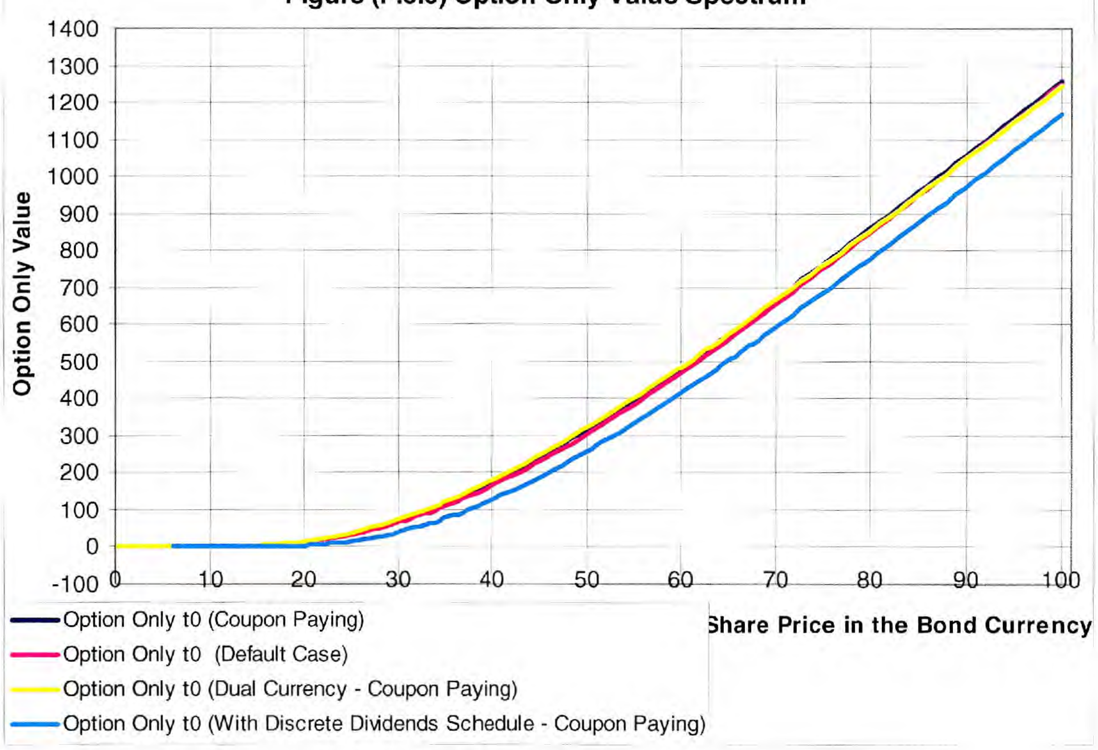


Figure (F.8.6) Delta spectrum

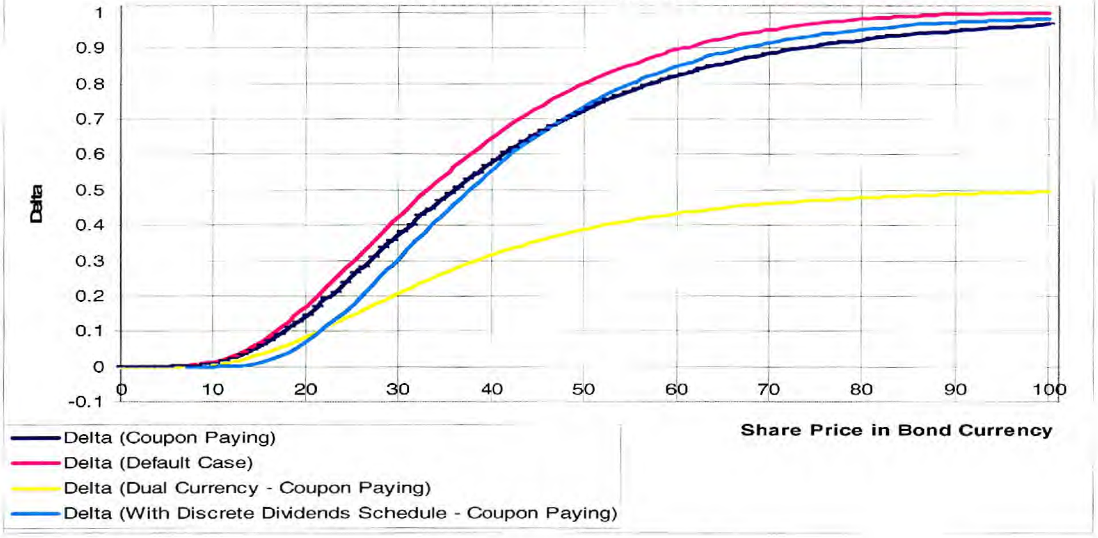


Figure (F.8.7) Gamma sprectrum

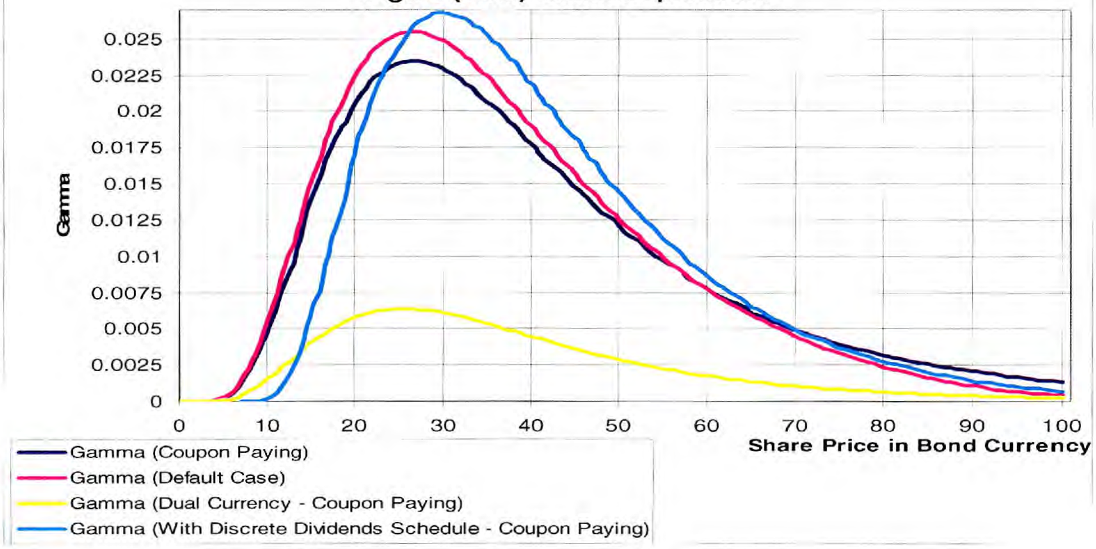
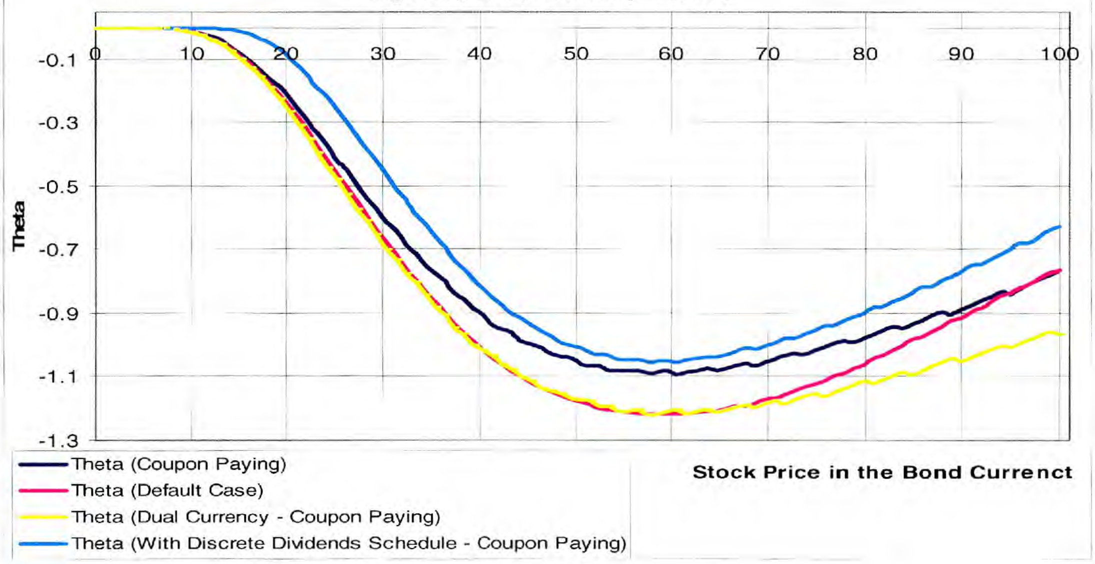


Figure (F.8.8) Theta spectrum



Call Schedule

Including a call schedule results in a negative portion of option value (positive for the issuer, negative for the investor). In the case of the single-call convertible bond, the call is a hard call at 1000 starting on the 15/01/2005 and ending on the 29/01/2008. For the case of the all hard calls convertible bond, the call schedule is used as defined previously with the difference that all calls are considered as hard calls.

Figure (F.8.9) Convertible Bond Price spectrum (Calls)

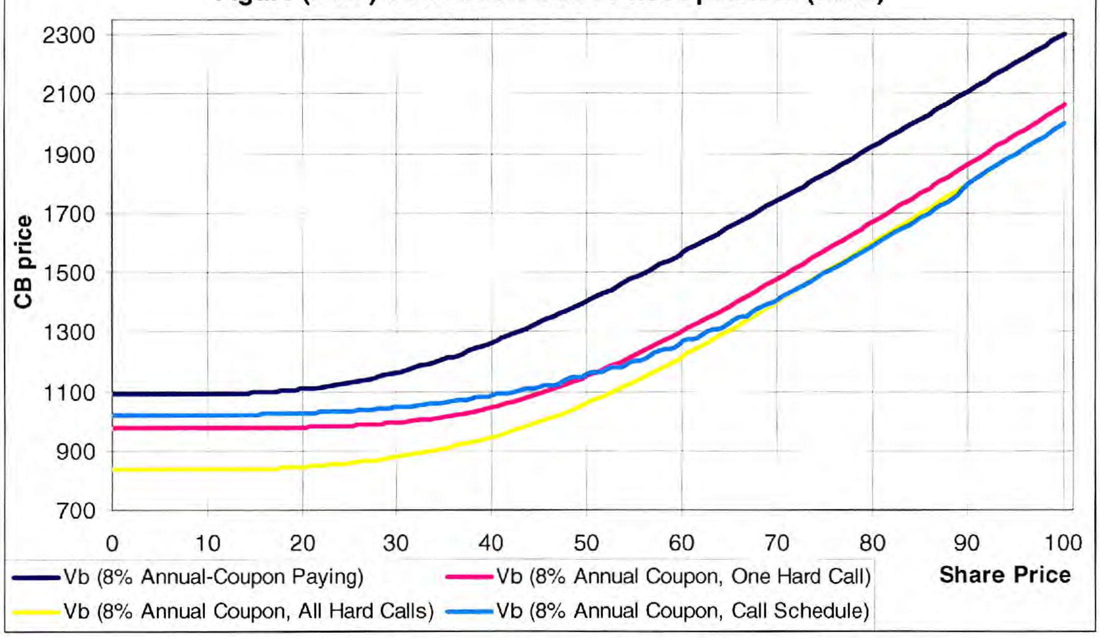
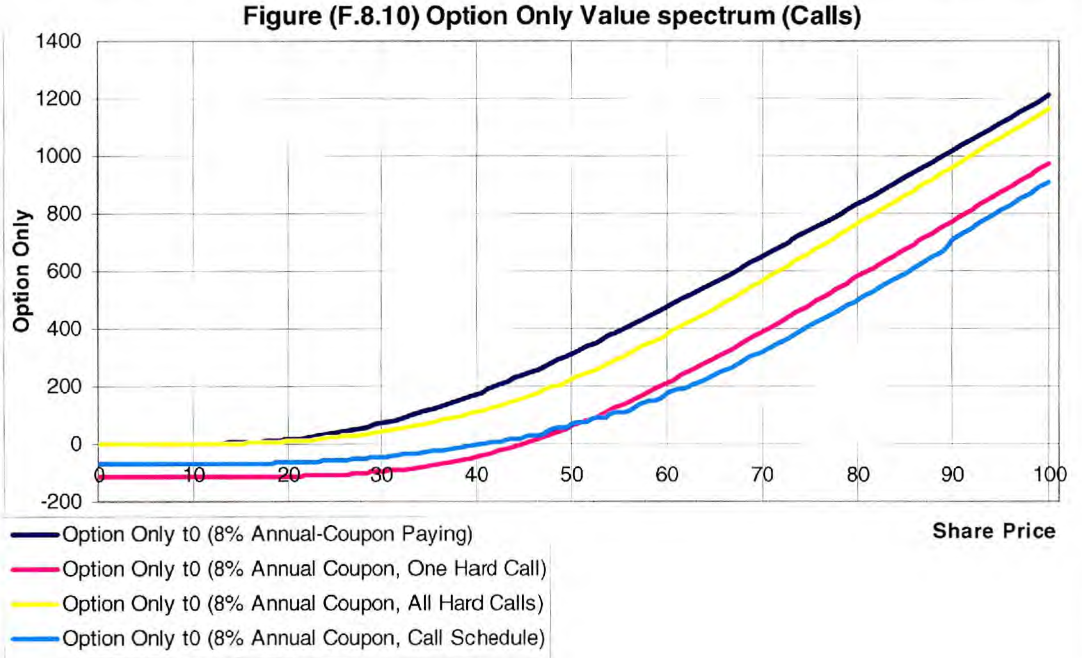


Figure (F.8.10) Option Only Value spectrum (Calls)



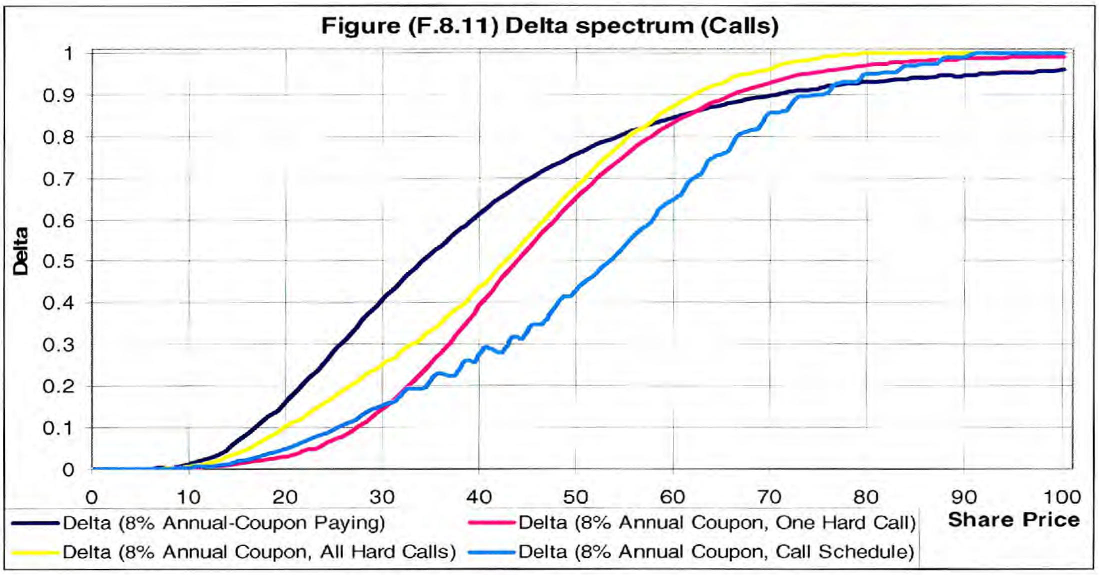


Figure (F.8.11) Delta spectrum (Calls)

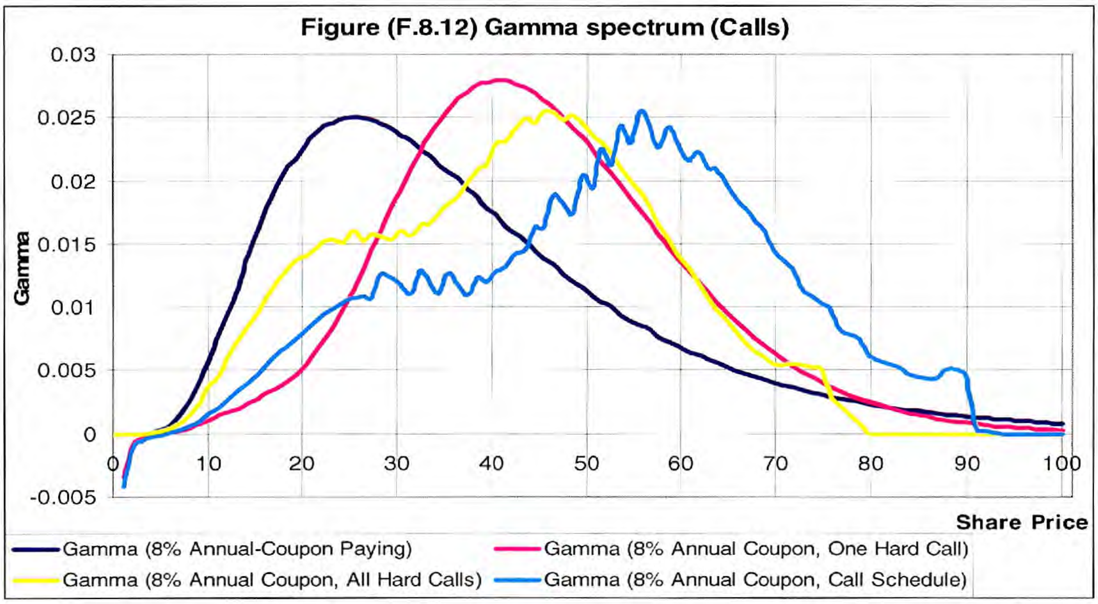


Figure (F.8.12) Gamma spectrum (Calls)

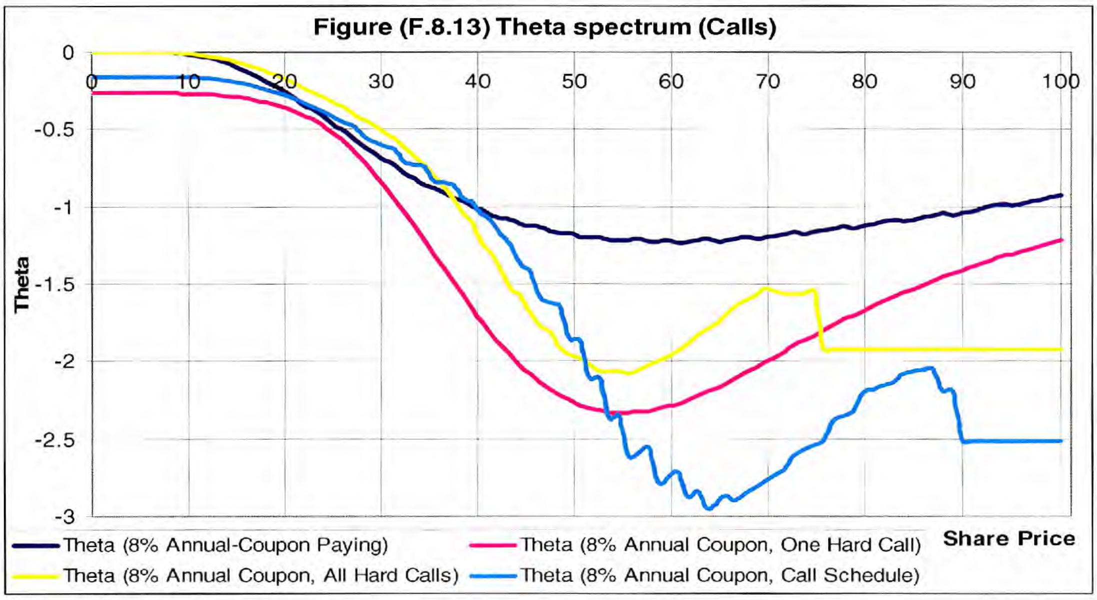


Figure (F.8.13) Theta spectrum (Calls)

Put Schedule

Including a put schedule results in an additional positive portion of option value (negative for the issuer, positive for the investor). In the case of the single-put convertible bond, the put is a hard put at 980 starting on the 15/01/2005 and ending on the 29/01/2008. For the case of the all hard puts convertible bond, the put schedule is used as defined previously with the difference that all puts are considered as hard puts.

Figure (F.8.14) Convertible Bond Price spectrum (Puts)

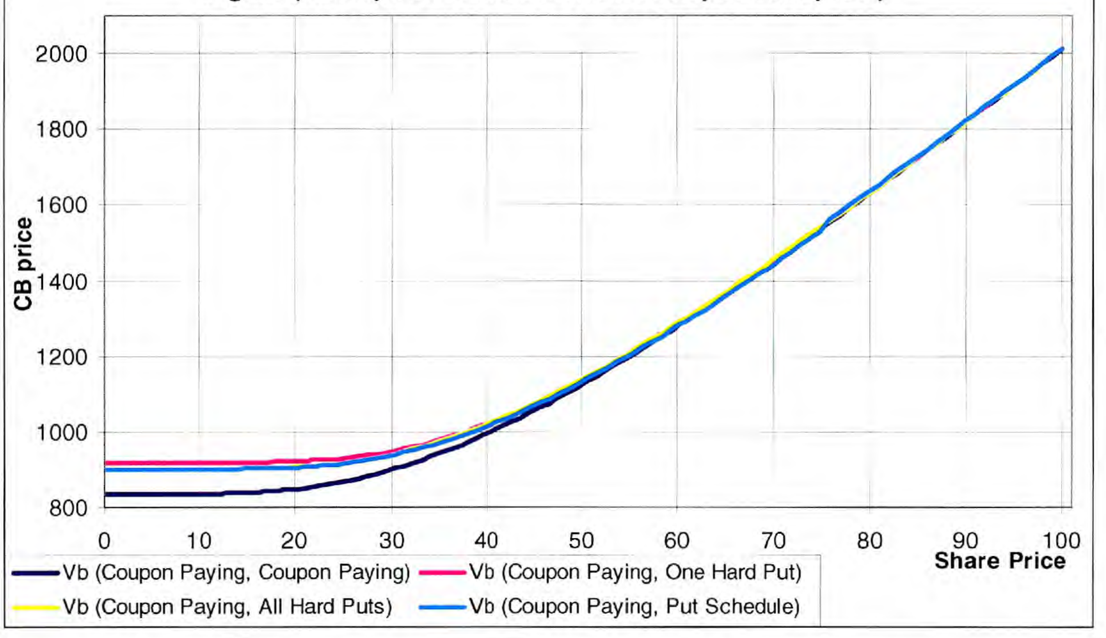
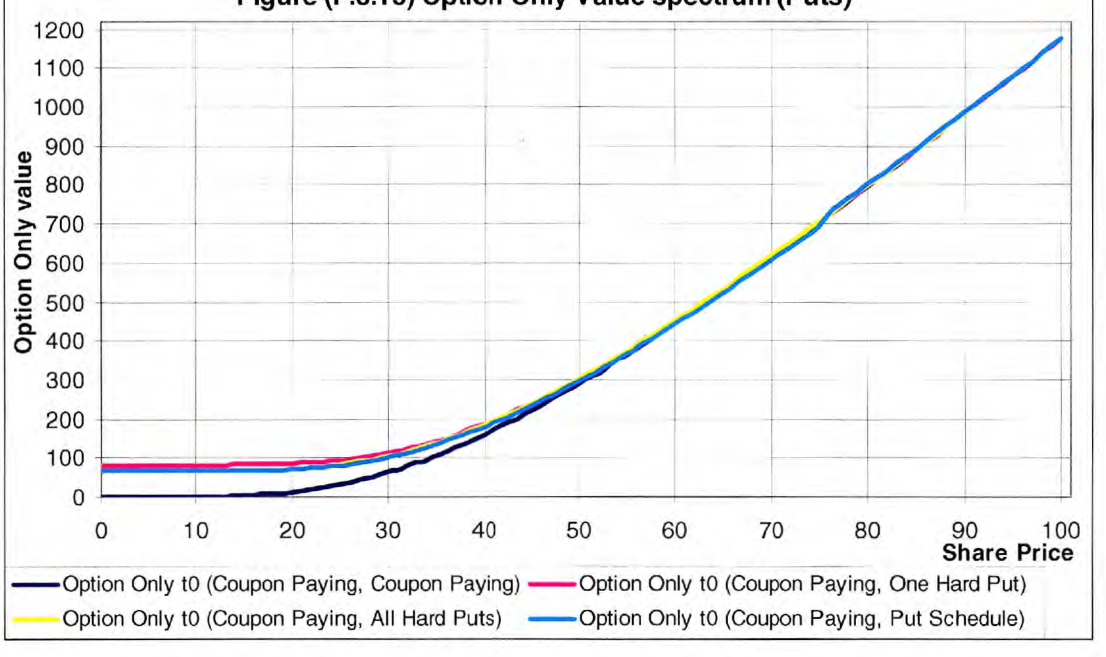


