G-STRUCTURES

AND

DUALITY

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To my parents,

for their constant support

Abstract

After introducing the basic structure of string theory, we review some of the geometric tools used to describe string backgrounds, specifically those endowed with a certain amount of supersymmetry, namely special holonomy, G-structures and Hitchin's Generalized Geometry. We also report on relevant recent developments using the latter two approaches. We then proceed to present our proposed extension of these approaches, building in particular on the language of Generalized Geometry, based on a new formalism (for M-theory compactifications on seven-dimensional manifolds) termed Exceptional Generalized Geometry owing to the central role of the exceptional Lie group $E_{7(7)}$. In this context we define an Exceptional Generalized Tangent (EGT) bundle endowed with a non-trivial twisted topology giving rise to a gerbe structure and a generalized Courant Bracket, the Exceptional Courant bracket (ECB). Further we introduce an Exceptional Generalized Almost Complex Structure defining an $SU(8)/\mathbb{Z}_2$ structure on the EGT allowing the definition of an Exceptional Generalized Metric (EGM). These results are the applied to eleven-dimensional supergravity where the $SU(8)/\mathbb{Z}_2$ structure arises from the bosonic sector. We then show how a reduced $SU(7)$ structure corresponds to backgrounds with effective $N = 1$ (off-shell) supersymmetry. Specifically, reformulating eleven-dimensional supergravity in an $N = 1$ $D = 4$ language, we identify the effective superpotential as an $SU(7)$ singlet in the gravitino variation. We then rewrite the resulting expression in a manifestly $E_{7(7)}$ invariant form. Finally we report on preliminary work aimed at formulating the supersymmetry variations in an $E_{7(7)}$ covariant way.

Declaration

The work presented in this thesis was carried out in the Theoretical Physics Group at Imperial College London under the supervision of Dr. Daniel J. Waldram. This thesis has not been submitted for a degree or diploma at any other university.

The work presented here is original, unless otherwise specified. Chapters 4 to 7 are based on work done in collaboration with Dr. Daniel J. Waldram to be reported in "M-theory, exceptional generalized geometry and superpotentials," (to appear shortly). Chapter 8 is similarly based on preliminary work done in collaboration with Dr. Daniel J. Waldram to be reported upon completion in "M-theory, exceptional generalized geometry and supersymmetry variations," (in preparation). Finally appendix B contains a brief description of early parallel research done in collaboration with Dr. Massimo Blasone and Dr. João Magueijo (and the associated publications) on neutrino oscillations and on modified Lorentz invariance.

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Contents

Abstract **3**

Acknowledgements **5**

1 Introduction **9**

1.1 A perturbative Quantum Gravity 9

1.2 String backgrounds: Compactification, supersymmetry and dualities . 13

1.3 Moduli fixing, fluxes and Generalized Geometries 25

2 Supersymmetric backgrounds **31**

2.1 Killing spinors and special holonomy 31

2.1.1 $SU(d/2)$ holonomy backgrounds in d even dimensions 36

2.1.2 Holonomy in seven dimensions: G_2 and beyond 40

2.2 G-Structures 42

2.2.1 $SU(3)$-structures 44

2.2.2 G_2-structures 48

2.2.3 Recent applications of G-structures 51

3 Generalized geometry **54**

3.1 Generalizing complex and Kähler geometry 54

3.2 Pure Spinors 60

3.3 Recent applications in String theory 65

3.3.1 On-shell supersymmetry 65
3.3.2 Off-shell supersymmetry 67

4 Mathematical Interlude: The exceptional Lie group $E_{7(7)}$ 73

4.1 Definition and the **56** representation 73
4.2 The **133** and **912** representations 75
4.3 A $GL(7)$ subgroup 77
4.4 Cliff$(7,0)$ and seven-dimensional spinors 79
4.5 The $SU(8)/\mathbb{Z}_2$ subgroup and spinor indices 81

5 Exceptional Generalized Geometry in seven dimensions 83

5.1 The exceptional generalized tangent space and $E_{7(7)}$ 84
5.2 The exceptional generalized metric and $SU(8)/\mathbb{Z}_2$ structures 89

6 Supersymmetric backgrounds and EGG 93

6.1 Conventions 94
6.2 Effective theories and field decompositions 95
6.3 $N=1$ M-theory backgrounds and $SU(7)$ structures 99
6.4 From G_2 to $SU(3)$ 103

7 The effective superpotential 109

7.1 Generic form of the effective superpotential 109
7.2 An $E_{7(7)}$ covariant superpotential 114

8 Towards manifestly $E_{7(7)}$ covariant supersymmetry variations 118

8.1 From spinors to forms 118
8.2 From forms to spinors 120
8.3 Elements of manifest $E_{7(7)}$ covariance 125

9 Conclusions and further Outlook 130

A Dualities **135**

A.1 Overview of the web of dualities 135

A.2 T-duality 136

A.2.1 T-duality and $O(d, d, \mathbb{Z})$ 136

A.2.2 T-duality and Buscher rules 142

A.3 S-duality and electromagnetic duality 144

A.4 U-duality 147

B Parallel research interests: Neutrino oscillations and modified Lorentz Invariance **151**

Chapter 1

Introduction

In this chapter we review the fundamentals of string theory. Starting from the appearance of a spin 2 particle in the string spectrum, implying the natural inclusion of General Relativity in the theory, we explain the role of string theory as the UV completion of point-particle field theories, particularly those including gravity. We then present the specifics of the theory, emphasizing the role of supersymmetry and dualities. We further describe the process of compactification and in particular we review the early geometric compactification schemes of the heterotic string and eleven-dimensional supergravity. This leads us to consider the moduli problem and its resolution through flux compactifications. Finally we consider the geometric constructs used to characterize the supersymmetric sector of the resulting "landscape", that is G-structures and Hitchin's Generalized Geometry, and outline how the original work presented here proposes to extend them.

1.1 A perturbative Quantum Gravity

String theory (reviewed in [1][2][3]) was introduced in 1968 [4] as an attempt to describe the strong interaction. The so-called Veneziano amplitude successfully reproduced the crossing-symmetry between interaction channels ($s - t$ duality) while maintaining the UV divergences under control by introducing an infinite tower of states (linked to the amplitude's poles) of increasing squared masses M^2 and spin J. These could then be seen to match the nearly linear relationship between M^2 and J for the many hadron resonances appearing in accelerators:

$$M^2 \simeq \frac{J}{\alpha'}$$

where $\alpha' \approx 1\ \mathrm{GeV}^{-2}$ is the Regge slope. It was then pointed out by Nambu [5], Nielsen [6] and Susskind [7] that this was consistent with the quantization of an extended elementary relativistic string.

Despite these successes, it was soon realized that strong interactions could more easily be described within the framework of non-Abelian (Yang-Mills) gauge theories, specifically quantum chromodynamics (QCD)[1].

One of the difficulties that had plagued string theories as models of the strong force was the systematic presence in the spectrum of a massless spin 2 particle [8][9]. This apparent drawback turns into a virtue if one identifies this excitation with the graviton. In fact the existence of this interacting spin 2 mode means that string theory necessarily encompasses Einstein Relativity. In the framework of General Relativity (GR) one tends to think of gravity as a geometric property of (pseudo-)Riemannian manifolds. Perturbatively however one may picture small departures from a fixed background metric as excitations of a spin two gauge field h:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

where κ^2 is proportional to Newton's gravitational constant. Writing down a Lagrangian for this perturbative gravity in analogy with $U(1)$ gauge (Maxwell) theory, h is expected to couple to a rank two symmetric (stress-energy) tensor $T_{\mu\nu}$ playing the role of the conserved current[2] ($\partial^\mu T_{\mu\nu} = 0$).

[1]Ultimately however QCD is expected to emerge from string models along with the other Standard Model (SM) gauge interactions.

[2]which is positive definite for normal matter in accordance with the attractive nature of gravity in absence of a negative-pressure cosmological constant

The most general equation of motion at most quadratic in derivatives and compatible with Lorentz invariance for a spin 2 particle is [10]:

$$-\partial^2 h_{\mu\lambda} - \partial_\mu \partial_\lambda h^\nu_\nu + \partial_\nu \partial_\lambda h^\nu_\mu + \partial_\mu \partial_\nu h^\nu_\lambda - \eta_{\mu\lambda}(-\partial^2 h^\nu_\nu + \partial_\alpha \partial_\beta h^{\alpha\beta}) = 2\kappa^2 T_{\mu\lambda}$$

Unitarity then requires gauge invariance (to remove unphysical non-transverse negative probability modes) under:

$$h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu$$

with λ an arbitrary one-form. So that by gauge-fixing on may set $h^\mu_\mu = 0$ and $\partial_\mu h^{\mu\nu} = 0$ and put the equation of motion in a more standard form:

$$-\partial^2 h_{\mu\nu} = 2\kappa^2 T_{\mu\nu}$$

However this picture is not fully consistent since the stress-energy tensor $T_{\mu\nu}$ does not, as written, capture all the physics involved. The issue is that the stress-energy tensor as it stands only contains contributions from matter fields. However one must include the energy carried by the emitted graviton itself as a quadratic term so that now: $T_{\mu\nu} = T^{matter}_{\mu\nu} + T^h_{\mu\nu}$. This in turn modifies both the gauge transformation and the equation of motion which are now incompatible with the new $T_{\mu\nu}$ which lacks a yet higher-order contribution from "gravity gravitating". Iteration of this process leads to an infinite expansion in h with the final (consistent) field equation being:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

which is none other than Einstein's equation (for a small perturbation around a fixed background) and the gauge transformation sums to a general coordinate transform. This derivation [11] also shows that GR follows uniquely from general covariance bar the possibility of higher (than two) order derivative corrections.

Of course such a field theoretic treatment of gravity is a priori non-renormalisable given simple power-counting arguments, since Newton's constant has a negative power mass dimension. And while pure Einstein gravity was calculated in [12] to be finite at one-loop, it was shown in the same paper that coupling gravity to a scalar field leads to a divergence at first order. One-loop divergences[3] were then shown to exist for gravity coupled to a Maxwell field [13], a Yang-Mills [14] field or a Dirac fermion [15]. What is more even pure gravity diverges at two-loop order [16]. Considering locally supersymmetric field theories[4], as introduced in [17] and [18], improved the situation somewhat [19]–[23] but only delayed divergences: first it was thought they would appear at three-loop order (for a review see [24]), but more recently, using a more refined power-counting based on harmonic superspace, the divergences were shown to arise at five-loop order [25]. Thus aside from the possibility of a non-trivial UV fixed point for Einstein gravity[5] with a corresponding non-perturbative description (see for example [26]) or supergravity[6] these difficulties seem to indicate the need to go beyond field theory and the point-particle paradigm. String theory in particular then acts as the UV completion of standard field theories: at an energy scale given by $E \simeq \frac{1}{\sqrt{\alpha'}}$ (most naturally identified with the Planck scale) it gives rise to an infinite tower of new modes which soften the divergences[7] rather like the infinite number of hadron resonances in the original application of the theory to the strong force with $\alpha' \approx \frac{1}{M^2_{\text{Planck}}}$ now replacing the Regge slope. String theory is thus a successful model for a perturbative quantum gravity on a fixed background.

[3]All divergence calculations usually refer to $D = 4$ but in fact the situation worsens as the dimension increases.

[4]so-called supergravities as they naturally include Einstein gravity

[5]Such a fixed point does exist for gravity in $2 + \epsilon$ dimensions for ϵ small but one may suspect that this relates critically to the remarkable properties of gravity in 2D.

[6]There are recent suggestions [27] that a series of cancellations might in fact make $N = 8$ supergravity UV finite. But this theory is unfortunately of questionable phenomenological use.

[7]In fact it is assumed that string theories are finite. The recent suggestions regarding the possible finiteness of $N = 8$ supergravity if proven true would be an interesting hint as to the truth of this assumption, since this theory may be obtained by compactifying string theory on a six-torus.

1.2 String backgrounds: Compactification, supersymmetry and dualities

Interestingly it turns out that the background on which strings propagate is subjected to a series of restrictions introduced both for mathematical consistency and phenomenological reasons. The original bosonic string theory for example can only be consistently defined in 26 space-time dimensions (i.e. the critical[8] dimension), as first suggested in [28]. The reason for this is best understood by considering string theory formulated as a two-dimensional conformal field theory (CFT) on the string's worldsheet as given by the Polyakov action[9]:

$$S_P = -\frac{T}{2}\int d^2\xi\sqrt{-\det g}\; g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \quad \text{with} \quad \alpha,\beta = 1,2 \quad \text{and} \quad \mu,\nu = 1\cdots 26$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension. Note that the action's name originates in Polyakov's emphasis on its advantages for quantization, but it was in fact introduced independently by Deser and Zumino [34] and Brink, Howe and Di Vecchia [35].

Given the conformal invariance of the action one can choose a gauge such that the worldsheet metric $g_{\alpha\beta} = \eta_{\alpha\beta}$ and we obtain the action for 26 bosonic fields with the space-time indices corresponding to internal symmetries (in fact Lorentz and Poincaré invariance in space-time) from the point of view of the worldsheet. This 2D CFT however suffers potentially from an anomaly of the conformal invariance. This is proportional to the central charge which vanishes if there are 26 fields thus fixing the dimension of space-time. Alternatively one may choose the so-called light-cone gauge quantization (first introduced for the Nambu-Goto action in [36]) procedure by exploiting the residual conformal invariance after the above choice of metric.

[8]There are also non-critical string theories but they exhibit undesirable phenomenological features such as a linearly growing dilaton.

[9]This action is classically equivalent (and quantum mechanically in the critical dimension but not in general) to the more intuitive Nambu-Goto action, proposed independently by Nambu [29], Goto [30] and Hara [31], which is based on extremizing the worldsheet area but is less amenable to the path-integral formalism as employed by Polyakov in [32][33].

The corresponding gauge choice is however non-covariant leading potentially to an anomaly of the Lorentz invariance which again only vanishes in 26 space-time dimensions.

Bosonic string theory remained however phenomenologically unattractive because its spectrum contained only bosonic space-time fields. More damagingly the ground state of this spectrum was a tachyon[10] which was impossible to remove. Both these difficulties are resolved by introducing supersymmetry on the worldsheet in the guise of massless (for conformal invariance) worldsheet-fermions. The remaining tachyon may then be removed using the so-called GSO projection [37][38] with the resulting spectrum now exhibiting space-time as well as worldsheet supersymmetry in the context of what is termed superstring theory (which is now consistent in 10 space-time dimensions).

It is important to emphasize that beyond resolving the technical issues of bosonic string theory, supersymmetry (more precisely space-time supersymmetry) was from its inception expected to play a significant role in any fundamental description of nature [39]: The Coleman-Mandula no-go theorem [40] was thought to limit symmetries of the S-matrix to a tensor product of the Poincaré group and internal symmetries. As a consequence it also implied that there could be no symmetry transformation linking the spin 2 graviton with the spin 1 vector boson carriers (photon, $W^{\pm}$, Z^0) of the other fundamental forces. It was then shown by Haag, Lopuszanski and Sohnius [41] that there was one possible extension scheme provided one allowed graded Lie algebras which were shown to correspond to transformations linking bosons and fermions (the first Super-Poincaré algebra had in fact been introduced for $D = 4$ in [42] and the first supersymmetric field theories were constructed in [43]). These act as intermediate stages for the transformations between vector bosons and graviton. Therefore supersymmetry (SUSY) would appear to be a natural ingredient in any unification

[10]The modern view is that this signifies an instability in the vacuum which would then roll via tachyon condensation to a stable ground-state. Whether this may make bosonic string theory phenomenologically useful is however unclear.

of the SM interactions[11]. Note however that there is a further subtlety to this argument in that the vector bosons only share a multiplet with the graviton in extended supersymmetry. Phenomenologically however only theories with $N = 1$ SUSY admit chiral fermions, which in the SM result for example in parity violating interactions. The above argument might however be retained provided one embeds $N = 1$ SUSY in extended SUSY and introduces a hierarchial breaking of supersymmetry at different scales. This is however only possible in the context of local supersymmetry, that is supergravity[12].

Further SUSY is one possible solution to the hierarchy problem [44]: The mass of the W or the expected mass of the Higgs can in principle not be maintained stable against loop-corrections that would renormalize it to higher energy scales (e.g. GUT scale $\approx 10^{15}$GeV or Planck scale $\approx 10^{19}$GeV) without unacceptable fine-tuning. By balancing contributions from fermion and bosons loops which have opposite sign SUSY makes the loop contributions vanish [45]. In particular quadratic divergences are removed, as for example in loop corrections to the Higgs mass :

$$\Delta m_H^2 = \frac{1}{8\pi^2}(\lambda_S - |\lambda_f|^2)\Lambda_{UV}^2 + \dots .$$

where m_H is the Higgs mass and Λ_{UV} the ultraviolet cut-off. For supersymmetric theories the coupling constants for a fermion and the corresponding scalar are equal ($\lambda = \lambda_S = |\lambda_f|^2$). Remarkably this mechanism still stabilizes the SM masses in the case of soft supersymmetry breaking. This consists in adding either mass terms or couplings with positive mass dimension to a supersymmetric Lagrangian explicitly breaking SUSY. This approach is somewhat unusual but can be understood in terms

[11]It is thus less surprising that coupling constants for 3 of the forces only converge properly at the GUT scale in the presence of SUSY if one assumes that Grand Unification embeds in a larger unification scheme including gravity

[12]The name originates in the fact that theories with local supersymmetry necessarily contain gravity: upon gauging the superalgebra the Poincaré group becomes the algebra of diffeomorphisms. This inevitability of gravity mirrors that of string theory and interestingly superstring theories admit supergravities as the limit where the typical geometric scales are much larger than the string length i.e. in the point-particle approximation.

of traditional spontaneous symmetry breaking in a hidden sector. The remaining divergences are then logarithmic (or polynomial in logarithms for higher order corrections) depending on the masses involved and correspond to the omitted terms in the above equation. If $m_{\rm soft}$ is the highest mass scale involved one finds:

$$\Delta m_H^2 = m_{\rm soft}^2 \left[\frac{\lambda}{16\pi^2} \ln(\Lambda_{UV}/m_{\rm soft}) + \dots \right].$$

Obviously since $m_{\rm soft}$ is of the order of the mass difference between SM particles and their superpartners, the masses of the latter are bounded from above if these models are to succeed. This is the reason behind the belief that superpartners should be accessible at the LHC.

Finally the lightest supersymmetric particle (LSP) is a strong cold dark matter candidate provided R-symmetry is conserved (making it the stable outcome of supersymmetric decay chains). Most models favor the neutralino (a mixed state of the photino, the zino and the higgsino) but a significant minority focusses on the gravitino.

Bearing in mind the relevance of supersymmetry, consider also that phenomenologically space-time is not manifestly ten-dimensional. One standard solution[13] to this problem is to consider space-time to be a (possibly warped) product $M^{1,3} \times X$ of Minkowski space $M^{1,3}$ and an internal manifold X whose volume is sufficiently small to give rise to an effective four-dimensional description. Remarkably this allows string theory to naturally include the Kaluza-Klein mechanism[14] whereby lower-dimensional gauge fields are obtained by dimensional reduction of higher-dimensional geometry (metric field) or internal fluxes (higher dimensional gauge fields).

[13] The Randall-Sundrum [46][47] (and other braneworld) models offer an alternative where a four-dimensional three-brane floats in a ten-dimensional bulk. Another attractive feature of these models is that they may contain an effective Planck scale circumventing the hierarchy problem.

[14] As an aside note that 7 years before Kaluza [48] extracted a (ultimately flawed) unified theory of electromagnetism and gravity by extending GR to five-dimensions (an idea refined by Klein's [49] proposal that the fifth dimension be compact), a similar split of 5D GR into 4D GR and Maxwell theory had been proposed by Nordström [50] but unfortunately largely overlooked.

Consequently the question arises as to how to extract and classify those compactifications which lead to supersymmetric theories in four dimensions. Another question is how to determine the duality relations between apparently distinct compactifications that give rise to the same physics and more specifically the same 4D effective theories. In fact after the discovery of five consistent superstring theories (type I, type IIA/B and heterotic with gauge group $E_8 \times E_8$ or $SO(32)$ [51][52]) in the early 80s in the course of the so-called first superstring revolution, it was realized in the second revolution (early 90s) that all five were linked, at least when suitably compactified[15], by a set of duality relations[53] (see appendix A.1). It was then conjectured that they were but different perturbative expansions around several vacua of a more fundamental non-perturbative[16] theory dubbed M-theory[54]. Interestingly it was proposed [55][56] that one of the duality relations (S-duality on type IIA) also pointed towards 11D $N = 1$ supergravity, the unique locally supersymmetric field theory in eleven dimensions, introduced in [57], which is assumed to be the low-energy limit of M-theory. Compactifications of this theory are referred to (somewhat abusively) as "M-theory compactifications" in the literature. It is these compactifications which are the object of the original work presented in this thesis. The internal manifold X is in this case seven- rather than six-dimensional.

Let us briefly comment on the specifics of these theories. Consider first closed strings: the left- and right-moving modes are decoupled and can be subjected to independent boundary conditions. For the worldsheet bosons these are necessarily periodic but for the worldsheet fermions one has the choice between periodic Ramond (R) boundary conditions or anti-periodic Neveu-Schwarz (NS) boundary conditions leading to four possible choices. Space-time bosons arise from the NS-NS and RR sectors, while space-time fermions arise from the NS-R and R-NS sectors.

[15]The two type II theories are linked by T-duality if compactified on toroidal backgrounds but not in 10D. There are however also dualities for non-toroidal compactifications such as the duality between type IIA compactified on a $K3$ surface and heterotic on a four-torus T^4.

[16]This non-perturbativeness opens up the possibility of making the theory manifestly background-independent i.e. treating space-time itself dynamically (maybe as a sea of strings).

For type IIB the two corresponding gravitini have the same chirality leading two $N = (2, 0)$ chiral supersymmetry in 10D and for type IIA they have opposite chirality with $N = (1, 1)$ non-chiral supersymmetry in 10D. The NS-NS sector is common to both theories and contains the 10D space-time metric G (corresponding to the graviton), the Kalb-Ramond B-field[17] and the scalar dilaton ϕ. The RR sector is made up of differential $p+1$-form potentials with $p+2$ field strength F_{p+2}, generalizations of the Maxwell gauge potential, which couple to charged $p+1$ dimensional hyperplanes known as Dp branes electrically and similarly to $D(6-p)$ branes magnetically[18]. The significance of D-branes as carriers of RR charge was first pointed out in [58] (with some earlier work in [59]) and soon further explored in [60][61] , while D-branes themselves were introduced in [62][63]). Further bosonic fields besides the metric are generically referred to as fluxes. Compactifications where they take non-zero values are consequently termed flux compactifications, otherwise one speaks of (purely) geometric compactifications.

For type IIB the RR fluxes are a 0-form, a 2-form and a 4-form (with self-dual 5-form field strength) and for type IIA a 1-form and a 3-form. The corresponding D-branes have $p = -1, 1, 3, 5, 7$ for type IIB and $p = 0, 2, 4, 6$ for type IIA. The case of $p = -1$ corresponds to a D-instanton, a localized point in space-time, while a $D0$ brane is a point-particle. Under special conditions this list may be completed by space-filling $D9$ branes for type IIB and $D8$ branes[19] for type IIA. There is also a magnetic source for the B-field, the NS five-brane which is not a D-brane. Importantly the endpoints of open strings with $p+1$ Neumann boundary conditions, or alternatively $9-p$ Dirichlet boundary conditions, which give the D(irichlet)-branes their name, are

[17]The B-field is a two-form potential with corresponding field strength H. It is the natural gauge potential associated with a fundamental string in the same way as point-particles naturally act as sources for the one-form gauge potential of Maxwell theory.

[18]The nomenclature follows from the fact that magnetic coupling to a given field strength may be interpreted as electric coupling to its image under Hodge duality, which for the Maxwell potential exchanges electric and magnetic fields.

[19]It does not appear in the analysis of the RR sector as the associated 10-form field strength is non-dynamical.

restricted to live on Dp-branes and the open-string massless modes define a gauge theory on the brane worldvolume (the generalization of the worldsheet).

Given this Type I may be obtained from type IIB by identifying it under its $\mathbb{Z}_2$ symmetry under worldsheet parity exchanging left- and right-movers through a so-called orientifold projection. The resulting theory contains unoriented closed and open strings and the bosonic modes surviving the projection are the metric and the dilaton in the NS-NS sector and the 2-form in the RR sector. Correspondingly type I admits $D1$ (the D-string) and $D5$ branes along with $D9$ branes accounting for open strings with 10 Neumann boundary conditions. The fundamental string is in fact unstable but its life-time is long enough at weak string coupling to appear in the perturbative spectrum. Further the projection leads to $N = 1$ 10D supersymmetry and anomaly cancellation implies an $SO(32)$ gauge group. Heterotic string theories finally combine 26D bosonic string theory for the left-movers with 10D superstring theory for the right-movers. The supplementary modes for the left-movers lead to one-form gauge potentials which are required to transform in $E_8 \times E_8$ or $SO(32)$ to guarantee anomaly cancellations. There are a priori no open strings and no D-branes in heterotic string theory as it seems impossible to set boundary conditions for the mismatched left- and right-movers, although there have been recent suggestions to the contrary in the context of cosmic strings [64].

As $\alpha' \to 0$ all superstring theories admit a supergravity as a limit[20]. In this context the D-branes appear as non-perturbative solitons in the field theory. They are in fact supersymmetric BPS states with masses (in appropriate units) equal to their conserved charges which appear as central charges in the supersymmetry algebra. Their masses go as $\sim \frac{1}{g_s}$ where g_s is the string coupling which is why they do not appear in the perturbative massless spectrum. They do however appear as the images

[20]The corresponding equations of motion are obtained by expanding the β function of the worldsheet action (which must vanish to guarantee conformal invariance) to lowest order in α'. Type IIB does not in fact admit an action corresponding to these equations because of the self-duality condition of the five-form field strength (if said field strength does not vanish).

of some string modes under strong-weak coupling dualities such as S-duality (reviewed in appendix A.3). Further the supergravity corresponding to type IIA can be obtained from 11D supergravity by dimensional reduction on a circle. The 11D supergravity spectrum contains the 11D gravitino, the 11D metric and a 3-form gauge potential A (with field strength F) which reduces to the type IIA B-field and RR 3-form. It also admits solitonic modes, namely the so-called M-theory $M2$ membrane and the $M5$ five-brane. The former gives rise to the fundamental string when wrapping the circle and a $D2$ brane otherwise, while the latter leads to a $D4$ brane and the NS five-brane respectively. A $D0$ brane corresponds to an $M2$ brane shrinking to zero size with momentum along the circle, while a $D6$ brane corresponds to a Kaluza-Klein monopole linked to the circle reduction. In most applications one is interested in purely bosonic supersymmetric backgrounds such that the only non-trivial SUSY variation is that of the gravitino. For later reference we will shall give the Type II gravitino variations explicitly. The former is compactly given in the democratic formulation [65].

$$\begin{aligned}\delta\Psi_M &= D_M\epsilon - \frac{1}{96}\mathrm{e}^{-\phi/2}\left(\Gamma_M{}^{PQR}H_{PQR} - 9\Gamma^{PQ}H_{MPQ}\right)P\epsilon \\ &- \sum_n \frac{\mathrm{e}^{(5-n)\phi/4}}{64\,n!}\left[(n-1)\Gamma_M{}^{N_1\ldots N_n} - n(9-n)\delta_M{}^{N_1}\Gamma^{N_2\ldots N_n}\right]F_{N_1\ldots N_n}\,P_n\,\epsilon\,, \qquad (1.2.1)\end{aligned}$$

with

$$n = 0,2,4,6,8, \quad P = \Gamma_{11} \quad \text{and} \quad P_n = -(\Gamma_{11})^{n/2}\sigma^1 \quad \text{for type IIA}$$

$$n = 1,3,5,7,9, \quad P = -\sigma^3 \quad \text{and} \quad P_n = i\sigma^2 \quad \text{for} \quad n = 1,5,9 \quad \text{and} \quad P_n = \sigma^1 \quad \text{for} \quad n = 3,7$$

for type IIB and where

$$F_n = dC_{n-1} - H \wedge C_{n-3}$$

are the modified RR field strengths and ϵ and Γ^M are the SUSY variation spinor and gamma matrices respectively. We use uppercase indices $M, N, \ldots = 0, \ldots, 9$ for curved ten-dimensional indices and Γ^{11} is the product of the 10D gamma matrices. The 11D equivalent is given in section 6.1 .

Returning to dualities one must add that strictly speaking some of these are merely conjectured, typically because they represent non-perturbative maps between strong and weak coupling regimes making them a priori difficult to test in perturbative string theory. These problems may however be circumvented in the presence of supersymmetry: by making use of SUSY non-renormalization theorems one may extrapolate certain quantities (generically associated with supersymmetric BPS states) from weak to strong coupling and thus submit the dualities to more stringent tests which were all successful (see for example [53] for the case of U-duality, itself reviewed in appendix A.4). More recently interest in generating and classifying supersymmetric backgrounds has been partially motivated by the need to rigorously test another important duality: the AdS-CFT conjecture. The proposed duality (first formulated in [66] with important refinements added in [67][68]) relates type IIB string theory or M theory on spaces which are asymptotically the product of Anti-de Sitter (AdS) space and a compact manifold to a conformal gauge theory.

More precisely in the simplest examples the above asymptotic geometry arises as the near horizon limit of a stack of N coincident branes, typically $D3$ branes for type IIB and $M2$ or $M5$ for M-theory leading to $AdS_5 \times S_5$ and $AdS_4 \times S_7$ or $AdS_7 \times S_4$. These simple examples have been significantly extended - see for example [69] for a configuration with branes at a conical singularity in string and M-theory. Let us however concentrate on the simple case of a $D3$ brane stack. At lowest order in α' (i.e. in the supergravity approximation) open strings stretching between those branes define a $U(N)$ superconformal field theory on their worldvolume. The duality is holographic in nature and links the dynamics in a bulk space-time with an AdS_{d+1} factor with a conformal gauge theory in d dimensions. For $D3$ branes the CFT is $D = 4$ $\mathcal{N} = 4$ super Yang-Mills (SYM) with effective ('t Hooft) coupling $\lambda = g_{YM}^2 N$ with $g_{YM}^2 = 4\pi g_s$ where g_s is the string coupling. The curvature radius of both the AdS factor and the sphere in the near-horizon limit is however given by $R^4 = 4\pi g_s N\alpha'^2$ so that:

$$\frac{l_s}{R} = \lambda^{-\frac{1}{4}}$$

where $l_s = \sqrt{\alpha'}$ is the string length. Assuming the conjecture holds true, it follows that in the supergravity approximation (where strings are point-like $l_s \ll R$ and calculations are quite tractable due to the absence of stringy α' corrections) type IIB string theory correctly describes a strongly coupled gauge theory. This has led to the hope that QCD, which with its large coupling constant is difficult to treat perturbatively, will be amenable to a similar treatment so that one might shed light on phenomena such as confinement and asymptotic freedom. Conversely highly curved string theories for which the supergravity limit no longer applies are expected to be described by gauge perturbation theory. The gauge theory further simplifies in the limit $N \to \infty$ at fixed λ (this is the 't Hooft limit [70] leading to purely planar diagrams) which on the string side corresponds to genus 0 (tree level) order in g_s. This theory has an infinite number of conserved charges and is conjectured to be integrable.

The AdS-CFT correspondence has already passed a series of non-trivial tests, the simplest of which is the agreement of the symmetries in both pictures. On the string side the compact space S^5 and the AdS factor AdS_5 have isometries $SO(6)$ and $SO(4,2)$ which have $SU(4)$ and $SU(2,2)$ as double covers. On the gauge side the former is identified with the R-symmetry group of $N = 4$ SUSY while the latter is the conformal symmetry of the field theory. Supersymmetry is critical in guaranteeing that this last symmetry is not broken by quantum corrections. Ultraviolet divergences then cancel to all orders and the theory is finite. Further adding fermionic generators on both sides one may check that the full symmetry group is $PSU(2,2|4)$. A similar discussion applies to other instances of the conjecture but the previous example is the most tractable. However in all cases supersymmetry plays a central role. Note finally that these are supersymmetric backgrounds with flux as for example the $D3$ branes couple to Ramond-Ramond five forms with N units of flux through the five-sphere.

Returning now to specific compactifications, note that the first supersymmetric backgrounds explored were however chosen to be purely geometric partially for simplicity and partially because of a series of no-go theorems involving flux compactifications. They were found to be special holonomy manifolds, with specifically $SU(3)$ for Calabi-Yau (complex) threefold string theory compactifications [71] and G_2 for M-theory compactifications (introduced, after the construction of the first compact G_2-holonomy manifolds by Joyce [72][73], in [74]). These holonomies lead to $N = 1$ effective theories in 4D for compactifications of heterotic string theory and M-theory (as needed because of the phenomenological requirement of chiral fermions).

Compactifications of heterotic $E_8 \times E_8$ in particular proved very promising in producing realistic models. Using the so-called standard embedding, whereby the $SU(3)$ holonomy group is embedded in E_8, one finds that one of the E_8 factors leads to an E_6 gauge group in four dimensions which may easily contain the standard model gauge group as well as provide an interesting GUT scenario. Further the number of chiral multiplets and thus the number of SM generations is fixed by topological constraints: in particular $h^{1,1}$ such multiplets contain scalars corresponding to the deformation of the (complexified) Kähler structure and $h^{2,1}$ contain scalars linked to complex structure deformations (we will come back to these moduli shortly), where $h^{p,q}$ is the (p,q) Hodge number[21]. The corresponding superpartners are chiral fermions and transform respectively in the **27** and $\overline{\mathbf{27}}$ while the gauge bosons transform in the adjoint **78**. The number of generations is then given by $h^{1,1} - h^{2,1} = \frac{\chi}{2}$ where χ is the Euler number of the manifold. In this scenario all the SM fields would be singlets under the other E_8 and decoupled from massless fields transforming in its adjoint which are in turn singlets of the E_6. Massive fields transforming under both

[21]Hodge numbers are the equivalents in Dolbeault cohomology of the Betti numbers in De Rham cohomology and count cohomology classes of closed (p,q)-forms.

E_8 factors may only appear at energies comparable to the string scale. The light E_6 singlets might however couple gravitationally to the SM and might form the hidden sector central in soft SUSY breaking.

G_2 compactifications were originally less successful owing to the fact that one may not obtain chiral 4D phenomenology from non-chiral 11D supergravity by compactifying on a smooth manifold [75]. Additionally the resulting gauge groups are necessarily Abelian [74]. A hint as to the probable resolution of this problem arises from the strong coupling limit of the a priori more promising heterotic $E_8 \times E_8$ which may be obtained by compactifying 11D supergravity on an interval $S^1/\mathbb{Z}_2$. This compactification is singular and in fact recent work [76, 77, 78] shows that non-Abelian gauge groups may arise from more general [22] singularities of co-dimension four while chiral fermions may live on singularities of co-dimension seven (such fermions are potentially charged under the gauge group). Such singularities may arise when a supersymmetric cycle is shrunk to a point. This has resulted in many efforts aimed at obtaining more realistic phenomenology from G_2 manifolds [79]–[91].

One such application for example relates G_2 holonomy seven-folds to intersecting $D6$ branes in type IIA (see for example [92]): from the M-theory point of view the corresponding background is a product of a flat Minkowski sevenfold and a multi-center Taub-NUT space. This configuration may preserve up to half the maximal supersymmetry (for parallel branes) depending on the angle between the branes. Further for certain brane arrangements and given some amount of residual supersymmetry this geometry may be seen as four-dimensional Minkowski space-time tensored with manifold of special holonomy[23]. Phenomenologically relevant models then correspond to G_2 holonomy giving $N = 1$ in 4D. Further one may interpret the open strings stretching between the branes from the M-theory point of view as membranes wrapping holomorphic embeddings of two-spheres in multi-center Taub-NUT. As the branes

[22] than that arising from the interval

[23] This manifold is not necessarily compact however: for the case of parallel branes this would be multi-center Taub-NUT times $\mathbb{R}^3$.

become coincident the now massless stretched strings define a non-Abelian gauge theory on the brane world-volume (in analogy to the discussion for AdS-CFT) which from an M-theory viewpoint results form the singularities linked to the collapsed cycles. Finally choosing angles carefully one may arrange for the GSO projection to selectively eliminate states of a given chirality leading to chiral fermions.

1.3 Moduli fixing, fluxes and Generalized Geometries

Just like the original Kaluza-Klein ansatz however all these compactifications are plagued by massless scalar fields in the four-dimensional spectrum corresponding to the moduli (adjustable deformation parameters) of the internal manifold. These are not consistent with phenomenology as for example they would lead to an unobserved fifth long-range force. Additionally [93][94] these fields would be at odds with cosmological models: in gravity-mediated hidden sector breaking of supersymmetry the moduli acquire masses of the order of the weak scale and would dominate the energy content after inflation at reheating until energy density dropped to values too low for nucleosynthesis.

However turning on the internal fluxes induces potentials for those moduli (e.g. mass terms [24]) decoupling them from the effective physics at low energies: this mechanism was first proposed in [95] and it now seems that for example in the supergravity limit of Type IIA all moduli may be fixed by fluxes [96]–[99]. The first flux compactifications (for a review see [100]) were proposed as extensions of the Calabi-Yau compactifications for the heterotic string [101, 102, 103] which allowed for a non-zero B-field (the first M-theory equivalent is [104]).

[24] at energy scales potentially significantly higher than the SUSY breaking scale

Initially however compactifications[25] to Minkowski or de Sitter space-times were thought to be ruled out by several no-go theorems [105]–[109]. These apply to general compactifications (more precisely without assuming supersymmetry) and rely solely on rewriting the 4D Einstein equations:

$$e^{-2E}R_4 + T_{\text{flux}} = 2\nabla^2 e^{2E}$$

where R_4 is the 4D Ricci scalar[26], T_{flux} parameterizes the flux contribution to the stress energy tensor and e^{2E} is a warp factor for the 4D metric. Integrating over a compact internal manifold, the right-hand side of the equation vanishes and since $T_{\text{flux}} > 0$, it follows that Minkowski and de Sitter solutions[27] are ruled out. Evasion of these no-go theorems then required the inclusion of localized negative tension sources, such as $O9$ orientifold planes providing a negative contribution to counteract T_{flux} or the addition higher-derivative (stringy) corrections modifying Einstein's equations[28]. The existence of these mechanisms rekindled interest in the rich phenomenology of flux compactifications.

