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# Theory and Practical Issues of Sigma Delta Modulators—Part I: Theory

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## Part 2 Main Results

- Stability analysis of SDMs
  - Conditions for the divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Rate of the divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Global stability of high order lowpass SDMs

# Contents

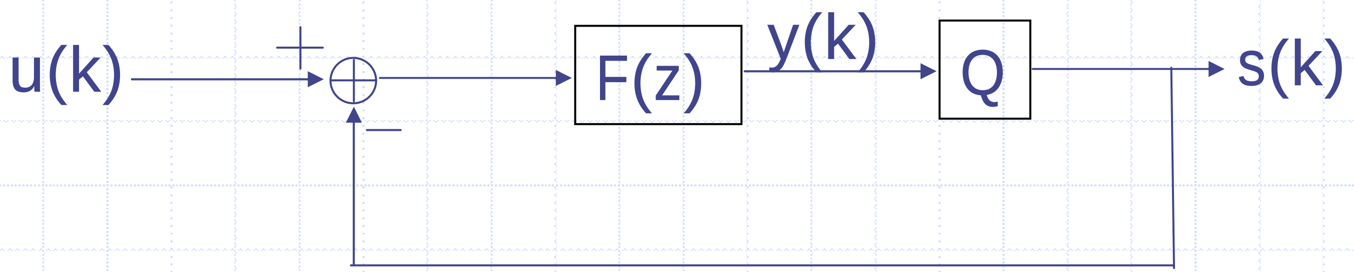
# Part 1:

# Introduction

# Introduction

- What are SDMs?
  - SDMs are systems implemented sigma delta modulation.
  - There are two types of SDMs: feedforward SDMs and feedbackward SDMs. In this presentation, we only concentrate on feedforward SDMs.
  - For feedforward SDMs, they consist of negative feedback of a loop filter and a quantizer.

# Introduction




- Order of SDMs is defined by the order of the filter.
- Number of bits of SDMs is defined by the number of bits in the quantizer.

# Introduction

- Why SDMs are so important?
  - SDMs can be used to convert analog inputs into low-resolution digital representations at very high sampling rates and to decode the output streams using several low precision digital signal representations to generate high precision representations.

# Introduction

- The advantages of using SDMs for analog to digital conversions are that they can achieve high quality analog input conversion by using simple and inexpensive circuits.
- Due to this reason, SDMs are found in many industrial and engineering applications, such as in capacitive flow sensors, microwave power amplifiers, etc.

# Introduction

- What are chaotic systems?

  Definition of chaotic systems (Devaney)

  - Sensitive to initial conditions

    An interval map $\Im : \wp \rightarrow \wp$ has sensitive dependence to initial conditions if there exists a $\delta > 0$, such that, for any $x \in \wp$, and any neighborhood $\wp_0$ of $x$, there exists $y \in \wp_0$, and an iteration $n \geq 0$ such that $\left| \Im^n(x) - \Im^n(y) \right| > \delta$

# Introduction

That is, no matter how small the neighborhood, the map has the ability that a small error can grow terribly large. This is often referred to as the Butterfly effect, due to the thought-experiment described by Edward Lorenz.

# Introduction

- Topological transitive

An interval map $\Im : \wp \to \wp$ is topologically transitive if for any two open sets $U, V \subset \wp$, there exists $k > 0$ such that $\Im^k(U) \cap V \neq \emptyset$.

This is equivalent to the statement that there exists a dense orbit. In other words, the state vector can end up almost anywhere in state space.

# Introduction

- The periodic orbits are dense.

Any 2 of the above forces the third to follow.

# Introduction

- Existing works on limit cycle analysis
  - It was found by Vladimir Friedman in 1988 that if the input step size of a lowpass SDM is a rational fraction of the quantizer step, then a limit cycle will occur and the period of the limit cycle will be a multiple of the denominator of that rational number.

# Introduction

- Existing works on limit cycle analysis
  - The maximum input step size that a second order lowpass SDM produces limit cycle behavior was found by Søren Hein in 1993.
  - He extended this result to higher order lowpass SDMs at the same year via the describing function approach. However, the computational complexity will certainly be increased if the order of SDMs is increased.

# Introduction

- Existing works on limit cycle analysis
  - An algorithm for searching admissible periodic output sequences of lowpass SDMs was proposed by Deokhwan Hyun in 2002.
  - Limit cycle behaviors should be avoided in some applications, such as in audio applications.
  - Methods to avoid the occurrence of limit cycles:
    - operate the SDMs in a chaotic regime (proposed by Richard Schreier in 1994).
    - employ a dithering approach to break down the limit cycles (proposed by A. J. Magrath in 1995).

# Introduction

- Existing works on limit cycle analysis
  - All the existing results are based on lowpass SDMs.
  - What are the necessary and sufficient conditions for bandpass SDMs to exhibit limit cycle behavior?
  - What are the periods of these limit cycles if they exist?
  - Are the limit cycles stable if they exist?

# Introduction

- Existing works on limit cycle analysis
  - What are the maximum value of the input step size and the range of filter parameter that give rise to fractal behavior?
  - This information is useful because SDM designers can avoid the occurrence of limit cycle behavior in the bandpass SDMs.