Figure (F.8.15) Option Only Value spectrum (Puts)



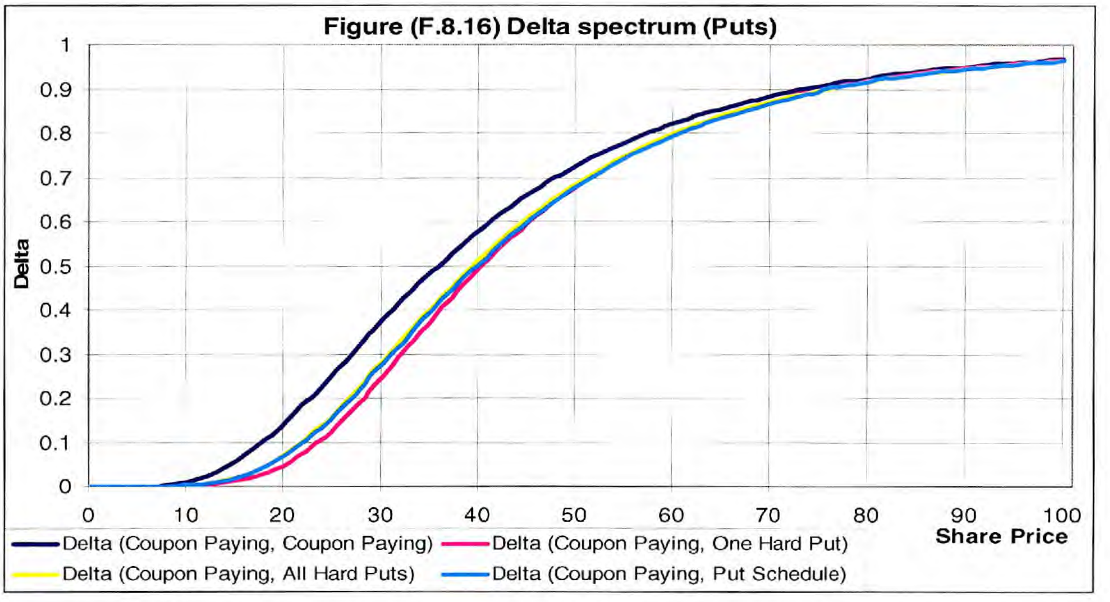


Figure (F.8.16) Delta spectrum (Puts)
Delta
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 10 20 30 40 50 60 70 80 90 100
Share Price
Delta (Coupon Paying, Coupon Paying)
Delta (Coupon Paying, One Hard Put)
Delta (Coupon Paying, All Hard Puts)
Delta (Coupon Paying, Put Schedule)

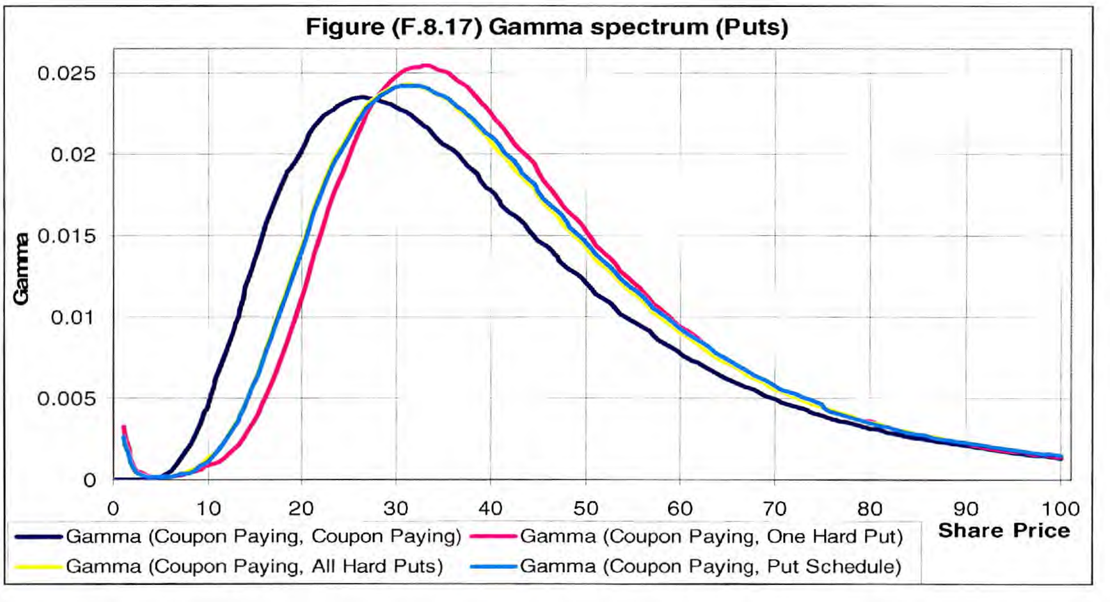


Figure (F.8.17) Gamma spectrum (Puts)
Gamma
0.025
0.02
0.015
0.01
0.005
0
0 10 20 30 40 50 60 70 80 90 100
Share Price
Gamma (Coupon Paying, Coupon Paying)
Gamma (Coupon Paying, One Hard Put)
Gamma (Coupon Paying, All Hard Puts)
Gamma (Coupon Paying, Put Schedule)

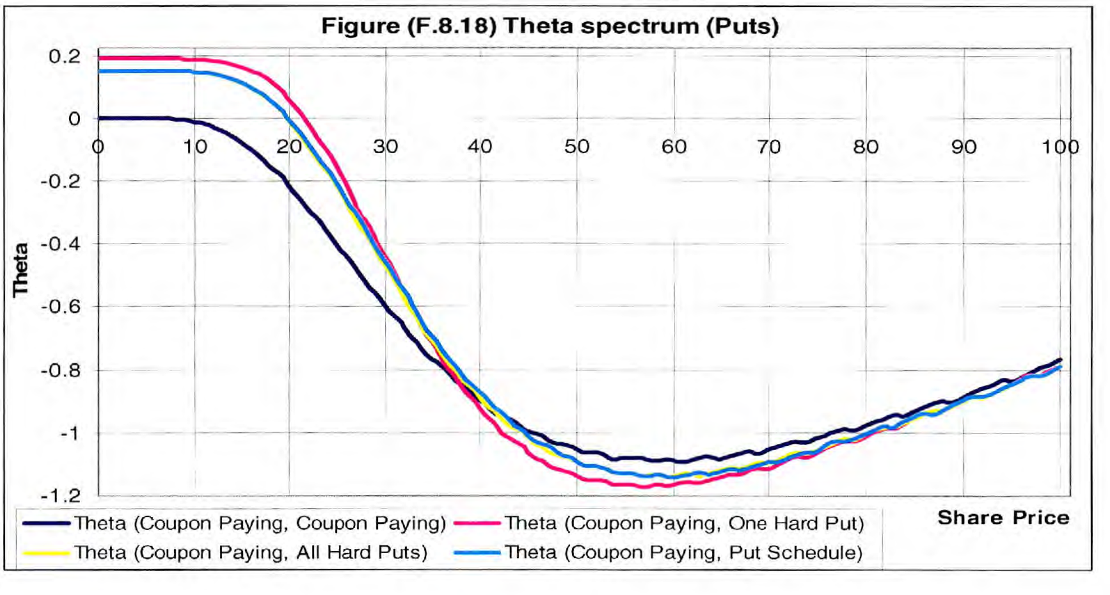
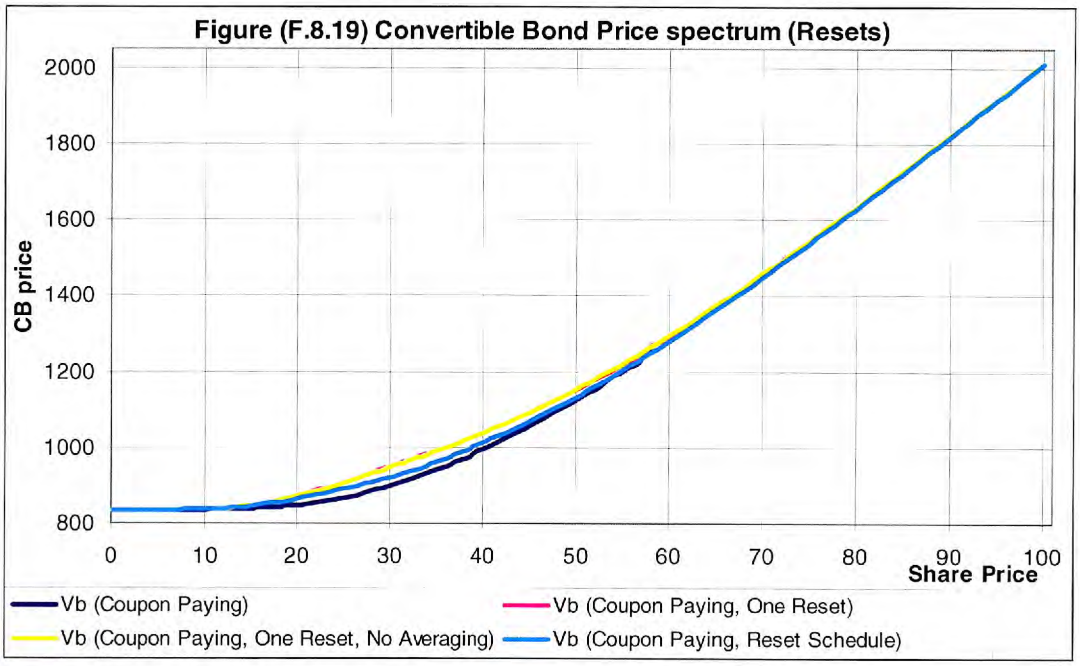
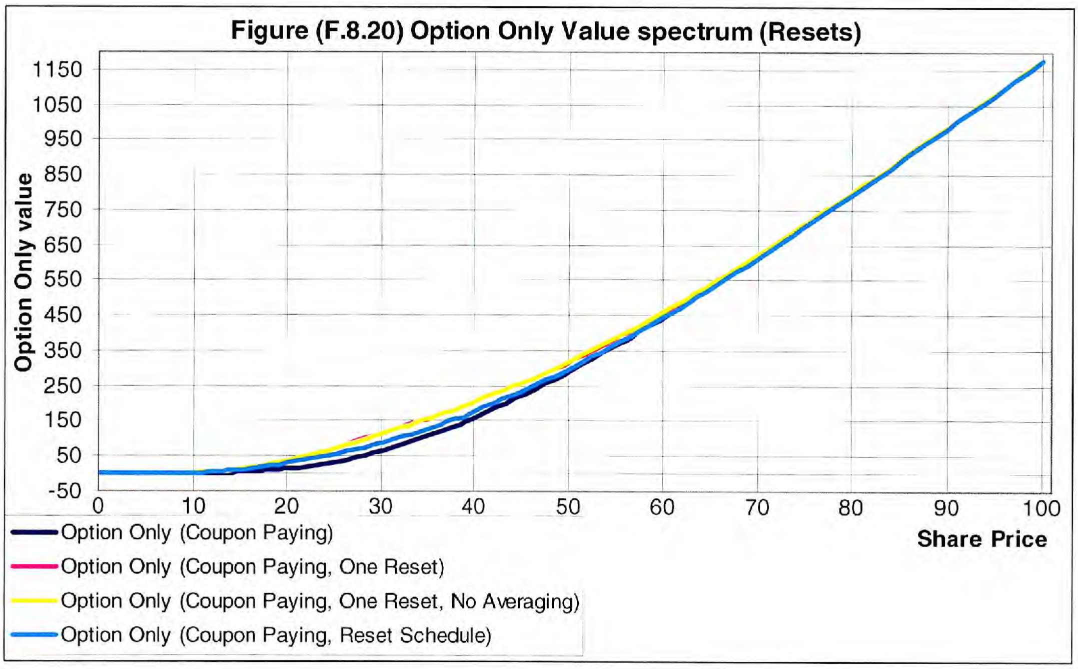


Figure (F.8.18) Theta spectrum (Puts)
Theta
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0 10 20 30 40 50 60 70 80 90 100
Share Price
Theta (Coupon Paying, Coupon Paying)
Theta (Coupon Paying, One Hard Put)
Theta (Coupon Paying, All Hard Puts)
Theta (Coupon Paying, Put Schedule)

Reset Schedule

Including a reset schedule results in an additional positive portion of option value (negative for the issuer, positive for the investor). In addition, the overall shape of the sensitivities is altered significantly. In the case of the single-reset convertible bond, the reset is a non-averaging reset on the 15/04/2005 resetting in the range 75% to 100%. For the case of the all non-averaging resets convertible bond, the reset schedule is used as defined previously with the difference that no averaging is performed for any of the resets.





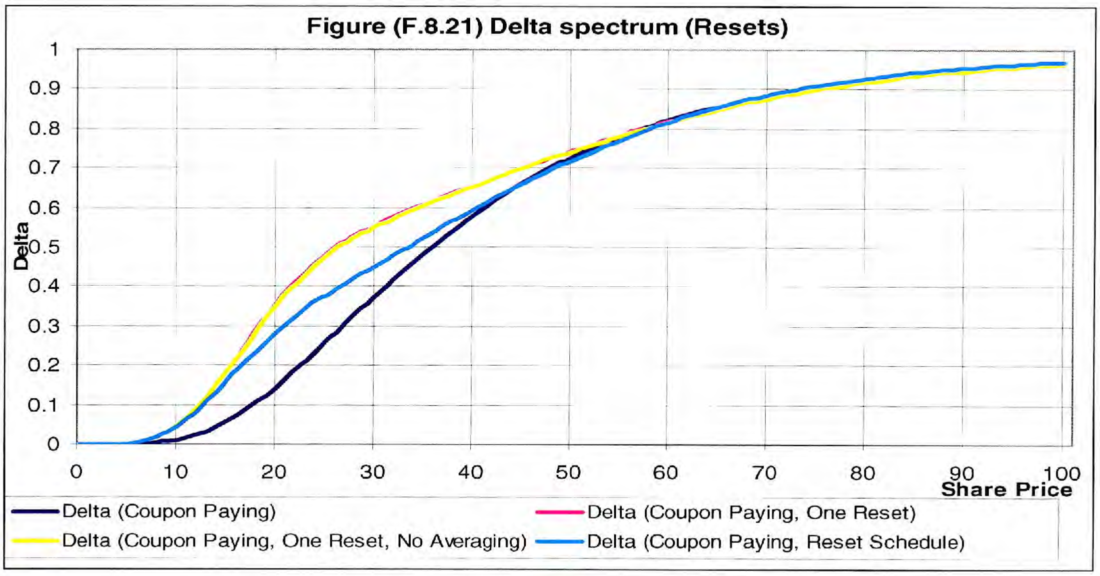


Figure (F.8.21) Delta spectrum (Resets)

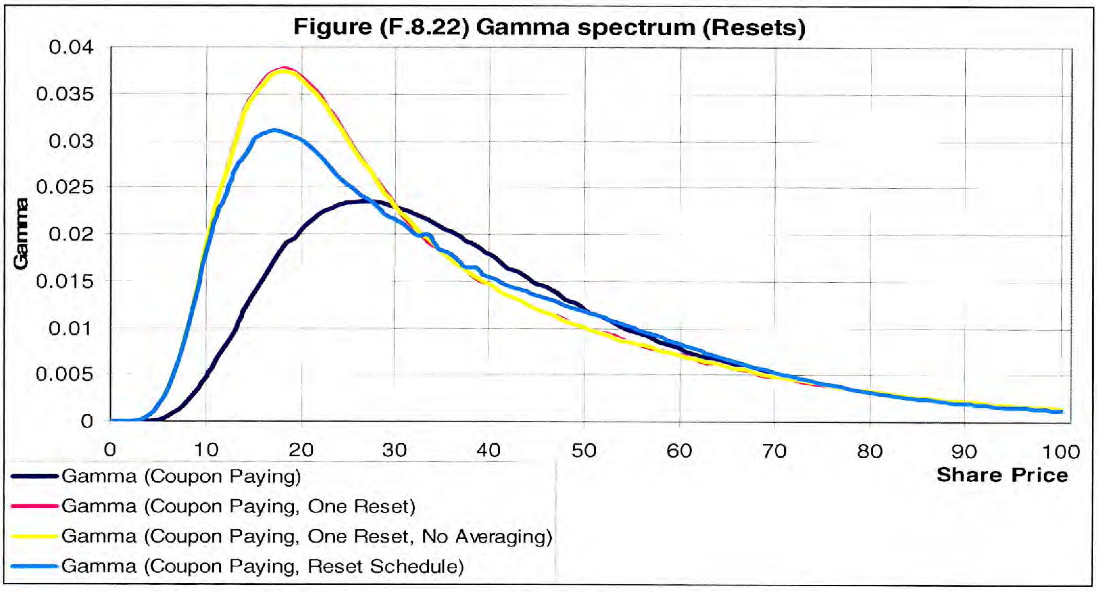


Figure (F.8.22) Gamma spectrum (Resets)

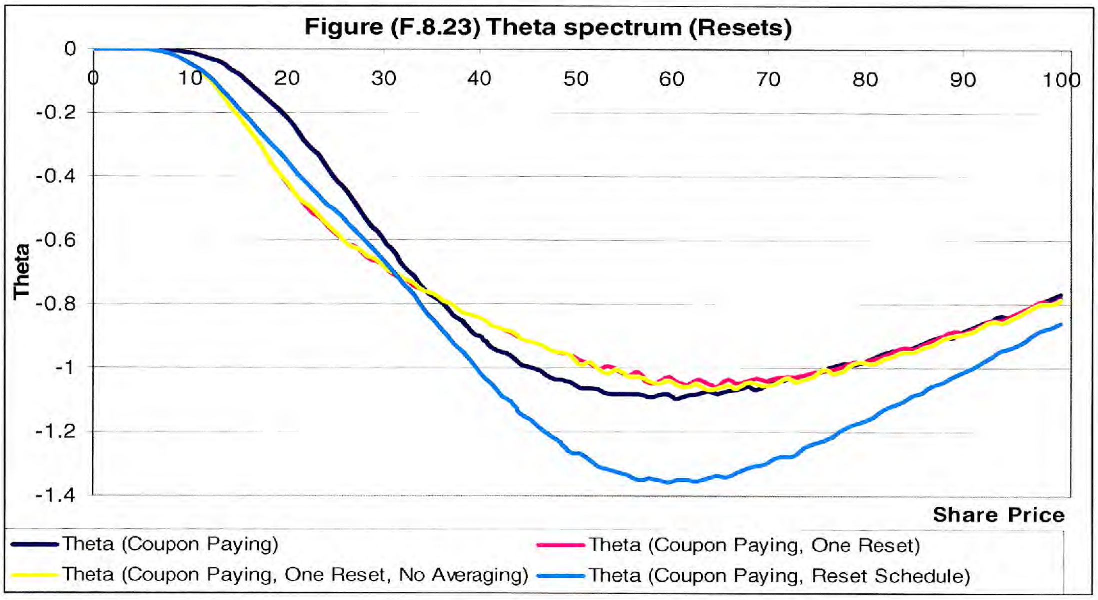


Figure (F.8.23) Theta spectrum (Resets)

All Features

The all features convertible bond includes the call schedule, the put schedule and the reset schedule as they were defined above. This is the most complicated configuration of all the convertible bonds simulated and demonstrated the precision limitations of the implemented model in calculating the sensitivities when the security is very complex.

Figure (F.8.24) Convertible Bond Price spectrum (All Features)

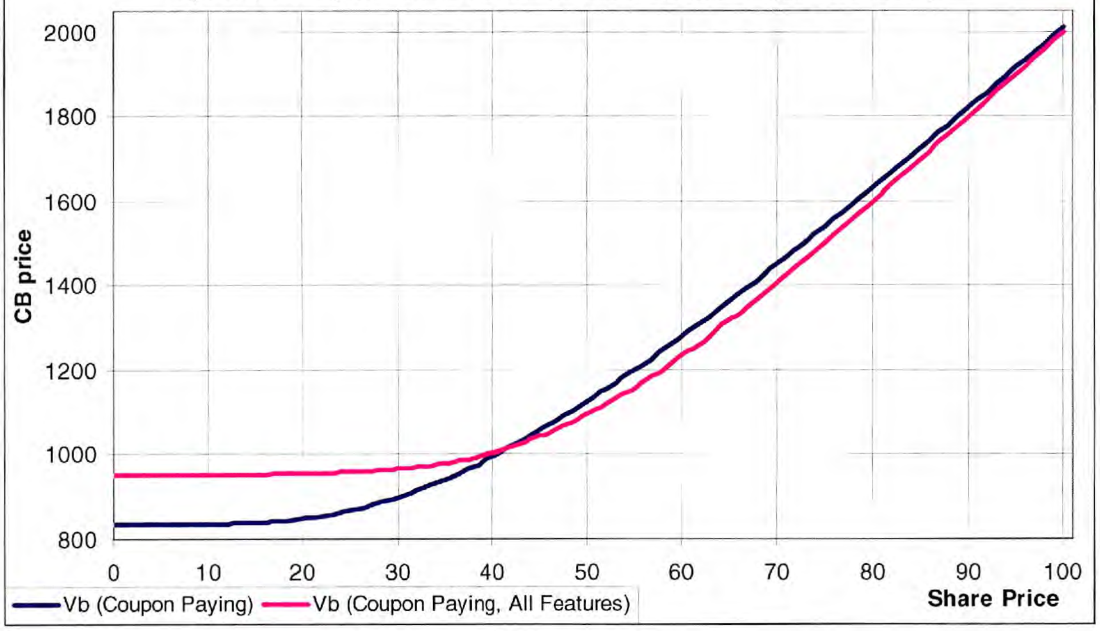
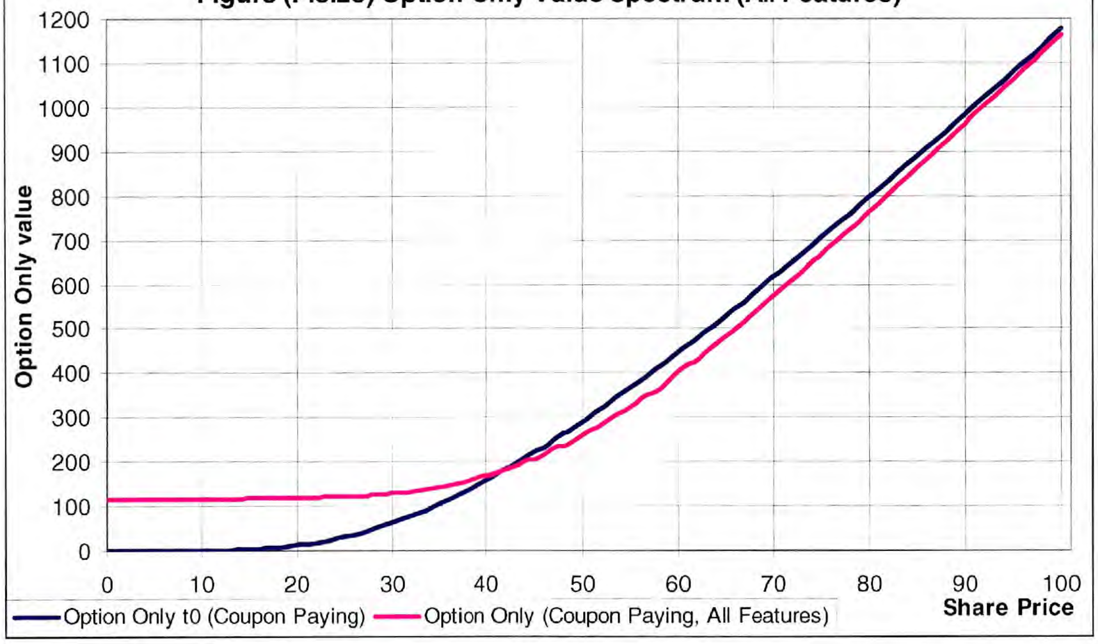


Figure (F.8.25) Option Only Value spectrum (All Features)