Fluxes tend to reduce the lower-dimensional gauge group and break supersymmetry which is important both to produce semi-realistic string models and open up novel avenues such as the possibility of $N = 1$ vacua from type II theories compactified on (flux-deformed) Calabi-Yau manifolds. Additionally they generically generate a warp factor for the non-compact space which may explain gauge hierarchies [107] (in a similar fashion to Randall-Sundrum models). Also from the point of view of the AdS-CFT correspondence the duals of flux compactifications turn out to be confining gauge theories [110, 111, 112] providing a first step towards the string theory description of QCD. In cosmology finally the flux KKLT proposal [113] realized metastable

[25] including warped compactifications

[26] corresponding to the unwarped metric

[27] Strictly speaking constructions with only one-form flux and $D7$ branes have a zero contribution and may lead to Minkowski solutions.

[28] and thus going beyond the supergravity limit

de Sitter vacua, thus circumventing the old problem of generating universes with positive cosmological constant from string theory. In a similar vein a model for inflation based on flux backgrounds was described in [114]. Despite these many successes flux compactifications are however an embarrassment of riches in that they produce a perplexingly large number of possible solutions, the now infamous "string landscape" [115]. Besides attempts at statistical study [116] one promising approach is to classify supersymmetric theories in the presence of fluxes.

This thesis[29] thus focusses on the description of general supersymmetric backgrounds through a series of geometric structures which extend those found for standard Calabi-Yau (and M-theory) compactifications. At this point one must note the difference between on-shell and off-shell supersymmetric backgrounds. An on-shell background is one for which the supersymmetry variations vanish. Often (for example for NS flux) this together with imposing the Bianchi identity for the fluxes automatically solves the equations of motion (see for example [108] or [117]) so that the background is also on-shell in the more traditional sense. Accordingly this configuration will be a solution corresponding to a given supersymmetric Lagrangian. However a generic solution may spontaneously break some or all of the supersymmetries a priori present in the Lagrangian which are referred to as off-shell supersymmetries. After the introduction of fluxes for example the condition for off-shell supersymmetry in 4D is that of G-structure on the internal manifold (rather than G-holonomy)[30] meaning that the structure group of the frame bundle (of the tangent bundle T) is reduced to G. The existence of a given G-structure is equivalent to that of globally defined G-invariant tensors or equivalently spinors, which translates into off-shell supersymmetry in the dimensionally reduced Lagrangians. On-shell supersymmetry on the other hand corresponds to differential conditions on the invariant

[29]For the remainder of this section we will mostly eschew bibliographic references as the topics addressed will be dealt with in more detail in the following chapters where the references will appear in a more appropriate context.

[30]For a given compactification dimension and number of supercharges in 4D the groups G agree since G-structures reduce to G-holonomy in the limit where the fluxes vanish.

tensors (spinors). In this context the fluxes can be understood as an obstruction (torsion) to the integrability of the G-structure to G-holonomy.

One may however recover integrability by embedding G-structures in structures defined on larger bundles than the tangent bundle T, as introduced in the context of Generalized Geometry. The original proposal - Generalized Complex Geometry (GCG) - was defined on $T \oplus T^*$ where T^* is the cotangent bundle. One of the most remarkable features of GCG is that the B-field is introduced on equal footing with the metric thus for example giving a natural description of the common NS sector of Type II (and Type I) string theories[31]. Further GCG interpolates between symplectic and complex geometry thus providing a natural extension of Kähler and Calabi-Yau backgrounds used for traditional compactifications. Finally GCG admits a natural $O(d,d)$ metric thus naturally incorporating $O(d,d,\mathbb{Z})$, the discrete subgroup appearing in the treatment of T-duality (see appendix A.2.1).

However in analogy with the case of simple G-structures it turns out that the introduction of Ramond-Ramond (RR) fluxes (or alternatively their D-brane sources) acts as an obstruction to the integrability of the new structures as encoded by the differential equations giving on-shell supersymmetric vacua. The original work presented in this thesis is based on the claim that by further extending the bundle considered one may define an integrable structure in the presence of all fluxes. More precisely this is done for M-theory compactifications from which point of view all the string theory fluxes arise from dimensional reduction. Importantly this description hints at a manifestly U-duality covariant formulation of M-theory by incorporating the exceptional group $E_{7,7}$ (see appendix A.4 for the group structure of U-duality). Thus this new approach is termed Exceptional Generalized Geometry (EGG).

The rest of this thesis is divided thus: In chapter 2 we review the geometric description of supersymmetry backgrounds in terms of special holonomy and G-structures, singling out the $SU(d/2)$ and G_2 cases. We also report on some recent applications

[31]The dilaton also appears, usually bundled with the warp factor, in the integrability conditions.

of G-structures. In chapter 3 we present Hitchin's Generalized Geometry, both in terms of Generalized Almost Complex Structures and the equivalent description in terms of pure spinors. Further we outline applications of this formalism to both on-shell and off-shell supersymmetric backgrounds, owing to their direct relevance to the original work presented here. In chapter 4 we summarize the properties of the exceptional Lie group $E_{7(7)}$ relevant to this work. In chapter 5 we define the structure of Exceptional Generalized Geometry (EGG) in seven dimensions. In particular we identify the Exceptional Generalized Tangent bundle (EGT) and its non-trivial twisted topology with the corresponding gerbe structure. Further we also define a Exceptional Courant Bracket (ECB). We then proceed to construct an Exceptional Generalized Metric (EGM) using an Exceptional Generalized Almost Complex Structure (EGACS) defining an $SU(8)/\mathbb{Z}_2$ structure on the EGT. In chapter 6 these results are specifically applied to eleven-dimensional supergravity, for which we arrange all the degrees of freedom in $E_{7(7)}$ or $SU(8)/\mathbb{Z}_2$ representations and explain how the bosonic sector accounts for the aforementioned $SU(8)/\mathbb{Z}_2$ structure in this case. We then show how, upon specializing to backgrounds leading to effective theories with $N = 1$ (off-shell) supersymmetry in four dimensions, one may reduce the structure on the EGT to an $SU(7)$ structure parameterized by a single object ϕ in the **912** representation of $E_{7(7)}$. Further we relate this $SU(7)$ structure to the underlying local $SU(3)$ structure on the ordinary tangent bundle. In chapter 7 we reformulate part of the bosonic sector of eleven-dimensional supergravity according to a structure reminiscent of $N = 1$ $D = 4$ supergravity. In particular we read off the effective superpotential by projecting out the $SU(7)$ singlet in the gravitino variation. We then rewrite the expression obtained in a manifestly $E_{7(7)}$ invariant form in terms of ϕ. In chapter 8 we report on preliminary work to find an $E_{7(7)}$ covariant formulation of the on-shell supersymmetry conditions. Specifically we show the equivalence of the supersymmetry variation equations to a set of constraints on differential forms representing the structure and flux degrees of freedom and suggest how these might be rewritten in a covariant form in terms of an appropriate coset description. Chapter 9 contains our conclusions and proposals for further work. Appendix A reviews string dualities with particular emphasis on the group structure, as relevant to the original work presented here. Finally Appendix B contains a brief summary of early work done on the topics of neutrino oscillations and modified Lorentz invariance and of the corresponding publications.

Chapter 2

Supersymmetric backgrounds

2.1 Killing spinors and special holonomy

A feature of the original Kaluza-Klein compactifications is that the existence of Killing vectors on the internal compact manifold (which thus admits isometries) translates into gauge symmetries in the non-compact space-time. Similarly [118] the existence of Killing spinors leads to various amounts of supersymmetry in the effective lower-dimensional theory[1]. A Killing spinor is one defined to be covariantly constant with respect to the spin connection ω:

$$\nabla_m \eta = \partial_m \eta - \frac{1}{4}\omega_{mnp}\gamma^{np}\eta = 0 \tag{2.1.1}$$

with $m, n, p = 1 \cdots d$ where d is the dimension of the internal space. If η is chosen to be the supersymmetric variation parameter this leads to a supersymmetric vacuum.

[1]Note however that these conditions are sufficient but not necessary. Modern Calabi-Yau compactifications for example lead to several gauged field theories in the absence of any internal isometry. Similarly when introducing G-structures it will become clear that supersymmetry may be achieved without Killing spinors. In particular Killing spinors are only relevant for purely geometric compactifications. Let us emphasize however that the analogy between Killing vectors and Killing spinors does not extend to these modern mechanisms: G-structures are unrelated to the standard methods for generating gauge theories.

Indeed recall that SUSY variations are in general of the form:

$$\delta(\text{fermions}) \sim \text{bosons} \quad ; \quad \delta(\text{bosons}) \sim \text{fermions} \tag{2.1.2}$$

Axiomatically the vacuum is Poincaré invariant and thus only scalar fields can have a non-zero expectation value. In particular all fermionic fields are set to 0 so that the variation of the bosonic fields vanishes identically. For definiteness let us now choose the example of 11-dimensional supergravity compactified to four dimensions. The only fermion is the spin $\frac{3}{2}$ gravitino (for its explicit variation and other conventions see section 6.1). Setting the flux to 0 in the case at hand, results in an eleven-dimensional Killing spinor equation as expected since 11D supergravity has N=1 supersymmetry. As a consequence of compactification the eleven-dimensional Lorentz group is broken from $Spin(1,10) \rightarrow Spin(1,3) \otimes Spin(7)$. It follows that the variation parameter must itself decompose as given in eq. (6.2.7) and further there is a related decomposition for the gamma matrices given in eq. (6.2.6).

Finally decomposing the eleven-dimensional [3] Killing spinor equation accordingly, the equation for the internal spinor η leads to an integrability condition:

$$[\nabla_m, \nabla_n]\eta = \frac{1}{4} R_{mnpq} \gamma^{pq} \eta = 0 \tag{2.1.3}$$

where γ^n are the internal gamma matrices. n solutions to this equation lead to $N = n$ supersymmetry in the four-dimensional theory. Contracting with γ^n on the left and using the fact that $\gamma^n \gamma^{pq} = \gamma^{npq} + g^{np}\gamma^q - g^{nq}\gamma^p$ (g being the internal metric) together with the Bianchi identity for the Riemann tensor $R_{m[npq]} = 0$ we obtain:

$$2R_{mnpq} g^{nq} \gamma^p \eta = 2R_{mp} \gamma^p \eta = 0 \tag{2.1.4}$$

It follows that[2]:

$$g^{mn}\bar{\eta}\gamma^q R_{nq} R_{mp}\gamma^p\eta = 0 \Rightarrow R^{mn}R_{mn} = 0 \tag{2.1.5}$$

which for an Euclidean internal manifold, that is one with positive definite metric, implies

$$R_{mn} = 0 \tag{2.1.6}$$

hence the internal manifold must be Ricci flat and thus a solution to Einstein's equation in vacuum, as expected in the absence of contributions from the fluxes (through the stress-energy tensor).

Geometrically spaces with one or more Killing spinors may be more completely characterized by their special (or reduced) holonomy (for a review see [119]). In general the holonomy group of a manifold $Hol(M)$ is generated by parallel transporting arbitrary spinors (or alternatively tensors) around closed loops $\mathcal{C}$ to generate the holonomy group:

$$U(\mathcal{C})\,\eta = e^{-\frac{1}{4}P\int_{\mathcal{C}} dy^m \omega_{mnp}\gamma^{np}\eta} = e^{P\int_{\mathcal{C}} dy^m \nabla_m \eta} \tag{2.1.7}$$

where P denotes path-ordering and the holonomy group is the set of the $U(\mathcal{C})$ for all closed loops $\mathcal{C}$ on the manifold. It can be further be shown that the holonomy does not depend on the starting point of the loop. Clearly the most general holonomy group in d dimensions will be $Spin(d)$ [3]. The existence of a Killing spinor then implies that the holonomy group is not maximal but must be reduced and lie within its stabilizer group in $Spin(d)$, that is the subgroup leaving the Killing spinor invariant[4]. The pos-

[2]using the fact that $\bar{\eta}\eta$ is constant for a Killing spinor

[3]provided the manifold is spin and spinors may be defined, otherwise the holonomy group will be that for vectors and other tensors i.e. $O(d)$ or, if the manifold is orientable, $SO(d)$ of which $Spin(d)$ is the double cover

[4]The particular spinor is irrelevant since different choices will be related by $Spin(d)$ rotations. The relevant point is the existence of a covariantly constant spinor irrespective of the basis chosen.

sible holonomy groups for irreducible, non-symmetric, simply-connected Riemannian manifolds fall into the 1955 Berger classification [120] (see table 2.1).

Hol(M)	dim(M)	manifold	other properties
$SO(d)$	d	orientable	
$U(n)$	$d = 2n$	Kähler	Kähler
$SU(n)$	$d = 2n$	Calabi-Yau	Kähler , Ricci flat
$Sp(n)$	$d = 4n$	Hyperkähler	Kähler , Ricci flat
$Sp(n)Sp(1)$	$d = 4n$	quaternionic Kähler	Einstein
G_2	$d = 7$	G_2	Ricci flat
$Spin(7)$	$d = 8$	$Spin(7)$	Ricci flat

Table 2.1: Berger classification

Note that Berger's proof relied on subjecting subgroups of $SO(d)$ (as arising in the classification of Lie groups) to a pair of algebraic tests: the groups in Berger's classification are thus only those which are not excluded as holonomy groups and for example it was only shown in 1996 that there exist compact manifolds with holonomy G_2 [72][73] and $Spin(7)$ [121]. In fact even in the non-compact case proof of existence for such spaces only dates back to 1985 [122] with the first explicit examples of complete metrics given in [123]. The restriction to irreducible manifolds i.e. those which are not locally isomorphic to a cartesian product guarantees that the holonomy group falls in an irreducible representation $SO(d)$. This restriction turns out to be somewhat too strong for some applications: for example while $d = 7$ is the canonical dimension for G_2 holonomy according to Berger's classification, a general simply connected Riemannian seven-fold in fact admits holonomies $\{1\} \subset SU(2) \subset SU(3) \subset G_2 \subset SO(7)$. The subgroups of G_2 arise when there is more than one solution to the internal Killing spinor equation and lead to extended supersymmetry for M-theory compactifications to four dimensions (see table 2.3). Geometrically the manifold must then be reducible: the most trivial case is that of M-theory compactified on $S^1 \times Y_6$ where Y_6 is a manifold of (now canonical in six dimensions) $SU(3)$ holonomy, which is equivalent to type IIA on Y_6 and is known to give $N = 2$ supersymmetry in 4D. Symmetric (or even locally symmetric) spaces on the other hand were excluded by Berger since they were fully classified in 1925 by Cartan as an extension of his classification of irreducible Lie algebras and their holonomies were accordingly categorized. In the special case were the symmetric space is a coset G/H the holonomy group was known to be isomorphic to H. This is the reason why the case with $Spin(9)$ holonomy originally proposed by Berger for $d = 16$ was later removed as it was proven that $Spin(9)$ manifolds are necessarily locally symmetric and in fact locally isomorphic to the Cayley projective plane $F_4/Spin(9)$ [124][125].

G_2 and $Spin(7)$ are so-called exceptional holonomies [119] because they do not fall in a generic class. The other classes can be seen as special cases of each other since a manifold of special holonomy G inherits the properties of manifolds with holonomy groups having G as a subset. Therefore since $Sp(n) \subseteq SU(2n) \subset U(2n) \subset SO(4n)$ every Hyperkähler manifold is a Calabi-Yau and every Calabi-Yau is Kähler (and thus complex) and all are orientable manifolds[5]. One may further deduce the special properties of some low-dimensional manifolds from accidental isomorphisms between the holonomy groups. Since $O(2) = U(1)$ every two-dimensional manifold is Kähler and the two-torus T^2 is the only (trivial) two-dimensional Calabi-Yau [6]. Also $Sp(1) = SU(2)$ so that every four-dimensional Calabi-Yau is necessarily Hyperkähler. In fact there is only one such manifold, namely the $K3$ surface. In addition $Sp(1)Sp(1) = SO(4)$ so that every orientable four-manifold is a quaternionic Kähler manifold. Note finally that the holonomy groups appear as automorphism groups of the division algebras where $O(n), U(n), Sp(n)Sp(1), G_2$ (with $SO(n), SU(n), Sp(n), G_2$ the corresponding subgroups of "determinant 1") correspond to $\mathbb{R}^n, \mathbb{C}^n, \mathbb{H}^n, \mathbb{O}^n$ respectively ($\mathbb{O}^n$ also admit $Spin(7)$ as automorphism).

The question now arises as to which holonomies may be associated with geometries admitting a Killing spinor. Given the integrability condition of the internal Killing

[5]Note that quaternionic Kähler manifolds are not in general Kähler but provided the scalar curvature is positive they admit a twistor space which is.

[6]considering only compact spaces: otherwise one must include the complex plane $\mathbb{C}$

spinor equation we must restrict ourselves to the Ricci flat holonomies in table 2.1. In fact it can be shown [126] that no further restriction is needed leading to a classification for orientable simply-connected spin manifolds of dimension $D \geq 3$ with at least one Killing spinor (see table 2.2 where $N(S_+)$ and $N(S_-)$ respectively denote the number of positive and negative chirality spinors given an orienatation[7]). Note

dim(M)	Hol(M)	$N(S_+)$	$N(S_-)$
$d = 4n \quad n \geq 1$	$SU(2n)$	2	0
$d = 4n \quad n \geq 2$	$Sp(n)$	$n+1$	0
$d = 4n+2 \quad n \geq 1$	$SU(2n+1)$	1	1
$d = 7$	G_2	1	0
$d = 8$	$Spin(7)$	1	0

Table 2.2: Killing spinors and holonomy

that the restriction to Ricci flat backgrounds also excludes irreducible Riemannian symmetric spaces (except the real line) which are known to be Einstein with non-zero scalar curvature. Further this classification is again for special holonomy groups in their canonical dimension.

We will now in the next two sections look at Calabi-Yau and G_2 holonomy manifolds in more detail due to their importance in string/M-theory.

2.1.1 $SU(d/2)$ holonomy backgrounds in d even dimensions

Calabi-Yau manifolds can be defined to be even-dimensional manifolds with $SU(d/2)$ holonomy. For definiteness we shall use the Calabi-Yau threefold[8] as an example [3]. For a six-dimensional manifold the maximal holonomy group is $Spin(6)$ which is isomorphic to $SU(4)$. A real spinor would transforms in the **8** of $Cliff(6)$ but this can be reduced under $SU(4)$ to a four-component complex spinor of definite chirality, transforming in the irreducible **4** of $SU(4)$ (the opposite chirality conjugate spinor

[7]reversing the orientation exchanges $N(S_+)$ and $N(S_-)$

[8]A threefold has 3 complex dimensions, hence it is a six-dimensional real manifold.

transforms in the $\bar{\mathbf{4}}$):

$$\mathbf{8} = \mathbf{4} + \bar{\mathbf{4}} \tag{2.1.8}$$

However the largest subgroup of $SU(4)$ admitting a singlet (for which it is the stabilizer group) is $SU(3)$ with the corresponding decomposition of the spinor:

$$\mathbf{4} = \mathbf{1} + \mathbf{3} \tag{2.1.9}$$

so that the holonomy group must lie within this subgroup in the presence of a Killing spinor in 6 dimensions. As already pointed out one immediate consequence is that Calabi-Yau manifolds are Kähler since their holonomy lies within $U(d/2)$, $U(3)$ in the case at hand. Recall that a Kähler manifold [127] is a complex manifold naturally endowed with an hermitian metric such the corresponding fundamental form is closed: The usual definition of a metric as a bilinear symmetric map from T_pX, the tangent plane at a point p, to $\mathbb{R}$ can be naturally extended to a map $g : T_pX^{\mathbb{C}} \times T_pX^{\mathbb{C}} \to \mathbb{C}$ by assuming the bilinearity can be analytically continued to the complex numbers:

$$g(w_{(1)}, w_{(2)}) = g(r + is, u + iv) = g(r, u) - g(s, v) + i\left[g(r, v) + g(s, u)\right] . \tag{2.1.10}$$

where u, v, r, s belong to T_pX. Given a complex structure with corresponding coordinates z^i for the manifold, one may write the components of the complex metric as:

$$g_{ij} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) \quad \text{and} \quad g_{i\bar{j}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^{\bar{j}}}\right) \tag{2.1.11}$$

with $g_{ij} = g_{ji}$, $g_{i\bar{\jmath}} = g_{\bar{\jmath}i}$, $\overline{g_{ij}} = g_{\bar{\imath}\bar{\jmath}}$ and $\overline{g_{i\bar{\jmath}}} = g_{\bar{\imath}j}$[9]. In those coordinates a hermitian metric satisfies the additional condition that $g_{ij} = g_{\bar{\imath}\bar{\jmath}} = 0$. [10] so that:

$$g = g_{i\bar{\jmath}}\, dz^i \otimes d\bar{z}^{\bar{\jmath}} + g_{\bar{\imath}j}\, d\bar{z}^{\bar{\imath}} \otimes dz^j \; . \tag{2.1.12}$$

One may then construct the so-called fundamental form

$$J = i g_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}} \quad \text{such that} \quad dJ = 0 \Leftrightarrow \text{Kähler} \tag{2.1.13}$$

i.e. the fundamental form J is closed for a Kähler manifold, in which case it is called the Kähler form. Note that a given Kähler form can be represented by any element of the corresponding cohomology class, as different choices are related by global holomorphic coordinate changes preserving the complex structure. On the other hand there are moduli linking different Calabi-Yau metrics corresponding to deformations which appear to make the metric non-hermitian. The point is that it can be put in hermitian form by a non-holomorphic coordinate change which thus changes the complex structure (i.e the choice of local complex coordinate patches covering the manifold) and the Kähler form (but still leaving it closed). Finally $dJ = 0$ implies

$$\frac{\partial g_{i\bar{\jmath}}}{\partial z^l} = \frac{\partial g_{l\bar{\jmath}}}{\partial z^i} \tag{2.1.14}$$

with the corresponding complex conjugate equations. Thus locally i.e. on the patch for which the z^i coordinates are valid we may define a complex function K (the Kähler

[9]which follows from the fact that the original metric is a symmetric real matrix

[10]This ensures that the metric is hermitian with respect to the (in this case integrable) almost complex structure $\mathcal{J}$ on T_pX i.e. $g(\mathcal{J}v_{(1)}, \mathcal{J}v_{(2)}) = g(v_{(1)}, v_{(2)})$ where $v_{(1)}, v_{(2)} \in T_pX$ and with $\mathcal{J} : T_pX \to T_pX$ such that $\mathcal{J}^2 = -I$.

potential) such that[11] :

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^{\bar{j}}} \ . \tag{2.1.15}$$

Consider now the of the Levi-Civita connection:

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}) \ . \tag{2.1.16}$$

In complex coordinates one finds that only the purely holomorphic and anti-holomorphic components survive:

$$\Gamma^l_{jk} = g^{l\bar{s}} \frac{\partial g_{k\bar{s}}}{\partial z^j} \quad \text{and} \quad \Gamma^{\bar{l}}_{\bar{j}\bar{k}} = g^{\bar{l}s} \frac{\partial g_{\bar{k}s}}{\partial \bar{z}^{\bar{j}}} \ . \tag{2.1.17}$$

with the corresponding complex Riemann and Ricci tensors:

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{s}} \frac{\partial \Gamma^{\bar{s}}_{\bar{j}\bar{l}}}{\partial z^k} \quad \text{and} \quad R_{\bar{i}j} = R^{\bar{k}}_{\ \bar{i}\bar{k}j} = -\frac{\partial \Gamma^{\bar{k}}_{\bar{i}\bar{k}}}{\partial z^j} \ . \tag{2.1.18}$$

The simplified holonomy of Kähler manifolds precisely follows from the fact that there are no mixed components Christoffel symbols: on a Kähler manifold the complex representation of a generic vector decouples into holomorphic and anti-holomorphic components in a way which is consistent with parallel transport leading to a holonomy group reduced from $SO(d)$ to its subgroup $U(d/2)$ namely $v = v^j \frac{\partial}{\partial z^j} + v^{\bar{j}} \frac{\partial}{\partial \bar{z}^{\bar{j}}}$.

A Calabi-Yau manifold finally is then one for which this is further reduced to $SU(d/2)$ by the topological restriction that the Kähler manifold admit a Ricci flat metric i.e. $R_{i\bar{j}} = -\frac{\partial}{\partial \bar{z}^{\bar{j}}} \Gamma^k_{ik} = 0$, from which one may deduce that the first Chern class c_1 vanishes. Chern classes are topological invariants usually defined in terms of polynomials of the curvature or Ricci form $\mathcal{R} = i R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$. The first Chern class in particular is the cohomology class of the Ricci form, more precisely $c_1 = \frac{1}{2\pi}[\mathcal{R}]$. For Kähler manifolds one may locally write $\mathcal{R} = i\partial\bar{\partial} ln \sqrt{detg}$. This being true globally,

[11]the non-mixed components being 0 by virtue of the metric being hermitian

meaning the Ricci form is exact and the Chern class vanishes, can be shown to be equivalent to the space being Ricci flat. Yau's theorem[128] then guarantees the converse namely that given a compact Kähler manifold with vanishing first Chern class one may always find metric with $SU(d/2)$ holonomy. This is technically very important as many examples of Kähler manifolds are known, which can then be arranged to have the first Chern class vanish, while it is non-trivial to construct a metric knowing solely its holonomy or Killing spinor content. In particular no explicit Calabi-Yau metrics are known but their Ricci-flat Kähler geometry has been widely used to derive a vast series of results concerning this class of manifolds such as the geometry of the corresponding moduli spaces.

2.1.2 Holonomy in seven dimensions: G_2 and beyond

The generic holonomy group in seven dimensions is $SO(7)$ which when restricted to its different subgroups $\mathcal{H}$ [129] leads to different amounts of supersymmetry N (see table 2.3):

$\mathcal{H}$	$8 \rightarrow$	N
$Spin(7)$	$\mathbf{8}$	0
G_2	$\mathbf{1}+\mathbf{7}$	1
$SU(3)$	$\mathbf{1}+\mathbf{1}+\mathbf{3}+\bar{\mathbf{3}}$	2
$SU(2)$	$\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{2}+\bar{\mathbf{2}}$	4
$\mathbf{1}$	$\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}+\mathbf{1}$	8

Table 2.3: Holonomy groups in seven dimensions

Thus M-theory compactifications on G_2 manifolds lead to phenomenologically interesting $N=1$ effective actions in 4D (first developed in [74] [12]). The mathematical

[12] If one generalizes the concept of holonomy to that of other connections beside the Levi-Civita connection the first examples where provided in [130] [131] for the case of the squashed seven-sphere, an analogue of the "round" seven-sphere but with isometry group broken from $SO(8)$ to $SO(5)\times SO(3)$. Strictly speaking one is however dealing with weak G_2 holonomy a special case (like traditional holonomy) of the G_2-structure description presented in the next section. Note also that the round seven-sphere has trivial holonomy with respect to the weak G_2 connection and thus leads to maximal SUSY in 4D like the seven-torus, but with added cosmological constant.

literature on manifolds with G_2 holonomy does not compare with that concerned with Calabi-Yau manifolds partly because there is no equivalent of Yau's theorem and more generally because odd-dimensional manifolds do not admit some of the rich mathematical structures found in even dimension (complex, symplectic or Kähler geometry for example). In fact as already mentioned the first examples of compact G_2 manifolds [72][73] are rather recent (slightly earlier work on the non-compact case may be found in [122][123]), while mathematicians have dealt with Calabi-Yau manifolds since the late 50s.

These G_2 manifolds were obtained by blowing-up singularities of orbifolds of the seven-torus T^7/Γ where Γ is a discrete symmetry group of the torus. These were in fact the first non-trivial examples of compact Ricci-flat (as expected from the Killing spinor integrability condition) odd-dimensional Riemannian manifolds. There is a series of topological restrictions on these manifolds. For example the holonomy will only be precisely G_2 if the fundamental group of the manifold $\pi_1(M)$ is finite. If the manifold admits a finite cover of the form $N_6 \times S^1$ or $N_4 \times T^3$ (where N_m is a compact and simply-connected m-fold) the holonomy will be $F \ltimes SU(3)$ or respectively $F' \ltimes SU(2)$ where F, F' are finite groups. These results are obviously relevant to M-theory compactifications to Calabi-Yau three-folds and the $K3$ surface (or T^4 i.e. the unique four-dimensional compact manifolds with $SU(2)$ holonomy). Further only the Betti numbers $b_0 = b_7 = 1$, $b_2 = b_5$ and $b_3 = b_4$ are non-zero. The number of vector multiplets in the $N = 1$ $D = 4$ theory (which is however non-chiral for smooth manifolds and thus phenomenologically uninteresting) is given by b_2, while the number of scalar multiplets is given by b_3. Additionally there will be $b_2 + b_3$ Majorana fermions (excluding the gravitino). Finally the first Pontryagin p_1 class[13] is known to be non-zero.

[13]Pontryagin classes p_k are related to Chern classes c_k by $p_k(T) = p_k(T, \mathbb{Z}) = (-1)^k c_{2k}(T \otimes \mathbb{C}) \in H^{4k}(M, \mathbb{Z})$ where $c_{2k}(T \otimes \mathbb{C})$ is the $2k$-th Chern class of the complexification of the tangent bundle T of the manifold and $H^{4k}(M, \mathbb{Z})$ is the $4k$-cohomology group with integer coefficients of the manifold.

2.2 G-Structures

G-holonomy can be understood as the integrability condition of a given G-structure (see for example [119] for a review). Technically a G-structure is defined as a principal G-subbundle of the tangent space frame bundle F. F itself is defined as the principal bundle[14] consisting of the set of all ordered bases of vectors one may place on the tangent space at a given point of a manifold. As such it has structure group $GL(d, \mathbb{R})$ for a d-dimensional manifold. The existence of a G-structure means one may consistently define a sub-bundle on which G acts transitively and freely (by construction G must be a subgroup of $GL(d, \mathbb{R})$). Typically this is equivalent to the existence of globally defined and nowhere vanishing G-invariant tensors or equivalently spinors. The existence of the G-structure is to be understood as a topological restriction since the structure group provides the transition functions in the tangent bundle.

The simplest example is that of an $O(d)$ structure with corresponds to a Riemannian metric g which allows to select an orthogonal frame bundle as a subset of F. Another example is that of an $SL(d, \mathbb{R})$ structure corresponding to orientable manifolds defined by the presence of a volume form Ω. The existence of both leads to an $SO(d)$ structure (simply by considering the common subgroup)[15]. Note also that any G-structure where $G \subset G'$ inherits the properties of a G'-structure: in particular any G-structure where $G \subset O(d)$ must admit a metric which can be constructed from the G-invariant tensor(s) although often trough a non-linear map.

In this case, when $G \subset O(d)$, a G-structure is said to be integrable to G-holonomy if its G-invariant tensors are compatible with the Levi-Civita connection i.e. if their covariant derivative vanishes. Note that the Levi-Civita is only one possible connection singled out by the requirement of being metric compatible ($\nabla g = 0$) and torsion-free. The holonomy is then specifically the holonomy of the Levi-Civita con-

[14]A principal bundle is a fiber bundle whose fiber is isomorphic to a group which acts freely and transitively on the fiber itself.

[15]Further if the manifold is a spin manifold the structure group is strictly $Spin(d)$.

nection. Obviously an $SO(d)$ structure is by the definition of the connection trivially integrable leading to the already mentioned result that $SO(d)$ is the generic holonomy group in d dimensions[16].

A slightly less trivial example is afforded by an almost complex structure which is essentially a $GL(d/2, \mathbb{C})$ structure, where d is obviously even in this case. The corresponding invariant tensor defines a map $\mathcal{J} : T_pX \to T_pX$ such that $\mathcal{J}^2 = -I$ on the tangent space at point p. This leads to a split of the complexified tangent bundle $T_{\mathbb{C}} = T_+ \oplus T_-$ where T_+ and T_- are respectively the $+i$ and $-i$ eigenbundle of $\mathcal{J}$. Accordingly differential forms may be decomposed into (p,q) forms i.e. antisymmetric maps on $T_+^p \otimes T_-^q$. If the structure is integrable complex coordinates may be defined on local patches with transition functions given by holomorphic functions and the tangent bundle split translates into a consistent split into holomorphic and anti-holomorphic coordinates and with (p,q) forms having p holomorphic and q anti-holomorphic indices. Locally one may then represent $\mathcal{J}$ in explicit coordinates:

$$J^z{}_{z'} = i\delta^z{}_{z'} \quad J^{\bar{z}}{}_{\bar{z}'} = -i\delta^{\bar{z}}{}_{\bar{z}'} \quad J^{\bar{z}}{}_{z'} = J^z{}_{\bar{z}'} = 0 \tag{2.2.1}$$

The integrability condition corresponds to the vanishing of the Nijenhuis tensor[17]:

$$N^m{}_{np} = J^q{}_n(\partial_q J^m{}_p - \partial_p J^m{}_q) - J^q{}_p(\partial_q J^m{}_n - \partial_n J^m{}_q) \tag{2.2.2}$$

Alternatively this may be phrased in terms of closure of the $\pm i$ eigenbundles of J under the Lie bracket given algebraically as[18]:

$$N(X,Y) = [X,Y] - [\hat{\mathcal{J}}X, \hat{\mathcal{J}}Y] + \hat{\mathcal{J}}[\hat{\mathcal{J}}X, Y] + \hat{\mathcal{J}}[X, \hat{\mathcal{J}}Y] \tag{2.2.3}$$

[16]More generally, including the case when $G \not\subset O(d)$, integrability is given by the vanishing of the intrinsic torsion associated with the structure.

[17]Note that of course $GL(d/2, \mathbb{C}) \not\subset O(d)$ so that the integrability of the almost complex in terms of the Nijenhuis tensor is given by the more general requirement of a vanishing intrinsic torsion mentioned earlier, which does not require referring to a metric.

[18]The previous tensorial expression for the Nijenhuis tensor is obtained by choosing the special case of basis vectors in the Lie bracket.

If an almost complex manifold is further endowed with an hermitian metric g, corresponding as already explained to the requirement $g(\mathcal{J}v_{(1)}, \mathcal{J}v_{(2)}) = g(v_{(1)}, v_{(2)})$ where $v_{(1)}, v_{(2)} \in T_pX$, one has a $U(d/2)$ structure. It then follows from eq. (2.2.1) that the fundamental form, a $(1,1)$ form, is naturally obtained by lowering indices on the almost complex structure with the hermitian metric. The integrability condition for $U(d/2)$ structure is equivalent to the Kähler condition that the fundamental form J (both indices down) be closed. Alternatively this may be phrased in terms of the integrability of two separate structures:

1. A nowhere vanishing two-form is the invariant tensor corresponding to $Sp(d,\mathbb{R})$ symplectic structure. Closure of the fundamental (Kähler) form corresponds to the integrability of the structure and thus every Kähler manifold is symplectic, in line with $U(d/2) \subset Sp(d,\mathbb{R})$.

2. Given the metric compatibility of the Levi-Civita connection the Kähler condition also implies the vanishing of the Nijenhuis tensor. Hence every Kähler manifold is also complex, as expected since $U(d/2) \subset GL(d/2,\mathbb{C})$.

It is worth noting that the integrability of the almost complex structure is not equivalent to the vanishing of its covariant derivative (as $GL(d/2,\mathbb{C}) \not\subset O(d)$), which is a stronger requirement equivalent to the Kähler condition.

2.2.1 $SU(3)$-structures

To describe Calabi-Yau manifolds (following the discussion in [109]) we need to introduce one new object, a $(d/2,0)$ form Ω which defines an $SL(d/2,\mathbb{C})$ structure. Together with a $(1,1)$ form which defines a $Sp(d,\mathbb{R})$ structure we obtain a structure defined by their common subgroup i.e. $SU(d/2)$ provided both structures are compatible, meaning their embeddings in $GL(d,\mathbb{R})$ must be chosen so the overlap is indeed $SU(d/2)$, which leads to algebraic constraints:

$$\frac{i^{\frac{d}{2}(\frac{d}{2}+2)}}{\frac{d}{2}!}J^{\frac{d}{2}} = \quad \Omega \wedge \bar{\Omega} \quad ; \quad J \wedge \Omega = \quad 0 \tag{2.2.4}$$

where $J^{\frac{d}{2}}$ denotes $\frac{d}{2}$ wedge products of the (1,1) form with itself.

Note that both structure groups are subgroups of $SL(d,\mathbb{R})$ and thus naturally define a volume form. The first equation above guarantees that both definitions agree.

The Calabi-Yau condition is then the integrability condition:

$$dJ = 0 \quad ; \quad d\Omega \quad = 0 \tag{2.2.5}$$

In principle one would expect the integrability condition to involve the covariant rather than just the exterior derivatives vanishing. However an $SU(d/2)$ structure implies an $SO(d)$ structure meaning that the pair (J, Ω) defines a metric (though not separately as the individual groups do not lie in $SO(d)$). As a result there is a non-trivial relationship between the pair and the Levi-Civita connection meaning that the above conditions are sufficient to ensure integrability. Note also that since $SL(d/2,\mathbb{C}) \subset GL(d/2,\mathbb{C})$ Ω defines an almost complex structure.

Note finally that the existence of a closed $(d/2, 0)$ form is a Dolbeault cohomology constraint, so that the restriction of Kähler manifolds to Calabi-Yau manifolds is indeed topological.

To link this picture to a description in terms of a Killing spinor η we introduce bilinears:

$$J_{mn} = -i\bar{\eta}\gamma_{mn}\eta, \quad ; \quad \Omega_{mnp} = -i\eta^T\gamma_{mnp}\eta \tag{2.2.6}$$

On can check that these obey the constraints for the pair (J, Ω) using Fierz identities. Further one may check that the integrability condition is equivalent to the Killing spinor equation for η. The existence of a nowhere vanishing η is further equivalent to that of a nowhere vanishing pair (J, Ω).

At this point it is important to note that non-integrable G-structures may still lead to on-shell supersymmetric backgrounds. We will again follow the description in [109] which builds on earlier work in [108] and [132] (together with results in the mathematics literature [133]). For a supersymmetric background with fluxes the SUSY parameter will no longer be covariantly constant with respect to the Levi-Civita connection but instead $\nabla_m^{(T)}\eta = 0$ [19] where $\nabla_m^{(T)}$ is a new connection with torsion. This torsion is then expressed in terms of the fluxes. It is always possible for a given spinor to find a connection with respect to which it vanishes. In the presence of supersymmetry the SUSY variation then implies that the flux terms must be equal to the torsion of that connection, which encodes its deviation with respect to the Levi-Civita connection. For three-form fluxes one further explicitly sees the flux entering as a correction to the spin connection (see for example eq. 1.2.1). Accordingly in the case of $SU(3)$ structure for example (J, Ω) will no longer be closed forms but the RHS of equation 2.2.5 is replaced by an expression in terms of the fluxes or alternatively the torsion. More explicitly we have:

$$\nabla_m^{(T)}\eta = \nabla_m\eta - \frac{1}{4}\kappa_{mnp}\Gamma^{np}\eta = 0 \ , \qquad (2.2.7)$$

where $\kappa_{mnp} \in \Lambda^1 \otimes \Lambda^2$ is the so-called contorsion related to the torsion by:

$$T_{mnp} = \frac{1}{2}(\kappa_{mnp} - \kappa_{nmp}) \qquad (2.2.8)$$

Now recall that

$$\Lambda^2 \simeq so(6) \simeq su(3) \oplus su(3)^{\perp} \qquad (2.2.9)$$

[19]Note that this implies that the norm of η is constant which is consistent with the G-structure requirement that it be nowhere vanishing.

where $so(6)$ and $su(3)$ are the Lie algebras of $SO(6)$ and $SU(3)$ respectively so that the contorsion may be further decomposed as

$$\kappa^{su(3)} + \kappa^0 \quad \text{where} \quad \kappa^{su(3)} \in \Lambda^1 \otimes su(3) \quad \text{and} \quad \kappa^0 \in \Lambda^1 \otimes su(3)^\perp \tag{2.2.10}$$

Now since we know that η is a $SU(3)$ singlet the above equation becomes:

$$\nabla_m \eta = \frac{1}{4}\kappa^0_{mnp}\Gamma^{np}\eta \tag{2.2.11}$$

Thus it is κ^0 the intrinsic contorsion or alternatively $T^0_{mnp} = \frac{1}{2}(\kappa^0_{mnp} - \kappa^0_{nmp})$ the intrinsic torsion which measures the obstruction to the manifold being Calabi-Yau. T^0 can then finally be decomposed into irreducible representations of $SU(3)$ as follows:

$$T^0 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5$$

where the representations $\mathcal{W}_i$ can be given in terms of J and Ω as shown in table 2.4, where the subscript 0 indicates primitivity conditions such as[20] $J \lrcorner \mathrm{d}J_0^{(2,1)} = 0$.

component	interpretation	$SU(3)$-representation
$\mathcal{W}_1$	$J \wedge \mathrm{d}\Omega$ or $\Omega \wedge \mathrm{d}J$	$\mathbf{1} \oplus \mathbf{1}$
$\mathcal{W}_2$	$(\mathrm{d}\Omega)_0^{2,2}$	$\mathbf{8} \oplus \mathbf{8}$
$\mathcal{W}_3$	$(\mathrm{d}J)_0^{2,1} + (\mathrm{d}J)_0^{1,2}$	$\mathbf{6} \oplus \bar{\mathbf{6}}$
$\mathcal{W}_4$	$J \wedge \mathrm{d}J$	$\mathbf{3} \oplus \bar{\mathbf{3}}$
$\mathcal{W}_5$	$\mathrm{d}\Omega^{3,1}$	$\mathbf{3} \oplus \bar{\mathbf{3}}$

Table 2.4: The five classes of the intrinsic torsion of a space with $SU(3)$ structure.

Explicitly the decomposition of $\Lambda^1 \otimes su(3)^\perp$ goes as:

$$(\mathbf{3} + \bar{\mathbf{3}}) \times (\mathbf{1} + \mathbf{3} + \bar{\mathbf{3}}) = (\mathbf{1} + \mathbf{1}) + (\mathbf{8} + \mathbf{8}) + (\mathbf{6} + \bar{\mathbf{6}}) + (\mathbf{3} + \bar{\mathbf{3}}) + (\mathbf{3} + \bar{\mathbf{3}}) \tag{2.2.12}$$

[20]where $\lrcorner$ denotes contraction of the form on the right over all the indices of the form on the left

and

$$\mathrm{d}J \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4, \quad ; \quad \mathrm{d}\Omega \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_5. \tag{2.2.13}$$

Specific classes of manifolds with $SU(3)$ structure are then selected by choosing T^0 to lie in a subset of these representations (see table 2.5 [100]) e.g. it can be shown that the Nijenhuis tensor $N^q_{mn} \in \mathcal{W}_1 \oplus \mathcal{W}_2$ so that choosing these representations to vanish leads to a complex manifold. So-called half-flat manifolds, which appear as mirror duals of Calabi-Yau spaces deformed by electric fluxes [134], are defined by:

$$d\Omega^- = 0 \quad ; \quad d(J \wedge J) = 0 \tag{2.2.14}$$

where $\Omega^\pm$ are the real and imaginary part of Ω and $d\Omega^{+\,2,2}$ is the equivalent on the mirror manifold of the NS 4-form giving the electric fluxes.

Another such class corresponds to manifolds that are conformally Calabi-Yau manifolds. Consistency requires that under a conformal rescaling of the metric $g \longrightarrow \mathrm{e}^{2f}g$ we have $J \longrightarrow \mathrm{e}^{2f}J$ and $\Omega \longrightarrow \mathrm{e}^{\frac{d}{2}f}\Omega$. This leaves W_1, W_2 and W_3 and the linear combination

$$(d-2)W_5 + (-1)^{\frac{d}{2}+1}2^{\frac{d}{2}-2}\frac{d}{2}W_4 \tag{2.2.15}$$

invariant. These must vanish for a conformal Calabi-Yau manifolds as they do for the Calabi-Yau manifolds themselves. This condition is however not sufficient and we must further require that W_4 and W_5 be exact.

2.2.2 G_2-structures

One may now repeat this analysis for G_2-structures [109][135] for M-theory $N = 1$ compactifications. A G_2 structure implies the existence of a nowhere vanishing three-form:

$$\varphi = e^{246} - e^{235} - e^{145} - e^{136} + e^{127} + e^{347} + e^{567}. \tag{2.2.16}$$

Manifold	Vanishing torsion class
Complex	$W_1 = W_2 = 0$
Symplectic	$W_1 = W_3 = W_4 = 0$
Half-flat	$\mathrm{Im}W_1 = \mathrm{Im}W_2 = W_4 = W_5 = 0$
Special Hermitian	$W_1 = W_2 = W_4 = W_5 = 0$
Nearly Kähler	$W_2 = W_3 = W_4 = W_5 = 0$
Almost Kähler	$W_1 = W_3 = W_4 = W_5 = 0$
Kähler	$W_1 = W_2 = W_3 = W_4 = 0$
Calabi-Yau	$W_1 = W_2 = W_3 = W_4 = W_5 = 0$
"Conformal" Calabi-Yau	$W_1 = W_2 = W_3 = 3W_4 - 2W_5 = 0$

Table 2.5: Vanishing torsion classes in special $SU(3)$ structure manifolds [100].

where $e^{abc} = e^a \wedge e^b \wedge e^c$ and a, b, c are frame indices. One may check that φ is invariant under a natural action of $M^a_b \in G_2 \subset SO(7)$ given by $M^a_b e^b$ so that φ defines a principal G_2 sub-bundle of frames $\{e^a\}$ of the frame bundle. This relates to an invariant spinor η through the bilinear:

$$\varphi = i\bar{\eta}\gamma^{pqr}\eta^c \, dx_p \wedge dx_q \wedge dx_r = i\bar{\eta}\gamma^{abc}\eta^c \, e_a \wedge e_b \wedge e_c \tag{2.2.17}$$

provided one chooses a basis such that:

$$\gamma^{1234}\eta = \gamma^{1256}\eta = \gamma^{1357}\eta = -\eta. \tag{2.2.18}$$

where the gamma matrices are those given in eq. (6.2.6) (see also section 6.1 for our conventions).

Since $G_2 \subset SO(7) \subset SL(7, \mathbb{R})$ φ defines a metric and a volume form [72][73][119]:

$$\begin{aligned} g_{mn} &= \det(h)^{-1/9} h_{nm} \, , \\ h_{mn} &= \frac{1}{144} \varphi_{mpq}\varphi_{nrs}\varphi_{tuv}\hat{\epsilon}^{pqrstuv} \, , \\ *_7 1 &= \varphi \wedge *_7 \varphi \end{aligned} \tag{2.2.19}$$

where $\hat{\epsilon}$ is the "pure-number" Levi-Civita pseudo-tensor, which in particular makes no reference to the determinant of the metric and only takes values $(\pm 1, 0)$. h_{AB} is the unique rank two symmetric object that can be constructed from φ up to a conformal rescaling. The factor in 2.2.19 is determined by the requirement that g transform properly as a tensor.

To determine the torsion classes note that the adjoint representation of $SO(7)$ decomposes as $\mathbf{21} \to \mathbf{7} + \mathbf{14}$ where $\mathbf{14}$ is the adjoint representation of G_2. The intrinsic torsion is then described by four classes [136],

$$T^0 \in \Lambda^1 \otimes g_2^\perp = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4, \\ \mathbf{7} \times \mathbf{7} = \mathbf{1} + \mathbf{14} + \mathbf{27} + \mathbf{7}. \tag{2.2.20}$$

Failure to reach integrability is encoded only by the exterior derivatives $d\varphi$ and $d *_7 \varphi$ which are the only non-trivial content of $\nabla_m \varphi$:

$$d\varphi = \mathcal{W}_1 * \varphi + \mathcal{W}_4 \varphi + \mathcal{W}_3 \quad , \quad d * \varphi = \frac{4}{3} \mathcal{W}_4 * \varphi + \mathcal{W}_2 \varphi \ . \tag{2.2.21}$$

which in turn is in line with the torsion classes corresponding to the decomposition of a four and a five-form as representations of $GL(7, \mathbb{R})$ under G_2:

$$\mathbf{35} \to \mathbf{1} + \mathbf{27} + \mathbf{7} \quad , \quad \mathbf{21} \to \mathbf{14} + \mathbf{7}. \tag{2.2.22}$$

where both $\mathbf{7}$s can be shown to be the same as there are four and not five classes.

Explicitly for example:

$$\mathcal{W}_4 \equiv \varphi \lrcorner \mathrm{d}\varphi = - * \varphi \lrcorner \mathrm{d} * \varphi \quad \text{and} \quad \mathcal{W}_1 \ \equiv *(\varphi \wedge \mathrm{d}\varphi) \tag{2.2.23}$$

In analogy with the $SU(3)$ case one may identify conformally G_2 holonomy manifolds by noting that under a conformal rescaling $g \to e^{2f} g$ one has respectively $\varphi \to e^{3f} \varphi$ and $*_7 \varphi \to e^{3f} *_7 \varphi$. This leaves $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$ invariant while

$\mathcal{W}_4 \to \mathcal{W}_4 - 12\mathrm{d}f$. Thus the desired manifolds correspond to choosing $\mathcal{W}_4$ to be exact and having the other classes vanish. Note that this is a special case of integrable manifolds with $\mathcal{W}_2 = 0$ on which one may define a G_2 Dolbeault cohomology [137].

Finally note that on a compact seven-dimensional spin manifold one is guaranteed to have at least an $SU(2)$-structure and thus implicitly an $SU(3)$- and a G_2-structure as was shown in [138] using the fact that any compact orientable seven-manifold naturally admits two linearly independent nowhere vanishing vector fields (which was proved in [139]).

2.2.3 Recent applications of G-structures

Let us now comment on some recent applications of this formalism. The main focus of interest in this area is the classification of supersymmetric solutions and the generation of new backgrounds on which to test the holographic AdS-CFT correspondence. Although we will not review it here one should note the existence of a related approach known as spinorial geometry [140]– [148].

In [132] and [108] the authors described supersymmetric solutions arising from NS five-branes wrapping SLAG cycles. Earlier work in [149] dealt with M-theory membranes wrapping holomorphic cycles. A complete classification of purely bosonic minimal supergravity solutions in five dimensions was then obtained in [150]. The most general type II compactifications on manifolds with $SU(3)$ structure admitting $N = 1$ vacua were constructed in [151, 152, 117, 153, 154]. Of high relevance to holographic duality were a series of works systematically classifying classes of solutions with AdS factors. All type IIB configurations with only the self-dual five-form flux and the metric as non-trivial field content leading to near horizon geometries of $D3$ branes with an AdS_3 factor were analyzed in [155]. An analogue for $M2$ branes with purely electric flux in 11D supergravity giving an AdS_2 factor can be found in [156]. (A class of M-theory compactifications with AdS_3 factors from wrapped M_5 branes were categorized in [157]). In both cases the internal manifold is found to be a warped $U(1)$ fibration over a Kähler manifold of dimension six and eight respectively. The corresponding supergravities preserve $\frac{1}{8}$ of the supersymmetry and are dual to an $N = (0, 2)$ CFT in two dimensions and supersymmetric conformal quantum mechanics with two supercharges respectively.

Methods for an explicit construction of the Kähler manifold were given in [158] (recovering some results from [159] which was itself extended in [160]). These were partially inspired by parallel developments aimed at finding supersymmetric backgrounds with $AdS_5 \times SE_5$ geometry where SE_5 is a five-dimensional Sasaki-Einstein manifold, defined such that a cone over an SE_5 base is a six-dimensional non-compact Calabi-Yau with conical singularity. Such backgrounds were introduced as an extension of the standard $AdS_5 \times S_5$ on which the original AdS-CFT conjecture was defined whereby the round five-sphere is replaced by its squashed equivalent. In the special case of Sasaki-Einstein manifolds one may identify the dual CFT and perform non-trivial checks of the correspondence. These spaces [3] all have $S^2 \times S^3$ topology and can be obtained as S^1 bundles over a four-dimensional Kähler base.

The first known non-trivial example (and for a long time only one besides the trivial case S^5 and quotients of both of these) was $T^{1,1} = SU(2) \times SU(2)/U(1)$ for which the base is simply $S^2 \times S^2$ and which has a metric with $SU(2) \times SU(2) \times U(1)$ isometry. It was only in [161] that a new (in fact infinite) class of such manifolds was found (denoted as $Y^{p,q}$ where p, q are coprime positive integers with $p > q$) with the isometry being now $SU(2) \times U(1) \times U(1)$ and whose $N = 1$ CFT duals were all identified. This result was obtained using the exhaustive classification of all supersymmetric compactifications with a warped AdS_5 factor (this includes the special case of a Minkowski factor) and non trivial flux of 11D supergravity given in [162] and which were found to be parameterized by a one-parameter family of 4D Kähler manifolds. The authors then explicitly described a large class of such solutions given by S^2 bundles over a four-dimensional base which can be either a Kähler - Einstein space with positive curvature or the product of two constant curvature Riemann surfaces. This last group includes spaces of topology $S^2 \times S^2 \times T^2$. $Y^{p,q}$ spaces are obtained by dimensionally reducing on one of the T^2 circles and doing a T-duality on the other (a generalized framework including $AdS_4 \times X_7$ M-theory compactifications was given in [163]). These results were finally extended in [164][165] to a new infinite class denoted $L^{p,q,r}$ with metric isometry $U(1) \times U(1) \times U(1)$.

Chapter 3

Generalized geometry

3.1 Generalizing complex and Kähler geometry

While generalized (complex) geometry (GCG) originally arose from an independent mathematical programme [166][167], its potential for application in physics was already clearly underlined in the seminal paper by Hitchin [168] (with the general formalism presented in detail in [169] ; see also [170] for a coordinate-based treatment). One of the main motivations was indeed to generalize Calabi-Yau manifolds.

The central tenet of GCG is that structures are no longer defined on the tangent bundle T but [1] on $T \oplus T^*$ where T^* is the (dual) cotangent bundle. Significantly this new bundle is endowed with a natural inner product :

$$(X + \xi, Y + \eta) = \frac{1}{2}(i_X \eta + i_Y \xi) \tag{3.1.1}$$

where $X, Y \in T$ and $\xi, \eta \in T^*$. This corresponds to a split signature metric in an

[1] in an even-dimensional space of dimension d

explicit coordinate base (dx^μ, ∂_μ) :

$$\mathcal{I} = \frac{1}{2}\begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \tag{3.1.2}$$

which naturally induces an $O(d,d)$ structure paralleling the discussion of structures defined on T in section 2.2. In analogy with traditional geometry a generalized almost complex structure (GACS) is defined as :

$$\begin{aligned} &\mathcal{J} : T \oplus T^* \to T \oplus T^* \\ &\text{such that} \quad \mathcal{J}^2 = -1_{2d} \\ &\text{and} \quad \mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I} \end{aligned} \tag{3.1.3}$$

where the last line makes the $O(d,d)$ metric hermitian with respect to the almost complex structure so that we now have a $U(d/2, d/2)$ structure. Integrability of this structure is established via a generalization of the Lie bracket known as the Courant bracket:

$$[X+\xi, Y+\eta]_c = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi). \tag{3.1.4}$$

where [,] is the Lie bracket. The GACS is integrable if one may define projectors compatible with the Courant bracket, that is:

$$\Pi_\mp[\Pi_\pm(X+\xi), \Pi_\pm(Y+\eta)]_c = 0 \quad \text{where} \quad \Pi_\pm = \frac{1}{2}(I \pm i\mathcal{J}) \tag{3.1.5}$$

or more explicitly:

$$[X+\xi, Y+\eta]_c - [\mathcal{J}(X+\xi), \mathcal{J}(Y+\eta)]_c + \mathcal{J}[\mathcal{J}(X+\xi), Y+\eta]_c + \mathcal{J}[X+\xi, \mathcal{J}(Y+\eta)]_c = 0$$

which is analogous to the definition of the Nijenhuis N tensor in terms of the Lie bracket as given in eq. (2.2.3).

One may isolate special cases by introducing a useful decomposition of $\mathcal{J}$:

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix}, \tag{3.1.6}$$

where $J : T\mathcal{M} \to T\mathcal{M}$, $P : T^*\mathcal{M} \to T\mathcal{M}$, $L : T\mathcal{M} \to T^*\mathcal{M}$ and $K : T^*\mathcal{M} \to T^*\mathcal{M}$ or in explicit indices $J^\mu{}_\nu$, $L_{\mu\nu}$, $P^{\mu\nu}$ and $K_\mu{}^\nu$. So that $\mathcal{J}^2 = -1_{2d}$ implies:

$$J^\mu{}_\nu J^\nu{}_\lambda + P^{\mu\nu} L_{\nu\lambda} = -\delta^\mu{}_\lambda, \tag{3.1.7}$$

$$J^\mu{}_\nu P^{\nu\lambda} + P^{\mu\nu} K_\nu{}^\lambda = 0, \tag{3.1.8}$$

$$K_\mu{}^\nu K_\nu{}^\lambda + L_{\mu\nu} P^{\nu\lambda} = -\delta^\mu{}_\lambda, \tag{3.1.9}$$

$$K_\mu{}^\nu L_{\nu\lambda} + L_{\mu\nu} J^\nu{}_\lambda = 0. \tag{3.1.10}$$

while the hermiticity of the $O(d,d)$ metric with respect to $\mathcal{J}$ is equivalent to:

$$J^\mu{}_\nu + K_\mu{}^\nu = 0, \qquad P^{\mu\nu} = -P^{\nu\mu}, \qquad L_{\mu\nu} = -L_{\nu\mu} \tag{3.1.11}$$

and finally integrability may be re-written as:

$$J^\nu{}_{[\lambda} J^\mu{}_{\rho],\nu} + J^\mu{}_\nu J^\nu{}_{[\lambda,\rho]} + P^{\mu\nu} L_{[\lambda\rho,\nu]} = 0 \tag{3.1.12}$$

$$P^{[\mu|\nu} P^{|\lambda\rho]}{}_{,\nu} = 0 \tag{3.1.13}$$

$$J^\mu{}_{\nu,\rho} P^{\rho\lambda} + P^{\rho\lambda}{}_{,\nu} J^\mu{}_\rho - J^\lambda{}_{\rho,\nu} P^{\mu\rho} + J^\lambda{}_{\nu,\rho} P^{\mu\rho} - P^{\mu\lambda}{}_{,\rho} J^\rho{}_\nu = 0 \tag{3.1.14}$$

$$J^\lambda{}_\nu L_{[\lambda\rho,\gamma]} + L_{\nu\lambda} J^\lambda{}_{[\gamma,\rho]} + J^\lambda{}_\rho L_{\gamma\nu,\lambda} + J^\lambda{}_\gamma L_{\nu\rho,\lambda} + L_{\lambda\rho} J^\lambda{}_{\gamma,\nu} + J^\lambda{}_\rho L_{\lambda\gamma,\nu} = 0 \tag{3.1.15}$$

Choosing

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix}. \tag{3.1.16}$$

we see for example that eq. (3.1.7) defines J as an almost complex structure and eq. (3.1.12) its integrability in terms of the corresponding Nijenhuis tensor. Similarly taking:

$$\mathcal{J} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{3.1.17}$$

embeds a symplectic structure ω in the GACS with the relevant integrability equations reducing to $d\omega = 0$. In general however a GACS interpolates between symplectic and complex geometry. Interestingly [100] once Calabi-Yau geometry is deformed by the introduction of H_3 NS fluxes to $SU(3)$ structure the resulting compactification manifolds will be mostly complex for type IIB and mostly symplectic for type IIA. Since both theories can be linked by T-duality the appearance of the $O(d,d)$ group structure becomes very suggestive. Further the A- and B-model in topological string theory, which are related by mirror symmetry (which in the Calabi-Yau context reduces to multiple T-duality), have observables depending exclusively on the symplectic and complex structures respectively.

These ingredients suggest the importance of GCG in providing a unified description of compactification beyond the Calabi-Yau case. This is further reinforced by the natural inclusion of the B-field. Note first that the Courant bracket admits a non-trivial (beyond diffeomorphisms which are already present for the Lie bracket) automorphism in terms of a closed two-form b:

$$[e^b(X+\xi), e^b(Y+\eta)]_c = e^b[X+\xi, Y+\eta]_c + i_X i_Y db = e^b[X+\xi, Y+\eta]_c \tag{3.1.18}$$

provided $db = 0$, with the action of b on $T \oplus T^*$ defined as:

$$e^b(X+\xi) = X + \xi + i_X b \tag{3.1.19}$$

and a resulting map on the GACS respecting integrability:

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{3.1.20}$$

The field strength $H = dB$ of the B-field may then be used to twist the Courant bracket:

$$[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta]_c + i_X i_Y H. \tag{3.1.21}$$

which under e^b transforms as:

$$[X + \xi, Y + \eta]_H \to [X + \xi, Y + \eta]_{H+db} \tag{3.1.22}$$

which is again an automorphism for closed b and corresponds to the local gauge ambiguity in choosing the B-field. Note that the twist is generated by an e^B action on $T \oplus T^*$.

This is even more salient if we consider the case of generalized Kähler geometry which is endowed with two GACS $\mathcal{J}_{1,2}$ and a positive definite metric $\mathcal{G}$ on $T \oplus T^*$ such that:

$$[\mathcal{J}_1, \mathcal{J}_2] = 0 \quad ; \quad \mathcal{G} = -\mathcal{J}_1 \mathcal{J}_2 \quad ; \quad \mathcal{G}^2 = 1 \tag{3.1.23}$$

The general form of $\mathcal{G}$ is in fact familiar from the study of T-duality (see eq. (A.2.10)):

$$\mathcal{G} = \begin{pmatrix} -G^{-1}B & G^{-1} \\ G - BG^{-1}B & BG^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} & G^{-1} \\ G & \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}.$$

but with G and B a priori merely symmetric and antisymmetric two-tensors. Choosing the embeddings of the complex and symplectic structure as given in eq. (3.1.16) and eq. (3.1.17) as the pair of GACS satisfies the conditions in eq. (3.1.23) in the special case of a traditional Kähler structure and confirms the interpretation of G as the Riemannian metric on the manifold. This is further clarified (along with the interpretation of B as the Kalb-Ramond field) by linking generalized Kähler geometry and the bi-hermitian target spaces of worldsheet models with $N = (2,2)$ supersymmetry [170]. Note finally that while the above equation suggest that NS-flux backgrounds may be generated by e^B transforms of purely geometric solutions, one has to bear in mind that since B is not closed this does not generate an automorphism of the (twisted) Courant bracket. Thus integrability of the geometric background does not guarantee that of the general one.

By construction the metric $\mathcal{G}$ is compatible[2] with the natural inner product $\mathcal{I}$ on $T \oplus T^*$.

Introducing such a metric (with the third condition in eq. (3.1.23)) reduces the structure group on $T \oplus T^*$ from $O(d,d)$ to $O(d) \times O(d)$. We already pointed out that a GACS reduces the structure to $U(d/2, d/2)$. Taken together we obtain a reduction to $U(d/2) \times U(d/2)$. Integrability of this structure reduces to that of each of the two GACS separately.

At this point let us note that given a topologically non-trivial B, the metric $\mathcal{G}$ cannot in fact be an inner product on sections $X + \xi \in T \oplus T^*$. Instead, the relevant objects are sections of an extension E

$$0 \longrightarrow T^* \longrightarrow E \longrightarrow T \longrightarrow 0 \tag{3.1.24}$$

where one identifies $(X + \xi)_{(\beta)} = \mathrm{e}^{\mathrm{d}\Lambda_{(\alpha\beta)}}(X + \xi)_{(\alpha)}$ on the intersection of two patches $U_\alpha \cap U_\beta$, or in components

$$X_{(\beta)} + \xi_{(\beta)} = X_{(\alpha)} + (\xi_{(\alpha)} + i_{X_{(\alpha)}}\mathrm{d}\Lambda_{(\alpha\beta)}). \tag{3.1.25}$$

[2]that is $\mathcal{I}(\mathcal{G}(X+\xi), \mathcal{G}(Y+\eta)) = \mathcal{I}(X+\xi, Y+\eta)$ or alternatively $\mathcal{G}^t\mathcal{I}\mathcal{G} = \mathcal{I}$

$\Lambda_{(\alpha\beta)}$ is determined by the gauge transformations allowing to globally patch B, which a priori is only locally defined. So on the overlap of two local patches $U_{(\alpha)} \cap U_{(\beta)}$ we have

$$B_{(\beta)} = B_{(\alpha)} + \mathrm{d}\Lambda_{(\alpha\beta)}, \tag{3.1.26}$$

so that $\Lambda_{(\alpha\beta)} = -\Lambda_{(\beta\alpha)}$, while on the triple overlap $U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)}$

$$\Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = \mathrm{d}\Lambda_{(\alpha\beta\gamma)}. \tag{3.1.27}$$

Mathematically this means B is a connection on a gerbe (see for instance [171]). If the flux is quantized $H \in H^3(M, \mathbb{Z})$ then one has $g_{(\alpha\beta\gamma)} = \mathrm{e}^{\mathrm{i}\Lambda_{(\alpha\beta\gamma)}} \in U(1)$ and these elements satisfy a cocycle condition on $U_{(\alpha)} \cap U_{(\beta)} \cap U_{(\gamma)} \cap U_{(\delta)}$

$$g_{(\beta\gamma\delta)} g^{-1}_{(\alpha\gamma\delta)} g_{(\alpha\beta\delta)} g^{-1}_{(\alpha\beta\gamma)} = 1. \tag{3.1.28}$$

Formally the $g_{(\alpha\beta\gamma)}$ define the gerbe, while the $\Lambda_{(\alpha\beta)}$ define a "connective structure" on the gerbe. Together they encode the analogue of the topological data of a $U(1)$ gauge bundle.

There exists an alternative, but equivalent, description of generalized geometry in terms of so-called *Cliff* (d, d) pure spinors which arises naturally from the space-time and particularly supergravity point of view. Given the emphasis on 11D supergravity in our original work, we shall now review this approach, following the treatment in [100].

3.2 Pure Spinors

The traditional representation of Clifford algebras involves its matrix action on spinors which form a representation of $Spin(d)$. However the algebra of *Cliff* (d, d) (for definiteness):

$$\{\tilde{\Gamma}^m, \tilde{\Gamma}^n\} = 0 \ , \qquad \{\tilde{\Gamma}^m, \Gamma_n\} = \delta^m_n \ , \qquad \{\Gamma_m, \Gamma_n\} = 0 \ . \tag{3.2.1}$$

(where $_{m,n} = 1 \ldots d$ and $^{m,n} = d + 1 \ldots 2d$ are not covariant and contravariant but independent indices) has another representation[3] in terms of an action on forms which are in one-to-one correspondence with the aforementioned spinors:

$$\tilde{\Gamma}^m = dx^m \wedge \ , \qquad \Gamma_n = i_n \tag{3.2.2}$$

with i_n: $\Lambda^p T^* \longrightarrow \Lambda^{p-1} T^*$, $i_n dx^{i_1} \wedge \ldots \wedge dx^{i_p} = p\, \delta_n^{[i_1} dx^{i_2} \wedge \ldots \wedge dx^{i_p]}$. Thus we may write the natural Clifford action of $X + \xi \in T \oplus T^*$ on a spinor Φ:

$$(X + \xi) \cdot \Phi = i_X \Phi + \xi \wedge \Phi \tag{3.2.3}$$

Consequently applying the anti-commutator of two such actions leads to:

$$(Y + \eta) \cdot [(X + \xi) \cdot \Phi] + (X + \xi) \cdot [(Y + \eta) \cdot \Phi] = 2(X + \xi, Y + \eta)\Phi \tag{3.2.4}$$

confirming that the generic inner product on the Clifford algebra coincides with the natural inner product on $T \oplus T^*$ with respect to $\mathcal{I}$ denoted here as (,) .

It follows that the inner product vanishes on L_Φ the sub-bundle of $T \oplus T^*$ containing the annihilators of Φ which is thus isotropic. If it is maximal isotropic (i.e. has maximal dimension d) Φ is a pure spinor. The existence of a maximal subbundle implies the existence of a projector and hence an endomorphism on $T \oplus T^*$ squaring to -1, while isotropy then implies the hermiticity condition in 3.1.3. There is thus a one to-one correspondence between a pure spinor and a GACS with the annihilator sub-bundle of the former being the $\pm i$ eigenbundle of the latter.

[3]Note that this is essentially the algebra of d pairs of fermionic creation and annihilation operators.

As a an example consider the (3,0) from $\Omega = dz^1 \wedge dz^2 \wedge dz^3$. It is annihilated by the sub-bundle with action:

$$\tilde{\Gamma}^{z^i} = dz^i \wedge \,, \qquad \Gamma_{\bar{z}_i} = i_{\frac{\partial}{\partial \bar{z}^i}} \tag{3.2.5}$$

with $i = 1 \ldots d/2$ for $d = 3$ and choosing obvious complex coordinates. The corresponding GACS is given by the embedding of a traditional almost complex structure as given in eq. (3.1.16), consistent with Ω defining such a structure.

Similarly the pure spinor e^{-iJ} where $J = dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3$ is a $(1,1)$ form and is equivalent to the GACS embedding of a symplectic structure given in eq. (3.1.17).

In terms of pure spinors integrability with respect to the Courant bracket translates into the condition:

$$\mathrm{d}\Phi = (i_v + \zeta\wedge)\Phi \tag{3.2.6}$$

for a given $v + \zeta \in T \oplus T^*$. A "generalized Calabi-Yau manifold" in the sense of [168] in particular is defined such that $v + \zeta = 0$ and Φ has a non-zero norm $||\Phi||$ given by:

$$||\Phi||^2 = \langle \bar{\Phi}, \Phi \rangle \qquad \text{where in general} \quad \langle \Phi, \Psi \rangle = (s(\Phi) \wedge \Psi)|_d \tag{3.2.7}$$

is the Mukai pairing with $s(\Phi) = \sum_n (-1)^{Int[n/2]} \Phi|_n$ where $|_n$ projects out the form of degree n. Note that in terms of the $Spin(d,d)$ representation of the spinors this is nothing but the natural bilinear $\bar{\Phi}\Psi$.

Integrability under the twisted Courant derivative is given in terms of the twisted exterior derivative:

$$\mathrm{d}_H \Phi' = (\mathrm{d} - H\wedge)\Phi' = (i_v + \zeta\wedge)\Phi' \tag{3.2.8}$$

leading to a definition of twisted generalized Calabi-Yau in analogy with the above. Note that $d_H^2 = 0$.

If Φ satisfies eq. (3.2.6) then $\Phi' = e^B \cdot \Phi = (1 + B\wedge + \frac{1}{2} B \wedge B \wedge + \ldots)\Phi$ satisfies eq. (3.2.8) with the corresponding GACS related by:

$$\mathcal{J}' = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}$$

This picture may be neatly related to ordinary G-structures on T described in section 2.2. Being a generalization of complex geometry GCG is most naturally applied to even-dimensional spaces so we shall use $SU(d/2)$ structures as example. Given an $SU(d/2)$ invariant spinor η we may define bi-spinors:

$$\Phi_+ = \eta\bar{\eta} \quad \text{and} \quad \Phi_- = \eta\eta^T \tag{3.2.9}$$

Recall now that the anti-symmetrized products of gamma matrices together with $I_{2^{d/2}\times 2^{d/2}}$ form an orthogonal basis for $GL(2^{d/2}, \mathbb{C})$ matrices (as the above bi-spinors) under the inner product $tr(A^\dagger B)$ for $A, B \in GL(2^{d/2}, \mathbb{C})$. The bi-spinors may thus be decomposed as (the usual Fierz identity):

$$\Phi_+ = \eta\bar{\eta} = s\sum_{k=0}^{d/2} \frac{1}{k!} \bar{\eta}\gamma_{i_1\ldots i_k}\eta\ \tilde{\Gamma}^{i_k\ldots i_1} \quad ; \quad \Phi_- = \eta\eta^T = s\sum_{k=0}^{d/2} \frac{1}{k!} \eta^T\gamma_{i_1\ldots i_k}\eta\ \tilde{\Gamma}^{i_k\ldots i_1} \tag{3.2.10}$$

where $s = \frac{1}{tr(I_{2^{d/2}\times 2^{d/2}})}$. Note that as a Clifford(d,d) spinor Φ_+ has positive chirality and Φ_- negative chirality. From the form point of view this means they are respectively sums of even and odd degree forms. Using eq. (3.2.2) to rewrite the gamma matrices as one-forms one finds:

$$\Phi_+ = \frac{s}{2}\, e^{-iJ} \quad \text{and} \quad \Phi_- = \frac{-is}{2}\, \Omega \tag{3.2.11}$$

where (J, Ω) is the usual pair defining an $SU(d/2)$ structure whose expression in terms of bilinears in η was given in eq. (2.2.6). It follows that an $SU(d/2)$ structure on T is equivalent to two GACS on $T \oplus T^*$ namely the natural embedding of a symplectic and an almost complex structure respectively. What is more both structures are compatible in the sense of the definition of a generalized Kähler structure as given in eq. (3.1.23) so that one has an $U(d/2) \times U(d/2)$ structure. From the point of view of annihilator spaces of the corresponding pure spinors it implies that their overlap must be of dimension $d/2$ i.e. they have $d/2$ annihilators in common. In fact since the pure spinors are globally defined one has an $SU(d/2) \times SU(d/2)$ Calabi-Yau structure[4]. It can be checked that integrability leads to the usual Calabi-Yau conditions for (J, Ω).

It can similarly be shown that an $SU(2)$ structure on T is also a special case of an $SU(3) \times SU(3)$ structure on $T \oplus T^*$ with pure spinors:

$$\Phi_+ = \eta_1 \bar{\eta}_2 \quad \text{and} \quad \Phi_- = \eta_1 \eta_2^T \tag{3.2.12}$$

with η^1, η^2 two nowhere vanishing spinors. $SU(3)$ structure can be understood as a special case with $\eta^1 = \eta^2$. An intermediate case is that of a so-called local $SU(2)$ structure which admits two nowhere vanishing spinors that are parallel (and thus not linearly independent) in at least one point so that the structure is not globally defined. This case is relevant for type II compactifications as the two ten-dimensional spinors need not be decomposed in terms of the same six-dimensional internal spinor. Requiring $N = 2$ SUSY [5] implies that the $SU(2)$ structure may only be local. Being midway between the above two special cases this unsurprisingly also leads to a manifold with $SU(3) \times SU(3)$ structure with the pure spinors given by eq. (3.2.12).

[4]The map between pure spinors and GACSs is not one-to one: the phases and overall scales of $\Phi_\pm$ may vary and still lead to the same GACS. However these are critical to determine whether the pure spinors are globally defined and must not vanish at any point hence the non-zero norm condition in eq. (3.2.7).

[5]Note that at this point we are talking about the amount of off-shell SUSY in the lower-dimensional effective action. In the following we shall however be interested in on-shell $N = 1$ solutions of $N = 2$ effective actions resulting from compactifying Type II on a manifold of $SU(3) \times SU(3)$ structure.

3.3 Recent applications in String theory

Having set the scene we may now describe some of the applications of this formalism. We shall here deal separately with on-shell and off-shell conditions.