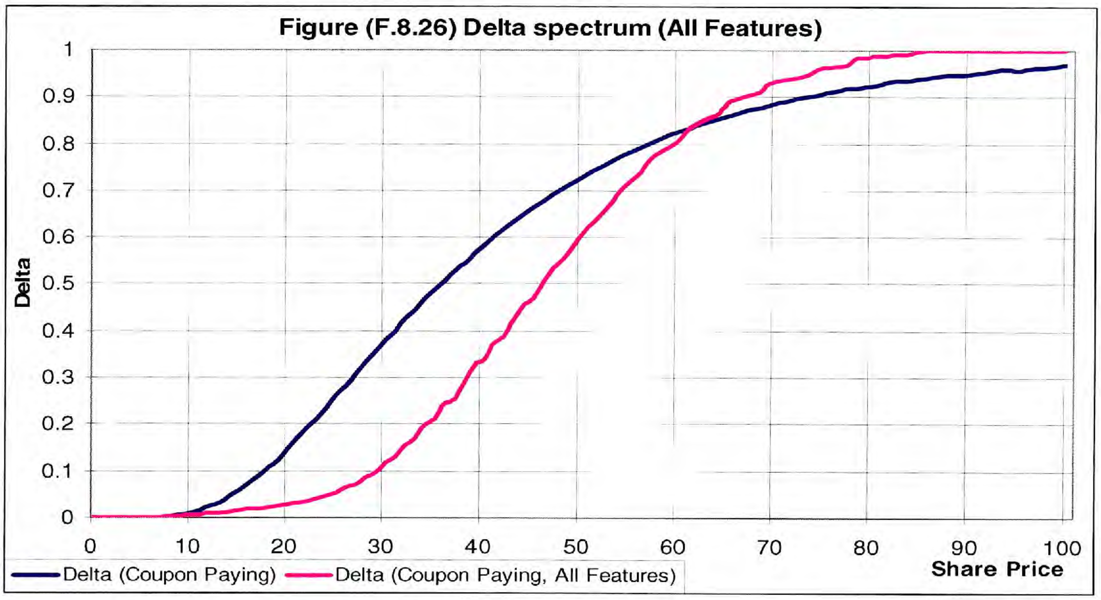


Figure (F.8.26) Delta spectrum (All Features)

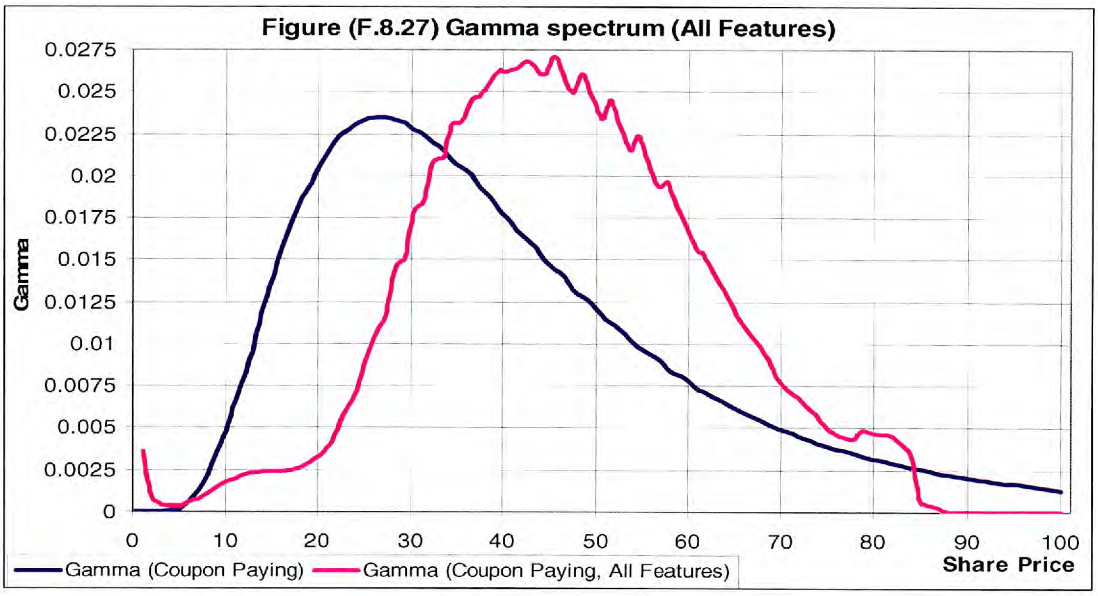


Figure (F.8.27) Gamma spectrum (All Features)

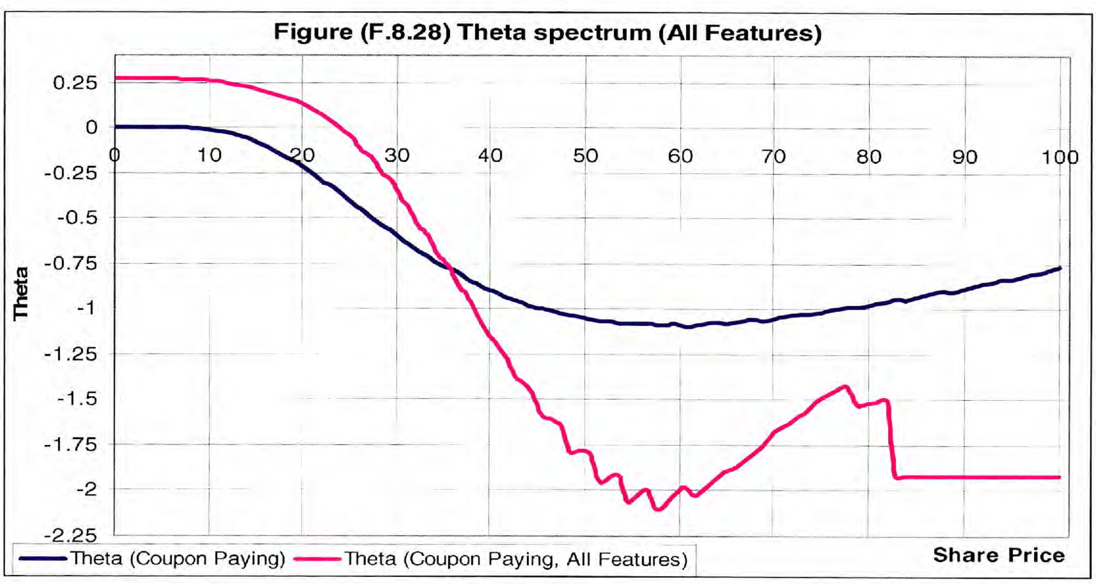


Figure (F.8.28) Theta spectrum (All Features)

Producing the CB price spectrum based on delta and gamma

Theoretically, the CB price spectrum can also be produced based on the delta and gamma sensitivities. So, instead of producing the CB price spectrum by running the pricing framework for each stock price in the spectrum as we did when producing the numbers for the previous figures, we employ the delta Δ and gamma Γ sensitivities, as well as the convertible bond price $V^{b,CD}$ corresponding to the current share price S^{CD}. We simply calculate the convertible bond price corresponding to the stock prices $\underline{S}^{CD,spectrum} = \left\{S^{CD,spectrum}_{m=0,1,2,\ldots,M-1}\right\}$ of the spectrum by using the sensitivities to calculate the change in the CB price when the current share price S^{CD} change to each stock price in the spectrum. We denote the actual convertible bond price spectrum (CB prices calculated by running the pricing framework for each stock price) as $\underline{V}^{b,spectrum} = \left\{V^{b,spectrum}_{m=0,1,2,\ldots,M-1}\right\}$, the convertible bond price spectrum produced based on the delta sensitivity as $\underline{V}^{b,delta} = \left\{V^{b,delta}_{m=0,1,2,\ldots,M-1}\right\}$, and the convertible bond price spectrum produced based on the delta and gamma sensitivities as $\underline{V}^{b,gamma} = \left\{V^{b,gamma}_{m=0,1,2,\ldots,M-1}\right\}$.

If the parameter R^{CR} denotes the prevailing conversion ratio on the calculations date, then the conversion values $\underline{V}^{conv} = \left\{V^{conv}_{m=0,1,2,\ldots,M-1}\right\}$ corresponding to each of the stock prices of the spectrum can be calculated.

$$V^{conv}_m = R^{CR} \times S^{CD,spectrum}_m \qquad m = 0,1,2,\ldots,M-1 \qquad \text{(Eq.8.1.6)}$$

The change in the conversion value $\underline{\delta V}^{conv} = \left\{\delta V^{conv}_{m=0,1,2,\ldots,M-1}\right\}$ with reference to the current share price can then be calculated as shown in the next equation.

$$\delta V^{conv}_m = V^{conv}_m - \left(R^{CR} \times S^{CD}\right) \qquad m = 0,1,2,\ldots,M-1 \qquad \text{(Eq.8.1.7)}$$

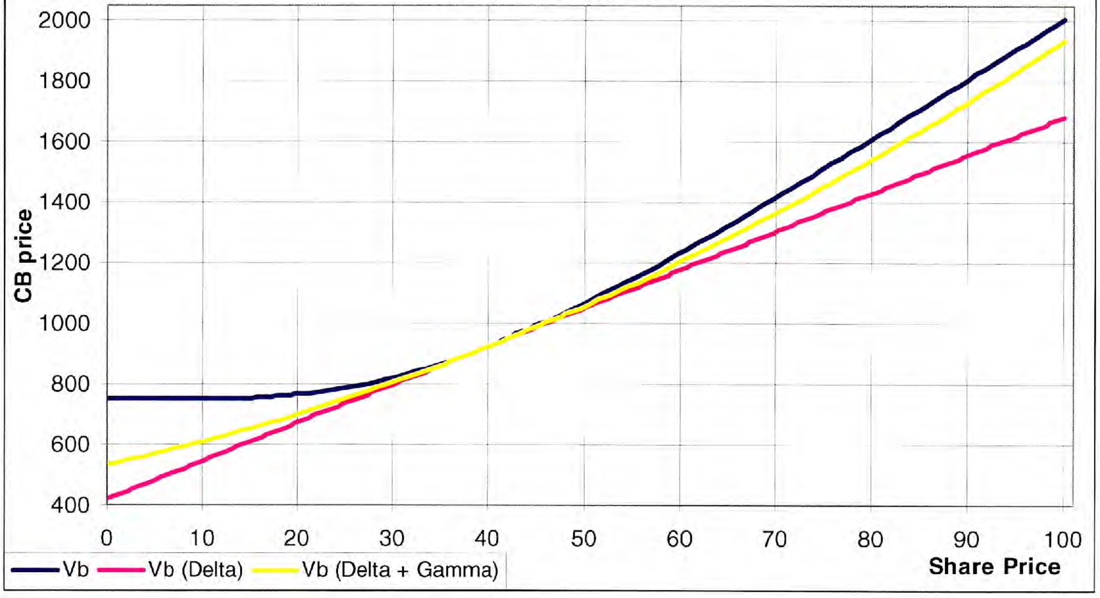
Based on the change in the conversion value for each point on the spectrum, the spectrum can now be produced based on the delta and gamma sensitivities.

$$V^{b,delta}_m = V^{b,CD} + \left(\Delta \times \delta V^{conv}_m\right) \qquad m = 0,1,2,\ldots,M-1 \qquad \text{(Eq.8.1.8)}$$

$$V^{b,delta}_m = V^{b,CD} + \left(\Delta \times \delta V^{conv}_m\right) + \left(\frac{1}{2} \times \left(\Gamma \times \delta V^{conv}_m\right)^2\right) \qquad m = 0,1,2,\ldots,M-1 \qquad \text{(Eq.8.1.9)}$$

The resultant numbers are presented graphically in the next figure. As it can be observed, if both sensitivities are employed, then the produced spectrum is closer to the actual spectrum. This also demonstrates that the pricing framework performs successfully in calculating the sensitivities of the convertible bond. Hence, this sensitivities output from the calculations can be successfully be used for hedging positions in the convertible bond priced. The performance of the pricing framework with respect to the calculation of the delta and gamma sensitivities is further studied in the following section.

Figure (F.8.29) Producing the convertible bond price spectrum based on the sensitivities



8.2. Scenario Analysis and Simulation

The resultant pricing framework was developed based on methods assuming continuous time parameters and conditions. A very strong and significant assumption was that the investing decisions, as well as the arbitrage-free conditions employed in the stochastic process of the stock, are part of a continuous time world. To be more precise, the initial derived equations governing the behaviour of the developed model are consisting of continuous time parameters, and then these equations and parameters are transformed to discrete form in order to carry out the numerical method – trinomial tree – for the pricing of the convertible method. This discretisation process mainly consists of the assumption that the continuous time parameter dt can be replaced by a very small time step Δt, and as Δt approaches zero, the calculations become more precise since the discrete form approaches better the continuous form of the model.

As a result of the continuous form of the model, the stochastic process of the underlying stock is based, among other assumptions, on the assumption that the replicating portfolio employed for calculating the transition probabilities is continuously re-balanced as a re-action to changes in the share price. This assumption is of course extended to the discrete form of the model, where it is assumed that the replicating portfolio employed for the stock process is re-balanced at each step. The purpose of the scenario analysis carried out in this section is to study the effect of this assumption in the cases where actual re-balancing is taken less frequently than assumed, which is also the case in the real-world. The effect of this assumption is quantified by calculating the "re-hedging" error at maturity when performing Monte Carlo simulation with discrete re-hedging at each step of the Monte Carlo paths.

The Monte Carlo employed in the calculations corresponding to both of the following sub-sections, involves simulating the stock price evolution from the calculation date to the expiration date of the convertible bond. In order to enhance the performance of the simulations, the Monte Carlo module was implemented based on the antithetic technique approach.

Theoretically – meaning based on the assumptions of the theoretical replicating portfolio governing the behaviour of the stock process - re-hedging takes place without affecting the bank account. This is the approach followed in the first series of simulations and presented in sub-section (8.2.1), while in sub-section (8.2.2) we consider the case where bank account can be altered in order to carry out perfect re-hedging at each re-hedging point, assuming that in this way the re-hedging error will be reduced.

We consider four different versions of the theoretical portfolio and denote these versions as TP1, TP2, TP3 and TP4, and four versions of the active trading approach and denote these as AT1, AT2, AT3 and AT4. Each version is explained as follows:

- TP1/AT1: For these versions we assume that the convertible bonds position is hedged with a position in stocks (underlying of the convertible bond) and that convertible bonds are held to maturity.
- TP2/AT2: For these versions we assume that the convertible bonds position is hedged with a position in stocks and that convertible bonds may be converted at any of the re-hedging points if that is considered profitable. This will result in unwinding the portfolio and all positions are translated to cash.
- TP3/AT3: For these versions we assume that the convertible bonds position is hedged with a position in stocks and an options position (same underlying), and that convertible bonds are held to maturity.
- TP4/AT4: For these versions we assume that the convertible bonds position is hedged with a position in stocks and an options position (same underlying), and that convertible bonds may be converted at any of the re-hedging points if that is considered profitable. This will result in unwinding the portfolio and all positions are translated to cash.

For all the versions, it is assumed that the position in convertible bonds on the calculation date includes a number of convertible securities equal to $n_{CBs} > 0$. First, the price of the convertible bond $V^{b,CD}$, and the delta $\Delta^{b,CD}$ and gamma $\Gamma^{b,CD}$ sensitivities corresponding to the current share price S^{CD} are calculated based on the pricing framework at hand. The price of the option $V^{w,CD}$, and the delta $\Delta^{w,CD}$ and gamma $\Gamma^{w,CD}$ sensitivities corresponding to the current share price S^{CD} are also calculated based on the same framework. It is assumed that the option is on a single share, hence the conversion ratio of the option in equal to one.

For versions TP1, TP2, AT1 and AT2, the number of shares n_{shares} for hedging the convertible bond positions on the calculations date is defined based on the delta sensitivity of the convertible bond. Then, the cash account $V^{cash,CD}$ on the calculations date is defined as the sum of the values of the convertible bond position and the equity position.

$$n_{shares} = -\Delta^{b,CD} \times R^{CR} \times n_{CBs} \qquad \text{(Eq.8.2.1)}$$

$$V^{cash,CD} = \left(n_{CBs} \times V^{b,CD}\right) + \left(n_{shares} \times S^{CD}\right) \qquad \text{(Eq.8.2.2)}$$

For versions TP3, TP4, AT3 and AT4, the number of options $n_{options}$ for gamma hedging the convertible bond positions on the calculations date is defined based on the gamma sensitivity of the convertible bond and the option. Then, the number of shares n_{shares} for hedging the convertible bond positions and the options positions on the calculations date is defined based on the delta sensitivity of the convertible bond and the option. Then, the cash account $V^{cash,CD}$ on the calculations date is defined as the sum of the values of the convertible bond position, the options position and the equity position.

$$n_{options} = -\frac{\Gamma^{b,CD} \times R^{CR} \times n_{CBs}}{\Gamma^{w,CD}} \qquad \text{(Eq.8.2.3)}$$

$$n_{shares} = -\left(\Delta^{b,CD} \times R^{CR} \times n_{CBs}\right) - \left(\Delta^{w,CD} \times n_{options}\right) \qquad \text{(Eq.8.2.4)}$$

$$V^{cash,CD} = \left(n_{CBs} \times V^{b,CD}\right) + \left(n_{options} \times V^{w,CD}\right) + \left(n_{shares} \times S^{CD}\right) \qquad \text{(Eq.8.2.5)}$$

If the number of re-hedging points is denoted as $N_{Hedging}$, then the dates and corresponding times of the re-hedging points are denoted as $\underline{t}^{a,Hedging} = \left\{t^{a,Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$ and $\underline{t}^{Hedging} = \left\{t^{Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$ respectively. All parameters related to the re-hedging points with index equal to zero, $i = 0$, correspond to the calculations date parameters. In other words, $t^{a,Hedging}_{i=0} = t^{a,CD}$. The re-hedging points are set in such a way that the time distances $\underline{dt}^{Hedging} = \left\{dt^{Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$ between the re-hedging points are roughly equal.

$$t_i^{Hedging} = \frac{t_i^{a,Hedging}}{NumDaysPerYear} \qquad i = 0,1,2,\ldots,N_{Hedging} \qquad \text{(Eq.8.2.6)}$$

$$\begin{aligned} dt_{i=0}^{Hedging} &= t_{i=0}^{Hedging} - t^{CD} = 0 \\ dt_i^{Hedging} &= t_i^{Hedging} - t_{i-1}^{Hedging} \qquad i = 1,2,\ldots,N_{Hedging} \end{aligned} \qquad \text{(Eq.8.2.7)}$$

If the number of simulation paths is equal to M, then for each simulation path, the share prices $\underline{S}^{Hedging}_{m=0,1,2,\ldots,M-1} = \left\{S^{Hedging}_{m,i=0,1,2,\ldots,N_{Hedging}}\right\}$ is established for each re-hedging point based on the diffusion process. Of course, it holds that $S^{Hedging}_{m=\{0,1,2,\ldots,M-1\},i=0} = S^{CD}$. The diffusion process employs the forward volatilities (established based on the volatility term structure), as well as the forward rates, which are denoted as $\underline{fr}^{Hedging} = \left\{fr^{Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$. Based on the later, the forward discount factors $\underline{fdf}^{Hedging} = \left\{fdf^{Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$, as well as the forward compounding factors $\underline{fcf}^{Hedging} = \left\{fcf^{Hedging}_{i=0,1,2,\ldots,N_{Hedging}}\right\}$, are calculated.

$$fdf_i^{Hedging} = e^{-r\times dt_i^{Hedging}} \qquad i = 0,1,2,\ldots,N_{Hedging} \qquad \text{(Eq.8.2.8)}$$

$$fcf_i^{Hedging} = e^{r\times dt_i^{Hedging}} \qquad i = 0,1,2,\ldots,N_{Hedging} \qquad \text{(Eq.8.2.9)}$$

For each realisation of the share price $\underline{S}^{Hedging}_{m=0,1,2,\ldots,M-1} = \left\{S^{Hedging}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, the following corresponding quantities are obtained by employing the convertible bond pricing framework: The CB prices $\underline{V}^{b}_{m=0,1,2,\ldots,M-1} = \left\{V^{b}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, the convertible bond delta sensitivities $\underline{\Delta}^{b}_{m=0,1,2,\ldots,M-1} = \left\{\Delta^{b}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, the convertible bond gamma sensitivities $\underline{\Gamma}^{b}_{m=0,1,2,\ldots,M-1} = \left\{\Gamma^{b}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, the option prices $\underline{V}^{w}_{m=0,1,2,\ldots,M-1} = \left\{V^{w}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, the option delta sensitivities $\underline{\Delta}^{w}_{m=0,1,2,\ldots,M-1} = \left\{\Delta^{w}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$, and the option gamma sensitivities $\underline{\Gamma}^{w}_{m=0,1,2,\ldots,M-1} = \left\{\Gamma^{w}_{m,i=1,2,\ldots,N_{Hedging}}\right\}$.

Based on the above values, re-hedging takes place at each re-hedging point of each path and the resultant quantities are the following: The number of shares $\underline{n}^{shares}_{m=0,1,2,...,M-1} = \left\{n^{shares}_{m,i=1,2,...,N_{Hedging}}\right\}$, the number of CBs $\underline{n}^{CBs}_{m=0,1,2,...,M-1} = \left\{n^{CBs}_{m,i=1,2,...,N_{Hedging}}\right\}$, the number of options $\underline{n}^{options}_{m=0,1,2,...,M-1} = \left\{n^{options}_{m,i=1,2,...,N_{Hedging}}\right\}$ and the cash account $\underline{V}^{cash}_{m=0,1,2,...,M-1} = \left\{V^{cash}_{m,i=1,2,...,N_{Hedging}}\right\}$.

Details on the re-hedging methods for each version are further presented in the proceeding sub-sections. However, for versions TP2, AT2, TP4 and AT4, conversion is allowed at the re-hedging points of the simulation paths and the approach is the same in the cases where unwind takes place. If the pricing framework returns a flag which denotes that is optimal to convert at a re-hedging point, then, the positions are unwind and everything is translated to cash as shown in the following equation. This means that at the proceeding points only simple cash compounding will be carried out. For versions TP2 and AT2 the number of options is always equal to zero.

$$V^{cash}_{m,i} = \left(fcf^{Hedging}_{i} \times V^{cash}_{m,i-1}\right) + \left(n^{CBs}_{m,i-1} \times V^{b}_{m,i}\right) + \left(n^{shares}_{m,i-1} \times S^{Hedging}_{m,i}\right) + \left(n^{options}_{m,i-1} \times V^{w}_{m,i}\right) \quad \text{(Eq.8.2.10)}$$

On maturity date, all the positions are translated to cash. The share prices on maturity date are denoted as $\underline{S}^{ED} = \left\{S^{ED}_{m=0,1,2,...,M-1}\right\}$, the respective options prices are denoted as $\underline{V}^{w,ED} = \left\{V^{w,ED}_{m=0,1,2,...,M-1}\right\}$, and the redemption value of the convertible bond is denoted as P^{Rd}. The final cash account values $\underline{V}^{cash,ED} = \left\{V^{cash,ED}_{m=0,1,2,...,M-1}\right\}$ on the maturity date are calculated as follows:

$$\left.\begin{aligned} V^{cash,ED}_{m} &= \left(e^{fr_{Mat} \times \left(t^{ED} - t^{Hedging}_{N_{Hedging}}\right)} \times V^{cash}_{m,N_{Hedging}}\right) + \left(n^{CBs}_{m,N_{Hedging}} \times P^{Rd}\right) \\ &+ \left(n^{options}_{m,N_{Hedging}} \times V^{w,ED}_{m}\right) + \left(n^{shares}_{m,N_{Hedging}} \times S^{ED}_{m}\right) \end{aligned}\right\} \quad m = \{0,1,2,...,M-1\}$$

(Eq.8.2.11)

The above equation holds in this exact form for the versions TP3 and AT3. In the case of the versions TP1, AT1, TP2 and AT2 no positions in options are included, hence the parameter $n^{options}_{m,N_{Hedging}}$ is zero. In addition, for versions TP2, AT2, TP4 and AT4, for some paths conversion may have taken place earlier at any of the re-hedging points, hence, on maturity, only simple compounding of the cash positions is involved.

8.2.1. Theoretical Portfolio Approach

For version TP1, re-hedging involves calculating the change in the value of the convertible bond position and then investing this amount in the shares position. In other words, the convertible bond position value is kept constant and unchanged and the number of CBs is re-calculated. Then, any profit (loss) is invested in (withdraw from) the shares position. For the cash account, simple compounding takes place. As a reminder, parameters with reference $i = 0$ correspond to the calculations date and are the same across all the simulations paths $m = 0,1,2,\ldots,M-1$. The same equations hold for version TP2, with the difference that unwinding can take place at the re-hedging points if it is optimal to do so.

$$n_{m,i}^{CBs} = \frac{n_{m,i=0}^{CBs} \times V_{m,i=0}^{b}}{V_{m,i}^{b}} \qquad m = \{0,1,2,\ldots,M-1\}, i = \{1,2,\ldots,N_{Hedging}\} \qquad \text{(Eq.8.2.12)}$$

$$n_{m,i}^{shares} = \frac{\left(n_{m,i-1}^{shares} \times S_{m,i}^{Hedging}\right) + \left[\left(n_{m,i}^{CBs} - n_{m,i-1}^{CBs}\right) \times V_{m,i}^{b}\right]}{S_{m,i}^{Hedging}} \qquad m = \{0,1,2,\ldots,M-1\}, i = \{1,2,\ldots,N_{Hedging}\}$$

(Eq.8.2.13)

$$V_{m,i}^{cash} = fcf_i^{Hedging} \times V_{m,i-1}^{cash} \qquad m = \{0,1,2,\ldots,M-1\}, i = \{1,2,\ldots,N_{Hedging}\} \qquad \text{(Eq.8.2.14)}$$

In the case of version TP3, the first step at the re-hedging points involves calculating the number of CBs based on the new convertible bond price and keeping constant the convertible bond position value. Then, the number of options required to delta hedge the convertible bond is calculated. Finally, the combined change in value of the convertible bond position and the warrant position is invested in (withdraw from) the shares position. For the cash account, simple compounding takes place. The same equations hold for version TP4, with the difference that unwinding can take place at the re-hedging points if it is optimal to do so.