3.3.1 On-shell supersymmetry

In [173] the authors rewrite the conditions for an $N = 1$ vacuum of the effective $N = 2$ theory resulting from compactifying an a background with $SU(3) \times SU(3)$ structure. Recall that the field content of type IIA and IIB has a common sector given by the metric G, the Kalb-Ramond field B and the dilaton ϕ and differs only in the Ramond-Ramond (RR) forms which in this context may be arranged as:

$$F_{\mathrm{IIA}\,\pm} = F_0 \pm F_2 + F_4 \pm F_6 \ , \qquad F_{\mathrm{IIB}\,\pm} = F_1 \pm F_3 + F_5 \ , \tag{3.3.1}$$

where F_5 is self-dual. Further consider the possibility of a warp factor for the metric parameterized by e^E:

$$ds^2_{10} = e^E g^{(4)}_{\mu\nu} dx^\mu dx^\nu + g_{mn} dy^m dy^n \ , \tag{3.3.2}$$

It was then shown that the SUSY equations are respectively:

$$\begin{aligned} e^{-2E+\phi}(d + H\wedge)(e^{2E-\phi}\tilde{\Phi}_+) &= 0 \, , \qquad (3.3.3) \\ e^{-2E+\phi}(d + H\wedge)(e^{2E-\phi}\tilde{\Phi}_-) &= dE \wedge \tilde{\Phi}^*_- - \frac{1}{16} e^\phi \Big[(|a|^2 - |b|^2) F_{\mathrm{IIA}\,-} - i(|a|^2 + |b|^2) * F_{\mathrm{IIA}\,+} \Big] \end{aligned}$$

for type IIA and

$$\begin{aligned} e^{-2E+\phi}(d - H\wedge)(e^{2E-\phi}\tilde{\Phi}_+) &= dE \wedge \tilde{\Phi}^*_+ + \frac{1}{16} e^\phi \Big[(|a|^2 - |b|^2) F_{\mathrm{IIB}\,+} - i(|a|^2 + |b|^2) * F_{\mathrm{IIB}\,-} \Big] \\ e^{-2E+\phi}(d - H\wedge)(e^{2E-\phi}\tilde{\Phi}_-) &= 0 \, , \qquad (3.3.4) \end{aligned}$$

for type IIB where

$$\tilde{\Phi}_+ = a\bar{b}\,\Phi_+ \ , \qquad \tilde{\Phi}_- = ab\,\Phi_- \tag{3.3.5}$$

are introduced to take into account the norm of the spinors (usually set to one for structure spinors):

$$\tilde{\eta}^1 = a\,\eta^1 \ , \qquad \tilde{\eta}^2 = b\,\eta^2 \ . \tag{3.3.6}$$

where η^1, η^2 are unit norm spinors. These norms are however not arbitrary and $N = 1$ SUSY requires that:

$$d|a|^2 = |b|^2 dE \ , \qquad d|b|^2 = |a|^2 dE \ , \tag{3.3.7}$$

We thus see that $N = 1$ vacua result from compactifications on manifolds with $SU(3) \times SU(3)$ structure, however only one pure spinor (and thus only the $SU(3,3)$ structure) is integrable. We thus have a generalized complex manifold but not a generalized Calabi-Yau manifold [6]. In particular it is the RR-fields who form the obstruction to full integrability. It was noted in [174] that turning them off [7] leads to an $N = 2$ vacuum instead. This is in line with the observation that two integrable and compatible GACS leads to generalized Kähler manifolds which are in correspondence with bi-hermitian geometries (as noted in [169]): these in turn are the natural target space for worldsheets with $N = (2,2)$ SUSY while for $N = 1$ SUSY in space-time one would expect $N = (2,1)$ SUSY on the worldsheet. Note finally that the integrability condition for $\tilde{\Phi}_+$ is not just closure under the exterior derivative but the more general form given in eq. (3.2.6) with $d(-2E + \phi)$ playing the role of the one-form.

The symmetry of the above equations also suggests the conjecture that one may

[6] Alternatively some authors call manifolds generalized Calabi-Yau even if only one structure is integrable, using the term generalized Calabi-Yau metric for the case where both structures are integrable.

[7] together with $E = 0$ and $\tilde{\Phi}_\pm = \Phi_\pm$

realize mirror symmetry as :

$$\Phi_+ \leftrightarrow \Phi_- \quad ; \quad F_{\text{IIA}} \leftrightarrow F_{\text{IIB}} \tag{3.3.8}$$

This is further underlined by the observation in [175][176] that the closure of one pure spinor is sufficient to define a topological model, Φ_+ for the A-model and Φ_- for the B-model.

3.3.2 Off-shell supersymmetry

Another important application was developed in [177] and further refined in [178]: following a methodology introduced in [179] in the study of hidden symmetries of 11D supergravity: the authors isolate 8 of the 32 supercharges present in type II and rewrite the ten-dimensional theory in a form reminiscent of $N = 2$ $D = 4$ supergravity without any compactification or truncation but merely by assuming that the Lorentz group in 10D is reduced from[8] $Spin(1,9) \to Spin(1,3) \otimes Spin(6)$. In our original research we aim to extend this work, both by achieving a geometrization of general flux backgrounds and by applying this description in a different dimensional setting to eleven-dimensional supergravity.

At this point it is worth noting that when we introduced Generalized Geometry we chose the historical angle favored in the mathematical literature. From the physical point of view this may however not be the most intuitive sequence, especially for the application at hand and our original work. Physically the metric $\mathcal{G}$ is specified by being the invariant under the maximal subgroup $O(d) \times O(d)$ of the natural $O(d,d)$ structure on $T \oplus T^*$. By considering a polarization map $T \oplus T^* \to T$ one may then isolate a $GL(d)$ subgroup of $O(d,d)$ which can be identified with traditional diffeomorphisms. Decomposing $\mathcal{G}$ in terms of this $GL(d)$ subgroup allows to express

[8]leading to a decomposition of the 10D spinor in a 4D spinor (which is an anti-commuting Grassmann variable) and a 6D (commuting) spinor

it in terms of G and B, thus neatly arranging the background fields in the common sector. From this viewpoint Generalized Complex structures are a second level of sophistication arising when one singles out certain supercharges in a supergravity background for the above mentioned reformulation. Similarly *Cliff*(d,d) spinors are more elementary than $\mathcal{G}$ and can be defined with reference to the $O(d,d)$ metric $\mathcal{I}$ only, while pure spinors arise again only in the context of supersymmetry.

Having clarified this point let us concentrate on the elements of [177] and [178] closest to the original work presented here. Recall first the nature of potential terms in $N=2$ $D=4$ supergravity (as reviewed in [180]): supersymmetric potential terms are typically quadratic in the scalar part of the fermion variations[9] and for most multiplets these are derivatives of the scalar part of the gravitino variation. For $N=2$ $D=4$ we have:

$$\delta\psi_{A\,\mu} = D_\mu\theta_A + i\gamma_\mu S_{AB}\theta^{cB} \;, \tag{3.3.9}$$

where $A=1,2$, $\theta_{1,2}$ are the SUSY parameters and S (such that $SS^\dagger = 1$) encodes the $SU(2)$ R-symmetry of the supergravity and the potential terms through the three prepotentials $\mathcal{P}^x, x=1,2,3$ via

$$S_{AB} = \frac{\mathrm{i}}{2}\,\mathrm{e}^{\frac{1}{2}K_V}\sigma^x_{AB}\mathcal{P}^x, \qquad \sigma^x_{AB} = \begin{pmatrix} \delta^{x1} - i\delta^{x2} & -\delta^{x3} \\ -\delta^{x3} & -\delta^{x1} - i\delta^{x2} \end{pmatrix}, \tag{3.3.10}$$

where K_V is the vector multiplet Kähler potential. As in [177] we specialize to the case of $SU(3)$ structure on T and give only a sketch outline of the derivation as we will in the following repeat a similar process for 11D supergravity. The existence of the aforementioned structure allows for the decomposition of all fields in terms of $SU(3) \subset Spin(6)$ representations. In particular the higher-dimensional precursor of

[9]This incidentally ensures that supersymmetric states minimize the Hamiltonian as expected from the SUSY algebra.

the four-dimensional gravitini can be identified as the singlets in the expansion of the ten-dimensional gravitini:

$$\hat{\Psi}^{\rm IIA}_{1\,\mu} = \psi_{1\,\mu+} \otimes \eta_+ + \psi_{1\,\mu-} \otimes \eta_- + \dots \; ; \; \hat{\Psi}^{\rm IIA}_{2\,\mu} \;\; = \psi_{2\,\mu-} \otimes \eta_+ + \psi_{2\,\mu+} \otimes \eta_- + \dots \tag{3.3.11}$$

in type IIA and

$$\hat{\Psi}^{\rm IIB}_{A\,\mu} = \psi_{A\,\mu+} \otimes \eta_- + \psi_{A\,\mu-} \otimes \eta_+ + \dots \; , \qquad A = 1, 2 \; . \tag{3.3.12}$$

in type IIB, corresponding to the terms involving the $SU(3)$ invariant structure spinor η_+ with $\eta_- = \eta_+^c$ (where the indices denote chirality). Note the shift of the 10D gravitino:

$$\hat{\Psi}_\mu \equiv \Psi_\mu + \frac{1}{2}\Gamma_\mu{}^m \Psi_m \; , \tag{3.3.13}$$

ensuring diagonal 4D gravitino kinetic terms. Consequently we may obtain the 10D analogue of eq. (3.3.9) by projecting out the singlet part of the variation $\delta\hat{\Psi}$ obtained from the 10D gravitino variation as given in eq. (1.2.1) in the democratic formulation. Turning on only fluxes consistent with 4D Poincaré invariance [10] we only retain F_n with $n = 0 \dots 6$ where $F_n = dC_{n-1} - H \wedge C_{n-3}$ are the field strengths including the Chern-Simons contribution [11]. The appropriate projector is given by:

$$\Pi_\pm = \mathbb{I} \otimes 2\left(\eta_\pm \otimes \bar{\eta}_\pm\right) \; . \tag{3.3.14}$$

with the signs indicating the chirality of the field acted upon. The prepotentials for type IIB can then be calculated to be :

[10] i.e. purely internal flux which is a scalar from the point of view of the compact space or its Hodge dual

[11] Note that the relation to the twisted exterior derivative d_H.

$$\mathcal{P}^1 = -2\,e^{\frac{1}{2}K_J+\phi^{(4)}}\langle\Phi_-, d\,\mathrm{Re}\,\Phi_+\rangle,$$
$$\mathcal{P}^2 = 2\,e^{\frac{1}{2}K_J+\phi^{(4)}}\langle\Phi_-, d\,\mathrm{Im}\,\Phi_+\rangle, \tag{3.3.15}$$
$$\mathcal{P}^3 = -\frac{1}{\sqrt{2}}e^{2\phi^{(4)}}\langle\Phi_-, G_{IIB}\rangle.$$

where physically K_J is the vector multiplet Kähler potential for type IIA (essentially obtained by mirror symmetry), $\phi^{(4)}$ is the 4D dilaton and G_{IIB} is defined in analogy to F_{IIB} such that $\langle e^{-iJ}, F_A\rangle = \langle e^{-(B+iJ)}, G_A\rangle$ which allows for J to be correctly complexified as $(B + iJ)$ giving the pure spinors:

$$\Phi_+ = e^{-B-iJ} \quad ; \quad \Phi_- = \Omega. \tag{3.3.16}$$

For type IIA we find:

$$\mathcal{P}^1 = -2e^{\frac{1}{2}K_\rho+\phi^{(4)}}\langle\Phi_+, d\,\mathrm{Re}\,\Phi_-\rangle,$$
$$\mathcal{P}^2 = -2e^{\frac{1}{2}K_\rho+\phi^{(4)}}\langle\Phi_+, d\,\mathrm{Im}\,\Phi_-\rangle, \tag{3.3.17}$$
$$\mathcal{P}^3 = \frac{1}{\sqrt{2}}e^{2\phi^{(4)}}\langle\Phi_+, G_{IIA}\rangle,$$

were K_ρ is now the vector multiplet for type IIB and G_{IIA} is defined in analogy to G_{IIB}.

Note that the vector multiplet Kähler potentials can themselves be written in terms of the structures since each pure spinor defines a special Kähler structure [167, 168, 182] in terms of deformations of the NS moduli $e^B\Phi_\pm$ leading to the Kähler potential :

$$K = -ln\ [i\int\langle e^B\Phi_\pm, e^B\bar{\Phi}_\pm\rangle] \tag{3.3.18}$$

which goes as $J\wedge J\wedge J$ for type IIA and $i\Omega\wedge\bar{\Omega}$ for type IIB which is exactly the same as for a Calabi-Yau compactification. This is unsurprising since a Calabi-Yau differs from a manifold with $SU(3)$ structure through differential conditions with respect to the internal manifold coordinates. However this data does not enter the derivation of the kinetic term in dimensional reduction and hence not the Kähler metric nor the Kähler potential from which it follows by differentiation with respect to the moduli.

At this point one may isolate $N = 1$ theories embedded in the $N = 2$ reformulation (essentially by breaking the R-symmetry from $SU(2)$ to $U(1)$) obtaining the superpotentials:

$$\begin{aligned}\mathcal{W}_{\rm IIA} &= +\cos^2\alpha\, \mathrm{e}^{\mathrm{i}\beta}\langle\Phi^+, \mathrm{d}\Phi^-\rangle - \sin^2\alpha\, \mathrm{e}^{-\mathrm{i}\beta}\langle\Phi^+, \mathrm{d}\bar{\Phi}^-\rangle + \sin 2\alpha\, \mathrm{e}^{\phi}\langle\Phi^+, G^+\rangle \\ \mathcal{W}_{\rm IIB} &= -\cos^2\alpha\, \mathrm{e}^{\mathrm{i}\beta}\langle\Phi^-, \mathrm{d}\Phi^+\rangle + \sin^2\alpha\, \mathrm{e}^{-\mathrm{i}\beta}\langle\Phi^-, \mathrm{d}\bar{\Phi}^+\rangle - \sin 2\alpha\, \mathrm{e}^{\phi}\langle\Phi^-, G^-\rangle\end{aligned} \tag{3.3.19}$$

where in going form $N = 2$ to $N = 1$ the two SUSY parameters $\theta_{1,2}$ are related as:

$$\theta_A = \theta n_A, \qquad n_A = \begin{pmatrix} a \\ b \end{pmatrix} , \quad a = \cos\alpha e^{-\frac{i}{2}\beta} , \quad b = \sin\alpha e^{\frac{i}{2}\beta} \tag{3.3.20}$$

and θ is the single $N = 1$ SUSY parameter and the R-symmetry corresponds to the $U(1)$ subgroup of $SU(2)$ leaving n_A invariant.

Several known superpotentials are special cases of these general expressions. For example the Gukov–Taylor–Vafa–Witten superpotential [95, 183] corresponds to $2\alpha = -\beta = \pi/2$:

$$\mathcal{W}_{\rm GTVW} = \mathrm{i}\ \mathrm{e}^{\phi}\langle(F_3 - \tau), \Omega\rangle , \tag{3.3.21}$$

where F_3 is related to G_3 in the usual way and $\tau = C_0 + \mathrm{i}\mathrm{e}^{-\phi}$ combines the dilaton and the RR 0-form.

The IIA superpotential arising from the RR fields (as proposed in [184]) is found by setting $\alpha = \pi/4$ and $\mathrm{d}\Phi^- = 0$:

$$\mathcal{W}_{\rm IIA,RR} = -\mathrm{i}\,\mathrm{e}^{\phi}\langle\mathrm{e}^{-(B+\mathrm{i}J)}, G_{IIA}\rangle . \tag{3.3.22}$$

Finally the superpotential proposed in [134] for the type IIA half-flat mirror manifold dual to the Calabi-Yau compactifications with type IIB electric NS fluxes is the special case with $\alpha = \pi/2$ and $\beta = -\pi/2$:

$$\mathcal{W}_{\text{half-flat}} = \langle \mathrm{e}^{-(B+\mathrm{i}J)}, \mathrm{d}\Omega \rangle \ . \tag{3.3.23}$$

The same authors generalized these results to the case of an $SU(3) \times SU(3)$ structure [178] showing they are unmodified except for the explicit form of $\Phi_{\pm}$. In particular Φ_- only contains the holomorphic 3-form Ω for an $SU(3)$ structure while in general it will be a generic sum of odd forms in 6D. Under mirror symmetry however, now given by:

$$\Phi^+ \leftrightarrow \Phi^-, \qquad G_{IIA} \leftrightarrow G_{IIB} \quad K_J \leftrightarrow K_\rho \tag{3.3.24}$$

the NS three-from flux H is decomposed in two : electric fluxes mapping to a four-form and magnetic fluxes mapping to a two-form. In [134, 185] $\mathrm{d}\,\mathrm{Re}\Omega$ (which when non-zero leads to half-flat manifolds) was identified as the dual of the electric fluxes but to define a similar construction for the magnetic fluxes had been an outstanding issue since one could not identify an appropriate two-form on the mirror side. From the above it follows that the mirrors of compactifications with magnetic fluxes must have a general $SU(3) \times SU(3)$ structure and the two-form in question relates to $d(\Phi_-|_1)$. Interestingly such manifolds turn out to be so-called non-geometric backgrounds and while in principle this means the supergravity approximation should break down the above formalism does reproduce correctly the corresponding low-energy effective description. This departure from geometry confirms suggestions in [186, 187, 188]. Similar methods were used to obtain $N = 1$ superpotentials directly from compactifications of the heterotic string [189] and M-theory [190].

This concludes our exposition of the literature, in the remainder of this thesis we shall present original work undertaken.

Chapter 4

Mathematical Interlude: The exceptional Lie group $E_{7(7)}$

In this section we review the properties of the group $E_{7(7)}$ relevant to our original work. A detailed definition of $E_{7(7)}$ and an exhaustive description of its properties can be found in Appendix B of [191] which itself refers to the original work by Cartan [192]. Relevant tables for the tensor products of representations of $E_{7(7)}$ and its subgroups may be found in [193].

4.1 Definition and the 56 representation

The group $E_{7(7)}$ can be defined by its action on the basic 56-dimensional representation: Let W be a real 56-dimensional vector space with a symplectic product Ω, then $E_{7(7)}$ is a subgroup of $Sp(56,\mathbb{R})$ leaving invariant a specific quartic invariant q.

Explicitly one can define Ω and q using the $SL(8,\mathbb{R}) \subset E_{7(7)}$ subgroup. If V is an eight-dimensional vector space, on which $SL(8,\mathbb{R})$ acts in the fundamental representation, then the **56** representation decomposes as

$$\mathbf{56} = \mathbf{28} + \mathbf{28}' \tag{4.1.1}$$

The **28** representation corresponds to $\Lambda^2 V$ and the **28**$'$ to $\Lambda^2 V^*$. Note that using $\epsilon \in \Lambda^8 V$, the totally antisymmetric form preserved by the $SL(8,\mathbb{R})$ action on V, one can identify $\Lambda^2 V^*$ with $\Lambda^6 V$. In summary, one identifies

$$W = \Lambda^2 V \oplus \Lambda^2 V^* \tag{4.1.2}$$

and writes $X \in W$ as the pair (x^{ab}, x'_{ab}) where $a, b = 1, \ldots, 8$.

The symplectic product Ω is then given by

$$\Omega(X,Y) = \Omega_{AB} X^A Y^B = x^{ab} y'_{ab} - x'_{ab} y^{ab}, \tag{4.1.3}$$

where $A, B = 1, \ldots, 56$ and the quartic invariant q is

$$\begin{aligned} q(X) &= q_{ABCD} X^A X^B X^C X^D \\ &= x^{ab} x'_{bc} x^{cd} x'_{da} - \frac{1}{4} x^{ab} x'_{ab} x^{cd} x'_{cd} \\ &+ \frac{1}{96} \left(\epsilon_{abcdefgh} x^{ab} x^{cd} x^{ef} x^{gh} + \epsilon^{abcdefgh} x'_{ab} x'_{cd} x'_{ef} x'_{gh} \right) \end{aligned} \tag{4.1.4}$$

or in explicit $SL(8,\mathbb{R})$ indices

$$\begin{aligned} q_{abcdefgh} &= \frac{1}{96} \epsilon_{abcdefgh} \\ q^{abcdefgh} &= \frac{1}{96} \epsilon^{abcdefgh} \\ q^{abcd}{}_{efgh} &= \frac{1}{24} \delta^{[a}_{[e} \delta^{b]}_{f]} \delta^{[c}_{[g} \delta^{d]}_{h]} - \frac{1}{4} \delta^{[a}_{[e} \delta^{b}_{f} \delta^{c}_{g} \delta^{d]}_{h]} \end{aligned} \tag{4.1.5}$$

where the last line applies to the five other permutation of indices with two pairs of indices up and two down, while all other entries vanish.

In what follows it will be useful to use a matrix notation where we write

$$X^A = \begin{pmatrix} x^{aa'} \\ x'_{aa'} \end{pmatrix} \tag{4.1.6}$$

such that the symplectic form

$$\Omega_{AB} = \begin{pmatrix} 0 & \delta_a^b \delta_{a'}^{b'} \\ \delta_b^a \delta_{b'}^{a'} & 0 \end{pmatrix} \tag{4.1.7}$$

Throughout it is assumed that *all pairs of primed and unprimed indices* (a, a') *etc are antisymmetrized.*

4.2 The 133 and 912 representations

There are two other representations of interest in this thesis, the first being the adjoint. By definition it is a 133-dimensional subspace A of the Lie algebra $sp(56, \mathbb{R})$. It decomposes under $SL(8, \mathbb{R})$ as

$$\begin{aligned} A &= (V \otimes V^*)_0 \oplus \Lambda^4 V^* \\ \mu &= ({\mu^a}_b, \mu_{abcd}) \\ \mathbf{133} &= \mathbf{63} + \mathbf{70}, \end{aligned} \tag{4.2.1}$$

where $(V \otimes V^*)_0$ denotes traceless matrices, so ${\mu^a}_a = 0$. The action on the **56** given by

$$\begin{aligned} \delta x^{ab} &= {\mu^a}_c x^{cb} + {\mu^b}_c x^{ac} + *\mu^{abcd} x'_{cd}, \\ \delta x'_{aa'} &= -{\mu^c}_a x'_{cb} - {\mu^c}_b x'_{ac} + \mu_{abcd} x^{cd}, \end{aligned} \tag{4.2.2}$$

with $*\mu^{a_1 \dots a_4} = \frac{1}{4!} \epsilon^{a_1 \dots a_8} \mu_{a_5 \dots a_8}$. In terms of the matrix notation we have $\delta X^A = {\mu^A}_B X^B$ with

$${\mu^A}_B = \begin{pmatrix} 2{\mu^a}_b \delta_{b'}^{a'} & *\mu^{aa'bb'} \\ \mu_{aa'bb'} & -2{\mu^b}_a \delta_{a'}^{b'} \end{pmatrix}. \tag{4.2.3}$$

Note that $\mu^{AB} = \mu^{A}{}_{C}\Omega^{-1CB}$ is a symmetric matrix. Taking commutators of the adjoint action gives the Lie algebra $\mu'' = [\mu, \mu']$

$$\begin{aligned} \mu''^{a}{}_{b} &= (\mu^{a}{}_{c}\mu'^{c}{}_{b} - \mu'^{a}{}_{c}\mu^{c}{}_{b}) + \frac{1}{3}(*\mu^{ac_1c_2c_3}\mu'_{bc_1c_2c_3} - *\mu'^{ac_1c_2c_3}\mu_{bc_1c_2c_3}), \\ \mu''_{abcd} &= 4(\mu^{e}{}_{[a}\mu'_{bcd]e} + \mu'^{e}{}_{[a}\mu_{bcd]e}). \end{aligned} \tag{4.2.4}$$

where the second term in the first line is obtained by use of the following identity:

$$\begin{aligned} (*\mu\mu' - *\mu'\mu)^{ab}{}_{cd} &= *\mu^{abef}\mu'_{efcd} - (\mu \leftrightarrow \mu') \\ &= \frac{-1}{6}(\delta^{a}{}_{c}E^{b}{}_{d} - \delta^{a}{}_{d}E^{b}{}_{c} - \delta^{b}{}_{c}E^{a}{}_{d} + \delta^{b}{}_{d}E^{a}{}_{c}) \\ &= \frac{-2}{3}\delta^{[a}{}_{[c}E^{b]}{}_{d]} \end{aligned} \tag{4.2.5}$$

where by taking the trace we find $E^{a}{}_{b} = *\mu^{acde}\mu'_{cdeb} - (\mu \leftrightarrow \mu')$

The other representation of interest in this thesis is the **912**. The representation space N decomposes under $SL(8,\mathbb{R})$ as

$$\begin{aligned} N &= S^2V \oplus (\Lambda^3V \otimes V^*)_0 \oplus S^2V^* \oplus (\Lambda^3V^* \otimes V)_0 \\ \phi &= (\phi^{ab}, \phi^{abc}{}_{d}, \phi'_{ab}, \phi'_{abc}{}^{d}) \\ \mathbf{912} &= \mathbf{36} + \mathbf{420} + \mathbf{36}' + \mathbf{420}', \end{aligned} \tag{4.2.6}$$

where S^nV denotes the symmetric product and $(\Lambda^3V \otimes V^*)_0$ denotes traceless tensors, so that $\phi^{abc}{}_{c} = 0$. The adjoint action of $E_{7(7)}$ on ϕ is given by

$$\begin{aligned} \delta\phi^{ab} &= \mu^{a}{}_{c}\phi^{cb} + \mu^{b}{}_{c}\phi^{ac} - \frac{1}{3}(*\mu^{acde}\phi'_{cde}{}^{b} + *\mu^{bcde}\phi'_{cde}{}^{a}), \\ \delta\phi^{abc}{}_{d} &= 3\mu^{[a}{}_{e}\phi^{bc]e}{}_{d} - \mu^{e}{}_{d}\phi^{abc}{}_{e} + *\mu^{abce}\phi'_{ed} + *\mu^{ef[ab}\phi'_{efd}{}^{c]} - *\mu^{efg[a}\phi'_{efg}{}^{b}\delta^{c]}_{d}, \\ \delta\phi'_{ab} &= -\mu^{c}{}_{a}\phi'_{cb} - \mu^{c}{}_{b}\phi'_{ac} - \frac{1}{3}(\mu_{acde}\phi^{cde}{}_{b} + \mu_{bcde}\phi^{cde}{}_{a}), \\ \delta\phi'_{abc}{}^{d} &= -3\mu^{e}{}_{[a}\phi'_{bc]e}{}^{d} + \mu^{e}{}_{e}\phi'_{abc}{}^{e} + \mu_{abce}\phi^{ed} + \mu_{ef[ab}\phi^{efd}{}_{c]} - \mu_{efg[a}\phi^{efg}{}_{b}\delta^{d}_{c]}. \end{aligned} \tag{4.2.7}$$

In terms of $Sp(56,\mathbb{R})$ indices, we have ϕ^{ABC}, corresponding to the Young tableau $\begin{array}{|c|c|}\hline A & C \\ \hline B & \\ \cline{1-1}\end{array}$

with $\phi^{ABC}\Omega_{AB} = 0$. The different components are given by

$$\begin{aligned}
\phi^{aa'bb'cc'} &= -\frac{1}{12}(\epsilon^{abb'cc'efg}\phi'_{efg}{}^{a'} - \epsilon^{baa'cc'efg}\phi'_{efg}{}^{b'}), \\
\phi^{aa'bb'}{}_{cc'} &= 2\phi^{ab}\delta^{a'}_{c}\delta^{b'}_{c'} - \phi^{aa'b}{}_{c}\delta^{b'}_{c'} + \phi^{bb'a}{}_{c}\delta^{a'}_{c'}, \\
\phi^{aa'}{}_{bb'}{}^{cc'} &= \phi^{ac}\delta^{a'}_{b}\delta^{c'}_{b'} - 2\phi^{aa'c}{}_{b}\delta^{c'}_{b'} - \phi^{cc'a}{}_{b}\delta^{a'}_{b'},
\end{aligned} \tag{4.2.8}$$

with $\phi^{aa'}{}_{bb'}{}^{cc'} = -\phi_{bb'}{}^{aa'cc'}$ and identical expressions for $\phi_{aa'bb'cc'}$ etc. but with raised and lowered indices reversed.

Finally we note the following tensor product containing a term in the adjoint for later reference:

$$\mathbf{56} \times \mathbf{912} = \mathbf{133} + \ldots. \tag{4.2.9}$$

In terms of $Sp(56,\mathbb{R})$ indices we have $\mu^{AB} = X^C\Omega_{CD}\phi^{D(AB)}$, while in terms of $SL(8,\mathbb{R})$ components one finds

$$\begin{aligned}
\mu^{a}{}_{b} &= \frac{3}{4}\left(x^{ac}\phi'_{cb} - x'_{bc}\phi^{ca}\right) + \frac{3}{4}\left(x^{cd}\phi'_{cdb}{}^{a} - x_{cd}\phi^{cda}{}_{b}\right), \\
\mu_{abcd} &= -3\left(\phi'_{[abc}{}^{e}x'_{d]e} + \frac{1}{4!}\epsilon_{abcdm_1\ldots m_4}\phi^{m_1m_2m_3}{}_{e}x^{m_4e}\right).
\end{aligned} \tag{4.2.10}$$

4.3 A $GL(7)$ subgroup

In this section we construct the embedding of the diffeomorphism group in $E_{7(7)}$. To make this embedding explicit we must identify a particular $GL(7,\mathbb{R}) \subset SL(8,\mathbb{R}) \subset E_{7(7)}$ subgroup. In this appendix we identify this group and give explicit expressions for part of the $E_{7(7)}$ action in terms of $GL(7,\mathbb{R})$, that is space-time tensor, representations.

We start with the embedding of $GL(7,\mathbb{R})$ in $SL(8,\mathbb{R})$ given by the matrix

$$\begin{pmatrix} (\det M)^{-1/4}M^{m}{}_{n} & 0 \\ 0 & (\det M)^{3/4} \end{pmatrix} \in SL(8,\mathbb{R}), \tag{4.3.1}$$

where $M \in GL(7,\mathbb{R})$. If $GL(7,\mathbb{R})$ acts linearly on the seven-dimensional vector space $\hat{F}$ this corresponds to the decomposition of the eight-dimensional representation space V as

$$V = (\Lambda^7 \hat{F})^{-1/4} \hat{F} \oplus (\Lambda^7 \hat{F})^{3/4}. \tag{4.3.2}$$

The **56** representation of $E_{7(7)}$ as given in 4.1.2 then decomposes as

$$W = (\Lambda^7 \hat{F}^*)^{-1/2} \left[\hat{F} \oplus \Lambda^2 \hat{F}^* \oplus \Lambda^5 \hat{F}^* \oplus (\Lambda^7 \hat{F}^*) \hat{F}^* \right]. \tag{4.3.3}$$

We can thus write a generic element of W as

$$X = x + \omega + \sigma + \tau, \tag{4.3.4}$$

where $x \in (\Lambda^7 \hat{F}^*)^{-1/2} \hat{F}$ etc. If we write the index $m = 1, \ldots, 7$ for the fundamental $GL(7,\mathbb{R})$ representation, note that, ignoring the tensor density factor $(\Lambda^7 \hat{F}^*)^{-1/2}$, τ has the index structure $\tau_{m,n_1 \ldots n_7}$, where n labels the $\hat{F}^*$ factor and $n_1 \ldots n_7$ the $\Lambda^7 \hat{F}^*$ factor. We can make the identification between $GL(7,\mathbb{R})$ indices and $SL(8,\mathbb{R})$ indices explicit by writing (again ignoring the $(\Lambda^7 \hat{F}^*)^{-1/2}$ factor)

$$\begin{aligned} x^{m8} &= x^m & x^{mn} &= \sigma^{mn}_{1\ldots7} \\ x'_{m8} &= \tau_{m,1\ldots7} & x'_{mn} &= \omega_{mn}, \end{aligned} \tag{4.3.5}$$

where $\sigma^{mn}_{p_1 \ldots p_7} = (7!/5!) \delta^m_{[p_1} \delta^n_{p_2} \sigma_{p_3 \ldots p_7]}$.

We can similarly decompose the **133** representation. We find

$$\begin{aligned} A &= (V \otimes V^*)_0 \oplus \Lambda^4 V^* \\ &= \hat{F} \otimes \hat{F}^* \oplus \Lambda^6 \hat{F} \oplus \Lambda^6 \hat{F}^* \oplus \Lambda^3 \hat{F} \oplus \Lambda^3 \hat{F}^*. \end{aligned} \tag{4.3.6}$$

We will be particularly interested in the action of the $\Lambda^3 \hat{F}^*$ and $\Lambda^6 \hat{F}^*$ parts of **133** on X. Identifying

$$\begin{aligned}\mu_{mnp8} &= \frac{1}{2}A_{mnp} \in \Lambda^3 V^* \\ \mu^m{}_8 &= -A'^m_{1\dots 7} \in \Lambda^6 V^*\end{aligned} \tag{4.3.7}$$

where $A'^m_{p_1\dots p_7} = (7!/6!)\delta^m_{[p_1}\tilde{A}_{p_2\dots p_7]}$ we have the action in the Lie algebra

$$(A+\tilde{A})\cdot X = i_x A + \big(i_x\tilde{A} + A\wedge\omega\big) + \big(jA\wedge\sigma - j\tilde{A}\wedge\omega\big). \tag{4.3.8}$$

Here we have introduced a new notation. The symbol j denotes the pure $\hat{F}^*$ index on sections of $(\Lambda^7\hat{F}^*)\hat{F}^*$. Hence

$$\big(j\alpha^{(p+1)}\wedge\beta^{(7-p)}\big)_{m,n_1\dots n_7} := \frac{7!}{p!(7-p)!}\alpha_{m[n_1\dots n_p}\beta_{n_{p+1}\dots n_7]} \tag{4.3.9}$$

4.4 Cliff(7, 0) and seven-dimensional spinors

In order to be able to define the $SU(8)/\mathbb{Z}_2$ subgroup in the next section, let us also fix our conventions for $Spin(7)$. The Clifford algebra Cliff$(7,0,\mathbb{R})$ is generated by the gamma matrices γ_m with $m = 1,\dots,7$ satisfying

$$\{\gamma_m,\gamma_n\} = 2g_{mn}1. \tag{4.4.1}$$

One finds Cliff$(7,0;\mathbb{R}) \simeq GL(8,\mathbb{C})$ and hence the spinor representation of the Clifford algebra is complex and eight-dimensional. We define the intertwiners A and C by

$$\gamma_M^\dagger = A\gamma_m A^{-1}, \qquad -\gamma_m^T = C^{-1}\gamma_m C, \tag{4.4.2}$$

with $A^\dagger = A$, $C^T = C$ and such that $-\gamma_m^* = D^{-1}\gamma_m D$ with $D = CA^T$. We also define the conjugate spinors

$$\bar{\eta} = \eta^\dagger A, \qquad \eta^c = D\eta^*. \tag{4.4.3}$$

Writing

$$\gamma_{m_1\dots m_7} = \gamma_{(7)}\epsilon_{m_1\dots m_7} \tag{4.4.4}$$

with $\epsilon_{1\dots7} = \sqrt{g}$, note that one can choose the gamma matrices such that $\gamma_{(7)} = i$. The intertwiner A provides an hermitian metric on the spinor space, which is invariant under the subgroup $SU(8) \subset \mathrm{Cliff}(7,0;\mathbb{R})$, with a Lie algebra spanned by $\{\gamma_{m_1m_2}, \gamma_{m_1m_2m_3}, \gamma_{m_1\dots m_6}, \gamma_{m_1\dots m_7}\}$.

The even part of the Clifford algebra generated by the γ_{mn} has $\mathrm{Cliff}(7,0;\mathbb{R})_{\mathrm{even}} \simeq GL(8,\mathbb{R})$ and hence a real spinor representation with $\eta = \eta^c$. Thus the spin group $Spin(7) \subset \mathrm{Cliff}(7,0;\mathbb{R})_{\mathrm{even}}$ similarly has a real spinor representation. For real spinors, $\bar{\eta} = \eta^T C^{-1}$, and C^{-1} provides metric on the spin space. This is invariant under a $Spin(8)$ group with Lie algebra spanned by $\{\gamma_{m_1m_2}, \gamma_{m_1\dots m_6}\}$. This can alternatively be described by, for $a, b = 1, \dots 8$

$$\hat{\gamma}_{ab} = \begin{cases} (\det g)^{-1/4}\gamma_{mn} & \text{if } a = m,\ b = n \\ (\det g)^{1/4}\gamma_m\gamma_{(7)} & \text{if } a = m,\ b = 8 \\ -(\det g)^{1/4}\gamma_n\gamma_{(7)} & \text{if } a = 8,\ b = n \end{cases} \tag{4.4.5}$$

which generate the $Spin(8)$ Lie algebra with metric

$$\hat{g}_{ab} = (\det g)^{-1/4} \begin{pmatrix} g_{mn} & 0 \\ 0 & \det g \end{pmatrix}. \tag{4.4.6}$$

Here we have introduced some factors of $\det g$ to match the form of $\hat{g}$ used elsewhere in this thesis (in particular the decomposition under $GL(7,\mathbb{R})$ given in section 4.3). With these conventions, the spinors η are of positive chirality with respect to $Spin(8)$.

If we make the spinor indices explicit writing η^α with $\alpha = 1, \dots, 8$ we can raise and lower spinor indices using the metric C^{-1} so, for instance,

$$\hat{\gamma}_{ab\,\alpha\beta} = C^{-1}_{\alpha\gamma}\hat{\gamma}_{ab}{}^{\gamma}{}_{\beta}, \qquad \hat{\gamma}_{ab}{}^{\alpha\beta} = \hat{\gamma}_{mn}{}^{\alpha}{}_{\gamma}C^{\gamma\beta}. \tag{4.4.7}$$

One also has the useful completeness relations, reflecting *Spin*(8) triality,

$$\begin{aligned} \hat{\gamma}_{ab}{}^{\alpha\beta}\hat{\gamma}^{ab}{}_{\gamma\delta} &= 16\delta^{[\alpha}_{[\gamma}\delta^{\beta]}_{\delta]}, \\ \hat{\gamma}_{ab}{}^{\alpha\beta}\hat{\gamma}^{cd}{}_{\alpha\beta} &= 16\delta^{[c}_{[a}\delta^{d]}_{b]}. \end{aligned} \tag{4.4.8}$$

4.5 The $SU(8)/\mathbb{Z}_2$ subgroup and spinor indices

The maximal compact subgroup of $E_{7(7)}$ is $SU(8)/\mathbb{Z}_2$. In the supersymmetry transformations, the spinor η transforms in the fundamental representation under (the double cover) $SU(8)$. Thus it is often useful to have the decomposition of the various $E_{7(7)}$ representations in terms of $SU(8)/\mathbb{Z}_2$.