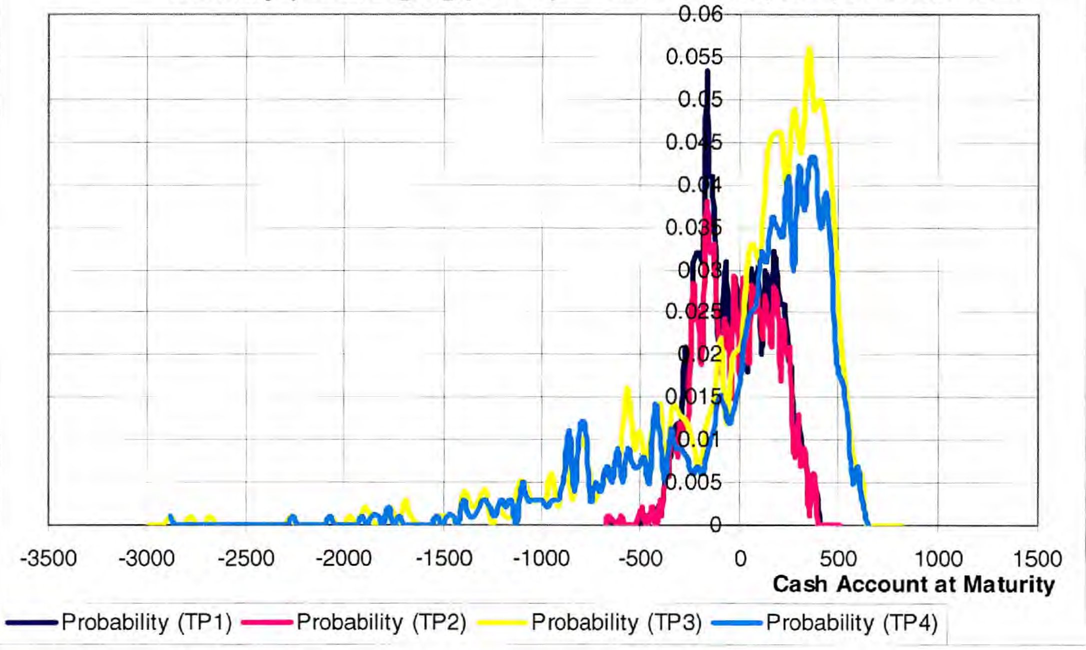
$$n_{m,i}^{CBs} = \frac{n_{m,i=0}^{CBs} \times V_{m,i=0}^{b}}{V_{m,i}^{b}} \qquad m = \{0,1,2,\ldots,M-1\}, i = \{1,2,\ldots,N_{Hedging}\} \qquad \text{(Eq.8.2.15)}$$

$$n_{m,i}^{options} = -\frac{\Delta^{b,CD} \times R^{CR} \times n_{CBs}}{\Delta^{w,CD}} \qquad m = \{0,1,2,...,M-1\}, i = \{1,2,...,N_{Hedging}\} \quad \text{(Eq.8.2.16)}$$

$$n_{m,i}^{shares} = \frac{\left(n_{m,i-1}^{shares} \times S_{m,i}^{Hedging}\right) + \left[\left(n_{m,i}^{CBs} - n_{m,i-1}^{CBs}\right) \times V_{m,i}^{b}\right] + \left[\left(n_{m,i}^{options} - n_{m,i-1}^{options}\right) \times V_{m,i}^{w}\right]}{S_{m,i}^{Hedging}} \quad \text{(Eq.8.2.17)}$$

$$V_{m,i}^{cash} = fcf_{i}^{Hedging} \times V_{m,i-1}^{cash} \qquad m = \{0,1,2,...,M-1\}, i = \{1,2,...,N_{Hedging}\} \quad \text{(Eq.8.2.18)}$$

Figure (F.8.30) Probability Distribution of the cash account at maturity (5 rehedging points) - Versions TP1, TP2, TP3 and TP4



8.2.2. Active Trading Approach

For version AT1, re-hedging is performed exactly in the same way that the initial hedging positions were set up. The number of convertible bonds is kept constant. The number of shares is defined based on the delta sensitivity of the convertible bond at each point. The cash needed to be invested in (withdraw from) the shares position is taken from the cash account (compounding is also carried out for the cash account). The same equations hold for version AT2, with the difference that unwinding can take place at the re-hedging points if it is optimal to do so.

$$n_{m,i}^{shares} = -\Delta_{m,i}^{b} \times R^{CR} \times n_{CBs} \quad \text{(Eq.8.2.19)}$$

$$V_{m,i}^{cash} = \left(fcf_i^{Hedging} \times V_{m,i-1}^{cash}\right) + \left[\left(n_{m,i}^{shares} - n_{m,i-1}^{shares}\right) \times S_{m,i}^{Hedging}\right] \qquad \text{(Eq.8.2.20)}$$

In the case of version AT3, the number of convertible bonds is also kept constant. Then, the number of options $n_{m,i}^{options}$ for gamma hedging the convertible bond positions on the calculations date is defined based on the gamma sensitivity of the convertible bond and the option. Then, the number of shares $n_{m,i}^{shares}$ for hedging the convertible bond positions and the options positions on the calculations date is defined based on the delta sensitivity of the convertible bond and the option. Finally, the cash account is adjusted based on the change in the value of the warrants and shares position. The same equations hold for version AT4, with the difference that unwinding can take place at the re-hedging points if it is optimal to do so.

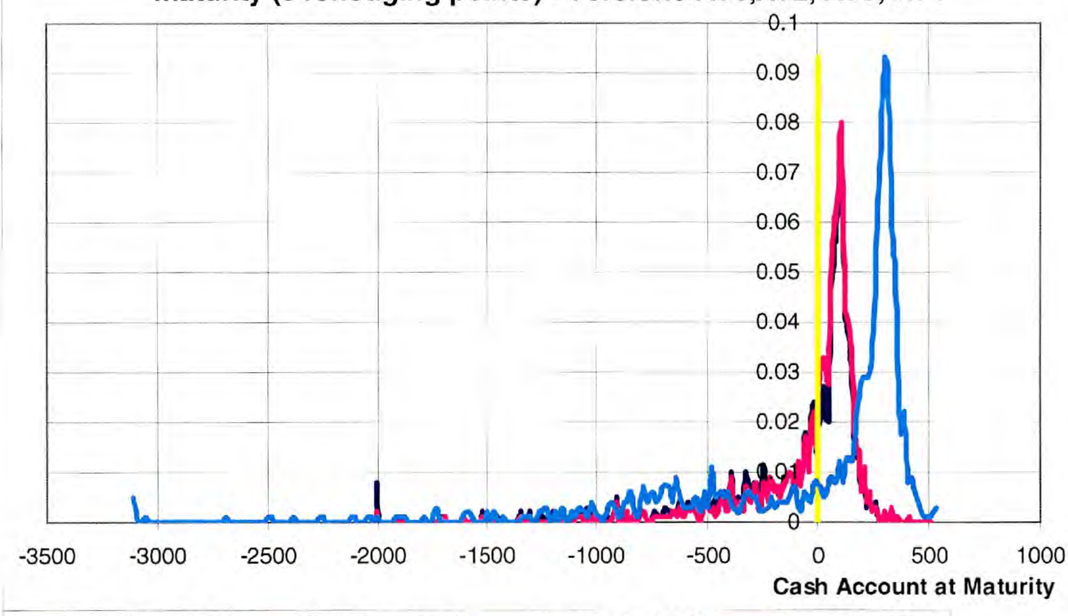
$$n_{m,i}^{options} = -\frac{\Gamma_{m,i}^{b} \times R^{CR} \times n_{CBs}}{\Gamma_{m,i}^{w}} \qquad \text{(Eq.8.2.21)}$$

$$n_{m,i}^{shares} = -\left(\Delta_{m,i}^{b} \times R^{CR} \times n_{CBs}\right) - \left(\Delta_{m,i}^{w} \times n_{m,i}^{options}\right) \qquad \text{(Eq.8.2.22)}$$

$$V_{m,i}^{cash} = \left(fcf_i^{Hedging} \times V_{m,i-1}^{cash}\right) + \left[\left(n_{m,i}^{options} - n_{m,i-1}^{options}\right) \times V_{m,i}^{w}\right] + \left[\left(n_{m,i}^{shares} - n_{m,i-1}^{shares}\right) \times S_{m,i}^{Hedging}\right]$$

(Eq.8.2.23)

Figure (F.8.31) Probability Distribution of the cash account at maturity (5 rehedging points) - Versions AT1,AT2, AT3, AT4



CHAPTER 9

TWO-DIMENSIONAL TREES

For dual currency convertible bonds, the exchange rate is another significant factor which has a future uncertainty to the same level like the stock price. This fact and the existence of liquid exchange rate derivatives are the main reasons that the dual currency convertible bonds participants are using two-factor models for pricing their CB positions, one factor for the underlying stock and one factor for the exchange rate.

Effort was made to employ a two-dimensional configuration that employs a correlation between the two stochastic processes. However, limitations are introduced when including the correlation and the employment of term structures and variable time steps as determined in chapter 5 is excluded. In order to maintain the framework structure as presented in the previous chapters, the decision was made to resort to a configuration with a zero correlation between the two stochastic processes. Both, the correlated and the uncorrelated structures are introduced in this chapter.

9.1. Correlated Stock and Exchange Rate Processes

Initially, the notation and indexing of the parameters is in continuous time form, and then changes into discrete form, following the notation used in the previous chapters. The following parameters are defined:

E_t The exchange rate for translating units in the foreign currency into units in the domestic currency, the bond currency.

σ_t^E The standard deviation of the log-normal returns of the exchange rate E_t.

S_t^F The asset price with dividend δ_t denominated in the foreign (equity) currency.

r_t^F The risk free rate of the foreign currency.

σ_t^S The standard deviation of the log-normal returns of the stock S_t^F in the foreign currency.

ρ_t^{ES} The correlation of the exchange rate E_t and the stock S_t^F.

r_t^D The risk free rate of the bond (domestic) currency.

S_t^D The asset price in the domestic currency.

Replacing the asset price S_t^F and the exchange rate E_t with their respective natural logarithms as shown in equations (9.1.5) and (9.1.6), equations (9.1.7) and (9.1.8) are obtained.

$$x_t^S = \ln\left(S_t^F\right) \qquad \text{(Eq.9.1.5)}$$

$$x_t^E = \ln\left(E_t\right) \qquad \text{(Eq.9.1.6)}$$

$$dx_t^S = \left(r_t^F - q_t - \rho_t^{ES}\,\sigma_t^S\,\sigma_t^E - \frac{1}{2}\left(\sigma_t^S\right)^2\right)dt + \sigma_t^S\,dw_t^S \qquad \text{(Eq.9.1.7)}$$

$$dx_t^E = \left(r_t^D - r_t^F - \frac{1}{2}\left(\sigma_t^E\right)^2\right)dt + \sigma_t^E\,dw_t^E \qquad \text{(Eq.9.1.8)}$$

Up to this point, the equations are in continuous form. The respective equations in discrete form are obtained by replacing dt with Δt, dx_t^S with Δx^S, and dx_t^E with Δx^E. For the rest of the parameters, the subscript t is replaced by the subscript i which will be used for the discrete form of the equations to denote the index of the step on the trinomial trees that will be employed here.

As in the previous chapters, the risk free curves in the equity (foreign) and bond (domestic) currencies are available, as well as the term structures of the volatility and the continuous dividend rate. In addition to these structures, we assume that the term structure of the volatility of the exchange rate and the term structure of the correlation of the stock and the exchange rate are also available. The only change in the notation from the previous chapters is in the notation for the stock volatility. In order to differentiate the stock volatility from the exchange rate volatility, the stock volatility is denoted as $\sigma_i^{S,steps}$ and the exchange rate volatility is denoted as $\sigma_i^{E,steps}$.

Two trinomial trees, one for the stock and the other for the exchange rate are employed. Representative node configurations of each tree are presented in the following two figures.

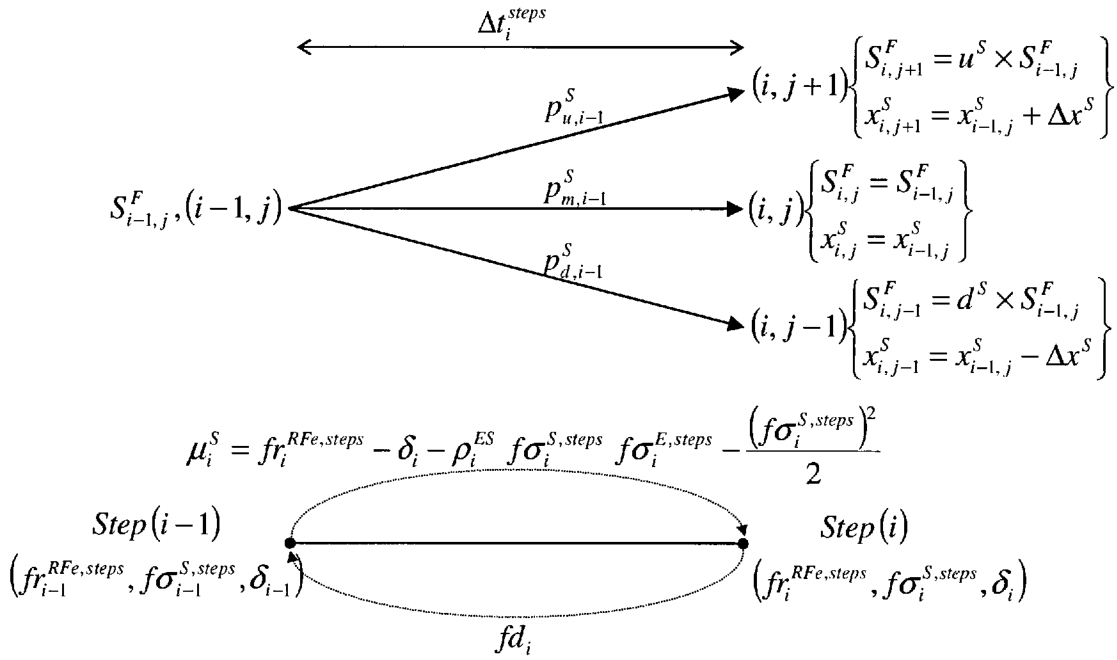


Figure (9.1) Representative Node Configuration of the Equity Tree

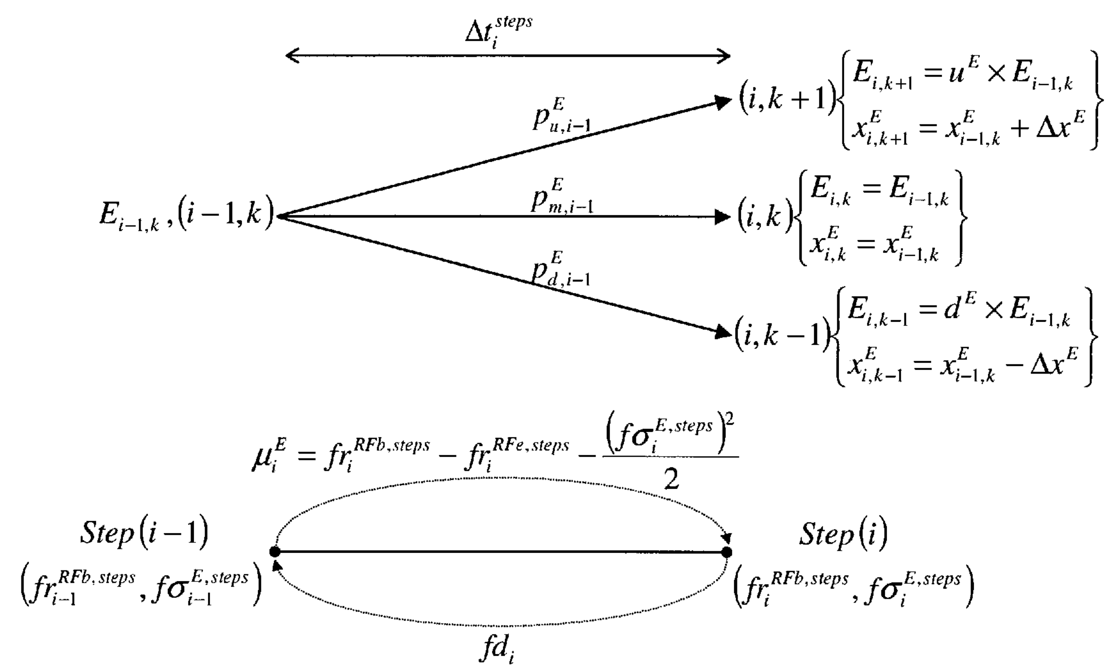


Figure (9.2) Representative Node Configuration of the Exchange Rate Tree

The next figure presents a representative node configuration of the two-dimensional tree resulted from the combination of two trinomial trees when there is a non-zero correlation between the two processes.

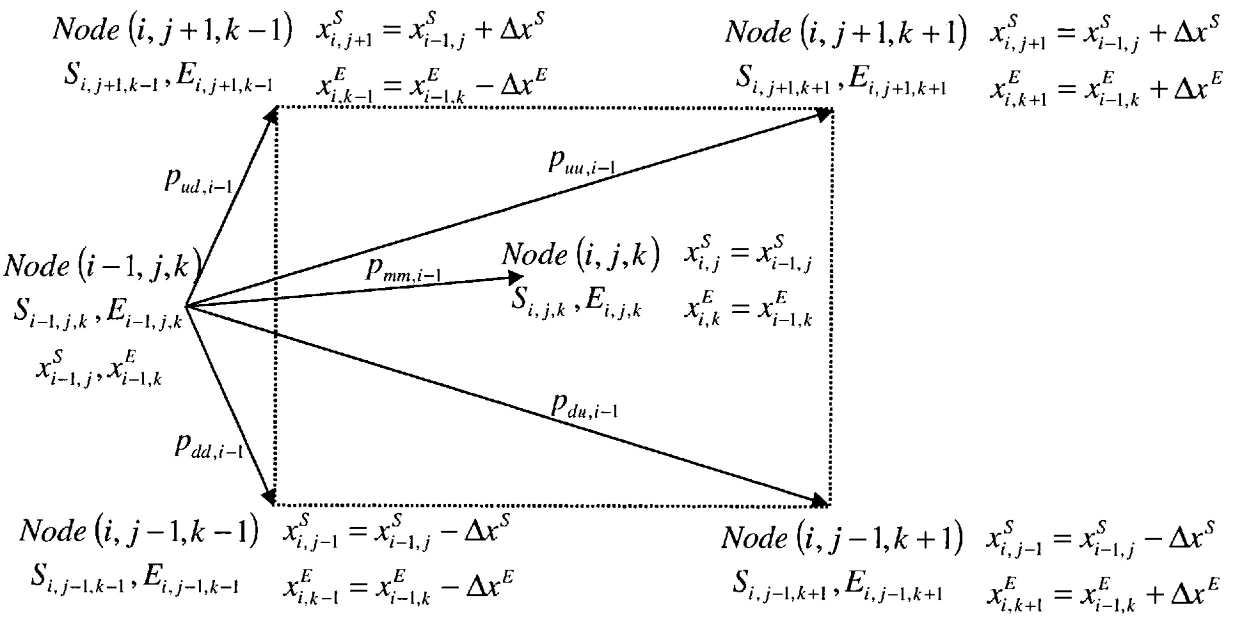


Figure (9.3) Representative Node Configuration

The equations determining the values of the transition probabilities must be established by accounting for the conditions that must hold for the expected values, the variances and the covariance of the changes of the natural logarithms of the stock price and the exchange rate.

$$E\left[\Delta x^S\right] = \mu_i^S\ \Delta t_i \qquad \text{(Eq.9.1.9)}$$

$$E\left[\Delta x^S \Delta x^S\right] = \left(f\sigma_i^S\right)^2\ \Delta t_i + \left(\mu_i^S\right)^2\ \Delta t_i^2 \qquad \text{(Eq.9.1.10)}$$

$$E\left[\Delta x^E\right] = \mu_i^E\ \Delta t_i \qquad \text{(Eq.9.1.11)}$$

$$E\left[\Delta x^E \Delta x^E\right] = \left(f\sigma_i^E\right)^2\ \Delta t_i + \left(\mu_i^E\right)^2\ \Delta t_i^2 \qquad \text{(Eq.9.1.12)}$$

$$E\left[\Delta x^S \Delta x^E\right] = \rho_i^{ES}\ f\sigma_i^S\ f\sigma_i^E\ \Delta t_i \qquad \text{(Eq.9.1.13)}$$

In addition, the sum of the probabilities must be equal to one.

$$p_{uu,i-1} + p_{ud,i-1} + p_{mm,i-1} + p_{du,i-1} + p_{dd,i-1} = 1 \qquad \text{(Eq.9.1.14)}$$

In order to simplify the notation, we also introduce the following parameters:

$$Q_i^S = \frac{\mu_i^S \, \Delta t_i}{\Delta x^S} \tag{Eq.9.1.15}$$

$$V_i^S = \frac{\left(f\sigma_i^S\right)^2 \, \Delta t_i + \left(\mu_i^S\right)^2 \, \Delta t_i^2}{\left(\Delta x^S\right)^2} \tag{Eq.9.1.16}$$

$$Q_i^E = \frac{\mu_i^E \, \Delta t_i}{\Delta x^E} \tag{Eq.9.1.17}$$

$$V_i^E = \frac{\left(f\sigma_i^E\right)^2 \, \Delta t_i + \left(\mu_i^E\right)^2 \, \Delta t_i^2}{\left(\Delta x^E\right)^2} \tag{Eq.9.1.18}$$

$$V_i^{ES} = \frac{\rho_i^{ES} \, f\sigma_i^S \, f\sigma_i^E \, \Delta t_i}{\Delta x^S \Delta x^E} \tag{Eq.9.1.18}$$

$$W_i^S = V_i^S + Q_i^S \tag{Eq.9.1.19}$$

$$\tilde{W}_i^S = V_i^S - Q_i^S \tag{Eq.9.1.20}$$

$$W_i^E = V_i^E + Q_i^E \tag{Eq.9.1.21}$$

$$\tilde{W}_i^E = V_i^E - Q_i^E \tag{Eq.9.1.22}$$

Based on equation (9.1.9):

$$\left(p_{uu,i-1} + p_{ud,i-1}\right)\left(+\Delta x^S\right) + p_{mm,i-1} \cdot 0 + \left(p_{du,i-1} + p_{dd,i-1}\right)\left(-\Delta x^S\right) = \mu_i^S \, \Delta t_i$$

$$p_{uu,i-1} + p_{ud,i-1} - p_{du,i-1} - p_{dd,i-1} = Q_i^S \tag{Eq.9.1.23}$$

Based on equation (9.1.10):

$$\left(p_{uu,i-1} + p_{ud,i-1}\right)\left(\Delta x^S\right)^2 + p_{mm,i-1} \cdot 0 + \left(p_{du,i-1} + p_{dd,i-1}\right)\left(\Delta x^S\right)^2 = \left(f\sigma_i^S\right)^2 \, \Delta t_i + \left(\mu_i^S\right)^2 \, \Delta t_i^2$$

$$p_{uu,i-1} + p_{ud,i-1} + p_{du,i-1} + p_{dd,i-1} = V_i^S \tag{Eq.9.1.24}$$

Based on equation (9.1.11):

$$\left(p_{uu,i-1} + p_{du,i-1}\right)\left(+\Delta x^E\right) + p_{mm,i-1} \cdot 0 + \left(p_{ud,i-1} + p_{dd,i-1}\right)\left(-\Delta x^E\right) = \mu_i^E \, \Delta t_i$$

$$p_{uu,i-1} + p_{du,i-1} - p_{ud,i-1} - p_{dd,i-1} = Q_i^E \tag{Eq.9.1.25}$$

Based on equation (9.1.12):

$$\left(p_{uu,i-1} + p_{du,i-1}\right)\left(\Delta x^E\right)^2 + p_{mm,i-1} \cdot 0 + \left(p_{ud,i-1} + p_{dd,i-1}\right)\left(\Delta x^E\right)^2 = \left(f\sigma_i^E\right)^2 \, \Delta t_i + \left(\mu_i^E\right)^2 \, \Delta t_i^2$$

$$p_{uu,i-1} + p_{du,i-1} + p_{ud,i-1} + p_{dd,i-1} = V_i^E \tag{Eq.9.1.26}$$

Based on equation (9.1.13):

$$\left(p_{uu,i-1}+p_{dd,i-1}\right)\left(\Delta x^S \Delta x^E\right)+p_{mm,i-1}\cdot 0+\left(p_{du,i-1}+p_{ud,i-1}\right)\left(\Delta x^S \Delta x^E\right)=\rho_i^{ES}\ f\sigma_i^S\ f\sigma_i^E\ \Delta t_i$$

$$p_{uu,i-1}+p_{dd,i-1}-p_{du,i-1}-p_{ud,i-1}=V_i^{ES} \qquad \text{(Eq.9.1.27)}$$

Equations (9.1.24) and (9.1.26) have the same left hand sides and this imposes a relationship as shown in the next equation:

$$V_i^S=V_i^E \qquad \text{(Eq.9.1.28)}$$

This last equation can further be elaborated as:

$$\frac{\left(f\sigma_i^S\right)^2\Delta t_i+\left(\mu_i^S\right)^2\Delta t_i^2}{\left(\Delta x^S\right)^2}=\frac{\left(f\sigma_i^E\right)^2\Delta t_i+\left(\mu_i^E\right)^2\Delta t_i^2}{\left(\Delta x^E\right)^2}$$

$$\frac{\left(\Delta x^S\right)^2}{\left(\Delta x^E\right)^2}=\frac{\left(f\sigma_i^S\right)^2\Delta t_i+\left(\mu_i^S\right)^2\Delta t_i^2}{\left(f\sigma_i^E\right)^2\Delta t_i+\left(\mu_i^E\right)^2\Delta t_i^2}$$

$$Q_i^{ES}=\frac{\Delta x^S}{\Delta x^E}=\sqrt{\frac{\left(f\sigma_i^S\right)^2\Delta t_i+\left(\mu_i^S\right)^2\Delta t_i^2}{\left(f\sigma_i^E\right)^2\Delta t_i+\left(\mu_i^E\right)^2\Delta t_i^2}} \qquad \text{(Eq.9.1.29)}$$

The ratio Q_i^{ES} which has been introduced in the last equation imposes a relationship between the space steps of the two trees. As it will be further discussed later, this relationship imposes limitations on the implementation of correlated processes in the frameworks employed in this thesis.

Combining equations (9.1.23) to (9.1.26) and employing equations (9.1.19) to (9.1.22), results in the new set of equations (9.1.30) to (9.1.33).

$$\left.\begin{array}{l}\left.\begin{array}{l}Eq.(9.1.23)\\+Eq.(9.1.24)\end{array}\right\}\Rightarrow 2\cdot\left(p_{uu,i-1}+p_{ud,i-1}\right)=Q_i^S+V_i^S\\ \qquad\qquad Eq.(9.1.19)\end{array}\right\}\Rightarrow p_{uu,i-1}+p_{ud,i-1}=\frac{W_i^S}{2}$$

$$p_{uu,i-1}+p_{ud,i-1}=\frac{W_i^S}{2} \qquad \text{(Eq.9.1.30)}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}Eq.(9.1.24)\\-Eq.(9.1.23)\end{array}\right\}\Rightarrow 2\cdot\left(p_{du,i-1}+p_{dd,i-1}\right)=V_i^S-Q_i^S\\ \qquad\qquad Eq.(9.1.20)\end{array}\right\}\Rightarrow p_{du,i-1}+p_{dd,i-1}=\frac{\tilde{W}_i^S}{2}$$

$$p_{du,i-1}+p_{dd,i-1}=\frac{\tilde{W}_i^S}{2} \qquad \text{(Eq.9.1.31)}$$

$$\left.\begin{array}{l} Eq.(9.1.25) \\ +Eq.(9.1.26) \end{array}\right\} \Rightarrow 2\cdot\left(p_{uu,i-1}+p_{du,i-1}\right)=Q_i^E+V_i^E \\ Eq.(9.1.21) \Bigg\} \Rightarrow p_{uu,i-1}+p_{du,i-1}=\frac{W_i^E}{2}$$

$$p_{uu,i-1}+p_{du,i-1}=\frac{W_i^E}{2} \qquad \text{(Eq.9.1.32)}$$

$$\left.\begin{array}{l} Eq.(9.1.26) \\ -Eq.(9.1.25) \end{array}\right\} \Rightarrow 2\cdot\left(p_{ud,i-1}+p_{dd,i-1}\right)=V_i^E-Q_i^E \\ Eq.(9.1.22) \Bigg\} \Rightarrow p_{ud,i-1}+p_{dd,i-1}=\frac{\tilde{W}_i^E}{2}$$

$$p_{ud,i-1}+p_{dd,i-1}=\frac{\tilde{W}_i^E}{2} \qquad \text{(Eq.9.1.33)}$$

Next step involves combining the last four derived equations and obtaining two new equations.

$$\left.\begin{array}{l} Eq.(9.1.30) \\ +Eq.(9.1.33) \end{array}\right\} \Rightarrow p_{uu,i-1}+2p_{ud,i-1}+p_{dd,i-1}=\frac{W_i^S+\tilde{W}_i^E}{2}$$

$$p_{uu,i-1}+p_{dd,i-1}+2p_{ud,i-1}=\frac{W_i^S+\tilde{W}_i^E}{2} \qquad \text{(Eq.9.1.34)}$$

$$\left.\begin{array}{l} Eq.(9.1.31) \\ +Eq.(9.1.32) \end{array}\right\} \Rightarrow 2p_{du,i-1}+p_{dd,i-1}+p_{uu,i-1}=\frac{\tilde{W}_i^S+W_i^E}{2}$$

$$p_{uu,i-1}+p_{dd,i-1}+2p_{du,i-1}=\frac{\tilde{W}_i^S+W_i^E}{2} \qquad \text{(Eq.9.1.35)}$$

Equation (9.1.27) can be re-written as follows:

$$p_{uu,i-1}+p_{dd,i-1}=p_{du,i-1}+p_{ud,i-1}+V_i^{ES}$$

Replacing equation (9.1.27) into equation (9.1.34):

$$p_{du,i-1}+p_{ud,i-1}+V_i^{ES}+2p_{ud,i-1}=\frac{W_i^S+\tilde{W}_i^E}{2}$$

$$p_{du,i-1}+3\cdot p_{ud,i-1}=\frac{W_i^S+\tilde{W}_i^E-2\cdot V_i^{ES}}{2} \qquad \text{(Eq.9.1.36)}$$

Replacing equation (9.1.27) into equation (9.1.35):

$$p_{du,i-1} + p_{ud,i-1} + V_i^{ES} + 2p_{du,i-1} = \frac{\tilde{W}_i^S + W_i^E}{2}$$

$$3 \cdot p_{du,i-1} + p_{ud,i-1} = \frac{\tilde{W}_i^S + W_i^E - 2 \cdot V_i^{ES}}{2} \qquad \text{(Eq.9.1.37)}$$

Based on the last two derived equations, equations (9.1.36) and (9.1.37), we can solve for the probabilities $p_{du,i-1}$ and $p_{ud,i-1}$.

$$\left.\begin{array}{l} 3 \times Eq.(9.1.36) \\ - Eq.(9.1.37) \end{array}\right\} \Rightarrow 8p_{ud,i-1} = \frac{3W_i^S + 3\tilde{W}_i^E - 6V_i^{ES} - \tilde{W}_i^S - W_i^E + 2V_i^{ES}}{2}$$

$$8p_{ud,i-1} = \frac{3V_i^S + 3Q_i^S + 3V_i^E - 3Q_i^E - 4V_i^{ES} - V_i^S + Q_i^S - V_i^E - Q_i^E}{2}$$

$$8p_{ud,i-1} = V_i^S + V_i^E + 2Q_i^S - 2Q_i^E - 2V_i^{ES}$$

$$p_{ud,i-1} = \frac{V_i^S + V_i^E - 2 \cdot V_i^{ES}}{8} + \frac{Q_i^S - Q_i^E}{4} \qquad \text{(Eq.9.1.38)}$$

$$\left.\begin{array}{l} 3 \times Eq.(9.1.37) \\ - Eq.(9.1.36) \end{array}\right\} \Rightarrow 8p_{du,i-1} = \frac{3\tilde{W}_i^S + 3W_i^E - 6V_i^{ES} - W_i^S - \tilde{W}_i^E + 2V_i^{ES}}{2}$$

$$8p_{du,i-1} = \frac{3V_i^S - 3Q_i^S + 3V_i^E + 3Q_i^E - 4V_i^{ES} - V_i^S - Q_i^S - V_i^E + Q_i^E}{2}$$

$$8p_{du,i-1} = V_i^S + V_i^E - 2Q_i^S + 2Q_i^E - 2V_i^{ES}$$

$$p_{du,i-1} = \frac{V_i^S + V_i^E - 2 \cdot V_i^{ES}}{8} - \frac{Q_i^S - Q_i^E}{4} \qquad \text{(Eq.9.1.39)}$$

The equations determining the probabilities $p_{uu,i-1}$ and $p_{dd,i-1}$ can be obtained by repeating the steps involved in equations (9.1.34) to (9.1.39) which resulted in the equations for the probabilities $p_{ud,i-1}$ and $p_{du,i-1}$. The first step involves combining the equations (9.1.30) to (9.1.33) in order to obtain two new equations.

$$\left.\begin{array}{l} Eq.(9.1.30) \\ + Eq.(9.1.32) \end{array}\right\} \Rightarrow 2p_{uu,i-1} + p_{ud,i-1} + p_{du,i-1} = \frac{W_i^S}{2} + \frac{W_i^E}{2}$$

$$2p_{uu,i-1} + p_{ud,i-1} + p_{du,i-1} = \frac{W_i^S + W_i^E}{2} \qquad \text{(Eq.9.1.40)}$$

$$\left.\begin{aligned} &Eq.(9.1.31) \\ +&Eq.(9.1.33) \end{aligned}\right\} \Rightarrow 2p_{dd,i-1} + p_{ud,i-1} + p_{du,i-1} = \frac{\tilde{W}_i^S}{2} + \frac{\tilde{W}_i^E}{2}$$

$$2p_{dd,i-1} + p_{ud,i-1} + p_{du,i-1} = \frac{\tilde{W}_i^S + \tilde{W}_i^E}{2} \qquad \text{(Eq.9.1.41)}$$

Equation (9.1.27) can be re-written as follows:

$$p_{du,i-1} + p_{ud,i-1} = p_{uu,i-1} + p_{dd,i-1} - V_i^{ES}$$

Replacing equation (9.1.27) into equation (9.1.40):

$$p_{uu,i-1} + p_{dd,i-1} - V_i^{ES} + 2p_{uu,i-1} = \frac{W_i^S + W_i^E}{2}$$

$$3 \cdot p_{uu,i-1} + p_{dd,i-1} = \frac{W_i^S + W_i^E + 2 \cdot V_i^{ES}}{2} \qquad \text{(Eq.9.1.42)}$$

Replacing equation (9.1.27) into equation (9.1.41):

$$p_{uu,i-1} + p_{dd,i-1} - V_i^{ES} + 2p_{dd,i-1} = \frac{\tilde{W}_i^S + \tilde{W}_i^E}{2}$$

$$p_{uu,i-1} + 3 \cdot p_{dd,i-1} = \frac{\tilde{W}_i^S + \tilde{W}_i^E + 2 \cdot V_i^{ES}}{2} \qquad \text{(Eq.9.1.43)}$$

Based on the last two derived equations, equations (9.1.42) and (9.1.43), we can solve for the probabilities $p_{uu,i-1}$ and $p_{dd,i-1}$.

$$\left.\begin{aligned} 3\times &Eq.(9.1.42) \\ -&Eq.(9.1.43) \end{aligned}\right\} \Rightarrow 8p_{uu,i-1} = \frac{3W_i^S + 3W_i^E + 6V_i^{ES} - \tilde{W}_i^S - \tilde{W}_i^E - 2V_i^{ES}}{2}$$

$$8p_{uu,i-1} = \frac{3V_i^S + 3Q_i^S + 3V_i^E + 3Q_i^E + 4V_i^{ES} - V_i^S + Q_i^S - V_i^E + Q_i^E}{2}$$

$$8p_{uu,i-1} = V_i^S + V_i^E + 2Q_i^S + 2Q_i^E + 2V_i^{ES}$$

$$p_{uu,i-1} = \frac{V_i^S + V_i^E + 2 \cdot V_i^{ES}}{8} + \frac{Q_i^S + Q_i^E}{4} \qquad \text{(Eq.9.1.44)}$$

$$\left.\begin{array}{l}3\times Eq.(9.1.43)\\ -Eq.(9.1.42)\end{array}\right\}\Rightarrow 8p_{dd,i-1}=\frac{3\tilde{W}_i^S+3\tilde{W}_i^E+6V_i^{ES}-W_i^S-W_i^E-2V_i^{ES}}{2}$$

$$8p_{dd,i-1}=\frac{3V_i^S-3Q_i^S+3V_i^E-3Q_i^E+4V_i^{ES}-V_i^S-Q_i^S-V_i^E-Q_i^E}{2}$$

$$8p_{dd,i-1}=V_i^S+V_i^E-2Q_i^S-2Q_i^E+2V_i^{ES}$$

$$p_{dd,i-1}=\frac{V_i^S+V_i^E+2\cdot V_i^{ES}}{8}-\frac{Q_i^S+Q_i^E}{4} \qquad \text{(Eq.9.1.45)}$$

The only equation that has not been employed up to this point is equation (9.1.14). This equation can be re-written as follows:

$$p_{mm,i-1}=1-p_{uu,i-1}-p_{ud,i-1}-p_{du,i-1}-p_{dd,i-1}$$

Substituting for the probabilities on the right hand side as these are defined by equations (9.1.38), (9.1.39), (9.1.44), and (9.1.45), we can obtain an equation for the probability of the middle branch.

$$p_{mm,i-1}=1-\frac{V_i^S+V_i^E+2\cdot V_i^{ES}}{8}-\frac{Q_i^S+Q_i^E}{4}-\frac{V_i^S+V_i^E-2\cdot V_i^{ES}}{8}-\frac{Q_i^S-Q_i^E}{4}$$
$$-\frac{V_i^S+V_i^E-2\cdot V_i^{ES}}{8}+\frac{Q_i^S-Q_i^E}{4}-\frac{V_i^S+V_i^E+2\cdot V_i^{ES}}{8}+\frac{Q_i^S+Q_i^E}{4}$$

$$p_{mm,i-1}=1-\frac{4V_i^S+4V_i^E}{8}$$

$$p_{mm,i-1}=1-\frac{V_i^S+V_i^E}{2} \qquad \text{(Eq.9.1.46)}$$

As it was demonstrated and summarised in equation (9.1.28), V_i^S and V_i^E must be equal, hence, equation (9.1.46) can be re-written as follows:

$$p_{mm,i-1}=1-V_i^S=1-V_i^E \qquad \text{(Eq.9.1.47)}$$

In figures (9.1) and (9.2), the two trees that constitute the two-dimensional configuration are presented diagrammatically, while their combination is presented in figure (9.3). The relationships between the probabilities presented in the three figures are summarised by the following equations.

$$p^S_{u,i-1} = p_{uu,i-1} + p_{ud,i-1} = \frac{W^S_i}{2} \tag{Eq.9.1.48}$$

$$p^S_{d,i-1} = p_{du,i-1} + p_{dd,i-1} = \frac{\tilde{W}^S_i}{2} \tag{Eq.9.1.49}$$

$$p^E_{u,i-1} = p_{uu,i-1} + p_{du,i-1} = \frac{W^E_i}{2} \tag{Eq.9.1.50}$$

$$p^E_{d,i-1} = p_{ud,i-1} + p_{dd,i-1} = \frac{\tilde{W}^E_i}{2} \tag{Eq.9.1.51}$$

$$p^S_{m,i-1} = p^E_{m,i} = p_{mm,i-1} = 1 - V^S_i = 1 - V^E_i \tag{Eq.9.1.52}$$

It is worth stating the number of nodes N_i involved at each step i in this configuration. This is determined according to the following equation.

$$N_i = 1 + 4 \cdot \sum_{m=1}^{i} m$$

$$N_i = 1 + 4 \cdot \frac{i \cdot (i+1)}{2}$$

$$N_i = 1 + 2 \cdot i \cdot (i+1) \tag{Eq.9.1.53}$$

It was shown that a relationship is imposed between the space steps of the two trees constituting the two-dimensional configuration. This relationship was summarised in equation (9.1.29) which is re-produced here as well.

$$Q^{ES}_i = \frac{\Delta x^S}{\Delta x^E} = \sqrt{\frac{\left(f\sigma^S_i\right)^2 \Delta t_i + \left(\mu^S_i\right)^2 \Delta t_i^2}{\left(f\sigma^E_i\right)^2 \Delta t_i + \left(\mu^E_i\right)^2 \Delta t_i^2}}$$

This equation means that in order to maintain the re-combining nature of the two trees and account for the correlation between the two processes, then only the following two approaches are plausible within this framework:

(i) We must not include any time-varying parameters and all parameters involved, like the volatilities, the risk-free rates, the continuous dividend yield and the correlation must be constant values throughout the two-dimensional configuration. In addition, the time step has to be constant and the same for all the steps in the tree.

(ii) We can include term structures in the configuration which could result in time varying volatilities, rates, dividend yield and correlation, but the time step Δt_i will be defined at each step based on equation (9.1.29). This will also mean that the last step, the maturity step most probably will have to be non-recombining since it is very unlike that the time steps resulting from this process will be such that the maturity is matched on one of the tree steps.

Both of the solutions for implementing the correlated two-dimensional model just discussed, deviate from the configuration developed in the previous chapters of this thesis. The implementations in the previous chapters were based on time varying parameters and some of the time steps were defined based on event dates. For this reason, the implementation of the two-dimensional configuration presented in this section was not implemented.

9.2. Uncorrelated Stock and Exchange Rate Processes

The two-dimensional configuration presented next assumes that the correlation between the two processes is equal to zero. In other words, it is assumed that the two processes are independent. As in the previous section, we consider two dimensional trees as presented in figures (9.1) and (9.2), included in the previous section. Since the two processes are totally independent, the equations determining the values of the transition probabilities of each tree can be derived separately. Then, the nine transition probabilities of the combined configuration can be calculated based on the transition probabilities of each tree.

Stock Tree

Basically, there are no changes in the equity (stock) tree configuration and it is the same as in the one-dimensional implementation for the dual currency convertible bonds presented in the previous chapters. This is due to the fact that the term in the drift equation of the stock which includes the correlation is eliminated since the correlation is assumed to be equal to zero. The equations governing the values of the parameters on the trinomial tree of the stock are outlined next.

$$\mu_i^S = fr_i^{RFe,steps} - \delta_i - \frac{\left(f\sigma_i^{S,steps}\right)^2}{2} \qquad \text{(Eq.9.2.1)}$$

$$p_{u,i-1}^S = \frac{1}{2}\left(\frac{\left(f\sigma_i^{S,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^S \Delta t_i^{steps}\right)^2}{\left(\Delta x^S\right)^2} + \frac{\mu_i^S \Delta t_i^{steps}}{\Delta x^S}\right) \qquad \text{(Eq.9.2.2)}$$

$$p_{m,i-1}^S = 1 - \frac{\left(f\sigma_i^{S,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^S \Delta t_i^{steps}\right)^2}{\left(\Delta x^S\right)^2} \qquad \text{(Eq.9.2.3)}$$

$$p_{d,i-1}^S = \frac{1}{2}\left(\frac{\left(f\sigma_i^{S,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^S \Delta t_i^{steps}\right)^2}{\left(\Delta x^S\right)^2} - \frac{\mu_i^S \Delta t_i^{steps}}{\Delta x^S}\right) \qquad \text{(Eq.9.2.4)}$$

$$u = e^{\Delta x^S} = \exp(\Delta x^S) \qquad \text{(Eq.9.2.5)}$$

$$d = e^{-\Delta x^S} = \exp(-\Delta x^S) \qquad \text{(Eq.9.2.6)}$$

$$S_{i,j}^F = S_{0,0}^F e^{j\Delta x^S} \qquad \forall i \in [1,N], \forall j \in [-i,i] \qquad \text{(Eq.9.2.7)}$$

<u>Exchange Rate Tree</u>

The equations governing the values of the parameters on the trinomial tree of the exchange rate are outlined next.

$$\mu_i^E = fr_i^{RFb,steps} - fr_i^{RFe,steps} - \frac{\left(f\sigma_i^{E,steps}\right)^2}{2} \qquad \text{(Eq.9.2.8)}$$

$$p_{u,i-1}^E = \frac{1}{2}\left(\frac{\left(f\sigma_i^{E,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^E \Delta t_i^{steps}\right)^2}{\left(\Delta x^E\right)^2} + \frac{\mu_i^E \Delta t_i^{steps}}{\Delta x^E}\right) \qquad \text{(Eq.9.2.9)}$$

$$p_{m,i-1}^E = 1 - \frac{\left(f\sigma_i^{E,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^E \Delta t_i^{steps}\right)^2}{\left(\Delta x^E\right)^2} \qquad \text{(Eq.9.2.10)}$$

$$p_{d,i-1}^E = \frac{1}{2}\left(\frac{\left(f\sigma_i^{E,steps}\right)^2 \Delta t_i^{steps} + \left(\mu_i^E \Delta t_i^{steps}\right)^2}{\left(\Delta x^E\right)^2} - \frac{\mu_i^E \Delta t_i^{steps}}{\Delta x^E}\right) \qquad \text{(Eq.9.2.11)}$$

$$u = e^{\Delta x^E} = \exp(\Delta x^E) \qquad \text{(Eq.9.2.12)}$$

$$d = e^{-\Delta x^E} = \exp(-\Delta x^E) \qquad \text{(Eq.9.2.13)}$$

$$E_{i,j} = E_{i,j} e^{j\Delta x^E} \qquad \forall i \in [1,N], \forall j \in [-i,i] \qquad \text{(Eq.9.2.14)}$$

Combination of the Stock and the Exchange Rate Trees

In order to combine the two trees we need to account for every possible combination of events, and this results in a 9-nomial configuration as presented in figure (9.4).