In particular, one can use the common *Spin*(8) subgroup to relate the decompositions under $SL(8,\mathbb{R})$ and $SU(8)/\mathbb{Z}_2$. Let $\hat{\gamma}^{ab}$ be the *Spin*(8) gamma matrices defined in 4.4.5. We can raise and lower $SL(8,\mathbb{R})$ indices using the metric $\hat{g}$. Similarly, spinor indices can be raised a lowered using C^{-1}.Under $SU(8)/\mathbb{Z}_2$ the decomposition of the **56** representation is thus explicitly:

$$\begin{aligned} X &= (x^{\alpha\beta}, \bar{x}_{\alpha\beta}) \\ \mathbf{56} &= \mathbf{28} + \bar{\mathbf{28}}. \end{aligned} \tag{4.5.1}$$

If the symplectic product takes the form

$$\Omega(X,Y) = \mathrm{i}\left(x^{\alpha\beta}\bar{y}_{\alpha\beta} - \bar{x}_{\alpha\beta}y^{\alpha\beta}\right) \tag{4.5.2}$$

then (x^{ab}, x'_{ab}) and $(x^{\alpha\beta}, \bar{x}_{\alpha\beta})$ are related by

$$
\begin{aligned}
x^{\alpha\beta} &= \frac{1}{4\sqrt{2}}(x^{ab} + \mathrm{i}x'^{ab})\hat{\gamma}_{ab}{}^{\alpha\beta}, \\
\bar{x}_{\alpha\beta} &= \frac{1}{4\sqrt{2}}(x^{ab} - \mathrm{i}x'^{ab})\hat{\gamma}_{ab\,\alpha\beta},
\end{aligned}
\tag{4.5.3}
$$

with $\hat{\gamma}_{ab}{}^{\alpha\beta}$ and $\hat{\gamma}_{ab\,\alpha\beta}$ given by 4.4.7, or equivalently

$$
\begin{pmatrix} x^{\alpha\beta} \\ \bar{x}_{\alpha\beta} \end{pmatrix} = \frac{1}{4\sqrt{2}} \begin{pmatrix} \hat{\gamma}_{ab}{}^{\alpha\beta} & \mathrm{i}\hat{\gamma}^{ab\,\alpha\beta} \\ \hat{\gamma}_{ab\,\alpha\beta} & -\mathrm{i}\hat{\gamma}^{ab}{}_{\alpha\beta} \end{pmatrix} \begin{pmatrix} x^{ab} \\ x'_{ab} \end{pmatrix}
\tag{4.5.4}
$$

Recall that there is a $SU(8)$ subgroup of $\mathrm{Cliff}(7,0;\mathbb{R})$ which leaves the spinor norm $\bar{\eta}\eta$ invariant. Since the defining **56** representation decomposes into objects with pair of $SU(8)$, both the 1 and -1 elements in $SU(8)$ leave X invariant and hence the subgroup of interest of $E_{7(7)}$ is actually $SU(8)/\mathbb{Z}_2$.

Viewing a 56-dimensional index either as a pair of $SL(8,\mathbb{R})$ indices or as a pairs of spinor indices, the relation 4.5.4 can be used to convert between $SL(8,\mathbb{R})$ and $SU(8)/\mathbb{Z}_2$ decompositions of any other $E_{7(7)}$ representations. In particular, decomposing the adjoint representation μ under $SU(8)/\mathbb{Z}_2$ as $\mathbf{133} = \mathbf{63} + \mathbf{35} + \bar{\mathbf{35}}$ and writing $\mu = (\mu^{\alpha}{}_{\beta}, \mu^{\alpha\beta\gamma\delta}, \bar{\mu}_{\alpha\beta\gamma\delta})$, with $\bar{\mu}_{\alpha\beta\gamma\delta} = *\mu_{\alpha\beta\gamma\delta}$ and $\mu^{\alpha}{}_{\alpha} = 0$, one finds

$$
\begin{aligned}
\delta x^{\alpha\beta} &= \mu^{\alpha}{}_{\gamma}x^{\gamma\beta} - \mu^{\beta}{}_{\gamma}x^{\alpha\gamma} + \mu^{\alpha\beta\gamma\delta}\bar{x}_{\gamma\delta}, \\
\delta\bar{x}_{\alpha\beta} &= -\mu^{\gamma}{}_{\alpha}\bar{x}_{\gamma\beta} + \mu^{\gamma}{}_{\beta}\bar{x}_{\alpha\gamma} + \bar{\mu}_{\alpha\beta\gamma\delta}x^{\gamma\delta},
\end{aligned}
\tag{4.5.5}
$$

with[1]

$$
\begin{aligned}
\mu^{\alpha}{}_{\beta} &= \frac{1}{4}\mu^{-}_{ab}\hat{\gamma}^{ab\,\alpha}{}_{\beta} + \frac{-\mathrm{i}}{48}\mu^{-}_{abcd}\hat{\gamma}^{abcd\,\alpha}{}_{\beta}, \\
\mu^{\alpha\beta\gamma\delta} &= \frac{1}{16}\left(\mathrm{i}\mu^{+}_{abcd} + 2\mu^{+}_{ac}g^{(8)}_{bd}\right)\hat{\gamma}^{ab\,\alpha\beta}\hat{\gamma}^{cd\,\gamma\delta}.
\end{aligned}
\tag{4.5.6}
$$

where $\mu^{\pm}_{ab} = \frac{1}{2}(\mu_{ab} \pm \mu_{ba})$ and $\mu^{\pm}_{abcd} = \frac{1}{2}(\mu_{abcd} \pm *\mu_{abcd})$.

[1] in the first line the minus superscript is not strictly necessary

Chapter 5

Exceptional Generalized Geometry in seven dimensions

In this chapter we discuss the structure of Exceptional Generalized Geometry (EGG) in seven-dimensions. Our motivation will be the compactification of eleven-dimensional supergravity to a four-dimensional effective theory.

Recall that Type II string backgrounds which include non-trivial fluxes have a natural description in terms of Hitchin's generalized geometry, where the metric and the NS–NS B-field are combined into a single geometrical object. The aim of the work presented here is to understand the details of how similar constructions based on the exceptional groups $E_{d(d)}$ can be used to describe supersymmetric M-theory, or more precisely, eleven-dimensional supergravity backgrounds.

We will concentrate on the physically important example of seven-dimensional backgrounds and, following [177, 178], consider the generic form of the superpotential in the corresponding $N = 1$ four-dimensional theory, or more generally, show partially how the eleven-dimensional supergravity can be rewritten in terms of $N = 1$-like structures. Specifically, we will show first how the structure of the supersymmetric background is characterized by a particular $E_{7(7)}$ object ϕ. From a four-dimensional perspective this encodes the scalar chiral multiplet degrees of freedom. We then show that the superpotential can be written as an $E_{7(7)}$ invariant function of ϕ.

The essential idea of the construction is that the $O(d,d)$ symmetries of generalized d-dimensional geometry, which in string theory are related to T-duality, should be replaced by the U-duality exceptional symmetry groups $E_{d(d)}$. Since U-duality connects all the degrees of freedom of eleven-dimensional supergravity, or for type II theories, both NS–NS and R–R degrees of freedom, this extension should provide a geometrization of generic flux backgrounds. These ideas have been described in some detail recently by Hull [194], who dubbed the corresponding geometry "extended" or more specifically "M-geometry". In addition, the fact that supergravity could be reformulated in terms of $E_{d(d)}$ objects was earlier pointed out by de Wit and Nicolai [179], with some recent connected work in [195], and motivated by this one might coin the term "exceptional generalized geometries" (EGG)[1].

5.1 The exceptional generalized tangent space and $E_{7(7)}$

We would thus like to describe an exceptional generalized geometry (EGG), analogous to the generalized geometry of Hitchin, but relevant for the description of eleven-dimensional supergravity (or type II supergravity including the RR fields) rather than simply the NS–NS sector of type II. We will concentrate on the case of a seven-dimensional manifold M. As already pointed out this construction has also been described, in general dimension, in [194] and is closely related to the work of [179, 195].

Introducing the generalized tangent space allowed us to construct objects with an $O(d,d)$ symmetry. The g and B degrees of freedom then encoded a $O(d,d)/O(d)\times O(d)$ coset, parameterizing the generalized metric G. This coset structure is well known from considering the compactification of the NS-NS sector of ten-dimensional

[1]This is the notation we will use in this thesis. The first letter of EGG can be viewed as standing for exceptional or extended or both

supergravity to $10-d$ dimensions. For instance, when the internal space is a torus T^d, the low-energy effective theory in $10-d$ dimensions has the scalar moduli parameterized by that same coset $O(d,d)/O(d)\times O(d)$, and a set of $U(1)$ gauge fields in the $2d$-dimensional representation of $O(d,d)$. In addition, there is, of course, a stringy $O(d,d;\mathbb{Z})$ T-duality symmetry. The string momentum and winding charges further couple to the $2d$ $U(1)$ gauge fields [53].

In constructing the EGG we use the analogous groups, that is, those related to U-duality. Eleven-dimensional supergravity has as boson fields a metric g and a three-form A. Compactifying on a T^d torus these lead to scalars parameterizing E_d/H (where E_d is a non-compact version of the exceptional group and H is its maximal compact subgroup) and $U(1)$ fields spanning a representation of E_d. The momentum, membrane, five-brane and, potentially, Kaluza–Klein monopole charges couple to the relevant $U(1)$ gauge fields. In particular in $d=7$, the scalars parameterize $E_{7(7)}/(SU(8)/\mathbb{Z}_2)$ and the gauge fields (together with their magnetic duals) or equivalently the M-theory charges, fill out the **56** representation of $E_{7(7)}$.

Given this analogy, since in generalized geometry the generalized tangent space is in the same representation of $O(d,d)$ as the $U(1)$ gauge fields it is natural to construct an exceptional generalized tangent space which transforms as the **56** representation of $E_{7(7)}$. As in generalized geometry, the $GL(7,\mathbb{R})$ diffeomorphism group should be a subgroup of $E_{7(7)}$, and we also expect the gauge transformations of A to be somehow embedded into $E_{7(7)}$. The construction is as follows.

As described in section 4.1, there is a natural $SL(8,\mathbb{R})$ subgroup of $E_{7(7)}$ under which the **56** representation space is given by

$$E=\Lambda^2V\oplus\Lambda^2V^* \tag{5.1.1}$$

where V is the eight-dimensional fundamental representation space of $SL(8,\mathbb{R})$. It is natural to embed the $GL(7,\mathbb{R})$ diffeomorphism group in $SL(8,\mathbb{R})$. Let us choose the embedding (see section 4.3) such that

$$V = \left[(\Lambda^7 T^*)^{1/4} \otimes T\right] \oplus (\Lambda^7 T^*)^{-3/4}. \tag{5.1.2}$$

One then finds

$$E = (\Lambda^7 T^*)^{-1/2} \otimes \left[T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*)\right]. \tag{5.1.3}$$

(Note that the final term in brackets can also be written as $(\Lambda^7 T^*)^2 \otimes \Lambda^6 T$). The bundle $(\Lambda^7 T^*)^{-1/2}$ is isomorphic to the trivial bundle, thus there is always a (non-canonical) isomorphism

$$E \simeq T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*). \tag{5.1.4}$$

Given such an isomorphism we can denote elements of E as

$$X = x + \omega + \sigma + \tau \in E, \tag{5.1.5}$$

where, in components, we have x^m, ω_{mn}, $\sigma_{m_1 \dots m_5}$ and $\tau_{m,n_1 \dots n_7}$.

The bundle E is what we define as the exceptional generalized tangent space (EGT)[2]. Except for the overall tensor density factor of $(\Lambda^7 T^*)^{-1/2}$, we see that we can identify it as a sum of vectors, two-forms, five-forms and one-forms tensored with seven-forms. Physically in M-theory we expect these to correspond to momentum, membrane, five-brane and Kaluza–Klein monopole charge respectively.

Recall that the generalized tangent space had a natural invariant metric defining the $O(d,d)$ group. As discussed in section 4.1, the group $E_{7(7)}$ is defined by, not a metric, but a symplectic structure Ω and symmetric quartic invariant q on the

[2]Note that there is a second possible way of embedding $GL(7,\mathbb{R})$, and hence choice for E, analogous to the choice of spin-structures of $O(d,d)$ [169], where E is defined as in 5.1.3 except with an overall factor of $\Lambda^7 T/|\Lambda^7 T|$. This bundle has a similar isomorphism to 5.1.4 but with T and T^* exchanged everywhere. We will ignore this second choice.

56-dimensional representation space. This are given explicitly in terms of $SL(8,\mathbb{R})$ representations in section 4.1. In EGG they are the analogues of the $O(d,d)$ metric $\mathcal{I}$.

Next we would like to identify the analogues of the B-shifts. We note that the 133-dimensional adjoint representation space of $E_{7(7)}$ decomposes as

$$A = (T \otimes T^*) \oplus \Lambda^6 T \oplus \Lambda^6 T^* \oplus \Lambda^3 T \oplus \Lambda^3 T^*. \tag{5.1.6}$$

Given there is a three-form potential A in eleven-dimensional supergravity, the analogue of B-shifts should be A-shifts generated by $A \in \Lambda^3 T^*$. In fact, we will also consider $\tilde{A}$-shifts with $\tilde{A} \in \Lambda^6 T^*$ corresponding to the dual form-potential. This will be described in more detail in the next section, for now we simply note that A and $\tilde{A}$ are both elements of the adjoint bundle given in eq. (5.1.6). Their action on $X \in E$ is given in eq 4.3.8. It exponentiates to

$$\begin{aligned} \mathrm{e}^{A+\tilde{A}}X = \\ & x + [\omega + i_x A] + \left[\sigma + A \wedge \omega + \frac{1}{2} A \wedge i_x A + i_x \tilde{A}\right] \\ & + \left[\tau + jA \wedge \sigma - j\tilde{A} \wedge \omega + jA \wedge i_x \tilde{A} + \frac{1}{2} jA \wedge A \wedge \omega + \frac{1}{6} jA \wedge A \wedge i_x A\right], \end{aligned} \tag{5.1.7}$$

where we are using a notation for elements of $T^* \otimes (\Lambda^7 T^*)$ defined in 4.3.9. Note that the action truncates at cubic order. Furthermore, the corresponding Lie algebra, unlike the case of B-shifts is not Abelian. We have the commutator

$$[A + \tilde{A}, A' + \tilde{A}'] = -A \wedge A'. \tag{5.1.8}$$

That is, two A-shifts commute to give a $\tilde{A}$ shift [196].

As in the case of generalized geometry, we would like to consider the case where we can make A and $\tilde{A}$ shifts corresponding to non-trivially patched form-field potentials.

Let us start with

$$X_0 \in T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*). \tag{5.1.9}$$

On a given patch $U_{(\alpha)}$ we define the shifted object

$$X_{(\alpha)} = \mathrm{e}^{A_{(\alpha)} + \tilde{A}_{(\alpha)}} X_0. \tag{5.1.10}$$

In passing from one patch to another we have, on $U_{(\alpha)} \cap U_{(\beta)}$,

$$X_{(\alpha)} = \mathrm{e}^{\mathrm{d}\Lambda_{(\alpha\beta)} + \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)}} X_{(\beta)}, \tag{5.1.11}$$

provided the connections A and $\tilde{A}$ patch as

$$\begin{aligned} A_{(\alpha)} - A_{(\beta)} &= \mathrm{d}\Lambda_{(\alpha\beta)}, \\ \tilde{A}_{(\alpha)} - \tilde{A}_{(\beta)} &= \mathrm{d}\tilde{\Lambda}_{(\alpha\beta)} - \frac{1}{2}\mathrm{d}\Lambda_{(\alpha\beta)} \wedge A_{(\beta)}. \end{aligned} \tag{5.1.12}$$

As we will see in the next section this is precisely the patching we get naturally from eleven-dimensional supergravity, and is necessary for the twisted EGT to depend only on the connective structure of the gerbes. We can define E more formally via a series of extensions

$$\begin{gathered} 0 \longrightarrow \Lambda^2 T^* \longrightarrow E'' \longrightarrow T \longrightarrow 0, \\ 0 \longrightarrow \Lambda^5 T^* \longrightarrow E' \longrightarrow E'' \longrightarrow 0, \\ 0 \longrightarrow T^* \otimes \Lambda^7 T^* \longrightarrow E \longrightarrow E' \longrightarrow 0. \end{gathered} \tag{5.1.13}$$

Above we have only given the first level of the connective structure of the gerbes. For instance on the corresponding multiple intersections of patches we have

$$\begin{gathered} \Lambda_{(\alpha\beta)} + \Lambda_{(\beta\gamma)} + \Lambda_{(\gamma\alpha)} = \mathrm{d}\Lambda_{(\alpha\beta\gamma)} \\ \Lambda_{(\beta\gamma\delta)} - \Lambda_{(\alpha\gamma\delta)} + \Lambda_{(\alpha\beta\delta)} - \Lambda_{(\alpha\beta\gamma)} = \mathrm{d}\Lambda_{(\alpha\beta\gamma\delta)}. \end{gathered} \tag{5.1.14}$$

For a quantized flux $F = \mathrm{d}A_{(\alpha)}$ we have $g_{(\alpha\beta\gamma\delta)} = \mathrm{e}^{\mathrm{i}\Lambda_{(\alpha\beta\gamma\delta)}} \in U(1)$ with the cocycle condition

$$g_{(\beta\gamma\delta\epsilon)}g^{-1}_{(\alpha\gamma\delta\epsilon)}g_{(\alpha\beta\delta\epsilon)}g^{-1}_{(\alpha\beta\gamma\epsilon)}g_{(\alpha\beta\gamma\delta)} = 1. \tag{5.1.15}$$

For $\tilde{\Lambda}_{(\alpha\beta)}$ there is a similar set of structures, with the final cocycle condition defined on a octuple intersection $U_{(\alpha_1)} \cap \cdots \cap U_{(\alpha_8)}$.

Finally we would like to identify the analogue of the Courant bracket for the EGT. We look for a pairing with A- and $\tilde{A}$-shifts as automorphisms when $\mathrm{d}A = \mathrm{d}\tilde{A} = 0$. One finds the unique "exceptional Courant bracket" (ECB)

$$\begin{aligned}
&[x + \omega + \sigma + \tau, x' + \omega' + \sigma' + \tau'] \\
&= [x, x'] + \mathcal{L}_x\omega' - \mathcal{L}_{x'}\omega - \frac{1}{2}\mathrm{d}\left(i_x\omega' - i_{x'}\omega\right) \\
&+ \mathcal{L}_x\sigma' - \mathcal{L}_{x'}\sigma - \frac{1}{2}\mathrm{d}\left(i_x\sigma' - i_{x'}\sigma\right) + \frac{1}{2}\omega \wedge \mathrm{d}\omega' - \frac{1}{2}\omega' \wedge \mathrm{d}\omega \\
&+ \frac{1}{2}\mathcal{L}_x\tau' - \frac{1}{2}\mathcal{L}_{x'}\tau + \frac{1}{2}\left(j\omega \wedge \mathrm{d}\sigma' - j\sigma' \wedge \mathrm{d}\omega\right) - \frac{1}{2}\left(j\omega' \wedge \mathrm{d}\sigma - j\sigma \wedge \mathrm{d}\omega'\right)
\end{aligned} \tag{5.1.16}$$

If G_{shift} is the group generated by the A and $\tilde{A}$ shifts, the ECB is invariant under the $\mathrm{Diff}(M) \ltimes G_{\text{shift}}$.

5.2 The exceptional generalized metric and $SU(8)/\mathbb{Z}_2$ structures

Having set up the EGT, its topology and the corresponding bracket, we now describe the analog of the generalized metric and how this encodes the fields of eleven-dimensional supergravity. Our motivation is that, when compactified on T^7, the moduli arising from the eleven-dimensional supergravity fields g and A parameterize a $E_{7(7)}/(SU(8)/\mathbb{Z}_2)$ coset space [179]. Rather than define the coset space and corresponding metric (see [194]), we will define the structure in a slightly different way.

Geometrically, the coset structure implies that g and A define an $SU(8)/\mathbb{Z}_2$ structure on E. Such a structure can be defined as follows. Recall that $E_{7(7)}$ preserved a symplectic structure Ω on E. As such it is a subgroup of $Sp(56,\mathbb{R})$. Suppose one has an (exceptional generalized) almost complex structure (EGACS) J which is compatible with Ω, that is $\Omega(JX, JY) = \Omega(X, Y)$. This defines a unitary $U(28)$ structure on E. If it is also compatible with the $E_{7(7)}$ quartic invariant, that is $q(JX) = q(X)$, then it is invariant under the common subgroup of $E_{7(7)}$ and $U(28)$ namely $SU(8)/\mathbb{Z}_2$. We will construct J explicitly below. To summarize, an $SU(8)/\mathbb{Z}_2$ structure is equivalent to an almost complex structure J on E such that

$$\Omega(JX, JY) = \Omega(X,Y) \quad ; \quad q(JX) = q(X), \tag{5.2.1}$$

or in other words $J \in E_{7(7)}$. Note that, in contrast to the generalized geometry case where the $O(d)\times O(d)$ structure was equivalent to a compatible product structure satisfying $\Pi^2 = 1$, for $SU(8)/\mathbb{Z}_2 \subset E_{7(7)}$ the structure is defined by a compatible complex structure satisfying $J^2 = -1$. Given J and Ω we can then define the corresponding exceptional generalized metric (EGM) G by

$$G(X,Y) = \Omega(X, JY), \tag{5.2.2}$$

which gives a positive definite metric on E. We now turn to how one constructs the generic form of J and hence G.

Given a metric $\hat{g}_{ab}$ on the $SL(8,\mathbb{R})$ representation space V, a natural way to define a particular almost complex structure J_0, using the conventions of section 4.1 (in particular pairs of indices aa' and bb' are antisymmetrized), is as follows

$$J_0X = \begin{pmatrix} 0 & -\hat{g}^{ab}\hat{g}^{a'b'} \\ \hat{g}_{ab}\hat{g}_{a'b'} & 0 \end{pmatrix} \begin{pmatrix} x^{bb'} \\ x'_{bb'} \end{pmatrix} = \begin{pmatrix} -\hat{g}^{ab}\hat{g}^{a'b'}x'_{bb'} \\ \hat{g}_{ab}\hat{g}_{a'b'}x^{bb'} \end{pmatrix}. \tag{5.2.3}$$

The corresponding EGM is

$$G_0(X,Y) = \hat{g}_{ab}\hat{g}_{a'b'}x^{aa'}y^{bb'} + \hat{g}^{ab}\hat{g}^{a'b'}x'_{aa'}y'_{bb'}. \tag{5.2.4}$$

From the definitions 4.1.3 and 4.1.4 of Ω and q it is clear that $J_0 \in E_{7(7)}$ provided $\det \hat{g} = 1$. Under an infinitesimal $E_{7(7)}$ transformation $\mu \in \mathbf{133}$ we have

$$\delta J_0 = [\mu, J_0] = \begin{pmatrix} \mu^{+aa'}{}_{bb'} & -2\mu^{+ab}\hat{g}^{a'b'} \\ -2\mu^{+}_{ab}\hat{g}_{a'b'} & -\mu^{+}_{aa'}{}^{bb'} \end{pmatrix} \tag{5.2.5}$$

where $\mu^{\pm}_{abcd} = \mu_{abcd} \pm *\mu_{abcd}$ and $\mu^{\pm}_{ab} = \mu_{ab} \pm \mu_{ba}$ and indices are raised and lowered using $\hat{g}$. Thus J_0 is invariant under the subgroup generated by μ^{-}_{ab} and μ^{-}_{abcd}. As discussed in section 4.5 this is precisely $SU(8)/\mathbb{Z}_2$ (see eq. (4.5.6)).

Given the embedding 5.1.2 of $GL(7,\mathbb{R}) \subset SL(8,\mathbb{R})$ discussed in detail in section 4.3, we can define $\hat{g}$ in terms of a seven-dimensional metric g as

$$\hat{g}_{ab} = (\det g)^{-1/4} \begin{pmatrix} g_{mn} & 0 \\ 0 & \det g \end{pmatrix}. \tag{5.2.6}$$

Acting on elements of $X = x + \omega + \sigma + \tau$ we have

$$G_0(X,X) = 2\left(|x|^2 + |\omega|^2 + |\sigma|^2 + |\tau|^2\right), \tag{5.2.7}$$

where $|\tau|^2 = \frac{1}{7!}\tau_{m,n_1\dots n_7}\tau^{m,n_1\dots n_7}$, $|\sigma|^2 = \frac{1}{5!}\sigma_{n_1\dots n_5}\sigma^{n_1\dots n_5}$ etc. and where we have dropped on overall factor of $(\det g)^{1/2}$ so that the result is a scalar, which is natural, since in writing $X = x + \omega + \sigma + \tau$ we are using the isomorphism given in eq. (5.1.4).

Given a seven-dimensional metric g we have thus been able to write a particular $SU(8)/\mathbb{Z}_2$ structure J_0. A generic structure, given all such structures lie in the same orbit, will be of the form $J = hJ_0h^{-1}$ where $h \in E_{7(7)}$, or equivalently $G(X,Y) =$

$G_0(h^{-1}X, h^{-1}Y)$. We write $h = e^{\mu}$ with the Lie algebra element $\mu = (\mu^a{}_b, \mu_{abcd}) \in$ **133**. The elements $\mu^m{}_n$ generate the $GL(7,\mathbb{R})$ subgroup and acting on J_0 simply change the form of the metric g. The additional components $\mu^8{}_m$ and $\mu^m{}_8$ modify the form of $\hat{g}$ in eq. (5.2.6). Since only μ^+_{ab} acts non-trivially on J_0, we need only consider transformations with, say, $\mu^m{}_8$. Similarly since only μ^+_{abcd} acts non-trivially we can generate a generic J using only, say μ_{mnp8}. However, $\mu^m{}_8$ and μ_{mnp8} transformations precisely correspond to the subgroup of A- and $\tilde{A}$-shifts. Thus, given a generic g defining G_0, the generic EGM can be written as

$$G(X,Y) = G_0(e^{-A-\tilde{A}}X, e^{-A-\tilde{A}}Y). \tag{5.2.8}$$

Note that this is analogous to the form of the generalized metric in Hitchin's generalized geometry. Note also that for non-trivial A and $\tilde{A}$, G_0 is an EGM on the untwisted EGT given by $T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^7 T^*)$, while G is an EGM on the twisted bundle E given by eq. (5.1.13).

Chapter 6

Supersymmetric backgrounds and EGG

In this chapter we will relate the EGG defined in the previous section to eleven-dimensional supergravity and in particular seven-dimensional supersymmetric backgrounds. The physical context we are interested in corresponds to the case where the eleven-dimensional space-time is topologically a product of a four-dimensional "external" and a seven-dimensional "internal" manifold.

$$M_{10,1} = M_{3,1} \times M_7 \tag{6.0.1}$$

If M_7 is compact we can consider compactifying eleven-dimensional supergravity to give an effective four-dimensional theory. In particular, we could focus on the case when the effective theory is supersymmetric. We could also look for particular examples of compactifications which are solutions of the supergravity field equations and preserve some number of supersymmetries. In either case, the geometry of M_7 is restricted. The goal here is to understand how this restricted geometry can be naturally described in terms of EGG structure on M_7.

6.1 Conventions

We adopt conventions where the eleven-dimensional supergravity action takes the form (note this matches, for instance, the conventions of [190])

$$\begin{aligned} S = & \frac{1}{2}\int_{M_{11}} \sqrt{-g}\, d^{11}x \Big[R - \overline{\Psi}_M \Gamma^{MNP} \nabla_N \Psi_P - \frac{1}{2} F_4 \wedge *_{11} F^4 \Big] \\ & - \frac{1}{192}\int_{M_{11}} \sqrt{-g}\, d^{11}x \overline{\Psi}_M \Gamma^{MNPQRS} \Psi_N (F_4)_{PQRS} - \frac{1}{2}\int_{M_{11}} F_4 \wedge *_{11} C_4 \\ & - \frac{1}{12}\int_{M_{11}} F_4 \wedge F_4 \wedge A_3 \,, \end{aligned}$$

where the eleven-dimensional Newton's constant is set to unity and

$$(C_4)_{MNPQ} = 3\overline{\Psi}_{[M}\Gamma_{NP}\Psi_{Q]} \,. \tag{6.1.1}$$

while the variation of the gravitino Ψ_M is given by

$$\delta\Psi_M = D_M\epsilon + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\delta_M^N \Gamma^{PQR}\right)(F)_{NPQR}\,\epsilon\, + \ldots, \tag{6.1.2}$$

We use uppercase indices $M, N, \ldots = 0, \ldots, 10$ for curved eleven-dimensional indices, $\mu, \nu, \ldots = 0, \ldots, 3$ for the four-dimensional (external) indices and $m, n, \ldots = 1, \ldots, 7$ for the indices on the internal manifold. F is the field strength of the eleven-dimensional three-from field.

The dots represent terms coupling Ψ_M and F. The metric g has signature $(-, +, \ldots, +)$ and Γ_M are the eleven-dimensional gamma matrices satisfying

$$\{\Gamma_M, \Gamma_N\} = 2g_{MN}1, \tag{6.1.3}$$

with $\Gamma_{M_1 \ldots M_{11}} = 1\epsilon_{M_1 \ldots M_{11}}$ where the volume form ϵ satisfies $\epsilon_{01\ldots10} = \sqrt{-g}$. The spinors ϵ and Ψ_M are Majorana. Given the intertwining relation $-\Gamma^T_{\hat{M}} = C^{-1}\Gamma_{\hat{M}}C$, we define the conjugate spinor $\bar{\Psi}_M = \Psi^T_M C^{-1}$. Further the equation of motion and

Bianchi identity for the flux F read

$$\mathrm{d} * F + \frac{1}{2} F \wedge F = 0 \quad ; \quad \mathrm{d}F = 0. \tag{6.1.4}$$

In this context all differential forms are defined as:

$$A_{(p)} = \frac{1}{p!} A_{m_1 \dots m_p} dx^{m_1} \wedge \dots dx^{m_p} \, . \tag{6.1.5}$$

Also

$$\begin{aligned} (dA)_{m_1 \dots m_{p+1}} &= (p+1) \nabla_{[m_1} A_{m_2 \dots m_{p+1}]} \\ (d^{\dagger} A)_{m_1 \dots m_{p-1}} &= -\nabla^n A_{n m_1 \dots m_{p-1}} \\ (*_D A)_{m_1 \dots m_{D-p}} &= \frac{(-1)^{p(D-p)}}{p!} \epsilon_{m_1 \dots m_{D-p}}{}^{n_1 \dots n_p} A_{n_1 \dots n_p} \end{aligned} \tag{6.1.6}$$

6.2 Effective theories and field decompositions

Here we will focus on the low-energy effective theory rather than the on-shell supersymmetric backgrounds. Given the product manifold structure in eq. (6.0.1), the tangent bundle decomposes as $T_{10,1} = T_{3,1} \oplus T_7$ and all the supergravity fields can be decomposed under a local $Spin(3,1) \times Spin(7) \subset Spin(10,1)$ symmetry. Normally one would derive a four-dimensional effective description by truncating the Kaluza–Klein spectrum of modes on M_7 to give a four-dimensional theory with a finite number of degrees of freedom. For instance, compactifying on a torus and keeping massless modes, one finds that the degrees of freedom actually arrange themselves into multiplets transforming under $E_{7(7)}$ for the bosons and $SU(8)/\mathbb{Z}_2$ for the fermions.

However, one can also keep the full dependence of all eleven-dimensional fields on both the position on $M_{3,1}$ and M_7. One can then simply rewrite the eleven-dimensional theory, breaking the local $Spin(10,1)$ symmetry to $Spin(3,1) \times Spin(7)$

so that it is analogous to a four-dimensional theory. This was done explicitly by de Wit and Nicolai in [179] retaining all 32 supersymmetries, where it was shown that in general the degrees of freedom fall into $E_{7(7)}$ and $SU(8)/\mathbb{Z}_2$ representations. In this work we will ultimately be interested in such reformulations focusing on only four of the supercharges so that the theory has a structure analogous to $N = 1$ four-dimensional supergravity. Note that formally the only requirement for making such rewritings is not that $M_{10,1}$ is topologically a product, but rather that the tangent space $T_{10,1}$ decomposes into a four- and seven-dimensional part

$$T_{10,1} = T_{3,1} \oplus \hat{F}. \tag{6.2.1}$$

For simplicity, here we will concentrate on the case of a product manifold, though all of our analysis actually goes through in the more general case, with the EGT defined in terms of $\hat{F}$ rather than T_7.

Let is briefly note how the fields decompose under $Spin(3,1) \times Spin(7)$. The degrees of freedom are the metric g_{MN}, three-form A_{MNP} and gravitino Ψ_M. Let us first consider the $Spin(3,1)$ scalars. The eleven-dimensional metric decomposes as a warped product

$$\mathrm{d}s^2(M_{11}) = \mathrm{e}^{2E} g^{(4)}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn} \mathrm{d}x^m \mathrm{d}x^n. \tag{6.2.2}$$

where $e^{-4E} = \det g_{mn}$, chosen so as to obtain the standard Einstein-Hilbert term upon dimensional reduction. In a conventional compactification, deformations of the internal metric g_{mn} lead to scalar moduli fields in the effective theory. Moduli fields can also arise from the flux F. Keeping only $Spin(3,1)$ scalar parts, one can decompose

$$F = *_7 \tilde{F} \wedge \mathrm{e}^{4E} \epsilon_{(4)} + F \tag{6.2.3}$$

where $F \in \Lambda^4 T^*_7$ and $\tilde{F} \in \Lambda^7 T^*_7$ and $\epsilon_{(4)} = \sqrt{-g^{(4)}} \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3$.

The eleven-dimensional equation of motion and Bianchi identity given in eq. (6.1.4) then decompose as

$$\begin{aligned} \mathrm{d}\tilde{F} + \frac{1}{2} F \wedge F = 0 \\ \mathrm{d}\big(\mathrm{e}^{4E} *_7 \tilde{F}\big) = 0, \\ \mathrm{d}F = 0 \\ \mathrm{d}\big(\mathrm{e}^{4E} *_7 F\big) + \mathrm{e}^{4E} F = 0 \end{aligned} \tag{6.2.4}$$

so one can introduce, locally,

$$\begin{aligned} F &= \mathrm{d}A, \\ \tilde{F} &= \mathrm{d}\tilde{A} - \frac{1}{2} A \wedge F, \end{aligned} \tag{6.2.5}$$

where on a patch $A \in \Lambda^3 T_7^*$ and $\tilde{A} \in \Lambda^6 T_7^*$. By definition, F and $\tilde{F}$ are globally defined. This means that the potentials A and $\tilde{A}$ must patch precisely as given by eq. (5.1.12). We thus see that the twisting of the EGT in eq. (5.1.11) is precisely that corresponding to the supergravity potentials.

Furthermore, given the discussion of the previous section 5.2 the scalar degrees of freedom g_{mn} and A_{mnp} and $\tilde{A}_{m_1 \dots m_6}$ scalars can be combined together as an EGM or equivalently an exceptional generalized almost complex structure J on E.

Turning briefly to the remaining fields, there are 28 bosonic $Spin(3,1)$ vector degrees of freedom coming from off-diagonal components of the metric $g_{\mu m}$ and from $A_{\mu mn}$. One usually also introduces the corresponding dual potentials giving a total of 56.

Finally for the fermionic degrees of freedom we decompose the eleven-dimensional gamma matrices as

$$\Gamma^\mu = \mathrm{e}^{-E} \gamma^\mu \otimes 1 \qquad \Gamma^m = \mathrm{i}\gamma_{(4)} \otimes \gamma^m \tag{6.2.6}$$

where the seven-dimensional gamma-matrices conventions are defined in section 4.4, while the four-dimensional gamma matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2g^{(4)}_{\mu\nu}1$ and $\gamma_{\mu_1\dots\mu_4} = \gamma_{(4)}\epsilon^{(4)}_{\mu_1\dots\mu_4}$. The real eleven-dimensional spinors correspondingly decompose as

$$\begin{aligned} \epsilon &= \mathrm{e}^{E/2}\theta_+ \otimes \eta + \mathrm{e}^{E/2}\theta_- \otimes \eta^c \\ \mathbf{32} &= (\mathbf{2}, \mathbf{8}) + (\bar{\mathbf{2}}, \bar{\mathbf{8}}) \end{aligned} \tag{6.2.7}$$

where $\mathrm{i}\gamma_{(4)}\theta_\pm = \pm\theta_\pm$ (with $\theta^c_+ = D\theta^*_+ = \theta_-$ and $-\gamma^*_\mu = D^{-1}\gamma_\mu D$) are chiral four-dimensional spinors and η is a complex $Spin(7)$ spinor. The factor of $\mathrm{e}^{E/2}$ is conventional. Thus Ψ_μ decomposes into eight spin-$\frac{3}{2}$ fermions, while Ψ_m gives 56 spin-$\frac{1}{2}$ fermions.

As discussed in section 4.4 there is a natural embedding of $SU(8)$ in the Clifford algebra $\mathrm{Cliff}(7,0;\mathbb{R})$ with the complex spinors in the fundamental representation. It turns out the all the degrees of freedom, fermionic and bosonic arrange as $SU(8)$ representations. Thus we can actually promote the $Spin(3,1) \times Spin(7)$ symmetry to $Spin(3,1) \times SU(8)$. This is summarized in the table 6.1 where $\mathbf{r_s}$ transforms as the $\mathbf{r}$ representation of $SU(8)$ with $Spin(3,1)$ spin $\mathbf{s}$.

g_{mn}, A_{mnp}, $\tilde{A}_{m_1\dots m_6}$:	$\mathbf{35_0} + \bar{\mathbf{35}}_\mathbf{0}$	Ψ_m :	$\mathbf{56_{1/2}}$
$g_{\mu m}$, $A_{\mu mn}$ + duals :	$\mathbf{28_1} + \bar{\mathbf{28}}_\mathbf{1}$	Ψ_μ :	$\mathbf{8_{3/2}}$
$g^{(4)}_{\mu\nu}$:	$\mathbf{1_2}$		

Table 6.1: Decomposition of eleven-dimensional supergravity fields under $Spin(3,1) \times SU(8)$

Note the familiar result that these representations precisely fill out the form of the $N = 8$ four-dimensional supergravity multiplet.

From an EGG perspective, the scalar degrees of freedom define a $SU(8)/\mathbb{Z}_2$ structure on the EGT. Given this structure (or rather the existence of a double cover $SU(8)$ structure, which is not always guaranteed) one can then define $SU(8)$ spinors and hence the fermionic degrees of freedom. This is the EGG analogue of requiring a metric, or $O(d)$ structure, and hence a set of vielbeins, before one can define ordinary spinors on a curved manifold.

Let us now turn to how this analysis changes when we pick out a single set of supersymmetry parameters, so that the four-dimensional effective theory has $N = 1$ supersymmetry.

6.3 $N = 1$ M-theory backgrounds and $SU(7)$ structures

We would now like to identify the analogue of the $N = 2$ $SU(3) \times SU(3)$ structure of type II theories for $N = 1$ compactifications of eleven-dimensional supergravity. This means picking out four preferred supersymmetries out of 32, or equivalently a fixed seven-dimensional spinor η in the general decomposition given in eq. (6.2.7). Note the this decomposition is that most general compatible with four-dimensional Lorentz invariance [197], and generically has complex η. Recall that the EGM G defines an $SU(8)/\mathbb{Z}_2$ structure on E, and η transforms in the fundamental representation $\mathbf{8}$ of the double cover $SU(8)$. As before, to define a generic low-energy effective theory the spinor η must be globally defined and nowhere vanishing. At each point the stabilizer of a fixed element of the $\mathbf{8}$ representation is $SU(7)$, thus we see it defines a special structure on the EGT E

$$N = 1 \text{ effective theory} \Leftrightarrow SU(7) \text{ structure on } E. \tag{6.3.1}$$

Note that projection $E \to T_7$ allows us to pick out a $GL(7, \mathbb{R})$ subgroup of $E_{7(7)}$, and similarly, given a EGM G it allows us to define a natural $Spin(7) \subset SU(8)$ subgroup. Under $Spin(7)$ we can choose real spinors so

$$\eta = \eta_1 - i\eta_2 \tag{6.3.2}$$

Each η_i is stabilized by a $G_2 \in Spin(7)$ subgroup. Thus from the point of the view of the ordinary tangent space T_7, if η_i are globally defined and non-vanishing we have a pair of G_2 structures. However, all we really require is a globally defined non-vanishing complex η. Thus in general we may not have either G_2 structure. Locally, the pair of η_i are preserved by a $SU(3)$ group.

One way to define the structure is as the pair of EGM and $SU(8)$ spinor (G, η), but as in the type II case, it is interesting to see if we can find a particular orbit in an $E_{7(7)}$ representation which can also be used to define the structure. Again we might expect that it will appear as a spinor bilinear. As we discuss in a moment, this space should also correspond to the $N = 1$ chiral multiplet space in the four-dimensional effective theory.

Decomposing under $SU(7)$, the **56** representation has no singlets so cannot have elements stabilized by $SU(7)$. The adjoint **133** does have a singlet. In terms of the spinor η, the singlet in $\mu \in \mathbf{133}$, using its decomposition $\mathbf{133} = \mathbf{63} + \mathbf{35} + \bar{\mathbf{35}}$ under $SU(8)$, can be written as

$$\begin{aligned} \mu_0 &= (\mu^\alpha{}_\beta, \mu_{\alpha\beta\gamma\delta}, \bar{\mu}^{\alpha\beta\gamma\delta}), \\ &= \left(\eta^\alpha \bar{\eta}_\beta - \frac{1}{8}(\bar{\eta}\eta) A^\alpha{}_\beta, 0, 0\right). \end{aligned} \tag{6.3.3}$$

However, it is easy to see that this is stabilized by $U(7)$ rather than $SU(7)$. In turns out the relevant representation is the **912**. We define the following $SU(7)$-singlet complex element in terms of its $SU(8)$ decomposition, that is $\mathbf{912} = \mathbf{36} + \mathbf{420} + \bar{\mathbf{36}} + \bar{\mathbf{420}}$,

$$\begin{aligned} \phi_0 &= (\phi^{\alpha\beta}, \phi^{\alpha\beta\gamma}{}_\delta, \bar{\phi}_{\alpha\beta}, \bar{\phi}_{\alpha\beta\gamma}{}^\delta) \\ &= (\chi^\alpha \chi^\beta, 0, 0, 0), \end{aligned} \tag{6.3.4}$$

where it turns out, in what follows, to be easier to use the conjugate spinor $\chi = \eta^c$.

Finally we can form the structure

$$\phi = \mathrm{e}^{A+\tilde{A}}\phi_0. \tag{6.3.5}$$

It turns out that ϕ does indeed define a generic $SU(7)$ structure.

To see this, we first note that under an infinitesimal $E_{7(7)}$ transformation we have

$$\begin{aligned}
\delta\phi^{\alpha\beta} &= (\mu^{\alpha}{}_{\gamma}\chi^{\gamma})\chi^{\beta} + \chi^{\alpha}(\mu^{\beta}{}_{\gamma}\chi^{\gamma}), \\
\delta\phi^{\alpha\beta\gamma}{}_{\delta} &= 0, \\
\delta\bar{\phi}_{\alpha\beta} &= 0, \\
\delta\bar{\phi}_{\alpha\beta\gamma}{}^{\delta} &= \mu_{\alpha\beta\gamma\epsilon}\chi^{\epsilon}\chi^{\delta}.
\end{aligned} \tag{6.3.6}$$

Thus ϕ_0 is stabilized by elements such that

$$\mu^{\alpha}{}_{\beta}\chi^{\beta} = 0, \qquad \mu_{\alpha\beta\gamma\delta}\chi^{\delta} = 0, \tag{6.3.7}$$

which implies, since $\bar{\mu} = *\mu$, that the stabilizer group is $SU(7)$ (and not as should be noted $SU(7)/\mathbb{Z}_2$). Since $\mathrm{e}^{A+\tilde{A}} \in E_{7(7)}$ the stabilizer of ϕ must also be $SU(7)$. Finally, note that the $SU(8)$ representations were defined using the gamma matrices γ^a defined using the seven dimensional metric g. Since the action of $\mathrm{e}^{A+\tilde{A}}$ generates a generic EGM G, we see that (when taken with the choice of generic g and spinor η, which are implicit in eq. (6.3.4)) the action of $\mathrm{e}^{A+\tilde{A}}$ must generate a generic element of the orbit under $E_{7(7)}$.

Note that we could also define a real object $\lambda = \mathrm{Re}\,\phi$

$$\lambda = \mathrm{e}^{A+\tilde{A}}(\chi^{\alpha}\chi^{\beta}, 0, \bar{\chi}_{\alpha}\bar{\chi}_{\beta}, 0), \tag{6.3.8}$$

which also manifestly defines the same $SU(7)$ structure. If $N(E)$ is the **912** representation space based on the EGT E, at each point $x \in M_7$, we can view λ as an embedding of the coset space

$$\lambda : \frac{E_{7(7)}}{SU(7)} \times \mathbb{R}^+ \hookrightarrow N(E), \tag{6.3.9}$$

where $\mathbb{R}^+$ just corresponds to a rescaling of λ. We will call this orbit subspace Σ. There should then be a natural complex structure on Σ which allows one to define the holomorphic ϕ. Note that λ far from fills out the whole of the **912** representation space. Rather we are considering a very particular orbit. It would be interesting to write down the particular non-linear conditions which define the orbit.

Finally let us also consider how the supergravity fields decompose under the $SU(7)$ subgroup and how these correspond to different $N = 1$ mutliplets. We have for $SU(7) \subset SU(8)$

$$\begin{aligned} \mathbf{8} &= \mathbf{7} + \mathbf{1}, \qquad & \mathbf{35} &= \mathbf{35}, \\ \mathbf{28} &= \mathbf{21} + \mathbf{7}, \qquad & \mathbf{56} &= \mathbf{35} + \mathbf{21}. \end{aligned} \tag{6.3.10}$$

This means we can arrange the degrees of freedom as in table 6.2.

multiplet	$SU(7)$ rep	fields
chiral	**35**	g_{mn}, A_{mnp}, $\tilde{A}_{m_1\ldots m_7}$, Ψ_m
vector	**21**	$g_{\mu m}$, $A_{\mu np}$, Ψ_m
spin-$\frac{3}{2}$	**7**	$g_{\mu m}$, $A_{\mu np}$, Ψ_μ
gravity	**1**	$g_{\mu\nu}$, Ψ_μ

Table 6.2: Multiplet structure under $SU(7)$

Note that the coset space $E_{7(7)}\,/\,SU(7)$ actually decomposes into $\mathbf{35} + \mathbf{7} + \bar{\mathbf{35}} + \bar{\mathbf{7}}$. Thus there are more degrees of freedom in λ than chiral degrees of freedom. The same phenomenon appears in the type II case and is associated to the gauge freedom of the extra spin-$\frac{3}{2}$ multiplets. On solution is to assume in a given truncation of the theory that there are no **7** degrees of freedom. Note that in this picture we expect there to be a natural Kähler metric on the coset space $\Sigma = E_{7(7)}\,/\,SU(7) \times \mathbb{R}^+$ corresponding to the Kähler potential of $N = 1$ theories. We do not address this problem here.

6.4 From G_2 to $SU(3)$

Before moving on to the calculation of the superpotential let us make the underlying $SU(3)$ structure arising in the presence of two G_2-invariant spinors explicit using an approach analogous to [109].

Define the fundamental $SU(7)$ four-form as

$$\psi = -\bar{\eta}\gamma^{npqr}\eta^c\, dx_n \wedge dx_p \wedge dx_q \wedge dx_r = -\bar{\eta}\gamma^{abcd}\eta^c\, e_a \wedge e_b \wedge e_c \wedge e_d \tag{6.4.1}$$

where a, b, c, d are internal frame indices. Now define χ_i $(i = 1, 2)$ such that:

$$\begin{aligned} \eta^c &= \alpha\chi_1 + \beta\chi_2 \\ \chi_i^c &= \chi_i \\ \bar{\chi}_i\chi_j &= \delta_{ij} \end{aligned} \tag{6.4.2}$$

with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$ given the normalization $\bar{\eta}\eta = 1$. Then

$$\psi = \alpha^2\psi_{11} + \beta^2\psi_{22} + \alpha\beta(\psi_{12} + \psi_{21}) \tag{6.4.3}$$

where we define

$$\psi_{ij} = -\bar{\chi}_i\gamma^{npqr}\chi_j\, dx_n \wedge dx_p \wedge dx_q \wedge dx_r = -\bar{\chi}_i\gamma^{abcd}\chi_j\, e_a \wedge e_b \wedge e_c \wedge e_d \tag{6.4.4}$$

The χ_i consequently define a plane normal (in spinor space) to the non-singlet components of any generic spinor. So we may choose:[1]

$$\gamma^{1234}\chi_i = \gamma^{1256}\chi_i = \gamma^{3456}\chi_i = -\chi_i. \tag{6.4.5}$$

[1]The chosen combinations of gamma matrices (where the indices are frame indices) all square to unity and commute with each other and can thus be used to define projection operators reducing the original eight-dimensional Majorana subspace to the aforementioned two-dimensional one labeled by a specific choice of both eigenvalues (note that the third condition follows from the first two by consistency).

Further since $\bar{\chi}_1\chi_2 = 0$ we have

$$\begin{aligned} \gamma^{1357}\chi_1 &= -\chi_1 \\ \gamma^{1357}\chi_2 &= +\chi_2. \end{aligned} \tag{6.4.6}$$

Before rewriting ψ in terms of $SU(3)$ structures we must present some useful identities. Note that:

$$\begin{aligned} \{\gamma^7, \gamma^{1357}\} &= 0 \\ (\gamma^7)^2 &= 1 \end{aligned} \tag{6.4.7}$$

From the first equality we deduce that

$$\begin{aligned} \gamma^7\chi_1 &= a\chi_2 \\ \gamma^7\chi_2 &= b\chi_1 \end{aligned} \tag{6.4.8}$$

where a, b are pure imaginary as:

$$\begin{aligned} \chi_i^c = \chi_i = &\quad D\chi_i^* \\ \gamma^{m*} &= -D^{-1}\gamma^m D \end{aligned} \tag{6.4.9}$$

Also

$$a = \bar{\chi}_2\gamma^7\chi_1 = (\bar{\chi}_1\gamma^7\chi_2)^* = b^* \tag{6.4.10}$$

which together with the second equality in eq. (6.4.7) leads to (up to a choice in sign):

$$\begin{aligned} -i\gamma^7\chi_1 &= \quad \chi_2 \\ -i\gamma^7\chi_2 &= -\chi_1 \end{aligned} \tag{6.4.11}$$

and then eq. (6.4.6) gives

$$\begin{aligned}\gamma^{135}\chi_1 &= i\chi_2\\ \gamma^{135}\chi_2 &= i\chi_1\end{aligned} \tag{6.4.12}$$

Remembering our convention that $-i\gamma_{(7)} = \gamma^{[1\cdots 7]} = 1$ we find:

$$\begin{aligned}\gamma^{12}\chi_1 = \gamma^{34}\chi_1 = \gamma^{56}\chi_1 = \chi_2\\ \gamma^{12}\chi_2 = \gamma^{34}\chi_2 = \gamma^{56}\chi_2 = -\chi_1\end{aligned} \tag{6.4.13}$$

Finally

$$\begin{aligned}\gamma^{235}\chi_1 = \gamma^{145}\chi_1 = \gamma^{136}\chi_1 &= \ \ i\chi_1\\ \gamma^{235}\chi_2 = \gamma^{145}\chi_2 = \gamma^{136}\chi_2 &= -i\chi_2\\ \gamma^{236}\chi_1 = \gamma^{146}\chi_1 = \gamma^{245}\chi_1 &= -i\chi_2\\ \gamma^{236}\chi_2 = \gamma^{146}\chi_2 = \gamma^{245}\chi_2 &= -i\chi_1\end{aligned} \tag{6.4.14}$$

and

$$\gamma^{246}\chi_1 = -i\chi_1 \quad ; \quad \gamma^{246}\chi_2 = i\chi_2 \tag{6.4.15}$$

We may now explicitly calculate the non-vanishing components of ψ i.e. those whose four-gamma matrix combinations commute with those in eq.(6.4.5). Further the non-vanishing components of ψ_{11} and ψ_{22} (ψ_{12} and ψ_{21}) will (anti-)commute with γ^{1357}.This leaves us with (commuting):

$$\gamma^{1234} \quad ; \quad \gamma^{1256} \quad ; \quad \gamma^{3456} \quad ; \quad \gamma^{1357} \quad ; \quad \gamma^{1467} \quad ; \quad \gamma^{2367} \quad ; \quad \gamma^{2457} \tag{6.4.16}$$

and (anti-commuting)

$$\gamma^{1367} \quad ; \quad \gamma^{1457} \quad ; \quad \gamma^{2467} \quad ; \quad \gamma^{2357} \tag{6.4.17}$$

It can then be shown that:

$$\begin{aligned}\psi_{11} =& \quad \text{Re}\,\Omega \wedge v + \frac{1}{2} J \wedge J \\ \psi_{22} =& -\text{Re}\,\Omega \wedge v + \frac{1}{2} J \wedge J \\ \psi_{12} =& -\text{Im}\,\Omega \wedge v = \psi_{21}\end{aligned} \tag{6.4.18}$$

where

$$\begin{aligned} &J = e^{12} + e^{34} + e^{56}, \\ &\Omega = (e^1 + \text{i}e^2)(e^3 + \text{i}e^4)(e^5 + \text{i}e^6) \\ &v = e^7. \end{aligned} \tag{6.4.19}$$

which given eq. (6.4.5) and eq. (6.4.6) can be shown to be the only non-vanishing components of a general $SU(3)$ structure in seven dimensions:

$$\begin{aligned} J_{mn} &= -\bar{\chi}_1 \gamma_{mn} \chi_2, \\ \Omega_{mnp} &= \text{i}\bar{\chi}_1 \gamma_{mnp} \chi_2 - \frac{1}{2} \left(\bar{\chi}_1 \gamma_{mnp} \chi_1 - \bar{\chi}_2 \gamma_{mnp} \chi_2 \right), \\ v_m &= -\text{i}\bar{\chi}_1 \gamma_m \chi_2. \end{aligned} \tag{6.4.20}$$

One can in fact check that:

$$\begin{aligned} J \wedge J \wedge J =& \ \frac{3i}{4} \Omega \wedge \bar{\Omega} \\ J \wedge \Omega =& \ 0 \\ i_v J = i_v \Omega =& \ 0 \end{aligned} \tag{6.4.21}$$

So in conclusion:

$$\psi = \frac{\alpha^2 + \beta^2}{2} J \wedge J + (\alpha^2 - \beta^2)\text{Re}\,\Omega \wedge v - 2\alpha\beta \text{Im}\,\Omega \wedge v \tag{6.4.22}$$

A particularly simple special case arises when $\eta = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$ as then $\psi = \Omega \wedge v$. In a similar fashion we define the fundamental $SU(7)$ three-form as

$$\varphi = *_7 \ \psi = i\bar{\eta}\gamma^{pqr}\eta^c \, dx_p \wedge dx_q \wedge dx_r = i\bar{\eta}\gamma^{abc}\eta^c \, e_a \wedge e_b \wedge e_c \tag{6.4.23}$$

where a, b, c are again internal frame indices. Then

$$\varphi = \alpha^2 \varphi_{11} + \beta^2 \varphi_{22} + \alpha\beta(\varphi_{12} + \varphi_{21}) \tag{6.4.24}$$

where we define

$$\varphi_{ij} = i\bar{\chi}_i\gamma^{pqr}\chi_j \, dx_p \wedge dx_q \wedge dx_r = i\bar{\chi}_i\gamma^{abc}\chi_j \, e_a \wedge e_b \wedge e_c \tag{6.4.25}$$

It can then be shown that:

$$\begin{aligned} \varphi_{11} &= \ -\text{Im } \Omega + J \wedge v \\ \varphi_{22} &= \quad \text{Im } \Omega + J \wedge v \\ \varphi_{12} &= \ -\text{Re } \Omega = \varphi_{21} \end{aligned} \tag{6.4.26}$$

Thus:

$$\varphi = (\alpha^2 + \beta^2) \ J \wedge v - (\alpha^2 - \beta^2) \ \text{Im } \Omega - 2\alpha\beta \ \text{Re } \Omega \tag{6.4.27}$$

Again there is a special case for $\eta = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2)$ where $\varphi = i \ \Omega$ and thus in this case $\psi = -i \ \varphi \wedge v$. More generally :

$$\eta = \frac{1}{\sqrt{2}} e^{i\delta}(\chi_1 \pm i\chi_2) \quad ; \quad \delta \in \mathbb{R} \Longrightarrow \psi = \mp \ i\varphi \wedge v \tag{6.4.28}$$

given the normalization $\bar{\eta}\eta = 1$ these are the only cases when $\bar{\eta}\eta^c = 0$.

One may also wish to calculate bilinears in terms of each of the individual G_2 structures. Note that the (complex) structure spinor is made up of two real structure spinors i.e. $\eta = \eta_1 - i\eta_2$ where:

$$\begin{aligned}\eta_1 &= (a\chi_1 + a'\chi_2)\\ \eta_2 &= (b\chi_1 + b'\chi_2)\end{aligned} \tag{6.4.29}$$

with $\alpha = a + ib$ and $\beta = a' + ib'$. It follows that:

$$\begin{aligned}\eta\otimes\bar{\eta} &= \frac{1}{8}\big[I_{8\times8} + \bar{\eta}\gamma^m\eta\gamma^m - \frac{1}{2!}\bar{\eta}\gamma^{mn}\eta\gamma^{mn} - \frac{1}{3!}\bar{\eta}\gamma^{mnp}\eta\gamma^{mnp}\big]\\
&= \frac{1}{8}\big[I_{8\times8} + i(\alpha\beta^* - \alpha^*\beta)v_m\gamma^m + \frac{1}{2!}J_{mn}\gamma^{mn}\\
&+ \frac{i}{3!}\big(|\alpha|^2\varphi_{11} + |\beta|^2\varphi_{22} + (\alpha\beta^* + \alpha^*\beta)\varphi_{12}\big)_{mnp}\gamma^{mnp}\big]\\
&= \frac{1}{8}\big[I_{8\times8} + i(\alpha\beta^* - \alpha^*\beta)v_m\gamma^m + \frac{1}{2!}J_{mn}\gamma^{mn}\\
&+ \frac{i}{3!}\big((|\alpha|^2 + |\beta|^2)J\wedge v - (|\alpha|^2 - |\beta|^2)\mathrm{Im}\Omega - (\alpha\beta^* + \alpha^*\beta)\mathrm{Re}\Omega\big)_{mnp}\gamma^{mnp}\big]\end{aligned} \tag{6.4.30}$$

From this we may deduce the expressions for $\eta_1\otimes\bar{\eta}_1$ $\eta_2\otimes\bar{\eta}_2$ by setting $\alpha = a\,;\ \beta = a'$ and $\alpha = ib\,;\ \beta = ib'$ respectively. Further we find that:

$$\begin{aligned}\eta_1\otimes\bar{\eta}_2 &= \frac{1}{8}\big[(ab + a'b')I_{8\times8} + i(ba' - ab')v_m\gamma^m + \frac{1}{2!}(ba' - ab')J_{mn}\gamma^{mn}\\
&+ \frac{i}{3!}\big((ab + a'b')J\wedge v - (ab - a'b')\,\mathrm{Im}\ \Omega - (ba' + ab')\,\mathrm{Re}\ \Omega\big)_{mnp}\gamma^{mnp}\big]\end{aligned} \tag{6.4.31}$$

from which one obtains $\eta_2\otimes\bar{\eta}_1$ by the substitution $a \leftrightarrow b\,;\ a' \leftrightarrow b'$. It is then trivial to obtain further relevant operators (and their expansion in terms of bilinears) $\chi_1\otimes\bar{\chi}_1, \chi_2\otimes\bar{\chi}_2, \chi_1\otimes\bar{\chi}_2$ and $\chi_2\otimes\bar{\chi}_1$ by nothing that :

$$\begin{aligned}a = 1; a' = 0 &\Rightarrow \eta_1 = \chi_1\\ b = 1; b' = 0 &\Rightarrow \eta_2 = \chi_2\end{aligned} \tag{6.4.32}$$

Chapter 7

The effective superpotential

In the previous chapter we found the objects defining the $SU(7)$ structure relevant to $N = 1$ reformulations of eleven-dimensional supergravity. The elements $\phi \in \Sigma$ should correspond to the chiral multiplet degrees of freedom. As such there should be an analogue of the four-dimensional superpotential W, appearing as a holomorphic function of ϕ. In this section, we will derive the generic structure of W and show that it can be written in an $E_{7(7)}$ covariant form. This is the analogue of the corresponding generalized geometry calculation in the case of type II given in [177, 178]. Note that the superpotential W, for the special case of a G_2 structure i.e. with a restricted Majorana spinor Ansatz, was previously derived using a somewhat different technique in [190] without however touching upon manifest covariance under the underlying symmetries.

7.1 Generic form of the effective superpotential

We will read off W from the variation of the four-dimensional gravitino. Recall that the $N = 1$ gravitino variations are given by

$$\delta\psi_{\mu+} = \nabla_\mu \theta_+ + \frac{1}{2}\mathrm{i}e^{K/2} W \gamma_\mu \theta_- + \dots, \qquad (7.1.1)$$

where W is the superpotential and K the Kähler potential. The expressions for $e^{K/2}W$ can then be derived directly from the eleven-dimensional gravitino variation (see section 6.1 for our conventions)

$$\delta\Psi_M = \nabla_M\epsilon + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\delta_M^N\Gamma^{PQR}\right)F_{MNPQ} + \ldots, \tag{7.1.2}$$

where the dots denote terms depending on Ψ_M. We must first identify the correctly normalized four-dimensional gravitino ψ_μ, which in his context appears as a $SU(7)$ singlet. A naive decomposition $\Psi_M = (\Psi_\mu, \Psi_m)$ identifying ψ_μ with Ψ_μ, leads to cross-terms in the kinetic energy, so instead we first need to diagonalize the four-dimensional gravitino kinetic energy term. This requires the following shift

$$\tilde{\Psi}_\mu := \Psi_\mu + \frac{1}{2}\Gamma_\mu\Gamma^m\Psi_m. \tag{7.1.3}$$

One must also introduce a rescaling to account for the warp factor in the metric Ansatz, and hence we identify the $SU(7)$ singlet part as

$$\tilde{\Psi}_\mu = e^{E/2}\psi_{\mu+}\otimes\eta + e^{E/2}\psi^c_{\mu+}\otimes\eta^c + \ldots \tag{7.1.4}$$

where the dots denote non-singlet terms. This rescaling by $e^{E/2}$ is the reason for adopting the conventions chosen in the spinor decomposition given in eq. (6.2.7). Given that we may rescale η by including factors in θ_+, we can always choose a normalization

$$\bar{\eta}\eta = 1. \tag{7.1.5}$$

This allows us to introduce the projectors

$$\begin{aligned}\Pi_+ &:= \frac{1}{2}(1 + i\gamma_{(4)})\otimes\eta\bar{\eta}\\ \Pi_- &:= \frac{1}{2}(1 - i\gamma_{(4)})\otimes\eta^c\bar{\eta}^c\end{aligned} \tag{7.1.6}$$

such that

$$e^{-E/2}\Pi_+\tilde{\Psi}_\mu = \psi_{\mu+} \otimes \eta. \tag{7.1.7}$$

It is now straightforward to calculate $\delta\psi_\mu$ in terms of η, F and $\tilde{F}$. By definition

$$\begin{aligned}\delta\psi_{\mu+} \otimes \eta = \mathrm{e}^{-E/2}\Pi_+\delta\tilde{\Psi}_\mu &= \mathrm{e}^{-E/2}\Pi_+\left(\delta\Psi_\mu + \frac{1}{2}\Gamma_\mu\Gamma^m\delta\Psi_m\right)\\ &= \frac{1}{2}\mathrm{i}\mathrm{e}^{K/2}W\gamma_\mu\theta_- \otimes \eta,\end{aligned} \tag{7.1.8}$$

which gives

$$\mathrm{e}^{K/2}W = \frac{\mathrm{i}}{4}\mathrm{e}^E\left(4\bar{\eta}\gamma^m\nabla_m\eta^c + \frac{1}{4!}F_{mnpq}\bar{\eta}\gamma^{mnpq}\eta^c - \mathrm{i} *_7 \tilde{F}\bar{\eta}\eta^c\right). \tag{7.1.9}$$

using the fact that $\bar{\eta}\gamma^m\eta^c = 0$ identically to remove $\nabla_m E$ terms. This expression can be put in a more standard form by writing[1]:

$$\eta^c\bar{\eta} = \frac{1}{8}\left[(\bar{\eta}\eta^c) - \frac{1}{3!}\bar{\eta}\gamma_{mnp}\eta\gamma^{mnp}\right]. \tag{7.1.10}$$

Note then that

$$\begin{aligned}\left[\nabla_m(\eta^c \otimes \bar{\eta}), \gamma^m\right] =& \frac{-i}{8\cdot 3!}\gamma^m\gamma^{pqr}\nabla_m\varphi_{pqr} + \frac{i}{8\cdot 3!}\gamma^{pqr}\gamma^m\nabla_m\varphi_{pqr}\\ =& -2\frac{i}{8\cdot 3!}\gamma^{mpqr}D_{[m}\varphi_{mpqr]}\\ =& -2\frac{i}{8\cdot 4!}\gamma^{mpqr}d\varphi_{mpqr}\end{aligned} \tag{7.1.11}$$

Further

$$\begin{aligned}\bar{\eta}\left[\nabla_m(\eta^c \otimes \bar{\eta}), \gamma^m\right]\eta^c =& -\bar{\eta}\gamma^m\nabla_m\eta^c\,\bar{\eta}\eta^c + \bar{\eta}\nabla_m\eta^c\,\bar{\eta}\gamma^m\eta^c + \bar{\eta}\eta^c\nabla_m\bar{\eta}\gamma^m\eta^c - \bar{\eta}\gamma^m\eta^c\,\nabla_m\bar{\eta}\eta^c\\ =& -2\bar{\eta}\eta^c\,\bar{\eta}\gamma^m\nabla_m\eta^c\end{aligned} \tag{7.1.12}$$

[1]We work in a basis where $\bar{\eta} = \eta^\dagger$

where again we used $\bar{\eta}\gamma^m\eta^c = 0$ and the corollary $\nabla_m\bar{\eta}\gamma^m\eta^c = -\bar{\eta}\gamma^m\nabla_m\eta^c$. Finally

$$\begin{aligned}(\bar{\eta}\gamma^m\nabla_m\eta^c)(\bar{\eta}\eta^c) &= -\frac{\mathrm{i}}{8\cdot 4!}(\mathrm{d}\varphi)_{mnpq}(*_7\varphi)^{mnpq} \\ &= -\frac{\mathrm{i}}{8} *_7 (\varphi\wedge \mathrm{d}\varphi),\end{aligned} \tag{7.1.13}$$

where we used

$$\varphi_{mnp} = \mathrm{i}\bar{\eta}\gamma_{mnp}\eta^c. \tag{7.1.14}$$

throughout. Similarly,

$$\frac{1}{4!}F_{mnpq}\bar{\eta}\gamma^{mnpq}\eta^c = - *_7 (F\wedge\varphi). \tag{7.1.15}$$

Hence

$$\begin{aligned}*_7\mathrm{e}^{K/2}W &= \frac{1}{8}\mathrm{e}^E\left[\frac{1}{\bar{\eta}\eta^c}\varphi\wedge\mathrm{d}\varphi - 2\mathrm{i}F\wedge\varphi + 2\tilde{F}\bar{\eta}\eta^c\right] \\ &= \frac{1}{8}\bar{\eta}\eta^c\mathrm{e}^E\left[\mathrm{d}(\tilde{\varphi}-\mathrm{i}A)\wedge(\tilde{\varphi}-\mathrm{i}A) + 2\mathrm{d}\tilde{A} - \mathrm{i}\mathrm{d}(A\wedge\tilde{\varphi})\right]\end{aligned} \tag{7.1.16}$$

where we have introduced the renormalized $\tilde{\varphi} = \varphi/\bar{\eta}\eta^c$. In the case where $\eta = \eta^c$ we have a global G_2 structure, our normalization convention implies that $\bar{\eta}\eta^c = 1$ and one finds that eq. (7.1.16) agrees with that derived in [190]. The generic $SU(7)$ case differs from the simple G_2 case through the pre-factor $\bar{\eta}\eta^c$ and the fact that $\tilde{\varphi}$ is no longer real. Note that of course the derivation of the first $\varphi\wedge\mathrm{d}\varphi$ term is problematic when $\bar{\eta}\eta^c = 0$ which happens when (in the notation of section 6.4) $\eta = \frac{1}{\sqrt{2}}e^{i\delta}(\chi_1 \pm i\chi_2)$ (with $\delta\in\mathbb{R}$). In this case on must use a slightly more refined approach.

Decomposing η in terms of χ_1, χ_2 one may show using the bilinears defined at the end of section 6.4 (with α, β constant) that:

$$
\begin{aligned}
\bar{\eta}\eta^c\,\bar{\eta}\gamma^m\nabla_m\eta^c = & \; (\alpha^2+\beta^2)\big[(\alpha^2+\beta^2)\;\; \bar{\chi}_1\gamma^7 D_7\chi_1 + (\alpha^2-\beta^2)\;\; \bar{\chi}_1\gamma^a D_a\chi_1 + 2\alpha\beta\bar{\chi}_1\gamma^a D_a\chi_2\big] \\
= & \; (\alpha^2+\beta^2)^2\;\; d\sigma\wedge\sigma - (\alpha^4-\beta^4)\;\; d\sigma\wedge\hat{\rho} - 2\alpha\beta(\alpha^2+\beta^2)\;\; d\sigma\wedge\rho \\
- & \; (\alpha^4-\beta^4)\;\; d\hat{\rho}\wedge\sigma + (\alpha^2-\beta^2)^2\;\; d\hat{\rho}\wedge\hat{\rho} + 2(\alpha^2-\beta^2)\alpha\beta\;\; d\hat{\rho}\wedge\rho \\
+ & \; 4\alpha^2\beta^2\;\; d\rho\wedge\rho + 2(\alpha^2-\beta^2)\alpha\beta\;\; d\rho\wedge\hat{\rho} - 2\alpha\beta(\alpha^2+\beta^2)\;\; d\rho\wedge\sigma
\end{aligned}
\tag{7.1.17}
$$

where $a = 1\cdots 6$ and we defined $\rho = \mathrm{Re}\,\Omega$, $\hat{\rho} = \mathrm{Im}\,\Omega$ and $\sigma = J\wedge v$.

Matching coefficients we then find that:

$$
d\hat{\rho}\wedge\hat{\rho} = d\rho\wedge\rho = d\rho\wedge\hat{\rho} + d\hat{\rho}\wedge\rho = 0 \tag{7.1.18}
$$

which implies that

$$
d\Omega\wedge\Omega = d\bar{\Omega}\wedge\bar{\Omega} = 0 \tag{7.1.19}
$$

Further

$$
\begin{aligned}
*_7\; \bar{\chi}_1\gamma^a D_a\chi_1 =& \frac{i}{8}[d\sigma\wedge\hat{\rho} + d\hat{\rho}\wedge\sigma] \\
*_7\; \bar{\chi}_1\gamma^a D_a\chi_2 =& \frac{i}{8}[d\sigma\wedge\rho + d\rho\wedge\sigma] \\
*_7\; \bar{\chi}_1\gamma^7 D_7\chi_1 =& \frac{-i}{8}[d\sigma\wedge\sigma]
\end{aligned}
\tag{7.1.20}
$$

Thus the torsion term will in general (with α, β no longer constants) be given by:

$$
\bar{\eta}\gamma^m\nabla_m\eta^c \;=\; *_7\; \frac{-i}{8}\big[(\alpha^2+\beta^2)d\sigma\wedge\sigma - (\alpha^2-\beta^2)(d\sigma\wedge\hat{\rho} + d\hat{\rho}\wedge\sigma) \tag{7.1.21}
$$

$$
-\; 2\alpha\beta(d\sigma\wedge\rho + d\rho\wedge\sigma)\big] + i[\beta(\partial_7\alpha) - \alpha(\partial_7\beta)] \tag{7.1.22}
$$

In particular when $\bar{\eta}\eta^c = 0$ implying $\alpha^2 + \beta^2 = 0$ and $\eta = \frac{1}{\sqrt{2}} e^{i\delta}(\chi_1 \pm i\chi_2)$ where $\delta \in \mathbb{R}$ we find [2]:

$$\begin{aligned}
\bar{\eta}\gamma^m \nabla_m \eta^c &= *_7 \ \frac{i}{8} 2\alpha^2 \big[(d\sigma \wedge \hat{\rho} + d\hat{\rho} \wedge \sigma) \mp i(d\sigma \wedge \rho + d\rho \wedge \sigma) \big] \\
&= *_7 \ \frac{e^{-i2\delta}}{8} \big[d\sigma \wedge \Omega + d\Omega \wedge \sigma \big] \\
&= *_7 \ \frac{e^{-i2\delta}}{8} \big[- dJ \wedge \Omega \wedge v + d\Omega \wedge J \wedge v \big] \\
&= *_7 \ \frac{e^{-i2\delta}}{8} 2\Omega \wedge dJ \wedge v
\end{aligned} \tag{7.1.23}$$

where in the last two lines we made use of the fact that $J \wedge \Omega = J \wedge \bar{\Omega} = 0$. Now recalling than in the special case we are looking at $\varphi = i\Omega$ we find collecting all the elements of the superpotential term:

$$\begin{aligned}
*_7 \ e^{K/2} W &= \frac{e^{-i2\delta}}{8} \Omega \wedge (2da + i dJ \wedge v) \\
&\text{or} \\
*_7 \ e^{K/2} W &= \frac{e^{-i2\delta}}{8} - \bar{\Omega} \wedge (2da + i dJ \wedge v)
\end{aligned} \tag{7.1.24}$$

for the case corresponding to the other sign. Note that the term in brackets suggests the equivalent in 6d $B + iJ$.

7.2 An $E_{7(7)}$ covariant superpotential

In this section we show that one may rewrite the superpotential term in a manifestly $E_{7(7)}$ invariant form using the $SU(7)$ structure $\phi \in \mathbf{912}$. We first need to introduce an embedding of the derivative operator into an $E_{7(7)}$ representation. Given the $GL(7)$ decomposition of the EGT given in eq. (5.1.3) we see that, assuming for the moment

[2] we do the calculation for the plus sign only; the other case is analogous

we have a metric g, we can introduce an operator

$$D = (D^{ab}, D_{ab}) \in \mathbf{56} \tag{7.2.1}$$

with

$$D^{mn} = D^{m8} = D_{mn} = 0, \qquad D_{m8} = (\det g)^{1/4}\nabla_m. \tag{7.2.2}$$

Consider now product between $D \in \mathbf{56}$ and $\phi \in \mathbf{912}$ and more precisely the projection thereof unto the component in the $\mathbf{133}$. Let us denote this by $(D\phi)^{AB} = D_C\phi^{C(AB)}$, where indices are raised and lowered using the symplectic structure Ω_{AB}. The expressions for $D\phi$ are given explicitly in eq. (4.2.10). One can then take the adjoint action of $D\phi$ on ϕ itself, which we denote as $(D\phi)\cdot\phi \in \mathbf{912}$. We will show that the superpotential can be written as

$$(D\phi)\cdot\phi = -\left(\frac{3}{4\sqrt{2}}\,\mathrm{e}^{-E}\mathrm{e}^{K/2}W\right)\phi. \tag{7.2.3}$$

That is the combination $(D\phi)\cdot\phi$ is an $SU(7)$ singlet proportional to ϕ while it is W, which is related to the constant of proportionality, which is in fact invariant under $E_{7(7)}$. Further we see that W is by definition a homogeneous holomorphic function of ϕ.