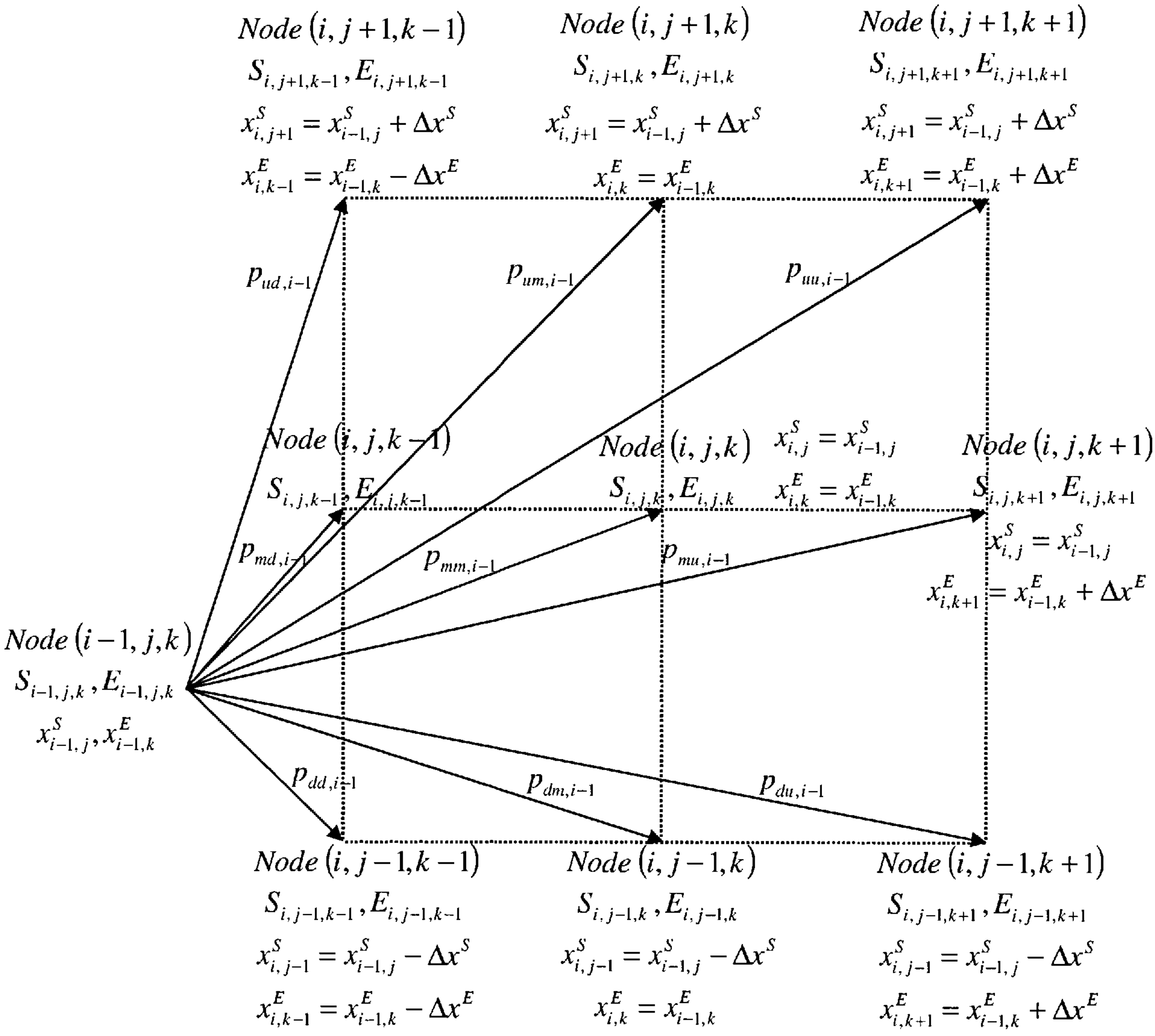


Figure (9.4) Representative Node Configuration

The node (i, j, k) on the combined configuration corresponds to the combination of node (i, j) on the Equity tree with node (i, k) on the exchange rate tree. The node (i, j) is the jth node at the ith step of the equity tree, and this holds for $\forall i \in [0, N], \forall j \in [-i, i]$. The node (i, k) is the kth node at the ith step of the exchange rate tree, and this holds for $\forall i \in [0,N], \forall k \in [-i,i]$. The set of parameters $\left(S^F_{i,j,k}, E_{i,j,k}, S^D_{i,j,k}\right)$ is defined for each node on the combination tree as shown in the next equations.

$$S^F_{i,j,k} = S^F_{i,j} \qquad \forall i \in [0,N], \forall j \in [-i,i], \forall k \in [-i,i] \qquad \text{(Eq.9.2.15)}$$

$$E_{i,j,k} = E_{i,k} \qquad \forall i \in [0,N], \forall j \in [-i,i], \forall k \in [-i,i] \qquad \text{(Eq.9.2.16)}$$

$$S^D_{i,j,k} = S^F_{i,j,k} \times E_{i,j,k} \qquad \forall i \in [0,N], \forall j \in [-i,i], \forall k \in [-i,i] \qquad \text{(Eq.9.2.17)}$$

The new nine transition probabilities are defined based on the following equations.

$$p_{uu,i-1} = p^S_{u,i-1} \times p^E_{u,i-1} \qquad \text{(Eq.9.2.18)}$$

$$p_{um,i-1} = p^S_{u,i-1} \times p^E_{m,i-1} \qquad \text{(Eq.9.2.19)}$$

$$p_{ud,i-1} = p^S_{u,i-1} \times p^E_{d,i-1} \qquad \text{(Eq.9.2.20)}$$

$$p_{mu,i-1} = p^S_{m,i-1} \times p^E_{u,i-1} \qquad \text{(Eq.9.2.21)}$$

$$p_{mm,i-1} = p^S_{m,i-1} \times p^E_{m,i-1} \qquad \text{(Eq.9.2.22)}$$

$$p_{md,i-1} = p^S_{m,i-1} \times p^E_{d,i-1} \qquad \text{(Eq.9.2.23)}$$

$$p_{du,i-1} = p^S_{d,i-1} \times p^E_{u,i-1} \qquad \text{(Eq.9.2.24)}$$

$$p_{dm,i-1} = p^S_{d,i-1} \times p^E_{m,i-1} \qquad \text{(Eq.9.2.25)}$$

$$p_{dd,i-1} = p^S_{d,i-1} \times p^E_{d,i-1} \qquad \text{(Eq.9.2.26)}$$

One of the basic relationships in both trees is that, for each tree independently, the transition probabilities at any node sum to one.

$$p^S_{u,i-1} + p^S_{u,i-1} + p^S_{d,i-1} = 1 \qquad \text{(Eq.9.2.27)}$$

$$p^E_{u,i-1} + p^E_{m,i-1} + p^E_{d,i-1} = 1 \qquad \text{(Eq.9.2.28)}$$

Based on equations (9.2.18) to (9.2.28), the nine defined transition probabilities sum to one and this is demonstrated as follows.

$$\begin{aligned} & p_{uu,i-1} + p_{um,i-1} + p_{ud,i-1} + p_{mu,i-1} + p_{mm,i-1} + p_{md,i-1} + p_{du,i-1} + p_{dm,i-1} + p_{dd,i-1} \\ & \quad = \left(p^S_{u,i-1} \times p^E_{u,i-1}\right) + \left(p^S_{u,i-1} \times p^E_{m,i-1}\right) + \left(p^S_{u,i-1} \times p^E_{d,i-1}\right) + \left(p^S_{m,i-1} \times p^E_{u,i-1}\right) \\ & \qquad + \left(p^S_{m,i-1} \times p^E_{m,i-1}\right) + \left(p^S_{m,i-1} \times p^E_{d,i-1}\right) + \left(p^S_{d,i-1} \times p^E_{u,i-1}\right) + \left(p^S_{d,i-1} \times p^E_{m,i-1}\right) + \left(p^S_{d,i-1} \times p^E_{d,i-1}\right) \end{aligned}$$

The last equation evolves as shown next.

$$\begin{aligned}
&p_{uu,i-1} + p_{um,i-1} + p_{ud,i-1} + p_{mu,i-1} + p_{mm,i-1} + p_{md,i-1} + p_{du,i-1} + p_{dm,i-1} + p_{dd,i-1} \\
&\quad = p^S_{u,i-1} \times \left(p^E_{u,i-1} + p^E_{m,i-1} + p^E_{d,i-1}\right) + p^S_{u,i-1} \times \left(p^E_{u,i-1} + p^E_{m,i-1} + p^E_{d,i-1}\right) \\
&\qquad + p^S_{d,i-1} \times \left(p^E_{u,i-1} + p^E_{m,i-1} + p^E_{d,i-1}\right) \\
&\quad = \left(p^S_{u,i-1} + p^S_{u,i-1} + p^S_{d,i-1}\right) \times \left(p^E_{u,i-1} + p^E_{m,i-1} + p^E_{d,i-1}\right) = 1 \times 1 = 1
\end{aligned}$$

So, the final result is equation (9.2.29).

$$p_{uu,i-1} + p_{um,i-1} + p_{ud,i-1} + p_{mu,i-1} + p_{mm,i-1} + p_{md,i-1} + p_{du,i-1} + p_{dm,i-1} + p_{dd,i-1} = 1 \qquad \text{(Eq.9.2.29)}$$

Parameters and Notation

It should be pointed out that all the parameters presented in the previous chapters for the one-dimensional trinomial tree, are still valid and exist in the stock trinomial tree of the two-dimensional configuration with the same notation exactly. For example, for the case of resetable convertible bonds, a parameter $K_{i,j}$ was introduced and defined as the prevailing strike (i, j). The same parameter $K_{i,j}$ is calculated in the same manner in the two-dimensional configuration and it holds that $K_{i,j}$ is the prevailing strike for all the nodes $(i, j, k = [-i, i])$. The same conditions would apply for example in the case of the flag $\xi^{Trigger,Call}_{i,j}$ when a conditional call is included in the features of the CB. In general, all the defined parameters in chapter 6 for the one-dimensional configuration are applicable for the two-dimensional configuration as well.

CHAPTER 10

EVALUATING THE TWO-DIMENSIONAL MODEL

In the previous chapter, two configurations for implementing a two-dimensional tree framework were presented. It was decided to follow the configuration with the uncorrelated processes because the configuration with the correlated processes could not be adjusted to the overall pricing framework presented in this thesis. In this chapter, the processes involved in the backward induction for pricing the convertible bond based on the two-dimensional configuration, are outlined. The calculation of the additional sensitivities, as well as the spectrum analysis for this pricing framework, is also presented.

10.1. Calculating the CB price and the Sensitivities

Forward Induction is carried out for each tree independently. The additional computations presented in section (6.5) are still available in the stock tree and employed if conditional calls/puts and/or resets are included in the features of the dual currency convertible bonds. In other words, nothing has changed from the previous chapters as far as the behaviour and the capabilities of the equity tree during the forward induction process are concerned.

However, the description of the Backward Induction has to be done in detail because this is the point where the two trees are combined. Next, the notation followed for the parameters included in this chapter is outlined:

- Node (i, j) is a node on the stock trinomial tree.
- Node (i,k) is a node on the exchange rate trinomial tree.
- Node (i, j,k) is a node on the combined configuration of the two trinomial trees.

At all the nodes of all the steps, the conversion flag $\xi_{i,j,k}^{ConvAllowed}$ is simply defined based on the following equation.

$$\left.\begin{array}{ll} Conversion\ Allowed\ at\ i: & \xi_{i,j,k}^{ConvAllowed}=1 \\ Conversion\ Not\ Allowed\ at\ i: & \xi_{i,j,k}^{ConvAllowed}=0 \end{array}\right\} \forall i \in [0,N], \forall j \in [-i,i], \forall k \in [-i,i]$$

(Eq.10.1.1)

Probability Weighted Discounting

For the cash-flows discounting, we follow the *probability weighted discounting* approach like in the previous chapters. Two parameters for each node on the tree, the flag $\xi_{i,j,k}^{Converted}$ and the probability $\pi_{i,j,k}^{Converted}$ are defined. During the backward induction, it is decided at each node whether to convert or not. If at a node it is decided to convert, then the flag $\xi_{i,j,k}^{Converted}$ is set equal to one, otherwise is set equal to zero. Then the probability of conversion $\pi_{i,j,k}^{Converted}$ is set as shown in the following equations. The initialisation of this process starts with setting both the flag and the probability for all the nodes at the maturity step equal to zero, since no conversion is allowed ever on the maturity date. The probabilities $\pi_{i,j,k}^{Converted}$ are employed in the discounting process, and this is demonstrated in the descriptions to follow for the rest of the calculations.

$$\left.\begin{array}{l} \xi_{i=N,j,k}^{Converted}=0 \\ \pi_{i=N,j,k}^{Converted}=0 \end{array}\right\} \forall j \in [-N,N], \forall k \in [-N,N] \qquad \text{(Eq.10.1.2)}$$

$$\left.\begin{array}{lll} \xi_{i,j,k}^{Converted}=0 & & if\ decided\ to\ hold \\ \xi_{i,j,k}^{Converted}=1 & & if\ decided\ to\ convert \end{array}\right\} \forall i \in [0,N-1], \forall j \in [-i,i], \forall k \in [-i,i]$$

(Eq.10.1.3)

$$\pi_{i,j,k}^{Converted}=\xi_{i,j,k}^{Converted} \times \begin{bmatrix} \left(p_{i,uu}\cdot\pi_{i+1,j+1,k+1}^{Converted}\right)+\left(p_{i,um}\cdot\pi_{i+1,j+1,k}^{Converted}\right)+\left(p_{i,ud}\cdot\pi_{i+1,j+1,k-1}^{Converted}\right) \\ +\left(p_{i,mu}\cdot\pi_{i+1,j,k+1}^{Converted}\right)+\left(p_{i,mm}\cdot\pi_{i+1,j,k}^{Converted}\right)+\left(p_{i,md}\cdot\pi_{i+1,j,k-1}^{Converted}\right) \\ +\left(p_{i,du}\cdot\pi_{i+1,j-1,k+1}^{Converted}\right)+\left(p_{i,dm}\cdot\pi_{i+1,j-1,k}^{Converted}\right)+\left(p_{i,dd}\cdot\pi_{i+1,j-1,k-1}^{Converted}\right) \end{bmatrix}$$

$$\forall i \in [0,N-1], \forall j \in [-i,i]$$

(Eq.10.1.4)

<u>Calculations at Maturity:</u> $(i = N)$

Backward Induction begins with the calculation of the convertible bond price at the last step, which is the step that corresponds to the maturity date. The holding value of the convertible bond is calculated as the sum of the redemption value and any possible coupon cash flows on that day.

$$V^{h}_{i,j,k} = P^{Rd} + q_i^{Cpn,steps} \qquad i = N, \forall j \in [-N, N], \forall k \in [-N, N] \qquad \text{(Eq.10.1.5)}$$

The final convertible bond value is simply calculated based on the following equation:

$$V^{b}_{i,j,k} = V^{h}_{i,j,k} \qquad i = N, \forall j \in [-N, N], \forall k \in [-N, N] \qquad \text{(Eq.10.1.6)}$$

<u>Calculations at the rest of the steps:</u> $(i = (N-1) \to 0)$

Starting from step $i = N-1$ and working backwards on the tree until step $i = 0$ (inclusively), the following calculations are repeated at each step.

First, the conversion value is calculated as depicted by equation (7.1.4) which is reproduced here as equation (10.1.7).

$$\left.\begin{array}{ll} V^{conv}_{i,j,k} = R^{CR} \times E_{i,k} \times S^{*}_{i,j} & \xi^{ConvAllowed}_{i,j,k} = 1 \\ V^{conv}_{i,j,k} = 0 & \xi^{ConvAllowed}_{i,j,k} = 0 \end{array}\right\} \forall j \in [-i, i], \forall k \in [-i, i] \qquad \text{(Eq.10.1.7)}$$

The holding value is calculated as the probability weighted sum of the connected nodes in the following in time step, plus any coupon values. Nine intermediate values $V^{h,uu}_{i,j,k}$, $V^{h,um}_{i,j,k}$, $V^{h,ud}_{i,j,k}$, $V^{h,mu}_{i,j,k}$, $V^{h,mm}_{i,j,k}$, $V^{h,md}_{i,j,k}$, $V^{h,du}_{i,j,k}$, $V^{h,dm}_{i,j,k}$ and $V^{h,dd}_{i,j,k}$, are calculated before the final value $V^{h}_{i,j,k}$ is established. The probability of converting is also taken into account.

$$V^{h,uu}_{i,j,k} = \left[\left(\pi^{Converted}_{i+1,j+1,k+1} \times fd^{RFe,steps}_{i+1}\right) + \left(\left(1 - \pi^{Converted}_{i+1,j+1,k+1}\right) \times fd^{Risky,steps}_{i+1}\right)\right] \times p_{uu,i} \times V^{b}_{i+1,j+1,k+1} \qquad \text{(Eq.10.1.8)}$$

$$V^{h,um}_{i,j,k} = \left[\left(\pi^{Converted}_{i+1,j+1,k} \times fd^{RFe,steps}_{i+1}\right) + \left(\left(1 - \pi^{Converted}_{i+1,j+1,k}\right) \times fd^{Risky,steps}_{i+1}\right)\right] \times p_{um,i} \times V^{b}_{i+1,j+1,k} \qquad \text{(Eq.10.1.9)}$$

$$V^{h,ud}_{i,j,k} = \left[\left(\pi^{Converted}_{i+1,j+1,k-1} \times fd^{RFe,steps}_{i+1}\right) + \left(\left(1 - \pi^{Converted}_{i+1,j+1,k-1}\right) \times fd^{Risky,steps}_{i+1}\right)\right] \times p_{ud,i} \times V^{b}_{i+1,j+1,k-1} \qquad \text{(Eq.10.1.10)}$$

$$V^{h,mu}_{i,j,k} = \left[\left(\pi^{Converted}_{i+1,j,k+1} \times fd^{RFe,steps}_{i+1}\right) + \left(\left(1 - \pi^{Converted}_{i+1,j,k+1}\right) \times fd^{Risky,steps}_{i+1}\right)\right] \times p_{mu,i} \times V^{b}_{i+1,j,k+1} \qquad \text{(Eq.10.1.11)}$$

$$V_{i,j,k}^{h,mm} = \left[\left(\pi_{i+1,j,k}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j,k}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{mm,i} \times V_{i+1,j,k}^{b} \quad \text{(Eq.10.1.12)}$$

$$V_{i,j,k}^{h,md} = \left[\left(\pi_{i+1,j,k-1}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j,k-1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{md,i} \times V_{i+1,j,k-1}^{b} \quad \text{(Eq.10.1.13)}$$

$$V_{i,j,k}^{h,du} = \left[\left(\pi_{i+1,j-1,k+1}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j-1,k+1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{du,i} \times V_{i+1,j-1,k+1}^{b} \quad \text{(Eq.10.1.14)}$$

$$V_{i,j,k}^{h,dm} = \left[\left(\pi_{i+1,j-1,k}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j-1,k}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{dm,i} \times V_{i+1,j-1,k}^{b} \quad \text{(Eq.10.1.15)}$$

$$V_{i,j,k}^{h,dd} = \left[\left(\pi_{i+1,j-1,k-1}^{Converted} \times fd_{i+1}^{RFe,steps}\right) + \left(\left(1-\pi_{i+1,j-1,k-1}^{Converted}\right) \times fd_{i+1}^{Risky,steps}\right)\right] \times p_{dd,i} \times V_{i+1,j-1,k-1}^{b} \quad \text{(Eq.10.1.16)}$$

$$\left.\begin{aligned} V_{i,j,k}^{h} &= q_i^{Cpn,steps} + V_{i,j,k}^{h,uu} + V_{i,j,k}^{h,um} + V_{i,j,k}^{h,ud} \\ &\quad + V_{i,j,k}^{h,mu} + V_{i,j,k}^{h,mm} + V_{i,j,k}^{h,md} + V_{i,j,k}^{h,du} + V_{i,j,k}^{h,dm} + V_{i,j,k}^{h,dd} \end{aligned}\right\} \quad \forall j \in [-i,i], \forall k \in [-i,i]$$

(Eq.10.1.17)

Then, according to the presence of calls and puts, the final convertible bond value at each node is calculated according to equation (Eq.10.1.18). Based on this equation, the convertible bond is initially set equal to the minimum of the holding value and the call value (if the convertible bond is callable at that step). Then, the result is compared to the converting value and the put value (if the convertible bond is puttable at that node), and the maximum of the three values is used as the final convertible bond value at that node. In the absence of any calls at a node (i,j), and consequently node (i,j,k), then the parameter $V_{i,j}^{c}$ is simply not included in the equation. In the same manner, in the absence of any puts at a node (i,j), then the parameter $V_{i,j}^{p}$ is simply not included in the equation. Finally, in the cases where no conversion is allowed at a node (i,j,k), then the parameter $V_{i,j,k}^{conv}$ is simply not included.

$$V_{i,j,k}^{b} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \qquad \forall j \in [-i,i], \forall k \in [-i,i] \qquad \text{(Eq.10.1.18)}$$

Once the above calculations have been performed at all the steps, the value of the convertible bond on the calculation date, V^{CD}, is set equal to the value $V_{0,0,0}^{b}$ which is the value of the convertible bond at the zero step of the tree.

$$V^{CD} = V_{0,0,0}^{b} \qquad \text{(Eq.10.1.19)}$$

Conditional Calls

When conditional calls are present, then there are some modifications in the calculations. In the previous chapter, it was described how the conditional calls are accounted for. Essentially, the probability of triggering a conditional call $\pi_{i,j}^{Trigger,Call}$ was calculated for all the nodes (i,j,k) that fell within the period of a conditional call. To complete the calculations for accounting for a conditional call, we need, for each node in the conditional call period, to calculate the value of the convertible bond in the case that the call is activated – triggered – and in the case that is not activated.

$$V_{i,j,k}^{b,CallTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \quad \text{(Eq.10.1.20)}$$

$$V_{i,j,k}^{b,CallNotTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, V_{i,j,k}^{h}\right) \quad \text{(Eq.10.1.21)}$$

The final value of the convertible bond at a node within the period covered by a conditional call is calculated as follows.

$$V_{i,j,k}^{b} = \pi_{i,j}^{Trigger,Call} \cdot V_{i,j,k}^{b,CallTriggered} + \left(1 - \pi_{i,j}^{Trigger,Call}\right) \cdot V_{i,j,k}^{b,CallNotTriggered} \quad \text{(Eq.10.1.22)}$$

Conditional Puts

During the Backward Induction process, the treatment of the conditional puts is in exactly the same fashion as that for the conditional calls. To complete the calculations for accounting for a conditional put, we need, for each node (i,j,k) in the conditional put period, to calculate the value of the convertible bond in the case that the put is triggered and in the case that is not.

$$V_{i,j,k}^{b,PutTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \quad \text{(Eq.10.1.23)}$$

$$V_{i,j,k}^{b,PutNotTriggered} = \max\left(V_{i,j,k}^{conv}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \quad \text{(Eq.10.1.24)}$$

The final value of the convertible bond at a node within the period covered by a conditional put is calculated as follows.

$$V_{i,j,k}^{b} = \pi_{i,j}^{Trigger,Put} \cdot V_{i,j,k}^{b,PutTriggered} + \left(1 - \pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j,k}^{b,PutNotTriggered} \quad \text{(Eq.10.1.25)}$$

Combining Conditional Puts and Calls

To cover the rare, but plausible, case where the period of a conditional call and the period of a conditional put overlap, we consider a general node (i, j, k) which falls in this overlapping period. We calculate four possible values: The convertible bond price $V_{i,j,k}^{b,CallPutTriggered}$ at that node with the conditional put and the conditional call both triggered, the convertible bond price $V_{i,j,k}^{b,CallTriggered}$ at that node with the call activated and the put deactivated, the convertible bond price $V_{i,j,k}^{b,PutTriggered}$ at that node with the put activated and the call deactivated, and, the convertible bond price $V_{i,j,k}^{b,NoneTriggered}$ at that node with both the put and the call deactivated.

$$V_{i,j,k}^{b,CallPutTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.10.1.26)}$$

$$V_{i,j,k}^{b,CallTriggered} = \max\left(V_{i,j,k}^{conv}, \min\left(V_{i,j,k}^{h}, V_{i,j}^{c}\right)\right) \qquad \text{(Eq.10.1.27)}$$

$$V_{i,j,k}^{b,PutTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j}^{p}, V_{i,j,k}^{h}\right) \qquad \text{(Eq.10.1.28)}$$

$$V_{i,j,k}^{b,NoneTriggered} = \max\left(V_{i,j,k}^{conv}, V_{i,j,k}^{h}\right) \qquad \text{(Eq.10.1.29)}$$

The final value of the convertible bond at a node within the period covered by both a conditional call and a conditional put is calculated as shown next.

$$\begin{aligned} V_{i,j,k}^{b} = &\left(\pi_{i,j}^{Trigger,Call} \cdot \pi_{i,j}^{Trigger,Put} \cdot V_{i,j,k}^{b,CallPutTriggered}\right) \\ &+ \left(\pi_{i,j}^{Trigger,Call} \cdot \left(1-\pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j,k}^{b,CallTriggered}\right) \\ &+ \left(\left(1-\pi_{i,j}^{Trigger,Call}\right) \cdot \pi_{i,j}^{Trigger,Put} \cdot V_{i,j,k}^{b,PutTriggered}\right) \\ &+ \left(\left(1-\pi_{i,j}^{Trigger,Call}\right) \cdot \left(1-\pi_{i,j}^{Trigger,Put}\right) \cdot V_{i,j,k}^{b,NoneTriggered}\right) \end{aligned} \qquad \text{(Eq.10.1.30)}$$

Resets

With very few changes in notation, the approach for Resets is the same as before.

$$R_{i,j}^{CR} = \frac{P^F}{K_{i,j}} \qquad \text{(Eq.10.1.31)}$$

$$\left.\begin{array}{ll} V_{i,j,k}^{conv} = R_{i,j}^{CR} \times E_{i,k} \times S_{i,j}^{*} & \xi_{i,j,k}^{ConvAllowed} = 1 \\ V_{i,j,k}^{conv} = 0 & \xi_{i,j,k}^{ConvAllowed} = 0 \end{array}\right\} \forall j \in [-i, i], \forall k \in [-i, i] \qquad \text{(Eq.10.1.32)}$$

Sensitivities

The same sensitivities are calculated as before, with the additional sensitivities to the exchange rate and the volatility of the exchange rate. Only the sensitivities based on the method of the tree embedded sensitivities are calculated for this two-dimensional framework.