We shall now proceed to prove this in two steps. First recall that we defined $\phi = \mathrm{e}^{A+\tilde{A}}\phi_0$. It is easy to see that $\mathrm{e}^{-A-\tilde{A}}D = D$. This follows from the fact that the only non-vanishing components of the potential acting on D are of the form (in explicit indices) $(*_7A)^{mnpq}$ and $(*_7\tilde{A})^m{}_8$. Hence the expression in eq. (7.2.3) can be rewritten as

$$(\mathrm{e}^{-A-\tilde{A}}D\mathrm{e}^{A+\tilde{A}}\phi_0)\cdot\phi_0 = -\left(\frac{3}{4\sqrt{2}}\,\mathrm{e}^{-E}\mathrm{e}^{K/2}W\right)\phi_0. \tag{7.2.4}$$

Typically this expression will involve a “connection” μ_m which is an element of the

$E_{7(7)}$ Lie algebra labeled by m entering as:

$$\mathrm{e}^{-A-\tilde{A}}\nabla_m \mathrm{e}^{A+\tilde{A}} := \nabla_m + \mu_m = \nabla_m + \mathrm{e}^{-A-\tilde{A}}\nabla_m(\mathrm{e}^{A+\tilde{A}}) \tag{7.2.5}$$

The connection μ_m may be put in a useful form by using an identity similar to the Baker-Campbell-Haussdorf formula:

$$N(s) = e^{-sL}\nabla_m(e^{sL}) = \sum_{n=1}^{\infty} \frac{s^n[\cdot, L]^{n-1}}{n!}\nabla_m L \tag{7.2.6}$$

for any matrix (or more generally operator) L. Here s is an auxiliary variable which has no background dependence and may be set to a convenient constant value. This relation is easily proven by iteration, noting that:

$$\frac{dN(s)}{ds} = [N(s), L] + \nabla_m L \quad \text{and} \quad N(0) = 0 \tag{7.2.7}$$

Given the commutator algebra in eq. (5.1.8) for the flux terms the above series will truncate at quadratic order and we have:

$$\begin{aligned} \mathrm{e}^{-A-\tilde{A}}\nabla_m \mathrm{e}^{A+\tilde{A}} &= \nabla_m + \nabla_m(A+\tilde{A}) + \frac{1}{2}[\nabla_m(A+\tilde{A}), A+\tilde{A}] \\ &= \nabla_m + \nabla_m A + \nabla_m \tilde{A} - \frac{1}{2}\nabla_m A \wedge A \end{aligned} \tag{7.2.8}$$

Now recall that in the $SU(8)/\mathbb{Z}_2$ basis the only non-vanishing component of ϕ_0 as given in eq. (6.3.4) is ${\phi_0}^{\alpha\beta} = \chi^\alpha\chi^\beta$ where $\chi = \eta^c$. This means

$$\begin{aligned} (\nabla_m + \mu_m)\phi^{\alpha\beta} &= \nabla_m(\chi^\alpha\chi^\beta) + ({\mu_m}^\alpha{}_\gamma\chi^\gamma)\chi^\beta + \chi^\alpha({\mu_m}^\beta{}_\gamma\chi^\gamma), \\ (\nabla_m + \mu_m)\phi^{\alpha\beta\gamma}{}_\delta &= 0, \\ (\nabla_m + \mu_m)\bar{\phi}_{\alpha\beta} &= 0, \\ (\nabla_m + \mu_m)\bar{\phi}_{\alpha\beta\gamma}{}^\delta &= \mu_{m\alpha\beta\gamma\epsilon}\chi^\epsilon\chi^\delta. \end{aligned} \tag{7.2.9}$$

Furthermore

$$D = (D^{\alpha\beta}, \bar{D}_{\alpha\beta}) = \frac{1}{2\sqrt{2}}(-\gamma^{m\,\alpha\beta}\nabla_m, \gamma^m_{\alpha\beta}\nabla_m). \tag{7.2.10}$$

Taken together we find

$$\begin{aligned}
&(\mathrm{e}^{-A-\tilde{A}} D\mathrm{e}^{A+\tilde{A}}\phi_0)^\alpha{}_\beta \\
&\quad = \frac{3i}{8\sqrt{2}}\left(\nabla_m(\chi^\alpha\chi^\gamma)\gamma^m_{\gamma\beta} + \mu_m{}^\alpha{}_\gamma\chi^\gamma\chi^\delta\gamma^m_{\delta\beta} - \chi^\alpha\gamma^m_{\beta\gamma}\mu_m{}^\gamma{}_\delta\chi^\delta - \chi^\alpha\gamma^{m\,\gamma\delta}\mu_{m\,\gamma\delta\beta\epsilon}\chi^\epsilon\right) \\
&(\mathrm{e}^{-A-\tilde{A}} D\mathrm{e}^{A+\tilde{A}}\phi_0)_{\alpha\beta\gamma\delta} \\
&\quad = \frac{-3i}{2\sqrt{2}}\mu_{m\,[\alpha\beta\gamma|\epsilon}\chi^\epsilon\gamma^m_{\delta]\theta}\chi^\theta.
\end{aligned} \tag{7.2.11}$$

Finally, again using the identity $\bar{\eta}\gamma^m\eta^c = 0$, or in this context $\chi^\alpha\gamma^m_{\alpha\beta}\chi^\beta = 0$, we find

$$\begin{aligned}
(\mathrm{e}^{-A-\tilde{A}} D\mathrm{e}^{A+\tilde{A}}\phi_0)\cdot\phi_0^{\alpha\beta} &= \frac{-3i}{4\sqrt{2}}\left(\chi^\gamma\gamma^m_{\gamma\delta}\nabla_m\chi^\delta + \chi^\gamma\gamma^m_{\gamma\delta}\mu_m{}^\delta{}_\epsilon\chi^\epsilon\right)\chi^\alpha\chi^\beta \\
&= \frac{-3i}{4\sqrt{2}}\left(\bar{\eta}^\gamma\gamma^{m\delta}{}_\gamma\nabla_m\eta^c_\delta + \bar{\eta}^\gamma\gamma^{m\delta}{}_\gamma\mu_{m\delta}{}^\epsilon\eta^c_\epsilon\right)\chi^\alpha\chi^\beta
\end{aligned} \tag{7.2.12}$$

with all other components vanishing. Finally we note that μ_m corresponds to an "A-shift" of $\nabla_m A$ and a "$\tilde{A}$-shift" of $\nabla_m\tilde{A} - \frac{1}{2}\nabla_m A\wedge A$. Using the decomposition in eq. (4.5.6) together with the definitions in eq. (4.3.7) we find

$$\chi^\gamma\gamma^m_{\gamma\delta}\mu_m{}^\delta{}_\epsilon\chi^\epsilon = \frac{1}{4\cdot 4!}F_{mnpq}\chi^\gamma\gamma_{\gamma\delta}^{mnpq}\chi^\delta - \frac{\mathrm{i}}{4}(*_7\tilde{F})\chi^\gamma C_{\gamma\delta}\chi^\delta \tag{7.2.13}$$

$$= \frac{1}{4\cdot 4!}F_{mnpq}\bar{\eta}^\gamma\gamma^{mnpq\delta}{}_\gamma\eta^c_\delta - \frac{\mathrm{i}}{4}(*_7\tilde{F})\bar{\eta}\eta^c \tag{7.2.14}$$

where we used the intertwiner C to lower and its inverse C^{-1} to raise $SU(8)/\mathbb{Z}_2$ indices and the fact that $\chi^\alpha = \bar{\eta}^\alpha$. Together these two expressions do indeed agree with the superpotential as given in eq. (7.1.9). Note however that there is an ambiguity here as to whether the derivative used to define D should be covariant or not. If it is the metric will enter through the spin connection making the expression for the superpotential in fact highly non-linear, and thus non-trivial, function of ϕ.

Chapter 8

Towards manifestly $E_{7(7)}$ covariant supersymmetry variations

In this chapter we describe the first elements of a proof that the modified SUSY Killing spinor equation in the presence of fluxes may be rewritten concisely in a fashion which is manifestly $E_{7(7)}$ covariant. To this effect we express the supersymmetry variations in terms of the spinor bilinears. Finally we comment briefly on a coset description which could potentially provide a path to manifest covariance. This work is the extension to 11D supergravity for general flux backgrounds of the results obtained in [173] for type II as reviewed in section 3.3.1.

8.1 From spinors to forms

As a starting point we use the equation derived in [197] (with due adaptation to our metric ansatz) in the case of a warped maximally symmetric external space with corresponding 4D Killing spinor equation:

$$\nabla_\mu \theta_+ = -\frac{i}{2}\Lambda_1 \gamma_\mu \mathrm{i}\gamma_{(4)}\theta_- + \frac{1}{2}\Lambda_2 \gamma_\mu \theta_-; , \tag{8.1.1}$$

where Λ_1 and Λ_2 are AdS factors relating the metric to the Ricci tensor such that:

$$R_{\mu\nu} = -3(\Lambda_1^2 + \Lambda_2^2)g_{\mu\nu} \tag{8.1.2}$$

If $\Lambda_1 = \Lambda_2 = 0$ we have warped Minkowski space. Consistency of the 11D SUSY variation with the 4D variation then leads to:

$$\begin{aligned} \nabla_m \chi &= -\frac{1}{2}\partial_m E\chi + \frac{i}{12} f e^{-4E}\gamma_m\chi & (8.1.3)\\ &+\frac{1}{288}F_{npqr}\gamma_m{}^{npqr}\chi - \frac{1}{36}F_{mnpq}\gamma^{npq}\chi, \\ \frac{i}{6}fe^{-4E}\chi &= \frac{1}{288}F_{mnpq}\gamma^{mnpq}\chi - \frac{1}{2}\partial^m E\gamma_m\chi + e^{-E}\frac{i}{2}\Lambda_1\chi^c + e^{-E}\frac{1}{2}\Lambda_2\chi^c & (8.1.4)\end{aligned}$$

where f is such that $F_{\mu\nu\lambda\rho} = f\epsilon_{\mu\nu\lambda\rho}$ and $\chi = \eta^c$ as before and the decomposition of the eleven-dimensional in terms of η and $\theta_\pm$ is given in eq. (6.2.7). In [197] the authors then derived a series of differential constraints on corresponding spinor bilinears:

$$\begin{aligned} \mathrm{d}(\hat\varphi_{(0)}) &= 0, & (8.1.5)\\ \mathrm{d}(e^{3E}\varphi_{(0)}) &= -2i\Lambda_1 e^{2E}\hat\varphi_{(1)} - 2\Lambda_2 e^{2E}\hat\varphi_{(1)}, & (8.1.6)\\ \mathrm{d}(e^{2E}\hat\varphi_{(1)}) &= 0, & (8.1.7)\\ \mathrm{d}(e^{4E}\hat\varphi_{(2)}) &= -e^{4E} * F + 3i\Lambda_1 e^{3E}\varphi_{(3)+} - 3i\Lambda_2 e^{3E}\varphi_{(3)-}, & (8.1.8)\\ \mathrm{d}(e^{3E}\varphi_{(3)-}) &= -i2\Lambda_1 e^{2E} * \hat\varphi_{(3)}, & (8.1.9)\\ \mathrm{d}(e^{3E}\varphi_{(3)+}) &= ie^{3E}\varphi_{(0)}F - i2\Lambda_2 e^{2E} * \hat\varphi_{(3)}. & (8.1.10)\end{aligned}$$

where a numerical subscript denotes a given bilinear's rank and a sign the real and imaginary part, with the bilinears themselves given by:

$$\begin{aligned} \hat\varphi_{(0)} &= \bar\eta\eta, & (8.1.11)\\ \varphi_{(0)} &= \bar\eta\eta^c, & (8.1.12)\\ \hat\varphi_m &= -\bar\eta\gamma_m\eta = \bar\eta^c\gamma_m\eta^c, & (8.1.13)\end{aligned}$$

$$\hat{\varphi}_{mn} = i\bar{\eta}\gamma_{mn}\eta = -i\bar{\eta}^c\gamma_{mn}\eta^c, \quad (8.1.14)$$

$$\hat{\varphi}_{mnp} = i\bar{\eta}\gamma_{mnp}\eta, \quad (8.1.15)$$

$$\varphi_{mnp} = i\bar{\eta}\gamma_{mnp}\eta^c, \quad (8.1.16)$$

Note that eq. (8.1.5) confirms the consistency of the normalization $\bar{\eta}\eta = 1$ up to a trivial overall constant. On the other hand eq. (8.1.6) implies $\bar{\eta}\eta^c = e^{-3E} = (\det g)^{3/4}$ in the absence of AdS factors thus implying that the special case $\bar{\eta}\eta^c = 0$ corresponds to singular solutions for the warped Minkowski case. Also for warped Minkowski eq. (8.1.10) implies that for the Majorana spinor ansatz $\eta = \eta^c$ corresponding to a single G_2 structure, as used in [190] for example, there are no supersymmetric solutions with internal 3-form flux as φ is real in this case and further one is in fact restricted to unwarped Minkowski space-time. The generalized spinor ansatz is thus essential to obtain non-trivial supersymmetric flux backgrounds.

8.2 From forms to spinors

At this point one may then ask if one can isolate a set of differential constraints, expressed in terms of the structure 3-form, which is equivalent to the flux-deformed SUSY variations. Let us specialize again to the case $\Lambda_1 = \Lambda_2 = 0$. Further let us again use[1] the rescaled form $\tilde{\varphi} = \frac{1}{\bar{\eta}\eta^c}\varphi$ and its Hodge dual $\tilde{\psi} = \frac{1}{\bar{\eta}\eta^c}\psi = *_7\tilde{\varphi}$ which have useful algebraic properties generalizing those of the unrescaled forms for the Majorana ansatz such as for example:

$$\frac{1}{7}\tilde{\varphi} \wedge *_7\tilde{\varphi} = \frac{1}{7}\tilde{\varphi} \wedge \tilde{\psi} = *_7 1 \quad (8.2.1)$$

[1]We assume $\bar{\eta}\eta^c \neq 0$.

or alternatively

$$\tilde{\varphi}_{mnp}\tilde{\varphi}^{mnp} = 42$$
$$\tilde{\psi}_{mnpq}\tilde{\psi}^{mnpq} = 168 \tag{8.2.2}$$

Since we are dealing with an $N = 1$ (off-shell) effective supergravity in 4D, the conditions for on-shell supersymmetric, purely bosonic backgrounds are given by:

$$W = 0 \tag{8.2.3}$$
$$\frac{\partial W}{\partial t^a} = 0 \tag{8.2.4}$$
$$D = 0 \tag{8.2.5}$$

where t^a are the moduli scalar fields and D is the D-term. Note that interestingly the first two equations are satisfied for W if they are satisfied by $W' = tW$ where t is a nowhere vanishing, but otherwise arbitrary, function. In analogy with these conditions we will now show that the supersymmetry variation equation is in one-to-one correspondence with the following expressions [2]:

$$*_7\big[(\mathrm{d}\tilde{\varphi} - i\mathrm{d}A) \wedge (\tilde{\varphi} - iA) + 2\,\lambda *_7 1 - i\mathrm{d}(A \wedge \tilde{\varphi})\big] = 0 \tag{8.2.6}$$
$$(d\tilde{\varphi} - idA) = 0 \tag{8.2.7}$$
$$\mathrm{d}(*_7\tilde{\varphi}) = -8\;\mathrm{d}E \wedge *_7\tilde{\varphi} \tag{8.2.8}$$

where the last two equations essentially reflect the decomposition of the internal gravitino variation under $SU(7)$.

The proof proceeds by construction. Define

$$\tilde{\chi} = \frac{1}{\sqrt{\bar{\eta}\eta^c}}\chi \tag{8.2.9}$$

[2] where $\lambda = *_7\mathrm{d}\tilde{A}$ locally but captures the global non-trivial topology of $\tilde{A}$

It then follows from:

$$\tilde{\chi}^T\gamma^m\tilde{\chi} = 0$$
$$\tilde{\chi}^T\gamma^{mn}\tilde{\chi} = 0 \tag{8.2.10}$$

that $\{\tilde{\chi}, \gamma^m\tilde{\chi}\}$ form an orthonormal basis for (complexified) eight-component spinors since $\tilde{\chi}^T\tilde{\chi} = 1$. In particular we have:

$$\frac{1}{4!}d\tilde{\varphi}_{mnpq}\gamma^{mnpq}\tilde{\chi} = u_0\tilde{\chi} + u_m\gamma^m\tilde{\chi} \tag{8.2.11}$$

with

$$u_0 = -\frac{1}{4!}d\tilde{\varphi}_{mnpq}\tilde{\psi}^{mnpq} = - *_7\left(d\tilde{\varphi}\wedge\tilde{\varphi}\right) \tag{8.2.12}$$
$$u_m = \frac{-i}{3!}d\tilde{\varphi}_{mnpq}\tilde{\varphi}^{npq} = -i\tilde{\varphi}\lrcorner\,\mathrm{d}\tilde{\varphi} \tag{8.2.13}$$

so that comparing with eq. (2.2.23) we find that the scalar u_0 and the one-form u correspond respectively to the torsion classes[3] $\mathcal{W}_1$ and $\mathcal{W}_4$. Torsion class $\mathcal{W}_3$ does not contribute as by construction $(\mathcal{W}_3)_{mnpq}\gamma^{mnpq}\tilde{\chi} = 0$. Replacing then $\tilde{\varphi}\wedge dA$ by $-i\tilde{\varphi}\wedge d\tilde{\varphi}$ in eq. (8.2.6) using eq. (8.2.7) we find:

$$u_0 = *_7\big[A\wedge dA - 2\,\lambda *_7 1)\big] = -2fe^{-4E} \tag{8.2.14}$$

Further given the properties of the torsion class $\mathcal{W}_4$ as given in eq. (2.2.21) we may read off u_m from eq. (8.2.8) with $u_m = 6i\,\partial_m E$, so that substituting into eq. (8.2.11) (using again eq. (8.2.7)) for $d\tilde{\varphi}$ and multiplying by $\frac{-i\sqrt{\eta\eta^c}}{12}$ we find:

$$\frac{i}{6}fe^{-4E}\chi = \frac{1}{288}F_{mnpq}\gamma^{mnpq}\chi - \frac{1}{2}\partial^m E\gamma_m\chi \tag{8.2.15}$$

[3]To be quite precise these torsion classes are, strictly speaking, defined for manifolds with G_2 structure. However the differential relations in eq. (2.2.23) are still valid for manifolds with a more general $SU(7)$ structure.

which is the algebraic part of the flux-deformed supersymmetry variation as given in eq. (8.1.4) for the warped Minkowski case.

To derive the differential constraint on the supersymmetry variation spinor note first that:

$$\frac{1}{\sqrt{\bar{\eta}\eta^c}}\nabla_m\chi = \nabla_m\tilde{\chi} + \frac{1}{2\bar{\eta}\eta^c}\partial_m(\bar{\eta}\eta^c)\tilde{\chi} \tag{8.2.16}$$

further

$$\nabla_m\tilde{\chi} = (\nabla_m\tilde{\chi})\tilde{\chi}^T\tilde{\chi} + \tilde{\chi}(\nabla_m\tilde{\chi})^T\tilde{\chi} = \nabla_m(\tilde{\chi}\tilde{\chi}^T)\tilde{\chi} = \frac{i}{8\cdot 3!}\nabla_m\tilde{\varphi}_{npq}\gamma^{npq}\tilde{\chi} \tag{8.2.17}$$

where we used

$$(\nabla_m\tilde{\chi})^T\tilde{\chi} = \frac{1}{2}[(\nabla_m\tilde{\chi})^T\tilde{\chi} + \tilde{\chi}^T(\nabla_m\tilde{\chi})] = \frac{1}{2}\partial_m(\tilde{\chi}^T\tilde{\chi}) = 0 \tag{8.2.18}$$

Since $\nabla_m\tilde{\varphi}$ is expressible solely[4] in terms of $d\tilde{\varphi}$ and $d^\dagger\tilde{\varphi}$ we may write:

$$\begin{aligned}\nabla_m\tilde{\chi} = {} & a_1(d^\dagger\tilde{\varphi})_{mn}\gamma^n\tilde{\chi} + a_2(d^\dagger\tilde{\varphi})_{np}\gamma_m{}^{np}\tilde{\chi} \\ + {} & a_3(d\tilde{\varphi})_{npqr}\gamma_m{}^{npqr}\tilde{\chi} + a_4(d\tilde{\varphi})_{mnpq}\gamma^{npq}\tilde{\chi}\end{aligned} \tag{8.2.19}$$

where a_1, a_3, a_3, a_4 are complex constants. Using eq. (8.2.8) we further have $(d^\dagger\tilde{\varphi})_{mn} = -8\,\partial^s E\tilde{\varphi}_{smn}$ from which we may deduce that the two first terms in eq. (8.2.19) only contain terms proportional to $\partial^s E\gamma_{ms}\tilde{\chi}$ and $\tilde{\chi}$. This follows by decomposing the relevant spinors using the basis $\{\tilde{\chi}, \gamma^r\tilde{\chi}\}$ such that:

$$\partial^s E\gamma_{ms}\tilde{\chi} = i\partial^s E\tilde{\varphi}_{smr}\gamma^r\tilde{\chi}; \tag{8.2.20}$$

$$\begin{aligned}(d^\dagger\tilde{\varphi})^{np}\gamma_{mnp}\tilde{\chi} &= 8i\,\partial_s E\tilde{\varphi}^{snp}\tilde{\varphi}_{mnp}\tilde{\chi} - 8\,\partial_s E\tilde{\varphi}^{snp}\tilde{\psi}_{mnpr}\gamma^r\tilde{\chi} \\ &= c\partial_m E\tilde{\chi} - c'\,\partial^s E\tilde{\varphi}_{smr}\gamma^r\tilde{\chi}\end{aligned} \tag{8.2.21}$$

[4]Again, strictly this a priori only applies to the case of simple G_2 structure where φ is real. One may however argue that this does not enter in the derivation of the equalities used here.

where in the last line we used the identities:

$$\tilde{\varphi}_{mnp}\tilde{\varphi}^{npr} = d\ \delta_m{}^r$$
$$\tilde{\varphi}_{mnp}\tilde{\psi}^{npqr} = d'\ \tilde{\varphi}_m{}^{qr} \tag{8.2.22}$$

whit c, c', d, d' being constants. It thus follows that:

$$\nabla_m\tilde{\chi} = b_1\partial^s E\gamma_{ms}\tilde{\chi} - b_2\partial_m E\tilde{\chi}$$
$$- ib_3(d\tilde{\varphi})_{npqr}\gamma_m{}^{npqr}\tilde{\chi} - ib_4(d\tilde{\varphi})_{mnpq}\gamma^{npq}\tilde{\chi} \tag{8.2.23}$$

upon substituting from eq. (8.2.7) and eq. (8.2.15) implies:

$$\nabla_m\chi = [\frac{-3}{2}\partial_m E - (b_1 + b_2)\partial_m E]\chi - \frac{ib_1}{3}fe^{-2A}\gamma_m\chi \tag{8.2.24}$$
$$+(b_3 + \frac{b_1}{144})F_{npqr}\gamma_m{}^{npqr}\chi + (b_4 + \frac{b_1}{36})F_{mnpq}\gamma^{npq}\chi$$

where[5] we defined $\chi = e^{\frac{-3E}{2}}\tilde{\chi}$ and b_1, b_2, b_3, b_4 are constants fixed by the consistency of eq. (8.2.23) which are expected to match the coefficients in eq. (8.1.3) with $b_1 = \frac{-1}{4}, b_2 = \frac{-3}{4}, b_3 = \frac{1}{192}$ and $b_4 = \frac{-1}{48}$. For example

$$\tilde{\chi}^T\nabla_m\tilde{\chi} = 0 \Rightarrow \frac{b_2}{36} = b_4$$
$$\tilde{\chi}^T\gamma^m\nabla_m\tilde{\chi} = \frac{-i}{192}d\tilde{\varphi}_{mnpq}\tilde{\psi}^{mnpq}\tilde{\chi} \Rightarrow 3b_3 + b_4 = \frac{-1}{192} \tag{8.2.25}$$

At this point a remark is in order. We could have made our discussion more general by leaving u_m, which represented the $\mathcal{W}_4$ torsion class, unspecified throughout, with its relationship with the derivative of the warp factor arising from consistency. From this perspective eq. (8.2.8) corresponds to the vanishing of the $\mathcal{W}_2$ torsion class, which thus seems to be equivalent to the vanishing of the D-term. Interestingly this means that in the case of a warped Minkowski non-compact external manifold, the internal

[5]The supersymmetry conditions then guarantee that this definition is consistent with that of χ as a normalized version of $\tilde{\chi}$ given in eq. (8.2.9).

manifold is necessarily an integrable manifold with G_2 Dolbeault cohomology for supersymmetric solutions.

We expect this no longer to be the case for the warped AdS external spaces. We will not explore in detail here how the previous calculations are modified in this case, but by consistency with eq. (8.1.5) - (8.1.10) and the gravitino variation, they should follow from the differential constraints:

$$\begin{aligned} *_7\big[(d\tilde{\varphi} - idA) \wedge (\tilde{\varphi} - iA) + 2\lambda *_7 1 - id(A \wedge \tilde{\varphi})\big] &= e_1 \frac{e^{-E}}{\bar{\eta}\eta^c}(i\Lambda_1 + \Lambda_2) \\ (d\tilde{\varphi} - idA) &= 2\frac{e^{-E}}{\bar{\eta}\eta^c}(i\Lambda_1 + \Lambda_2)[\hat{\varphi}_{(1)} \wedge \tilde{\varphi} - i *_7 \hat{\varphi}_{(3)}] \\ d(*_7\tilde{\varphi}) &= -8\, dE \wedge *_7\tilde{\varphi} + X_5 \end{aligned}$$

where e_1 is a constant and $*_7X_5$ is non-vanishing two-from representing $\mathcal{W}_2$. The first equation follows from the $\nabla_\mu\theta_+$ term in the gravitino variation which vanishes for Minkowski space but shifts $e^{K/2}W$ by the AdS factors in the superpotential calculation for Anti-de-Sitter space where we identify $e^{-K/2}$ with $\frac{e^{-E}}{\bar{\eta}\eta^c}$. The second equation follows from substituting eq. (8.1.6) in eq. (8.1.10). In the last one we merely parameterized calculational ambiguities which we expect to be lifted in the context of further work with a reasonable starting point being the Hodge dual of eq. (8.1.8). Note in passing that in the presence of non-vanishing AdS factors eq. (8.1.7) follows trivially from eq. (8.1.6).

8.3 Elements of manifest $E_{7(7)}$ covariance

Let us now discuss the possible manifest $E_{7(7)}$ covariance of the on-shell supersymmetry constraints. Ideally one would expect all three conditions eq. (8.2.6) to eq. (8.2.8) to be equivalent to a single $E_{7(7)}$ covariant expression in terms of ϕ, in analogy to the type II case in the absence of RR-fluxes as given by the integrability condition[6] in

[6]although in this case there are two conditions, namely one for each pure spinor

eq. (3.2.6). Here we take a less ambitious intermediary step by trying to make each equation separately manifestly covariant. The D-term not yet having been calculated, we will may not include eq. (8.2.8) in this analysis. On the other hand eq. (8.2.6) has already been shown to be covariant in the context of the superpotential calculation. This thus leaves us with eq. (8.2.7). Since we expect it to appear as a derivative of W with respect to the moduli, as hinted in eq. (8.2.4), we will start by rewriting the superpotential in an alternative form. Note that at this point this is introduced mostly as a convenient construct, without a strong reference to a specific interpretation. Define thus Φ^- transforming in the **63** of $SU(8)/\mathbb{Z}_2$:

$$\Phi^-{}_\alpha{}^\gamma = \frac{8\eta^c_\alpha\bar{\eta}^\gamma}{\bar{\eta}\eta^c} - \delta_\alpha{}^\gamma = \frac{8\phi_{\alpha\beta}C^{\beta\gamma}}{\bar{\eta}\eta^c} - \delta_\alpha{}^\gamma = \frac{-1}{48}\tilde{\varphi}^-_{abcd}\hat{\gamma}^{abcd\gamma}{}_\alpha \tag{8.3.1}$$

whose embedding in $E_{7(7)}$ is given by:

$$2\Phi^-{}_{[\alpha}{}^{[\gamma}\delta_{\beta]}{}^{\delta]} = \frac{1}{16}\tilde{\varphi}^-_{abcd}(\gamma^{ab})_\alpha{}^\beta(\gamma^{cd})_\gamma{}^\delta \tag{8.3.2}$$

where $\tilde{\varphi}^- = \tilde{\varphi}\wedge y^8 - *_7\tilde{\varphi}$. Further define its orthogonal complement within $E_{7(7)}$ as:

$$\Phi^{+\alpha\beta\gamma\delta} = \frac{1}{16}\tilde{\varphi}^+_{abcd}(\hat{\gamma}^{ab})^{\alpha\beta}(\hat{\gamma}^{cd})^{\gamma\delta} \tag{8.3.3}$$

where $\tilde{\varphi}^+ = \tilde{\varphi}\wedge y^8 + *_7\tilde{\varphi}$. Taken together these give the $SU(8)/\mathbb{Z}_2$ decomposition of the $E_{7(7)}$ Lie algebra element Φ given by $\tilde{\varphi}\wedge y^8$ in terms of $SL(8)$ representations. One may then show that the superpotential may be written as:

$$\begin{aligned} W &= \mathrm{tr}\left(\gamma^m e^{-\mathcal{T}}\nabla_m e^{\mathcal{T}}\right) \\ &= \mathrm{tr}\left(\gamma^m[e^\Phi e^{\mathcal{A}} e^{\mathcal{A}^6}]^{-1}\nabla_m[e^\Phi e^{\mathcal{A}} e^{\mathcal{A}^6}]\right) \end{aligned} \tag{8.3.4}$$

where the trace is taken in the adjoint of $E_{7(7)}$ and

$$e^{\mathcal{T}} = e^\Phi e^{\mathcal{A}} e^{\mathcal{A}^6} = e^{\Phi+\mathcal{A}+\frac{1}{2}[\Phi,\mathcal{A}]+\mathcal{A}^6} \tag{8.3.5}$$

with $\mathcal{A}$ and $\mathcal{A}^6$ being respectively the embeddings of the fluxes A and $\tilde{A}$ in the Lie algebra of $E_{7(7)}$. Varying now with respect[7] to $\mathcal{T}$ one finds (up to a surface term):

$$\begin{aligned} \delta W = & \ \mathrm{tr}\left(\gamma^m \delta(e^{-\mathcal{T}}) \nabla_m e^{\mathcal{T}}\right) - \mathrm{tr}\left(\gamma^m \nabla_m e^{-\mathcal{T}} \delta(e^{\mathcal{T}})\right) \\ = & -\mathrm{tr}\left(\gamma^m e^{-\mathcal{T}} \delta(e^{\mathcal{T}}) e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}}\right) + \mathrm{tr}\left(\gamma^m e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}} e^{-\mathcal{T}} \delta(e^{\mathcal{T}})\right) \\ = & \ \mathrm{tr}\left([\gamma^m, e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}}] e^{-\mathcal{T}} \delta(e^{\mathcal{T}})\right) \\ = & \ \mathrm{tr}\left([\gamma^m, e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}}] \delta\mathcal{T}\right) - \frac{1}{2}\,\mathrm{tr}\left([[\gamma^m, e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}}], \mathcal{T}] \delta\mathcal{T}\right) \end{aligned} \tag{8.3.6}$$

where, in going from the first to the second line, we used the identities

$$\delta(e^{-\mathcal{T}}) = -e^{-\mathcal{T}} \delta(e^{\mathcal{T}}) e^{-\mathcal{T}} \quad ; \quad \nabla_m e^{-\mathcal{T}} = -e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}} e^{-\mathcal{T}} \tag{8.3.7}$$

together with the cyclicity of the trace. Further in the last line we used an analogue of the previously derived identity

$$e^{-\mathcal{T}} \delta(e^{\mathcal{T}}) = \delta\mathcal{T} + \frac{1}{2}[\delta\mathcal{T}, \mathcal{T}] \tag{8.3.8}$$

The second term in the last line of eq. (8.3.6) vanishes identically. Projecting out the parts vanishing identically in the first term one finds:

$$\delta W = 0 \Leftrightarrow [\gamma^m, e^{-\mathcal{T}} \nabla_m e^{\mathcal{T}}] - (\tilde{\varphi} \leftrightarrow -\tilde{\varphi}, A \leftrightarrow -A, \tilde{A} \leftrightarrow \tilde{A}) = 0 \tag{8.3.9}$$

which is equivalent to eq. (8.2.7). This expression is covariant, as well as the superpotential invariant, under a rigid transformation $S \in E_{7(7)}$ acting as:

$$\gamma^m \to S\gamma^m S^{-1} \quad \Phi \to S\Phi S^{-1} \quad \mathcal{A} \to S\,\mathcal{A}\,S^{-1} \quad \mathcal{A}^6 \to S\,\mathcal{A}^6\,S^{-1} \tag{8.3.10}$$

[7] More precisely the variation is with respect to either $\tilde{\varphi}$ or A. The variation with respect to $\tilde{A}$ vanishes identically, in line with the fact that the six-form only contributes to the superpotential trough its non-trivial topology; such a contribution does not appear in a variational derivation.

A priori the projection in eq. (8.3.9) seems problematic in the sense that it appears difficult to phrase it in a covariant way independent of the underlying degrees of freedom. A generic $E_{7(7)}$ transformation will mix the different $GL(7)$ representations, so the projection will no longer be effected by simple sign inversion. A general background, which might no longer be described by 11D supergravity and may in fact be non-geometrical, would probably be characterized by $\mathcal{T}' = S\,\mathcal{T}\,S^{-1}$. This $\mathcal{T}'$ would then need to be decomposed into the images of $\Phi, \mathcal{A}, \mathcal{A}^6$ under S using a covariant algorithm based on group theoretic arguments. For example note that the statement that $\mathcal{A}^6$ commutes with $\Phi, \mathcal{A}$ is in fact covariant. As an aside note also that $\Phi, \mathcal{A}, \mathcal{A}^6$ live in an isotropic subspace of the **133** as defined by the natural inner product on the Lie algebra.

Another problem with this approach is that physically the theory is most naturally formulated in terms of $\phi \in \mathbf{912}$ rather than $\tilde{\varphi} \in \mathbf{133}$. Although we defined a map[8] between these it involves the intertwiner C, whose definition relies on the existence of Clifford algebra and thus a metric, making it a highly non-linear function of ϕ a priori. This in turn implies that the derivation presented here would lead to a very involved picture in terms of ϕ although a priori one might expect a rather simple expression such as $D\phi = 0$. To explain this one must remember that ϕ is not a generic element of the **912** but is highly constrained, nor is $\tilde{\varphi}$ a generic element of the **133**. The constrained orbit of ϕ in the **912** would generically be linked to a projector, which supposedly acts trivially on the superpotential (as do the sign changes in eq. (8.3.9)) while leading to a complicated form for the susy variations in terms of ϕ. Nevertheless this route is most promising when attempting a fully covariant formulation. The description in terms of $\tilde{\varphi}$ is however useful as a $E_{7(7)}\,/\,SU(7)$ coset description of the orbit of ϕ, although there is the outstanding issue of defining a fiducial point on the orbit which is invariant under $SU(7)$. As defined Φ has in fact $U(7)$ as stabilizer (see the discussion of eq. (6.3.3)).

[8]which also maps the differential operator from the **56** to the **133**

A different, but equally promising, approach would be to phrase the SUSY variations in terms of the integrability of the EGACS. More precisely one would need to show the equivalence of the supersymmetry equations and the closure of the $\pm i$ eigenbundles of the EGACS under the action of the generalized Courant bracket.

Since we introduced the coset construction let us comment on its possible relevance to the calculation of the Kähler potential. An important point to emphasize is that we expect the Kähler potential in this case to coincide with that of eleven-dimensional supergravity compactified on manifolds of G_2 holonomy. This follows first because the flux does not enter the metric on the Kähler moduli space resulting from dimensional reduction. Secondly the G_2 holonomy compactifications are a special case of a G_2 structure manifold, singled out by differential conditions on the structure forms. However in obtaining the aforementioned Kähler metric, and by extension the Kähler potential, from dimensional reduction such differential constraints with respect to the internal manifold's coordinates do not come into play. Thus we expect the Kähler potential to be proportional to $\ln \det g_{mn}$. It then seems probable that the Kähler potential is in fact proportional to the determinant of the 28×28 upper left sub-matrix of G_0 as given in eq. (5.2.7), when thought of as a 56×56 matrix. This procedure may seem extremely ad hoc but is in fact a generic feature of Kähler potentials emerging from coset constructions (see for example [198, 199, 200] or [201] for a more recent description).

Chapter 9

Conclusions and further Outlook

There has always been a rich interplay between string theory and geometry. We reviewed for example the connection between supersymmetric low-energy phenomenology in four dimensions and special holonomy internal manifolds in early purely geometric compactification schemes of the heterotic string or eleven-dimensional supergravity. One might equally have focussed on the description of mirror symmetry in terms of the Dolbeault cohomology of Calabi-Yau manifolds or the importance of complex and symplectic compactification manifolds in topological string theories.

One common thread throughout our discussion is the role of fluxes as obstructions to the integrability of a given geometric structure, introduced to describe supersymmetric backgrounds. More specifically we focussed on how the obstruction is lifted through an embedding in a larger structure, which is now integrable. Alternatively, given that said integrability is generically phrased in terms of the on-shell supersymmetry conditions, one finds, in off-shell supersymmetric formulations, that the fluxes are geometrized. In the first set of examples given, fluxes appear as the torsion of G-structures defined on the tangent bundle T. More precisely the fluxes parameterize their departure from integrable special G-holonomy. Such an example, is afforded by the deformation of Calabi-Yau ($SU(3)$ holonomy) three-folds to manifolds with non-integrable $SU(3)$ structure. However in the case of NS B-field flux, this $SU(3)$

structure maybe embedded in a $SU(3)\times SU(3)$ structure defined on the bundle $T\oplus T^*$, which turns out to be integrable. The main tool in this context is Generalized (Complex) Geometry as developed by Hitchin and his collaborators. Remarkably this formalism not only naturally includes the B-field and introduces it on equal footing with the metric, but does so in a form manifestly covariant under the $O(d,d)$ in terms of which T-duality is formulated, for example by reproducing the coset structure of the $\frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R})\times O(d,\mathbb{R})}$ moduli space appearing for toroidal compactifications.

However even the $SU(3)\times SU(3)$ structure on $T\oplus T^*$ fails to integrate in the presence of RR-fluxes or put differently, in the context of off-shell supersymmetry, these fluxes fail to geometrize. The original work presented here remedies this problem by considering Exceptional Generalized Geometry (EGG) with structures defined on a yet larger bundle $E = (\Lambda^7 T^*)^{-1/2}\,[T\oplus\Lambda^2 T^*\oplus\Lambda^5 T^*\oplus(\Lambda^7 T^*)T^*]$ in the context of eleven-dimensional supergravity. Physically each component of the bundle may be associated to a conserved charge of the supergravity, such as the membrane charges. It was then shown that the bosonic degrees of freedom may be arranged in a manifestly $E_{7(7)}$ covariant way combining the metric and the three-form potential on equal footing (via an exceptional generalized almost complex structure) in an exceptional generalized metric (EGM), which is invariant under $SU(8)/\mathbb{Z}_2$. This approach thus automatically incorporates the RR potentials (as well as the NS B-field) given they arise from the three-form through dimensional reduction on a circle for type IIA, from which type IIB is obtained via T-duality. Further, provided this setting is preserved under quantum corrections, it could give rise to a manifestly $E_{7(7)}(\mathbb{Z})$ covariant formulation in eleven dimensions without resorting to specific compactification Ansätze, thus providing a first step towards manifest U-duality covariance. In this context we explicitly calculated the superpotential to show that it could be expressed solely in terms of the $E_{7(7)}$ covariant objects and is in fact invariant under $E_{7(7)}$ transformations.