Delta And Gamma

The delta sensitivity to the stock price is defined as Δ^S.

$$\Delta^S = \frac{\partial V^b}{\partial S} = \frac{V^b_{1,1,0} - V^b_{1,-1,0}}{V^{conv}_{1,1,0} - V^{conv}_{1,-1,0}} \quad \text{(Eq.10.1.33)}$$

The gamma sensitivity to the stock price is defined as Γ^S and is calculated in equation (10.1.36).

$$\Delta^{S,up} = \frac{V^b_{1,1,0} - V^b_{1,0,0}}{V^{conv}_{1,1,0} - V^{conv}_{1,0,0}} \quad \text{(Eq.10.1.34)}$$

$$\Delta^{S,down} = \frac{V^b_{1,0,0} - V^b_{1,-1,0}}{V^{conv}_{1,0,0} - V^{conv}_{1,-1,0}} \quad \text{(Eq.10.1.35)}$$

$$\Gamma^S = \frac{\partial^2 V^b}{\partial S^2} = \frac{\Delta^{S,up} - \Delta^{S,down}}{\left(\frac{V^{conv}_{1,1,0} - V^{conv}_{1,-1,0}}{2}\right)} = \frac{\frac{V^b_{1,1,0} - V^b_{1,0,0}}{V^{conv}_{1,1,0} - V^{conv}_{1,0,0}} - \frac{V^b_{1,0,0} - V^b_{1,-1,0}}{V^{conv}_{1,0,0} - V^{conv}_{1,-1,0}}}{\left(\frac{V^{conv}_{1,1,0} - V^{conv}_{1,-1,0}}{2}\right)} \quad \text{(Eq.10.1.36)}$$

The delta sensitivity to the exchange rate is defined as Δ^E.

$$\Delta^E = \frac{\partial V^b}{\partial E} = \frac{V^b_{1,0,1} - V^b_{1,0,-1}}{V^{conv}_{1,0,1} - V^{conv}_{1,0,-1}} \quad \text{(Eq.10.1.37)}$$

The gamma sensitivity to the exchange rate is defined as Γ^E and is calculated in equation (10.1.40).

$$\Delta^{E,up} = \frac{V^b_{1,0,1} - V^b_{1,0,0}}{V^{conv}_{1,0,1} - V^{conv}_{1,0,0}} \quad \text{(Eq.10.1.38)}$$

$$\Delta^{E,down} = \frac{V^b_{1,0,0} - V^b_{1,0,-1}}{V^{conv}_{1,0,0} - V^{conv}_{1,0,-1}} \qquad \text{(Eq.10.1.39)}$$

$$\Gamma^E = \frac{\partial^2 V^b}{\partial E^2} = \frac{\Delta^{E,up} - \Delta^{E,down}}{\left(\frac{V^{conv}_{1,0,1} - V^{conv}_{1,0,-1}}{2}\right)} = \frac{\frac{V^b_{1,0,1} - V^b_{1,0,0}}{V^{conv}_{1,0,1} - V^{conv}_{1,0,0}} - \frac{V^b_{1,0,0} - V^b_{1,0,-1}}{V^{conv}_{1,0,0} - V^{conv}_{1,0,-1}}}{\left(\frac{V^{conv}_{1,0,1} - V^{conv}_{1,0,-1}}{2}\right)} \qquad \text{(Eq.10.1.40)}$$

If the correlation of the stock price and the exchange rate was included in the pricing framework, then the sensitivity $\Gamma^{S,E}$ could be calculated based on the second derivative as shown in the next equations.

$$\Gamma^{S,E} = \frac{\partial^2 V^b}{\partial S\, \partial E} = \frac{\frac{V^b_{1,1,1} - V^b_{1,1,-1}}{V^{conv}_{1,1,1} - V^{conv}_{1,1,-1}} - \frac{V^b_{1,-1,1} - V^b_{1,-1,-1}}{V^{conv}_{1,-1,1} - V^{conv}_{1,-1,-1}}}{\left(V^{conv}_{1,1,-1} + \frac{V^{conv}_{1,1,1} - V^{conv}_{1,1,-1}}{2}\right) - \left(V^{conv}_{1,-1,-1} + \frac{V^{conv}_{1,-1,1} - V^{conv}_{1,-1,-1}}{2}\right)}$$

$$\Gamma^{S,E} = \frac{\frac{V^b_{1,1,1} - V^b_{1,1,-1}}{V^{conv}_{1,1,1} - V^{conv}_{1,1,-1}} - \frac{V^b_{1,-1,1} - V^b_{1,-1,-1}}{V^{conv}_{1,-1,1} - V^{conv}_{1,-1,-1}}}{\frac{V^{conv}_{1,1,1} + V^{conv}_{1,1,-1}}{2} - \frac{V^{conv}_{1,-1,1} + V^{conv}_{1,-1,-1}}{2}} \qquad \text{(Eq.10.1.41)}$$

Theta

The rate of change of the convertible bond price with respect to the passage of time is referred to as the ***theta*** sensitivity.

$$\Theta = \frac{V^b_{1,0,0} - V^b_{0,0,0}}{\Delta t^{steps}_{i=1}} \qquad \text{(Eq.10.1.42)}$$

Vega

Vega is defined as the sensitivity of the convertible bond price to changes in the volatility of the underlying stock price. The value $V^{CD,vega1\%,S}$ of the convertible bond is obtained when we re-calculate the price after we have shifted the term structure of the volatility of the stock price by 1% upwards.

$$Vega^S = V^{CD,vega1\%,S} - V^{CD} \qquad \text{(Eq.10.1.43)}$$

We can also calculate a Vega as the sensitivity of the convertible bond price to changes in the volatility of the exchange rate. The value $V^{CD,vega1\%,E}$ of the convertible bond is obtained when we re-calculate the price after we have shifted the term structure of the volatility of the exchange rate by 1% upwards.

$$Vega^{E} = V^{CD,vega1\%,E} - V^{CD} \qquad \text{(Eq.10.1.44)}$$

10.2. Spectrum Analysis

The same approach as in section (8.1) is followed here for the evaluation of the performance of the model. Spectrum analysis for a quantity is initially performed by calculating the value of the quantity for a given range of share price values. However, in addition to the analysis of section (8.1), for the case of the two-dimensional model evaluated in this section, additional spectrum analysis is performed for the same quantities for a given range of exchange rate values.

Share Price Spectrums

The resultant spectrums of the convertible bond price, the option only value, the delta and gamma sensitivities to the stock and the theta sensitivity are presented in figures (F.10.1) to (F.10.5).

Figure (F.10.1) Convertible Bond Price Spectrum

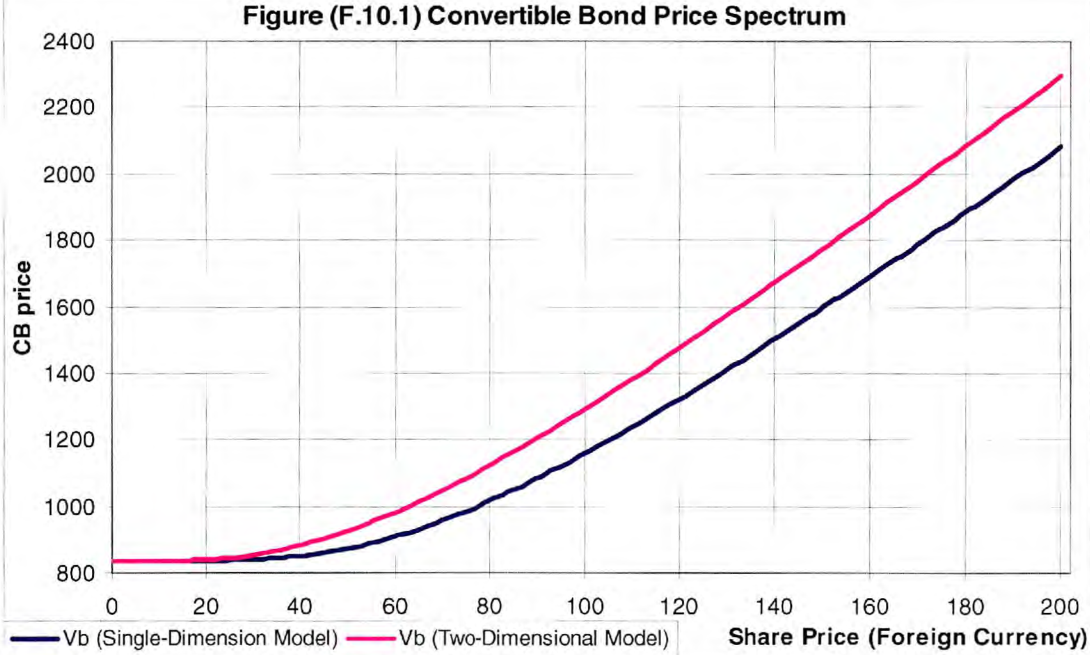


Figure (F.10.2) Option Only Value spectrum

Figure (F.10.3) Delta (to the share price) spectrum

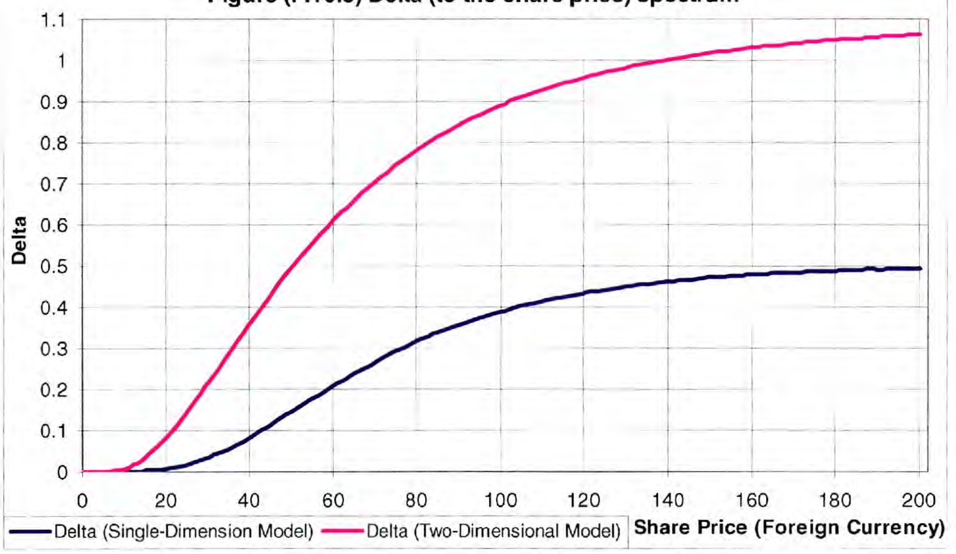


Figure (F.10.4) Gamma (to the share price) spectrum

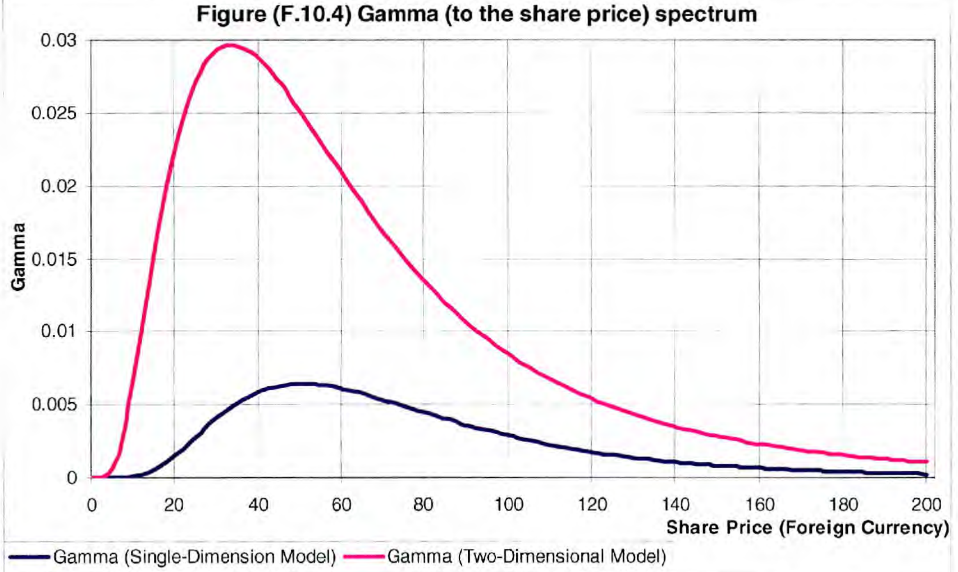
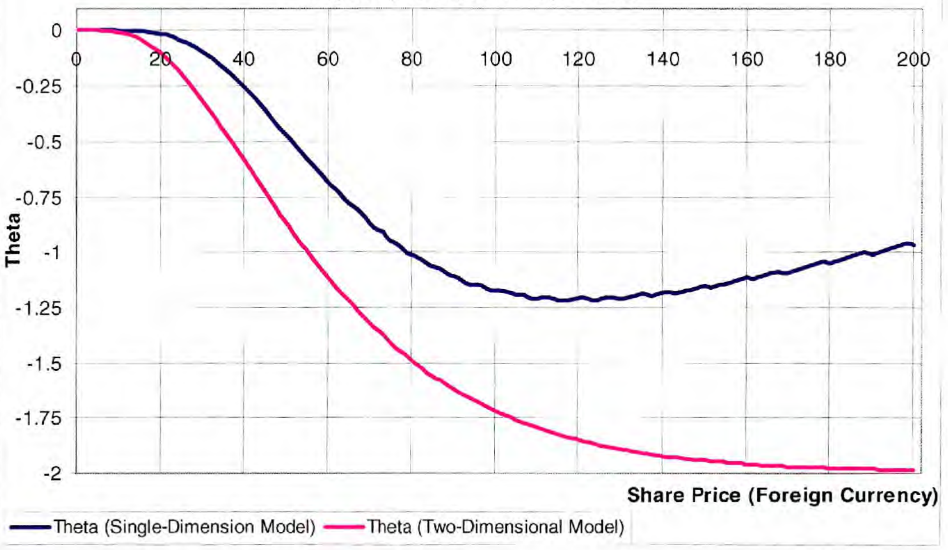


Figure (F.10.5) Theta spectrum



The inclusion of the exchange rate tree has clearly an effect on the spectrums of all the quantities with respect to the share price. The resultant figure for option only value demonstrates clearly the transformation of the additional optionality into option value. This additional optionality corresponds to the possible rewards coming from the upside of the exchange rate. Realisations of the future exchange rate which are greater than the current forward exchange rate result in profits for the investor of the dual currency convertible bond.

The upside of the exchange rate was not accounted for in the one-dimensional configuration. In the case of the one-dimensional model introduced in the previous part of the thesis, Part B, the forward exchange rate was calculated and employed at each step of the tree. However, this configuration did not take into account the volatility of the exchange rate, or, in other words, the fluctuations of the exchange rate.

Here, in the two-dimensional configuration, the fluctuations of the exchange rate and the possible realisations of the exchange rate other than those depicted by the interest rate differentials are accounted for. Realisations of the exchange rate smaller than the forward exchange rate do not result in any additional returns to the investor, but they do not result into losses since the investor will not choose to convert if he could realise losses due to the exchange rate realisation. On the other hand, he can enjoy the upside of the exchange rate. Additional profits could be realised if the future exchange rate realisations are greater than the forward exchange rate. These additional profits are translated into an additional option value. Hence, dual currency convertible bonds inherit some FX derivatives characteristics.

Exchange Rate Spectrums

The resultant spectrums of the convertible bond price, the option only value, the delta sensitivity to the exchange rate and the gamma sensitivity to the exchange rate are presented in figures (F.10.6) to (F.10.9).

Figure (F.10.6) Convertible Bond price spectrum

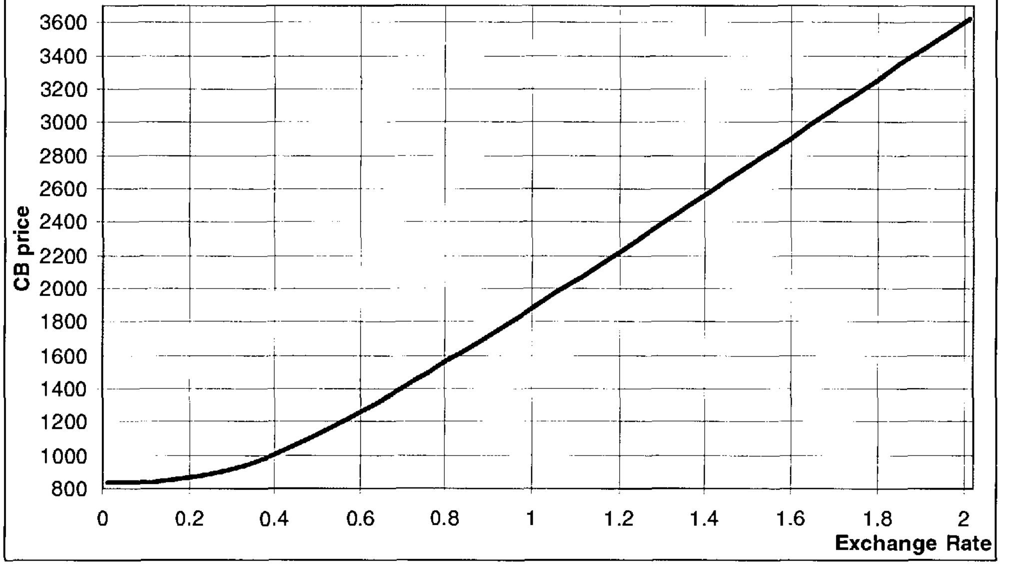
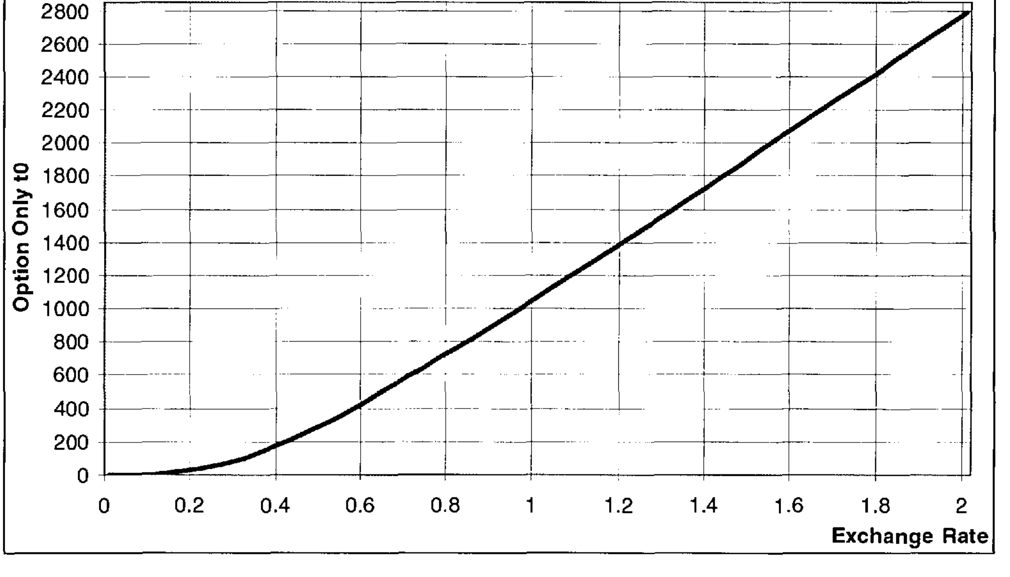
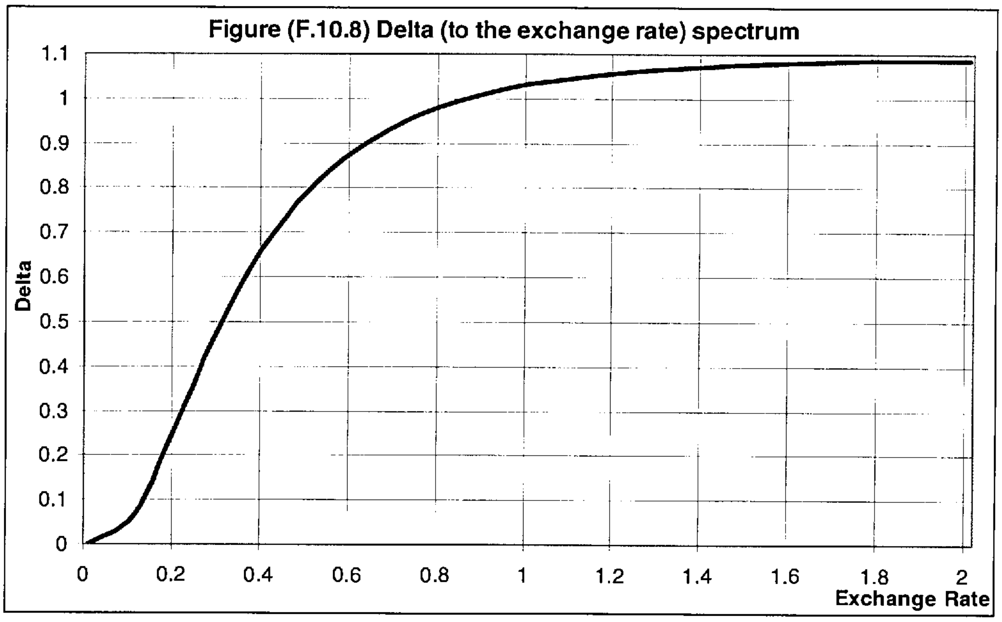
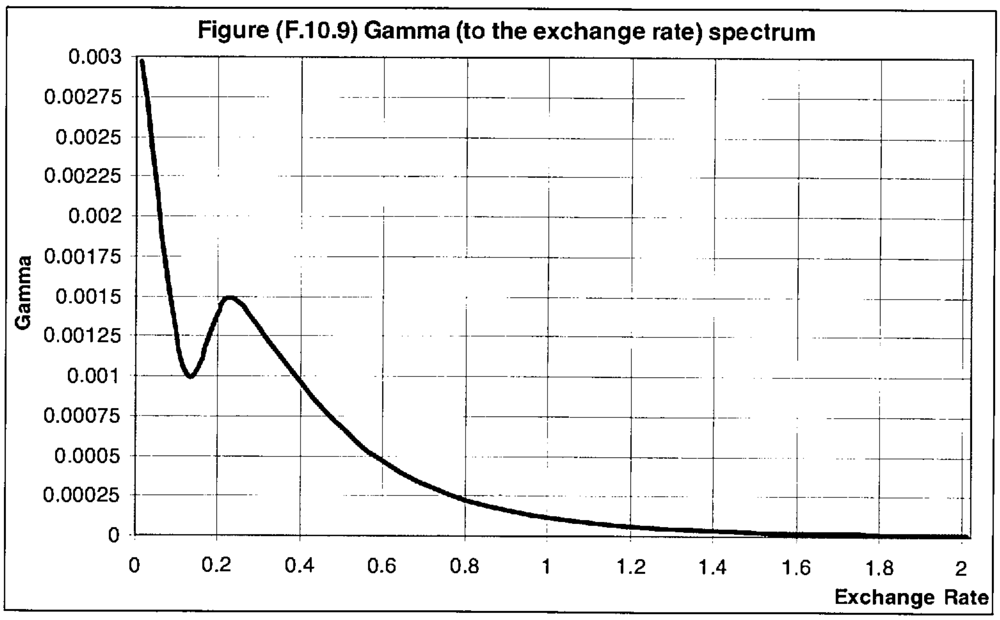


Figure (F.10.7) Option Only Value spectrum







These graphs demonstrate in a more direct way the effect of the exchange rate on the price of the convertible bond. The levels of the sensitivity of the convertible bond price to the exchange rate are shown to the last two graphs. Based on these results, it can be stated that a dual currency convertible bond has a significant sensitivity to the movements of the exchange rate and a position in a dual currency convertible bond should be related to a hedging FX position, in addition to the hedging of the equity exposure which is required for all the types of convertible bonds.

Finally, the surfaces of the convertible bond price and option only value as resulted based on a grid defined by the stock price and the exchange rate, are included in the next two figures.

Figure (10.2.10) The Convertible Bond price surface

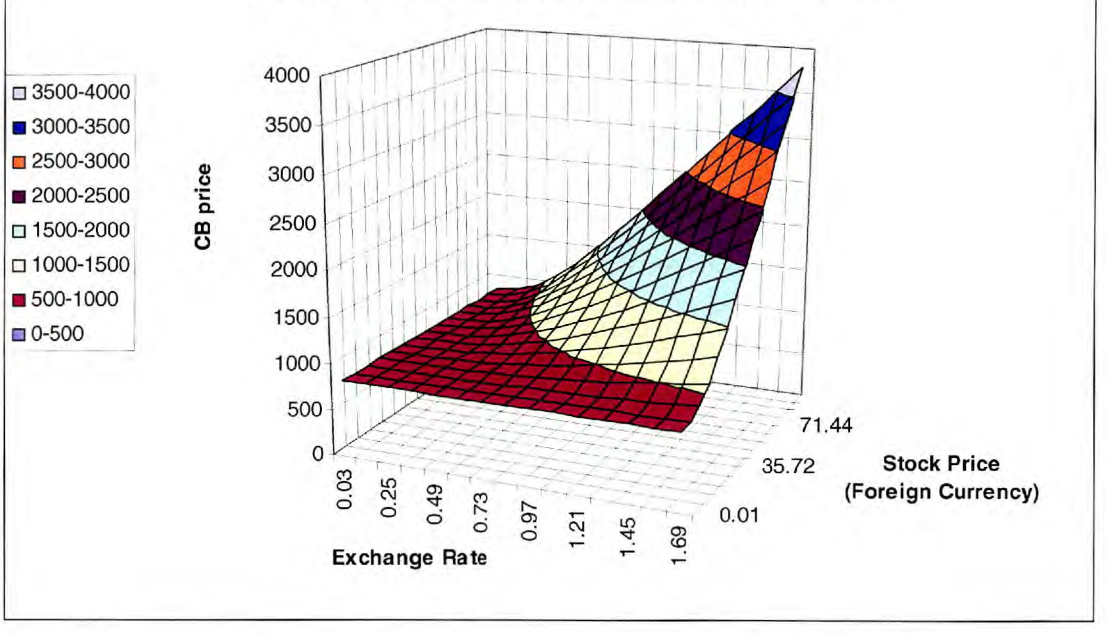
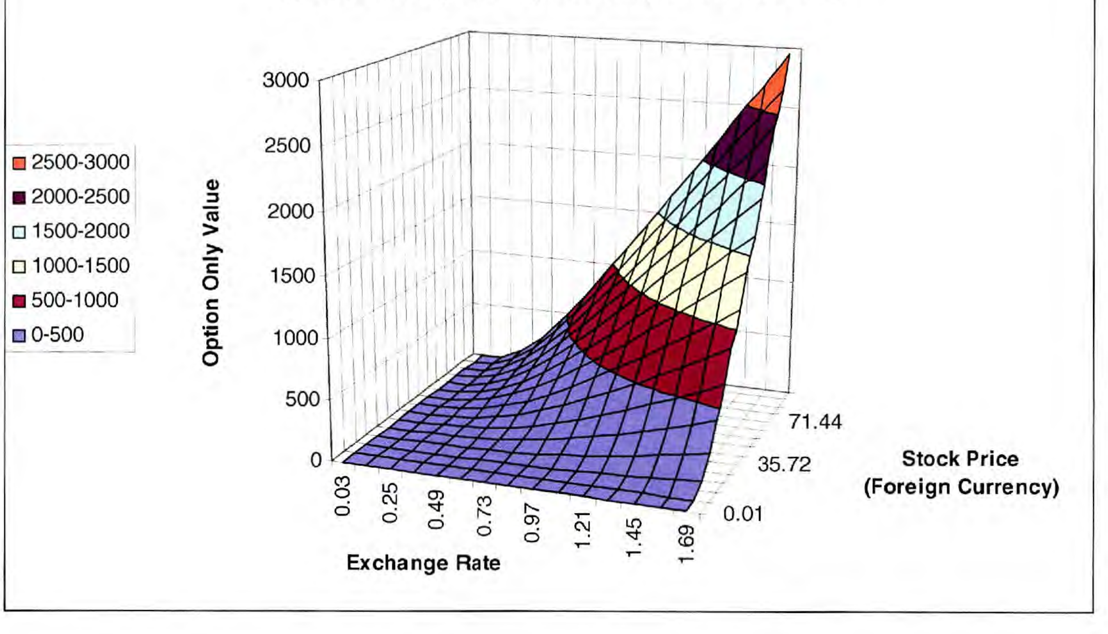


Figure (10.2.11) The Option Only Value Surface



CHAPTER 11

CONCLUSION

The material presented in the previous chapters is summarised in this concluding chapter of the thesis. Then, an effort is made to evaluate the contribution of the work of this thesis and to identify the limitations of the proposed pricing frameworks. The main conclusions are also discussed and recommendations for future work are made.

11.1. Thesis Summary

The convertible bond instrument and the respective market were presented in the introductory chapters. The various convertible structures were listed and categorised based on their attributes. Convertible bond specifications were outlined and an example demonstrated the structure of a standard convertible bond. Chapter 2 finished with the identification of the market sources of risk inherited in a convertible bond structure.

The theoretical foundations of the work presented in this thesis are based on arbitrage pricing. This allows employing market information based on a relative pricing approach. For the purposes of this thesis, market information was required for the risk-free curves, the risky curves, the Equity and FX implied volatilities, and other market based parameters. In chapter 3, it was demonstrated in detail the derivation of the risk free curve for the Bond currency based on market data on the GBP currency. The aim of this chapter was not to present the best or the most popular approach to curve construction. On the contrary, the objective was to demonstrate the importance of having knowledge of the origins and the procedures involved in the extraction of market quantities necessary for the pricing models, and all the assumptions and approximations these procedures introduce. Nevertheless, a technique was chosen that returns smooth zero-coupon continuous rates curves which are well-fitted to market data. In addition, the polynomials used throughout the thesis for interpolating on the Bond and Equity currency rates were derived and presented in this chapter.

Part A of the thesis finished with chapter 4 where the inputs to the CB pricing framework were recognised and the notation of the thesis for the input parameters was defined. The input parameters were also categorised into input parameters based on contractual terms and input parameters based on market parameters.

In the first chapter of Part B, chapter 5, the methodology for defining the number of steps and the respective dates was presented. This methodology aimed in improving the sampling quality of the tree. Having determined the step dates, the calculations for the realisations at the tree steps of all the deterministic parameters that are non-dependent to the stock price realisations, were formulated.

The most volatile factor involved in the pricing of a convertible bond is the price of the underlying equity. The stochastic process for the underlying equity was implemented in the form of a recombining trinomial tree. A trinomial tree implementation involves two basic processes, the forward induction and the backward induction. Chapter 6 was devoted to the forward induction of the trinomial tree process developed and implemented for the purposes of this thesis. The conditions under which the calculations result in valid probabilities were established, because the structure of trinomial trees does not guarantee valid probabilities. Finally, the conditional probabilities were introduced in order to enable accounting for the conditional calls and puts, as well as the resets.

In Chapter 7, the introduction of the one-dimensional pricing framework was completed by presenting the procedures involved in the Backward Induction on the tree and the methods for the calculation of the sensitivities of the instrument. Then, in Chapter 8, the evaluation of the one-dimensional model developed in this thesis was evaluated based on spectrum analysis and scenario analysis.

Two possible configurations for implementing a two dimensional-tree framework were presented in chapter 9. It was decided to follow the configuration with the uncorrelated processes for the stock price and the exchange rate because the configuration with the same correlated processes could not be adjusted to the overall pricing framework presented in this thesis. In chapter 10, the calculations involved in the backward induction for pricing the convertible bond based on the two-dimensional configuration, were outlined. The calculation of the additional sensitivities, as well as the spectrum analysis for this pricing framework, was also presented.

11.2. Contribution

The contributions of this thesis in the area of pricing and analysis of convertible bonds are as follows:

- The traditional trinomial tree configuration which is consistent with the Black-Scholes equation for the stochastic process of a stock price was employed. As part of the contribution of this thesis, this trinomial-tree configuration (which allows for a variable time step and use of term structures of the market parameters) was employed in a unified framework for pricing convertible bonds.
- Methods for establishing the number of steps and the step dates were introduced in this thesis. These methods aim in improving the sampling quality of the pricing numerical technique, with emphasis to the inclusion of the event dates.
- The traditional trinomial tree configuration employed was extended by introducing the conditional tree probabilities in order to deal with path dependency in cases where resets and conditional calls and puts are part of the contract specifications of the instrument.
- Methods for analysing the performance of the model and for studying the behaviour of the convertible bond were introduced in the form of spectrum analysis and scenario analysis. Graphical results were obtained from the implementation of the one dimension model.
- The two-dimensional tree configuration for two correlated processes as described in relevant literature (reference [35]) was considered. As part of this thesis contribution, the equations determining the calculation of the transition

probabilities were re-derived based on the approach followed in the literature for the one-dimensional approach (the traditional trinomial-tree configuration). Most importantly, as part of this thesis, the relationship between the space steps of the two combined trinomial trees and the time step, as well as with the market parameters, was derived as presented in equation (9.1.29). This relationship does not allow variation in the time step or use of term structures of market parameters.

- A two-dimensional structure of two uncorrelated processes was employed. As part of the contribution of this thesis, this configuration which allows for a variable time step and use of term structures of the market parameters was employed in a unified framework for pricing convertible bonds.
- Like in the case of the one-dimensional configuration and as part of this thesis contribution, the two-dimensional tree configuration employed was extended by introducing the conditional tree probabilities in order to deal with path dependency in cases where resets and conditional calls and puts are part of the contract specifications of the instrument.
- Graphical results were obtained from the implementation of the two-dimensional model.

11.3. Conclusions

Based on the results of the spectrum analysis of the one-dimensional configuration many observations were made on the behaviour of the convertible bond and these are discussed next. The inclusion of a coupon schedule affects the valuation of the convertible bond since it introduces discontinuities in the Bond Floor time line to maturity. Discontinuities are also introduced when a discrete dividends schedule is employed, this time on the stock price tree. The resultant spectrums demonstrated that the coupon and discrete schedules must be accounted for if they are part of a convertible bond security.

Calls have a negative effect for the investor's value since they reduce the investor's optionality, while puts have a positive effect for the investor's value since they increase the investor's optionality. These results were anticipated and verified from the respective spectrum analysis. In the case of out of the money convertibles, it was able to specify the negative option value introduced due to the inclusion of the call schedule, while in the case of puts, a positive option value was introduced. If conditionality was present in the schedules, then this reduced the effect of each schedule – less negative option value from the call schedule, less positive option value from the put schedule.

Resets schedules have positive value for the investor if they are resetting the strike downwards and negative value if they are resetting the strike upwards. This was also observed in the results of the spectrum analysis. Introducing a downwards resetting date added value to the investor. However, introducing another reset date which resets upwards and downwards reduced the positive effect of the first reset date.

The resultant spectrums for the extreme cases where all the features were activated demonstrated the limitations of the implementation. The obtained spectrums were not smooth as those resulted from the previous calculations. Increase in the number of steps in order to increase the precision could be considered. However, in all numerical-techniques based pricing frameworks there is always a point where we have to trade-off between precision and speed.

The scenario analysis had two objectives. The first objective was to demonstrate the error introduced because of the fact that the discrete-form implemented configuration was based on conditions derived from continuous-form assumptions and equations. This was demonstrated through the simulations of the theoretical portfolio approach and the result was the probability distribution of the re-hedging error on the maturity. The observed shape of the distribution, which has a significant portion of noise because the number of paths of the Monte Carlo was restricted to 1000 due to very heavy computationally simulations, showed that there is a bias in favour to the investor of the convertible bond. The second objective was to demonstrate the fact that when allowing perfect re-hedging at the re-hedging points, the bias in favour of the investor slightly increases and the probability distribution is less dispersed.

The derivation of the two-dimensional configurations, the correlated and the un-correlated, demonstrated the restrictions introduced when the correlation between the two processes is non-zero. The resultant graphs from the spectrum analysis performed based on the implemented two-dimensional configuration, demonstrated how significant is the effect of exchange rate in the valuation of dual currency convertible bonds.

Finally, it was recognised through this work that instruments like convertible bonds involve complicated processes with the employment of various numerical techniques within the same framework. Consequently, convertible bonds model risk is not negligible and CB market participants should ensure that they comprehend the assumptions and the limitations of the models employed.

11.4. Limitations

Precision of the calculated parameters in numerical techniques is usually a trade off with computational efficiency.

In the case of the one-dimensional configuration, no significant limitations are identified with respect to computational efficiency. Even in the case of embedded conditionality where there are additional computations requirements, the pricing configuration can provide results efficiently and enables the repetitively use of the pricing function in spectrum analysis and scenarios simulation, or in a portfolio with a number of convertible bonds positions.

In the case of the two-dimensional configuration, the computational burden is greater than in the case of the one-dimensional. In the case of embedded conditionality where there are additional computations requirements, this becomes even more evident. Even though it was possible to use the two-dimensional pricing configuration in spectrum analysis efficiently and without sacrificing on precision, it was found that the same pricing function could not be used successfully in scenario simulation because computationally it was very demanding.

11.5. Future Work

Extensions of this work and in general on the existing market approaches for pricing convertible bonds, can be performed. Some of the possible extensions are identified as follows:

- The sampling quality of the tree can always be improved. Methods that are already employed in pricing frameworks of other instruments for defining the optimum number of steps could be employed for maximising the sampling quality of the pricing configurations introduced in this thesis for pricing convertible bonds.
- Robustness of the trinomial tree is reduced by the possibility that negative probabilities could be calculated and employed for some sets of input parameters. This is an issue that all practitioners are faced with when employing trinomial trees and there is on-going research for overcoming this limitation of trinomial trees.
- A solution to overcome the restrictions in the two-dimensional configurations with correlated processes where the employment of variable time step or term structures of market parameters is not allowed.
- Interest rate risk and credit spread risk (or simply credit risk) in convertible bonds were issues not dealt with in this work. There is available research in these areas of convertible bonds. Other techniques could also be considered, like Monte Carlo approaches where the computational requirements increase linearly with the addition of more processes.

In general, a spectrum of techniques and approaches are being considered and studied in the industry for pricing convertible bonds. However, it has to be recognised that the magnitude of the published research and overall literature on this subject is not representative of the volume of the trading activity, and what's more important, the complexity of this instrument. The main reason for this situation could be the fact that valuation of this instrument involves a combination of a number of numerical techniques, constituting this area of quantitative research a challenging subject.

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APPENDIX I

Cubic Splines Implementation

The material presented in this Appendix is extension of the respective material included in references [24], [31] and [32].

A *spline* function provides a solution to situations where continuity of derivatives is a concern when interpolating. A polynomial between each pair of table points and whose coefficients are not determined strictly locally is a spline. The non-locality is not complete but it provides smoothness in the interpolated function up to some order of derivative. Splines have less possibility of wild oscillation between the tabulated points than other polynomial interpolation methods, hence they are considered more stable. Here, we are concerned with the most popular and most implemented spline methods, the *cubic splines*. The interpolated function for a cubic spline is smooth in the first derivative and continuous through the second derivative.

Natural Cubic Spline

Given the known values of equation (Eq.I.1), we want to construct the set of functions shown in equation (Eq.I.2).

$$\begin{array}{ll} x & x_0, x_1, x_2, x_3, \ldots, x_{N-2}, x_{N-1} \\ y & y_0, y_1, y_2, y_3, \ldots, y_{N-2}, y_{N-1} \\ or & \\ \multicolumn{2}{l}{(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_{N-2}, y_{N-2}), (x_{N-1}, y_{N-1})} \end{array} \qquad \text{(Eq.I.1)}$$

$$S(x) = \begin{cases} S_0(x) & x_0 \le x \le x_1 \\ S_1(x) & x_1 \le x \le x_2 \\ \vdots & \\ S_{N-2}(x) & x_{N-2} \le x \le x_{N-1} \end{cases} \qquad \text{(Eq.I.2)}$$

In equation (Eq.I.2), $S_j(x)$ denotes the cubic polynomial that will be used on the subinterval $\lfloor x_j, x_{j+1} \rfloor$.

The conditions are summarised as follows:

a. $S(x)$ is a cubic polynomial, denoted as $S_j(x)$ on the subinterval $[x_j, x_{j+1}]$ for each $j = 0,1,2,\ldots,N-2$.

b. $S(x_j) = y_j$ for each $j = 0,1,2,\ldots,N-1$.

c. $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$ for each $j = 0,1,2,\ldots,N-3$.

d. $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ for each $j = 0,1,2,\ldots,N-3$.

e. $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ for each $j = 0,1,2,\ldots,N-3$.

f. $S''(x_0) = S''(x_{N-1}) = 0$ (free or natural boundary).

We define the numbers $z_{j=0,1,2,\ldots,N-1}$ as follows.

$$z_j = S''(x_j) \qquad j = 0,1,2,\ldots,N-1 \tag{Eq.I.3}$$

Based on condition (f) above, the following holds:

$$z_0 = z_{N-1} = 0 \tag{Eq.I.4}$$

We also define the numbers $h_{j=0,1,2,\ldots,N-2}$.

$$h_j = x_{j+1} - x_j \qquad j = 0,1,2,\ldots,N-2 \tag{Eq.I.5}$$

On the interval $[x_j, x_{j+1}]$, S'' is a linear polynomial taking the values z_j and z_{j+1} at the endpoints. Hence,

$$S''_j(x) = \frac{z_{j+1}}{h_j}(x - x_j) + \frac{z_j}{h_j}(x_{j+1} - x) \qquad [x_j, x_{j+1}] \tag{Eq.I.6}$$

Equation (Eq.I.7) is obtained by integrating twice (Eq.I.6).

$$S_j(x) = \frac{z_{j+1}}{6h_j}(x - x_j)^3 + \frac{z_j}{6h_j}(x_{j+1} - x)^3 + c_j x + d_j \qquad [x_j, x_{j+1}] \tag{Eq.I.7}$$

Another form of the last equation is the following (E_j and F_j are two constants):

$$S_j(x) = \frac{z_{j+1}}{6h_j}(x - x_j)^3 + \frac{z_j}{6h_j}(x_{j+1} - x)^3 + E_j(x - x_j) + F_j(x_{j+1} - x) \qquad [x_j, x_{j+1}] \tag{Eq.I.8}$$

The conditions that $S_j(x_j) = y_j$ and $S_j(x_{j+1}) = y_{j+1}$ are applied to obtain the values of the constants E_j and F_j. The result is shown in equation (Eq.I.9).

$$S_j(x) = \frac{z_{j+1}}{6h_j}(x - x_j)^3 + \frac{z_j}{6h_j}(x_{j+1} - x)^3 + \left(\frac{y_{j+1}}{h_j} - \frac{h_j z_{j+1}}{6}\right)(x - x_j) + \left(\frac{y_j}{h_j} - \frac{h_j z_j}{6}\right)(x_{j+1} - x) \qquad [x_j, x_{j+1}] \tag{Eq.I.9}$$

The last condition to be imposed is that of the continuity in the first derivative. Differentiating equation (Eq.I.9), we obtain equation (Eq.I.10).

$$S_j'(x)=\frac{z_{j+1}}{6h_j}(x-x_j)^2-\frac{z_j}{6h_j}(x_{j+1}-x)^2+\frac{y_{j+1}}{h_j}-\frac{h_jz_{j+1}}{6}-\left(\frac{y_j}{h_j}-\frac{h_jz_j}{6}\right)\qquad [x_j,x_{j+1}] \tag{Eq.I.10}$$

Introducing the parameter b_j as shown in equation (Eq.I.11), then equation (Eq.I.10) can be also be written in the form shown in equation (Eq.I.12).

$$b_j=\frac{1}{h_j}(y_{j+1}-y_j) \tag{Eq.I.11}$$

$$S_j'(x_j)=-\frac{h_jz_{j+1}}{6}-\frac{h_jz_j}{3}+b_j\qquad [x_j,x_{j+1}] \tag{Eq.I.12}$$

In the same fashion we can obtain $S_{j-1}'(x)$ for the interval $[x_{j-1},x_j]$.

$$S_{j-1}'(x_j)=-\frac{h_{j-1}z_j}{6}-\frac{h_{j-1}z_{j-1}}{3}+b_{j-1}\qquad [x_{j-1},x_j] \tag{Eq.I.13}$$

Setting the first derivatives of equations (Eq.I.12) and (Eq.I.13) equal to each other, then equation (Eq.I.14) is obtained (after re-arranging). This equation holds for all $j=1,2,\ldots,N-2$ since we want continuity in the first derivative across all the ranges.

$$h_{j-1}z_{j-1}+2(h_{j-1}+h_j)z_j+h_jz_{j+1}=6(b_j-b_{j-1})\qquad [x_{j-1},x_j] \tag{Eq.I.14}$$

For obtaining further simplification of the last equation, we introduce another two sets of numbers, u_j and v_j for $j=1,2,\ldots,N-2$. The new form is shown in (Eq.I.17).

$$u_j=2(h_{j-1}+h_j) \tag{Eq.I.15}$$

$$v_j=6(b_j+b_{j-1}) \tag{Eq.I.16}$$

$$h_{j-1}z_{j-1}+u_jz_j+h_jz_{j+1}=v_j\qquad [x_{j-1},x_j],\ j=1,2,\ldots,N-2 \tag{Eq.I.17}$$

All the previous steps and equations result in a tridiagonal system of equations that must be solved simultaneously. The system of equations is shown in (Eq.I.18) where the simplicity of the first and last equations of this system is a result of condition (f).

$$\begin{aligned}
&z_1=0\\
&h_0z_0+u_1z_1+h_1z_2=v_1\\
&h_1z_1+u_2z_2+h_2z_3=v_2\\
&\vdots\\
&h_{j-1}z_{j-1}+u_jz_j+h_jz_{j+1}=v_j\\
&\vdots\\
&h_{N-3}z_{N-3}+u_{N-2}z_{N-2}+h_{N-2}z_{N-1}=v_{N-2}\\
&z_{N-1}=0
\end{aligned} \tag{Eq.I.18}$$

The system of equations can also be written in the matrix and vector form of equation (Eq.I.19).

$$\begin{bmatrix} 1 & 0 & & \cdots & & & & & 0 \\ h_0 & u_1 & h_1 & 0 & & \cdots & & & 0 \\ 0 & h_1 & u_2 & h_2 & 0 & \cdots & & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \cdots 0 & h_{j-1} & u_j & h_j & 0 & \cdots & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ 0 & \cdots & & 0 & h_{N-2} & u_{N-1} & h_{N-1} & & 0 \\ 0 & \cdots & & & 0 & h_{N-3} & u_{N-2} & & h_{N-2} \\ 0 & & \cdots & & & & 0 & & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ z_j \\ \vdots \\ z_{N-3} \\ z_{N-2} \\ z_{N-1} \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 \\ v_2 \\ \vdots \\ v_j \\ \vdots \\ v_{N-3} \\ v_{N-2} \\ 0 \end{bmatrix} \qquad \text{(Eq.I.19)}$$

The above system is simplified and reduced into the following form:

$$\begin{bmatrix} u_1 & h_1 & 0 & & \cdots & & & 0 \\ h_1 & u_2 & h_2 & 0 & & \cdots & & 0 \\ 0 & h_2 & u_3 & h_3 & 0 & \cdots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots 0 & h_{j-1} & u_j & h_j & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & h_{N-5} & u_{N-4} & h_{N-4} & & 0 \\ 0 & \cdots & & 0 & h_{N-4} & u_{N-3} & & h_{N-3} \\ 0 & & \cdots & & & h_{N-3} & & u_{N-2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_j \\ \vdots \\ z_{N-4} \\ z_{N-3} \\ z_{N-2} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_j \\ \vdots \\ v_{N-4} \\ v_{N-3} \\ v_{N-2} \end{bmatrix} \qquad \text{(Eq.I.20)}$$

After the above system of equations has been solved, the values z_j will be available. This could be the end of the calculations. However, some additional computations are carried out to bring the results in a more desirable form, the form presented in equation (Eq.I.21) which is the Taylor expansion of $S_j(x)$ about the point x_j.

$$S_j(x) = A_j + B_j(x - x_j) + C_j(x - x_j)^2 + D_j(x - x_j)^3 \qquad \text{(Eq.I.21)}$$

Since the above equation corresponds to the Taylor expansion, then the parameters A_j, B_j, C_j, D_j are defined as follows:

$$A_j = S_j(x_j) \qquad B_j = S_j'(x_j) \qquad C_j = S_j''(x_j) \qquad D_j = S_j'''(x_j) \qquad \text{(Eq.I.22)}$$

So, the first coefficient and the third coefficient are simply:

$$A_j = y_j \qquad \text{(Eq.I.23)}$$

$$C_j = \frac{z_j}{2} \qquad \text{(Eq.I.24)}$$

Working out the coefficient of x^3 in equation (Eq.I.9) will result in finding the fourth coefficient of equation (Eq.I.21) which is the coefficient of the respective parameter.

$$D_j = \frac{1}{6h_j}\left(z_{j+1} - z_j\right) \qquad \text{(Eq.I.25)}$$

For the second coefficient in equation (Eq.I.21), the solution can be found through equation (Eq.I.12).

$$B_j = -\frac{h_j}{6} z_{j+1} - -\frac{h_j}{3} z_j + \frac{1}{h_j}\left(y_{j+1} - y_j\right) \qquad \text{(Eq.I.26)}$$

The form of the solution in equation (Eq.I.21) enables the formulation of the nested form of $S_j(x)$, which is the following:

$$S_j(x) = y_j + (x - x_j)\left(B_j + (x - x_j)\left(\frac{z_j}{2} + \frac{1}{6h_j}(x - x_j)(z_{j+1} - z_j)\right)\right) \qquad \text{(Eq.I.27)}$$

Natural Cubic Splines Implementation Steps

Input values are the N sets of points (x_j, y_j), $j = 0,1,2,\ldots,N-1$ and $x_{j+1} > x_j$.

Step 1

$h_j = x_{j+1} - x_j \qquad j = 0,1,2,\ldots,N-2$

$b_j = \frac{1}{h_j}\left(y_{j+1} - y_j\right) \qquad j = 0,1,2,\ldots,N-2$

Step 2

Setting the triadiagonal system.

$u_0 = v_0 = 0$

$u_1 = 2(h_1 + h_0)$

$v_1 = 6(b_1 - b_0)$

Then, for $j = 2,3,4,\ldots,N-2$:

$$u_j = 2\left(h_j + h_{j-1}\right) - \frac{h_{j-1}^2}{u_{j-1}}$$

$$v_j = 6\left(b_j - b_{j-1}\right) - \frac{h_{j-1}v_{j-1}}{u_{j-1}}$$

Step 3

Solving the triadiagonal system.

$z_0 = z_{N-1} = 0$

Then, for $j = N-2, N-3, N-4, \ldots, 2, 1$:

$$z_j = \frac{v_j - h_j z_{j+1}}{u_j}$$

Step 4

Then, for $j = 0,1,2,\ldots,N-2$:

$$A_j = y_j$$

$$B_j = -\frac{h_j}{6} z_{j+1} - \frac{h_j}{3} z_j + \frac{1}{h_j}(y_{j+1} - y_j)$$

$$C_j = \frac{z_j}{2}$$

$$D_j = \frac{1}{6h_j}(z_{j+1} - z_j)$$

In the four steps above, there is no danger of running into a division by zero. The parameters h_j will always be greater than zero since $x_{j+1} > x_j$, hence $u_j > h_j + h_{j+1} > 0$.

Interpolating

If it is desired to obtain the *y*-value that corresponds to a point (x_m, y_m), $x_0 \le x_m \le x_{N-1}$, then the following steps can be carried out.

Step 1

Establish which equation $S_j(x)$ to use. The answer is in the form $S_k(x)$ and $x_k \le x_m \le x_{k+1}$.

Step 2

Calculate $S_k(x_m)$.

$$h_m = x_m - x_k$$

$$S_k(x_m) = A_k + h_m\left(B_k + h_m\left(\frac{z_k}{2} + \frac{1}{6h_k} h_m (z_{k+1} - z_k)\right)\right)$$