In the course of this work a rich new geometry was uncovered. Specifically, besides the extension of Generalized Geometry already described above, we identified a non-trivial twisting of the Exceptional Generalized Tangent bundle (EGT) E. Further the Chern-Simons term was naturally integrated in this formalism. Additionally we defined the analogue of the Courant bracket appearing in Generalized Geometry, which we termed the Exceptional Courant Bracket (ECB). Given the ECB one may explicitly check the integrability of the Exceptional Generalized Almost Complex Structure.

At this point let us note some a priori limitations of this approach to be resolved in the course of further work. First of all, while there is a clearly defined structure on the bundle $E = (\Lambda^7 T^*)^{-1/2} [T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus (\Lambda^7 T^*) T^*]$ with the corresponding $E_{7(7)}$ covariant generalized geometric objects, only the superpotential in the $N = 1$ $D = 4$ rearrangement of eleven dimensional supergravity was explicitly shown to be invariant, the most obvious omission being the Kähler potential. More significantly our discussion is rooted in the supergravity approximation and thus in particular does not guarantee that the manifestly $E_{7(7)}$ formulation would be preserved if α' corrections where introduced. Of course this is an issue when considering the long term goal of understanding M-theory within this language but might also be an obstacle to semi-realistic phenomenology. Indeed the flux backgrounds studied here would be subjected to the no-go theorems we reported on earlier, with higher order corrections being precisely one possible resolution besides the introduction of localized sources. Another such caveat is that the backgrounds presented here are smooth and thus would fail to reproduce chiral fermions or non-Abelian gauge groups in lower dimensions. On a more formal note, it is as yet unclear whether the derivative appearing in the superpotential calculation should be a covariant or a simple partial derivative. In the former case there is an implicit reference to the internal metric, which is a related to the $SU(7)$ structure through a non-linear map.

These considerations naturally point to several avenues for further work. The obvious first step consists in obtaining a covariant rewriting of the remaining bosonic terms in the Lagrangian to be identified with the Kähler potential and the D-term in the $N = 1$ $D = 4$ reformulation. Further one would like to extend the new formalism from off-shell to on-shell supersymmetry by formulating the supersymmetry variations in a manifestly $E_{7(7)}$ covariant form. Some preliminary work for this project was presented here which recast the variation, originally given in terms of spinors, in terms of the forms appearing in EGG. Another approach would consist in phrasing the supersymmetry variations as the integrability conditions of the Exceptional Generalized Almost Complex Structure given on terms of the closure of its $\pm i$ eigenbundles under the Exceptional Courant Bracket.

In the context of phenomenology one could then try to adapt our results to model building by including the effect of singularities and that of negative tension objects allowing the evasion of the no-go theorems. Another promising line of research would lie in the rewriting of known higher-order derivative corrections of eleven-dimensional supergravity in the language of EGG, which might further also be of significance in circumventing the no-go theorems. In parallel one might extend EGG to compactifications to other dimensions with a corresponding different group structure. Compactifying to five dimensions for example, one would expect the EGG to be based on $E_{6(6)}$. At this point EGG might be employed as a tool for the generation and systematic classification of supersymmetric backgrounds, potentially by developing an analogue of Yau's theorem, and it would be natural to apply it to testing the AdS/CFT correspondence. Although this formalism would not yield an explicit form for the background fields, it is probable that general conclusions may be drawn about, for example, the dual CFT in analogy with the many results known for Calabi-Yau manifolds in absence of an explicit metric. One would expect to be able to construct new examples of the duality based on as yet unknown backgrounds.

Finally one could explore connections between EGG and potentially related approaches. One such formalism is generalized holonomy where a supercovariant derivative, arising from the supersymmetry variations in generic backgrounds with fluxes, with holonomy lying within in $SL(32, \mathbb{R})$ is used to classify supersymmetric solutions of 11-dimensional supergravity [202][203]. Another interesting approach, which similarly to the work presented here involves $E_{7(7)}$ in a central way, is the so-called embedding tensor technique used to obtain non-trivial gaugings of maximal supergravity (see [204] for a review). Finally we might try to relate EGG to the reformulations of eleven-dimensional supergravity in terms of the Kac-Moody algebras E_9, E_{10}, E_{11} (see for example [206][205]).

Appendix A

Dualities

A.1 Overview of the web of dualities

As noted before the five different superstring theories are linked by a web of dualities. In particular the type II theories are linked (after compactification on at least a circle S_1) by T-duality as are the two heterotic theories. This duality relates two equivalent theories defined on different but dual backgrounds. It is perturbative in nature and can be shown to hold order by order in g_s, the string coupling. The formalism introduced by Buscher [207][208] further permits to understand it as the gauging of isometries on the worldsheet where they translate into internal symmetries of the 2D CFT. A closely related generalization which is applicable in absence of isometries, as is the case on generic Calabi-Yau manifolds, was dubbed mirror symmetry and relates a type IIA theory defined on a given Calabi-Yau to a type IIB theory on a mirror Calabi-Yau . Interestingly mirror symmetry is supposed to be reducible to multiple T-dualities within the context of SYZ conjecture[209], where the Calabi-Yau is seen as a T^3 fibration [1].

There exists further a conjectured[2] non-perturbative S-duality which relates the

[1] a major subtlety being that generically there will be degenerate fibers locally

[2] It is only conjectured because formulations of string theory are mostly perturbative. It has however passed a series of non-trivial tests based on for example non-renormalized BPS multiplets.

strong coupling limit of one theory to the weak coupling limit of another one. It is a symmetry for Type IIB (which is self-dual) and links heterotic $SO(32)$ to type I. The S-dual of type IIA and heterotic $E_8 \times E_8$ is in fact 11D supergravity compactified on the circle S^1 and the interval $S^1/\mathbb{Z}_2$ respectively. The dilaton, whose expectation value $\langle\phi\rangle$ is related to the string coupling as $g_s = e^{\langle\phi\rangle}$, can be interpreted as a modulus ($e^{\langle\phi\rangle}$ giving roughly the circle radius or interval length) in the context of the dimensional reduction form 11d to 10d thus explaining the decompactification at strong coupling. The heterotic $E_8 \times E_8$ case is the focus of Hořava-Witten theory[210][211] which posits two "end-of-the world" 9-branes at each end of the interval on which respectively one of the E_8 gauge theories lives. An interesting phenomenological approach consists in embedding the SM in one of the E_8 while the other gives the hidden sector possibly responsible for SUSY breaking.

Both these dualities have a group theoretic interpretation. In fact S- and T-duality are conjectured to be part of a much larger U-duality group[53]. In the following we will describe these dualities in more technical detail focussing particularly on the group structure as this is the aspect most apparent in the Generalized Geometry description which is the focus of this thesis.

A.2 T-duality

A.2.1 T-duality and $O(d, d, \mathbb{Z})$

T(arget space)-duality (for a review see [212]) relates different backgrounds in the ten-dimensional target space which all give rise to the same spectrum and CFT correlation functions and are thus physically equivalent. The typical example is that of a string propagating on a circular subspace of radius R. T-duality asserts that this will lead to the same physics as a string propagating along a circle of radius $\frac{\alpha'}{R}$. The reason for this are the enhanced degrees of freedom of extended objects: A point particle can only propagate along the circle, whose periodicity results in a quantization of the momentum and thus the energy such that it goes as $\frac{n^2}{R^2}$. On the other hand a string may wind around m times leading to an energy m^2R^2. Another example is familiar from the context of the SYZ conjecture for mirror symmetry (which can be understood as multiple T-duality under well defined conditions) where the momentum modes of a 0-brane are exchanged with the winding modes of a 3-brane.

Let us now look at the specific case of d-dimensional toroidal backgrounds and how T-duality can be implemented as an $O(d,d,Z)$ transformation. Consider the usual sigma model action for the bosonic string:

$$S = \frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \left[\sqrt{g}g^{\alpha\beta}G_{ij}\partial_\alpha X^i\partial_\beta X^j + \epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j + \sqrt{g}\Phi R^{(2)}\right] \quad \text{(A.2.1)}$$

with the periodicity condition $X^i \approx X^i + 2\pi m^i$. Now introduce the canonical conjugate momenta of the X^i :

$$2\pi P_i = G_{ij}\dot{X}^j + B_{ij}X^{j\prime} = p_i + \text{oscillators}, \quad \text{(A.2.2)}$$

with $p_i = n_i$ where the n_i must be integers because of the periodicity of the torus. From this we may the calculate the Virasoro constraints:

$$\begin{aligned} H &= L_{0L} + L_{0R} \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \left[(2\pi)^2(P_iG^{ij}P_j) + X^{i\prime}(G - BG^{-1}B)_{ij}X^{j\prime} + 4\pi X^{i\prime}B_{ik}G^{kj}P_j\right] \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma(P_L^2 + P_R^2), \\ P_{La} &= [2\pi P_i + (G-B)_{ij}X^{j\prime}]e_a^{*i}, \quad P_{Ra} = [2\pi P_i - (G+B)_{ij}X^{j\prime}]e_a^{*i}, \\ \int_0^{2\pi} d\sigma P X' &= L_{0L} - L_{0R} = 0 \end{aligned} \quad \text{(A.2.3)}$$

with e and e^* such that $\sum_{a=1}^d e_i^a e_j^a = 2G_{ij}, \quad \sum_{a=1}^d e_i^a e_a^{*j} = \delta_i^j, \quad \sum_{a=1}^d e_a^{*i} e_a^{*j} = \frac{1}{2}(G^{-1})^{ij}$. The former represent a choice of einbeins coinciding with the basis of the compactification lattice, the latter are just the dual to this basis. By then truncating to the terms containing no oscillators [3] we find:

$$\begin{aligned} H &= L_{0L} + L_{0R} = \frac{1}{2}(p_L^2 + p_R^2) \\ &= \frac{1}{2}\left[n_i(G^{-1})^{ij}n_j + m^i(G - BG^{-1}B)_{ij}m^j + 2m^iB_{ik}(G^{-1})^{kj}n_j\right] \end{aligned} \quad \text{(A.2.4)}$$

with n_i and m_i being the momentum and winding modes respectively and $p_R = [n^t + m^t(B - G)]e^*$, and $p_L = [n^t + m^t(B + G)]e^*$. Note that the pair (p_R, p_L) thus encodes all the physics of the problem with G and B determining the local geometrical/gauge nature of the manifold, while (m_i, n_i) describes its topological properties (the periodicity). Now we may ask whether we may build the moduli space corresponding to different choices of metric and Kalb-Ramond field at fixed (m_i, n_i) through the action of a given group. The relevant quantity to parameterize the moduli is the length of the lorentzian vector (p_R, p_L):

$$p_L^2 - p_R^2 = 2m^in_i \in 2\mathbb{Z}, \quad \text{(A.2.5)}$$

The group conserving this relation *a priori* is $O(d, d, \mathbb{R})$. However since only the norm of $p_R(p_L)$ enter physical quantities, we must mod out a pair of $O(d, \mathbb{R})$ rotation so that the required moduli space is [213][214] $\frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \times O(d,\mathbb{R})}$.

We must now identify the symmetry group i.e. the subgroup of the group sweeping out the moduli space leaving the Hamiltonian invariant and surviving after quantum corrections. This turns out be $O(d, d, \mathbb{Z})$ as shown in [215][216][217][218]. Before we can give a justification for this statement we must find an appropriate action of $O(d, d, \mathbb{R})$ on the background which may be represented by $E = G + B$.

[3]since we are only interested in the relation between winding and momentum modes which encompass the geometrical degrees of freedom of the string

In this context recall the standard matrix representation of $O(d,d,\mathbb{R})$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{A.2.6}$$

where a, b, c, d are $d \times d$ matrices such that given

$$J = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \tag{A.2.7}$$

we have [4]

$$g^t J g = J \quad \Rightarrow \quad a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \qquad a^t d + c^t b = I. \tag{A.2.8}$$

To describe the action of $O(d,d,\mathbb{R})$ we parameterize the hamiltonian as:

$$H = \frac{1}{2}(p_L^2 + p_R^2) = \frac{1}{2} Z^t M Z, \quad Z = (m_a, n_b). \tag{A.2.9}$$

where

$$M(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{A.2.10}$$

Note that M (first introduced in [215]) is itself an element of the group and can in fact be decomposed as $M(E) = g_E g_E^t$ [5] if we take

$$g_{E=G+B} = \begin{pmatrix} e & B(e^t)^{-1} \\ 0 & (e^t)^{-1} \end{pmatrix}, \tag{A.2.11}$$

where $g_E \in O(d,d,\mathbb{R})$

[4]Note that $J^{-1} = J$, and thus $g^{-1}J(g^t)^{-1} = J$ leading to $J = gJg^t$ so that $g^t \in O(d,d,\mathbb{R})$ if $g \in O(d,d,\mathbb{R})$

[5]This decomposition is not unique: in fact $M(E) = \tilde{g}_E \tilde{g}_E^t$ with $\tilde{g}_E = g_E A$ where $A \in O(d,\mathbb{R}) \times O(d,\mathbb{R})$ as expected given the coset structure of the moduli space.

M then transforms as $M(E') = gM(E)g^t$ where $E' = g(E)$ from which we deduce the action of $O(d,d,\mathbb{R})$ to be:

$$E' \equiv g(E) = (aE+b)(cE+d)^{-1}. \tag{A.2.12}$$

Looking now at $O(d,d,\mathbb{Z})$ i.e. the subgroup with integer entries we can substantiate the claim that it is the symmetry group of the moduli. The whole group can be generated by three sets of transformations:

- constant shifts in the Kalb-Ramond field

$$g_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}, \tag{A.2.13}$$

  where $\Theta_{ij} \in \mathbb{Z}$ and $\Theta_{ji} = -\Theta_{ij}$ which correspond to the transformation $B_{ij} \longrightarrow B_{ij} + \Theta_{ij}$. This shift leads to the addition a total divergence to the Lagrangian:

$$\begin{aligned} S &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \epsilon^{\alpha\beta}\Theta_{ij}\partial_\alpha X^i \partial_\beta X^j \\ &= \frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \epsilon^{\alpha\beta}\Theta_{ji} X^i \partial_\alpha \partial_\beta X^j + \text{surface term} \\ &= 0 + \text{surface term} \end{aligned} \tag{A.2.14}$$

  where in the penultimate line we use the fact that Θ_{ij} is constant. The topology being non-trivial the surface term will in general not vanish but provided $\Theta_{ij} \in \mathbb{Z}$ the shift is a multiple of 2π and does not change the path integral $\sim e^{iS}$.

- Base changes

$$g_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}, \tag{A.2.15}$$

  where $A \in GL(d,\mathbb{Z})$. This corresponds to $E' = AEA^t$ or $G' = AGA^t$ and $B' = ABA^t$ i.e. coordinate changes which preserve the compactification lattice.

These can be decomposed into permutations and reflections of the coordinates and the modular transformations $SL(d,\mathbb{Z})$ of T^d the d-dimensional torus.

- Factorized duality

$$g_{D_i} = \begin{pmatrix} I - e_i & e_i \\ e_i & I - e_i \end{pmatrix}. \tag{A.2.16}$$

Here e_i is zero, except for the ii component which is 1, and I is a d-dimensional identity matrix. In the case of a pure toroidal background ($B = 0$) each g_{D_i} inverts the radius of one of the cycles i.e $R_i \longrightarrow \frac{1}{R_i}$ The generalization of T-duality to higher dimensions is in fact background inversion whereby $E' = E^{-1}$ corresponding to the $O(d,d,\mathbb{Z})$ transformation

$$g_D = \begin{pmatrix} O & I \\ I & O \end{pmatrix}. \tag{A.2.17}$$

corresponding to the background transformations:

$$\begin{aligned} G &\rightarrow G' = (G - BG^{-1}B)^{-1}, \quad B \rightarrow B' = (B - GB^{-1}G)^{-1}, \\ G^{-1}B &\rightarrow -BG^{-1}. \end{aligned} \tag{A.2.18}$$

Clearly the Hamiltonian is invariant under this transformation if combined with an exchange of winding and momentum modes. Note also that when $B = 0$ the transformation corresponds to $G \rightarrow G^{-1}$ generalizing the one-dimensional result as claimed.

Here we did not include the transformations of the oscillators [6] and refer the reader to [212] for further details.

[6] we also left aside an additional symmetry corresponding to worldsheet parity $\sigma \rightarrow -\sigma$ equivalent to $B \rightarrow -B$ which $\notin O(d,d,\mathbb{Z})$

A.2.2 T-duality and Buscher rules

An alternative way to look at T-duality is by linking it to isometries of the background in a formalism known as Buscher[208][207] duality (which has in fact wider applicability than the $O(d, d, \mathbb{Z})$ formalism which is a priori restricted to toroidal backgrounds). Consider again the sigma model action but now introducing complex worldsheet coordinates:

$$S = \frac{1}{2\pi} \int d^2z \, [(G_{\mu\nu}(x) + B_{\mu\nu}(x))\partial x^\mu \bar{\partial} x^\nu + \frac{1}{2}\phi(x) R^{(2)}] \tag{A.2.19}$$

with

$$z \equiv \frac{1}{\sqrt{2}}(\tau + i\sigma), \quad \partial \equiv \frac{1}{\sqrt{2}}(\partial_\tau - i\partial_\sigma), \tag{A.2.20}$$

so that $d^2z = d\sigma d\tau$, and $\partial x \bar{\partial} x = \frac{1}{2}((\partial_\tau x)^2 + (\partial_\sigma x)^2)$. Define also $H = dB$. Then a transformation

$$\delta x^\mu = \epsilon \mathbf{k}^\mu \tag{A.2.21}$$

will be an isometry of the action if

$$\mathcal{L}_{\mathbf{k}} G_{\mu\nu} = \mathbf{k}_{\mu;\nu} + \mathbf{k}_{\nu;\mu} = 0 \quad ; \quad \mathcal{L}_{\mathbf{k}} H = 0 \Rightarrow \mathcal{L}_{\mathbf{k}} B = d\omega \quad ; \quad \mathcal{L}_{\mathbf{k}} \phi = \mathbf{k}^\mu \phi_{,\mu} = 0$$

for some one-form ω. If the chosen isometry is abelian[7] one may choose a coordinate system $\{x^\mu\} = \{x^0, x^a\}$ where the isometry is represented as a translation of $x^0 \equiv \theta$.

[7]Non-abelian isometries give rise to several complications such as obstructions to T-duality.

Consequently there will be no explicit θ dependence in the action which may be rewritten as:

$$S[x^a,\theta] = \frac{1}{2\pi}\int d^2z\,[G_{00}(x^c)\partial\theta\bar{\partial}\theta + (G_{0a}(x^c) + B_{0a}(x^c))\partial\theta\bar{\partial}x^a + (G_{a0}(x^c) + B_{a0}(x^c))\partial x^a\bar{\partial}\theta$$
$$+(G_{ab}(x^c) + B_{ab}(x^c))\partial x^a\bar{\partial}x^b + \frac{1}{2}\int d^2z\phi(x^c)R^{(2)}].$$

The abelian isometry can then be made to hold locally by minimal gauge coupling $\partial\theta \to \partial\theta + A$. We can then make A pure gauge i.e. unphysical by adding a Lagrange multiplier $\tilde{\theta}F$, where $F = \partial\bar{A} - \partial A$ is the field strength of the abelian gauge field. We may now choose a gauge where $\theta = 0$ such that

$$S[x^a,A,\tilde{\theta}] = \frac{1}{2\pi}\int d^2z\,[G_{00}A\bar{A} + (G_{0a} + B_{0a})A\bar{\partial}x^a + (G_{a0} + B_{a0})\partial x^a\bar{A} +$$
$$(G_{ab} + B_{ab})\partial x^a\bar{\partial}x^b + \frac{1}{2}\int d^2z\phi R^{(2)} + \tilde{\theta}F]. \quad (A.2.22)$$

In fact the $\theta = 0$ gauge corresponds to $A = \partial\theta$, $\bar{A} = \bar{\partial}\theta$ so that both models are equivalent. Computing the equations of motion for $A, \bar{A}$ one finds that they are algebraic so that they maybe inserted back into the Lagrangian giving a new dual action:

$$S' = \frac{1}{2\pi}\int d^2z\,[(G'_{\mu\nu}(x^a) + B'_{\mu\nu}(x^a))\partial y^\mu\bar{\partial}y^\nu - \frac{1}{4}\int d^2z\phi'(x^a)R^{(2)}], \quad (A.2.23)$$

where $\{y^\mu\} = \{\tilde{\theta}, x^a\}$ with

$$G'_{00} = \frac{1}{G_{00}}, \quad G'_{0a} = \frac{B_{0a}}{G_{00}}, \quad G'_{ab} = G_{ab} - \frac{G_{a0}G_{0b} + B_{a0}B_{0b}}{G_{00}}$$
$$B'_{0a} = \frac{G_{0a}}{G_{00}}, \quad B'_{ab} = B_{ab} - \frac{G_{a0}B_{0b} + B_{a0}G_{0b}}{G_{00}}. \quad (A.2.24)$$

These are the so-called Buscher rules. The dilaton on its part gets a correction at one loop ensuring the dual theory is conformally invariant provided the original theory is:

$$\phi' = \phi - \frac{1}{2}\log G_{00}. \tag{A.2.25}$$

A.3 S-duality and electromagnetic duality

S-duality was first described in the context of $N = 4$ $D = 4$ Super Yang-Mills (SYM) in terms of the Montonen-Olive duality introduced in [219] for $SO(3)$ Yang-Mills and embedded in SYM [8] in [220]. Consider the Lagrangian:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4g_{\text{YM}}^2}F^{\mu\nu}F_{\mu\nu} - \frac{\theta}{32\pi^2}F^{\mu\nu} * F_{\mu\nu} - \frac{1}{2}D^\mu\Phi D_\mu\Phi \\ &\equiv -\frac{1}{32\pi}\text{Im}\left[\tau(F^{\mu\nu} + i * F^{\mu\nu})(F_{\mu\nu} + i * F_{\mu\nu})\right] - \frac{1}{2}D^\mu\Phi D_\mu\Phi\end{aligned} \tag{A.3.1}$$

where $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}$. This action can be shown to possess an $SL(2,\mathbb{Z})$ symmetry under which τ transforms according to a fractional linear transformation as:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1. \tag{A.3.2}$$

A general such $SL(2,\mathbb{Z})$ transformation can be generated by a succession of only two operations:

$$\tau \rightarrow \tau + 1 \quad \text{and} \quad \tau \rightarrow \frac{-1}{\tau} \tag{A.3.3}$$

The first one corresponds to a 2π shift in the vacuum angle θ parameterizing the topological term and leaves the path integral invariant. The second one is most easily understood in absence of a topological term when $\theta = 0$ where it can be seen to correspond to a strong-weak coupling duality, the hallmark of S-duality.

[8] The introduction of supersymmetry resolved some of the caveats raised by the authors of the first paper such as the control of quantum corrections.

This last transformation is sometimes referred to as electro-magnetic duality in analogy with Abelian Maxwell theory. The latter has long been known to be invariant under exchange of electric and magnetic fields in the absence of sources. In the presence of electric charges however the symmetry only holds provided there are magnetic monopoles. In this case the Dirac-Schwinger-Zwanziger quantization condition relating electric charge q_e and magnetic charge q_m :

$$q_e q_m = 2\pi n, \qquad n \in \mathbb{Z} \tag{A.3.4}$$

implies the strong-weak duality under exchange of magnetic and electric degrees of freedom.

S-duality appears in an analogous fashion in string theory. In fact this is not a coincidence: the AdS-CFT correspondence links $N = 4$ SYM (which is conformal) to string theory on an $AdS_5 \times S_5$ background and in particular relates their coupling constants[9]. Consider for example type IIB for which S-duality is a symmetry. Defining $\tau = C_0 + ie^{-\phi}$ (where C_0 is the RR 0-form sometimes referred to as the axion and ϕ is the dilaton) the above fractional linear transformation [10] can be written as a matrix action by $\Lambda \in SL(2, \mathbb{Z})$ on a matrix $\mathcal{M}$ such that:

$$\mathcal{M} \to (\Lambda^{-1})^T \mathcal{M} \Lambda^{-1} \quad \text{with} \quad \mathcal{M} = \frac{1}{\text{Im}\tau} \begin{pmatrix} |\tau|^2 & -\text{Re}\tau \\ -\text{Re}\tau & 1 \end{pmatrix} \tag{A.3.5}$$

where

$$\Lambda = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \tag{A.3.6}$$

[9] It may in fact be shown that given some assumptions $\tau_{\text{YM}} = \tau_{\text{IIB}}$.

[10] In this context the $\tau \to \tau + 1$ shift arises from the periodicity of C_0 in the space perpendicular to the 7-brane to which it couples magnetically. The other transformation is a strong-weak coupling duality in terms of g_s .

One may then put the kinetic terms for $\mathcal{C}_0$ and ϕ in a manifestly $SL(2,\mathbb{Z})$ invariant form:

$$\frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi + e^{2\phi}\partial^\mu\mathcal{C}_0\partial_\mu\mathcal{C}_0\right) = \frac{1}{2(\mathrm{Im}\tau)^2}\partial^\mu\tau\partial_\mu\bar{\tau} = -\frac{1}{4}\mathrm{tr}\left(\partial^\mu\mathcal{M}\partial_\mu\mathcal{M}^{-1}\right) \quad \text{(A.3.7)}$$

All the other fields in the IIB action transform under the S-duality group [11] and in fact the whole action can be shown to be invariant. Note that in principle the symmetry is $SL(2,\mathbb{R})$ in the classical supergravity approximation, which is however broken to the discrete subgroup with integer entries by quantum corrections. In addition to this global invariance there is also a $SO(2)$ or $U(1)$ local symmetry. The two moduli in fact parameterize a $SL(2,\mathbb{R})/U(1)$ coset or rather $[SL(2,\mathbb{R})/U(1)]/SL(2,\mathbb{Z})$ taking into account identifications under S-duality. This construction is quite generic for supergravity theories with global symmetries and in fact plays an important role in describing U-dualities to which we turn in the next section (in fact we encountered a similar construction for the moduli of toroidal compactifications when discussing T-duality with the equivalent of $\mathcal{M}$ being $M(E)$ defined in eq. (A.2.10)).

But before that let us comment on the obvious resemblance between the above action of $SL(2,\mathbb{Z})$ and the modular transformations of a two-torus T^2 with complex structure τ. This has prompted the conjecture [221] (inspired by work in [222]) that type IIB may be a T^2 compactification of a 12D theory known as F-theory thus giving S-duality a geometric interpretation.

[11] at least strictly speaking as some are singlets like the Einstein-frame metric and the four-form, while the RR two-form and the B-field form a doublet transforming linearly.

A.4 U-duality

As pointed out before T- and S-duality embed into a larger U-duality group relating different backgrounds[12]. In particular compactifying a higher-dimensional supergravity on these backgrounds results in an lower-dimensional effective supergravity having the U-duality group as an internal symmetry. Consider the canonical form of the bosonic part of a $D = 4$ supergravity Lagrangian in the absence of a potential for the moduli and non-Abelian Yang-mills fields:

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{4}R - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}m_{IJ}(\phi)F^{\mu\nu I}F^J_{\mu\nu} - \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}a_{IJ}(\phi)F^I_{\mu\nu}F^J_{\rho\sigma}\right) \quad \text{(A.4.1)}$$

with $I = 1 \ldots k$ and where g_{ij} is the metric on the space of scalar moduli ϕ_i resulting from the metric and fluxes of the internal manifold (including the dilaton), while $F^I_{\mu\nu}$ are the $U(1)$ abelian field strengths corresponding to gauge potentials A^I_μ and $-N_{IJ} = a_{IJ} + im_{IJ}$ is a $k \times k$ complex symmetric matrix with $ImN < 0$ to guarantee a positive definite kinetic term for the gauge vectors. The symmetry group of the abelian field strengths and thus that of the whole Lagrangian must be contained [223] in $Sp(2k, \mathbb{R})$. To illustrate this define:

$$G_{\mu\nu I} \equiv 2\frac{\partial\mathcal{L}}{\partial F^{\mu\nu I}} = m_{IJ}{*}F^J_{\mu\nu} + a_{IJ}F^J_{\mu\nu} \quad \text{with} \quad *F^I_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma I} \qquad \text{(A.4.2)}$$

One may then succinctly write both the equations of motion and the Bianchi identity for the gauge fields as:

$$\mathrm{d}\mathcal{F} = \mathrm{d}\begin{pmatrix} F^I \\ G_I \end{pmatrix} = 0 \qquad \text{(A.4.3)}$$

[12] by which now we mean not just geometry but also general field content and coupling constants

A priori this system of equations is invariant under an $GL(2k, \mathbb{R})$ symmetry:

$$\mathcal{F} \rightarrow \mathcal{F}' = \begin{pmatrix} F'^I \\ G'_I \end{pmatrix} = \Lambda \mathcal{F} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mathcal{F} \tag{A.4.4}$$

where A, B, C, D are arbitrary non-singular $k \times k$ matrices. For these equations to result from a Lagrangian of the above form we must however require:

$$G'_{\mu\nu I} \equiv 2\frac{\partial \mathcal{L}}{\partial F'^{\mu\nu I}} = m'_{IJ} {*F'}^J_{\mu\nu} + a'_{IJ} {F'}^J_{\mu\nu} \tag{A.4.5}$$

Requiring m', a' to be symmetric implies $\Lambda \in Sp(2k, \mathbb{R})$.

Many of the known supergravities do in fact possess a global symmetry group G whose maximal compact subgroup H acts as a local symmetry [53]. Clearly from the above $G \subset Sp(2k, \mathbb{R})$ for a theory with k $U(1)$ gauge fields transforming linearly under G. The moduli on the other hand transform non-linearly and parameterize a coset G/H described by a G-valued matrix $\mathcal{V}$ which transforms as:

$$V(x) \rightarrow h(x)V(x)\Lambda^{-1} \qquad h \in H \ , \ \Lambda \in G \tag{A.4.6}$$

Typically H will be orthogonal or unitary so that one may define a coset representative as $M = \mathcal{V}^T\mathcal{V}$ or respectively $M = \mathcal{V}^\dagger\mathcal{V}$ (which is the equivalent of the $\mathcal{M}$ we described in the context of S-duality or $M(E)$ in that of T-duality) in terms of which the kinetic term will typically be proportional to $\mathrm{tr}\,(\partial^\mu M \partial_\mu M^{-1})$.

This is the case for all supergravities with $N \geq 4$ in 4D and in particular maximal $N = 8$ supergravity which has 28 $U(1)$ gauge fields for which $G = E_{7(7)} \subset Sp(56, \mathbb{R})$ and $H = SU(8)$. For $N = 4$ supergravity with m vector multiplets $k = 6 + m$ and $G = SL(2, \mathbb{R}) \times O(6, m)$ and $H = U(1) \times O(6) \times O(m)$. The so-called "exceptional" $N = 2$ supergravity[224][225], another theory with 28 gauge bosons, has $G = E_{7(-25)}$ and $H = E_6 \times U(1)$.

n	$E_{n(n)}$	H_n	$\dim(E_{n(n)})$	$\dim(E_{n(n)}/H_n)$
2	$SL(2,\mathbb{R}) \times \mathbb{R}$	$SO(2)$	4	3
3	$SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$SO(3) \times SO(2)$	11	7
4	$SL(5,\mathbb{R})$	$SO(5)$	24	14
5	$Spin(5,5)$	$(Sp(2) \times Sp(2))/\mathbb{Z}_2$	45	25
6	$E_{6(6)}$	$Sp(4)/\mathbb{Z}_2$	78	42
7	$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$	133	70
8	$E_{8(8)}$	$Spin(16)/\mathbb{Z}_2$	248	128

Table A.1: The U-duality groups E_n, their maximal compact subgroups H_n, and the dimensions of E_n and the cosets E_n/H_n.

However as in the case of S-duality the actual symmetry group is broken by quantum to a discrete subgroup $G(\mathbb{Z}) = G \cap Sp(2k, \mathbb{Z})$ which can be interpreted as resulting from the Dirac-Schwinger-Zwanziger quantization condition on the electric and magnetic charges corresponding to the abelian gauge fields.

In particular for $N = 8$ supergravity one has $G(\mathbb{Z}) = E_{7(7)}(\mathbb{Z})$ which has a subgroup $O(6,6,\mathbb{Z}) \times SL(2,\mathbb{Z})$. Noting that the theory may be obtained as a six-torus T^6 compactification of type IIB these subgroups are interpreted respectively as the T-duality group of the torus and the S-duality group of type IIB. This is part of a pattern for toroidal compactifications that extends to other dimensions: in general the effective theory obtained from compactifying type IIB to d dimensions where $d = 11 - n$ [13] has a symmetry group $G(\mathbb{Z}) = E_{n(n)}(\mathbb{Z})$ where $E_{n(n)}$ is the nth element in a formal class completing the set of maximal non-compact forms of the exceptional groups E_6, E_7, E_8. Further there is a corresponding coset structure $E_{n(n)}/H_n$ where H_n is the maximal compact subgroup of $E_{n(n)}$ (see table A.1. [194]) Note the dimensionality of the coset for $n = 7$ corresponding to a T^7 compactification of M-theory giving maximal $N = 8$ supergravity in four dimensions. This corresponds to the 70 moduli arising from the internal metric g_{mn}, the internal part of the 11D SUGRA 3-form a_{mnp} and an internal 6-form arising from the 3-form through the non-trivial

[13] or alternatively using T-duality and the fact that M-theory compactified on a circle is Type IIA, this also corresponds to 11D supergravity compactified on a n-torus T^n

equation of motion linked to the Chern-Simons term. Further the above table maybe completed by including IIB supergravity for $n = 1$ and it is conjectured that the theories corresponding to $n = 9, 10, 11$ have the symmetry groups E_9, E_{10}, E_{11} whose generators do not form a Lie but a Kac-Moody algebra.

Note also that for $n = 2$ the interplay between T- and S-duality is trivial[14]. In fact the factor $\mathbb{R}$ is isomorphic to the T-duality group for compactification of Type II on a circle, namely $SO(1,1,\mathbb{R})$. Recall that $SO(2)$ is isomorphic to $U(1)$ which is itself isomorphic to the circle S^1. $SO(1,1,\mathbb{R})$ being the non-compact form of $SO(2)$ it follows that it is isomorphic to the real line and thus $\mathbb{R}$. The non-trivial embeddings of T- and S-duality for compactification on higher-dimensional tori may be traced to the requirement that $E_{n(n)}(\mathbb{Z})$ must contain both $SL(n,\mathbb{Z})$, the geometric modular group for T^n expected from the M-theory point of view, and $SO(n-1, n-1, \mathbb{Z})$, the T-duality group resulting from the type II compactifications. The U-duality groups are generated non-trivially from a combination of both these subgroups. This is most easily seen for $n = 3$ where $E_{3(2)}(\mathbb{Z}) = SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$ is the minimal group containing $SL(3,\mathbb{Z})$ and $SO(2,2,\mathbb{Z}) = SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})$.

Finally let us remark that since in our original work we focus on seven-dimensional compactifications of 11D supergravity, we see the group structure for $n = 7$ emerge from a exceptional generalized geometry approach.

[14]More so for $n = 1$ corresponding to uncompactified type II which has only S-duality.

Appendix B

Parallel research interests: Neutrino oscillations and modified Lorentz Invariance

This work forms part of much larger programme initiated by Blasone and Vitiello to give neutrino oscillations a description within Quantum field theory[1] (QFT). They consistently defined a Hilbert space for eigenstates of definite flavor (circumventing an earlier no-go theorem by Giunti, Kim et al.) which led to a series of surprises such as, amongst others, the existence of a new "flavor" vacuum (unitarily equivalent to the naive QFT "mass" vacuum), corrections to the usual Pontecorvo oscillation formula and non-trivial propagators.

As far as the PhD is concerned, this parallel strand of research originated as an MSci project[2] and eventually led to three articles:

- M. Blasone, P. Pires Pacheco and H.W.C. Tseung, "Neutrino oscillations from relativistic flavor currents," Phys. Rev. D **67** (2003) 073011 [arXiv:hep-ph/0212402]

[1]The original Pontecorvo formalism within non-relativistic Quantum Mechanics (QM) is in fact inconsistent since the required superposition of energy eigenstates with different masses is forbidden by the Bargmann QM superselection rule.

[2]That is a 4th year undergraduate project.

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In this article [226] flavor four-currents, defined by Blasone and Vitiello using a Noether-type formalism, are used to solve a longstanding phenomenological problem in the literature, namely the direct calculation of a space oscillation formula by defining a flux through the detector surface[3]. It was shown to reproduce the standard results in the literature in the highly-relativistic limit, while numerically a deviation was predicted in the low-energy regions, allowing the theory to be tested.

- M. Blasone, J. Magueijo and P. Pires Pacheco, "Neutrino mixing and Lorentz invariance," Europhys. Lett. **70** (2005) 600 [arXiv:hep-ph/0307205] ;
  M. Blasone, J. Magueijo and P. Pires Pacheco, "Lorentz invariance for mixed neutrinos," Braz. J. Phys. **35** (2005) 447 [arXiv:hep-ph/0504141].
  The central idea of these articles [227][228] is to derive a dispersion relation for the neutrino from the expectation value of a suitably defined Hamiltonian on the flavor states[4]. It was further shown that the covariance of the expression could only be guaranteed if one defined a Poincaré algebra deformed by the generator of (neutrino) mixing transformations.[5] Finally an experimental test was suggested to apply these results to the end-point of beta decay to settle the controversy in the literature as to which of the flavor or mass eigenstates constitute the fundamental physical objects.

[3]This replaces a conceptually dubious shortcut through the more easily obtained time oscillation formula. The obtained formula describes neutrino oscillation in fully three-dimensional terms and allows naturally for the inclusion of a wave-packet (as necessary for a realistic description).

[4]In fact this represents only a classical limit on which quantum fluctuations should in principle superimposed.

[5]The dispersion relation was indeed shown to be expressible in the parametrization introduced by Magueijo et al. for non-linear representations of the Lorentz algebra.

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