# Introduction

- Existing works on stability analysis
  - Kirk C. H. Chao have conducted extensive simulations in 1990 and found that the forth order SDM should have the SNR transfer function less than two at high frequency.
  - However, these empirical results have not been supported theoretically.

# Introduction

- Existing works on stability analysis
  - Rex T. Baird performed the stability analysis by modeling the quantizer as an amplifier with a variable gain in 1994. Then the stability of the SDM can be derived via an examination of the root locus of the SDM.
  - Although this method is simple, this approach cannot explain the occurrence of various behaviors of SDMs for different initial conditions.

# Introduction

- Existing works on stability analysis
  - Gang Feng modelled the SDM as a piecewise discrete-time linear system in 2002 and performed the stability analysis via deriving the existence of a piecewise smooth Lyapunov function.
  - However, this result only explained the exponentially stable behavior, that is, the state variables converge to the origin exponentially for all initial conditions. This approach cannot explain the occurrence of various behaviors of SDMs for different initial conditions.

# Introduction

- Existing works on stability analysis
  - Søren Hein have employed a describing function approach to perform the stability analysis in 1993.
  - Although this approach is good for the analysis of limit cycle behavior, it still cannot be further applied to explain the chaotic behavior because chaotic signals involve very rich frequency spectra.

# Introduction

- Existing works on stability analysis
  - Richard Schreier found in 1994 that the chaotic behavior of SDMs depends on the zeros of the error transfer function.
  - Lars Risbo reported in 1995 that if one or more than one of the poles of the loop filter are outside the unit circle, the limit cycle will be unstable and chaotic behavior may occur.

# Introduction

- Existing works on stability analysis
  - However, these results only provided the relationship between the loop filter characteristics and the stability of the SDM. The relationship between the initial condition of the loop filter and the stability of the SDM has not been exploited.

# Introduction

- Existing works on stability analysis
  - Ngai Wong studied the stability of SDMs based on state space diagonalization and decomposition, continuous-time embedding and Poincaré map techniques in 2004.
  - This method has integrated and generalized several approaches for the stability analysis, so this method is rather complicated and has a high computational complexity.

# Introduction

- Existing works on stability analysis
  - Richard Schreier proposed an algorithm for computing the invariant set in 1997 and investigated the global stability of high order feedforward SDMs.
  - However, the existence of the invariant set only implies for local stability, it does not imply for the global stability.

# Introduction

- Existing works on stability analysis
  - Nguyen T. Thao investigated the tiling phenomenon of SDMs in 2004. He found that the global stability of second order double loop SDMs depends on whether the range of a mapping forms a partition on the invariant set or not.
  - However, this result is only applied for second order double loop SDMs.

# Part 2:

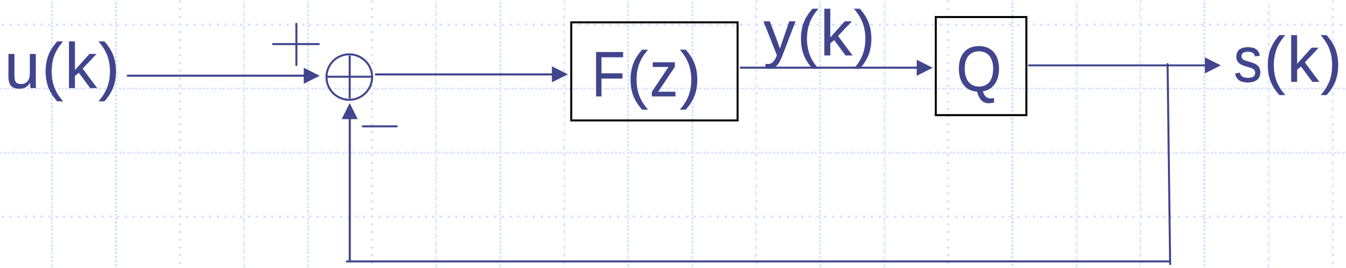
# Main Results

# Main results

- Analysis of limit cycle behavior in SDMs
  - Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter equal to one
  - Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Second order strictly stable bandpass SDMs
  - High order lowpass SDMs

# Main results

- Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter equal to one
  - Formulation (based on Orla Feely's work)



# Main results

- Formulation (based on Orla Feely's work)

$$F(z)=\frac{2\cos\theta\, z^{-1}-z^{-2}}{1-2\cos\theta\, z^{-1}+z^{-2}}$$

$$\Rightarrow y(k)-2\cos\theta\, y(k-1)+y(k-2)$$

$$=2\cos\theta\left(u(k-1)-Q(y(k-1))\right)-\left(y(k-2)-Q(y(k-2))\right)$$

$$\Rightarrow y(k)=2\cos\theta\, y(k-1)-y(k-2)+$$

$$2\cos\theta\left(u(k-1)-Q(y(k-1))\right)-\left(y(k-2)-Q(y(k-2))\right)$$

$$\mathbf{u}(k)\equiv\left[u(k-2)\quad u(k-1)\right]^T$$

$$\mathbf{x}(k)\equiv\left[y(k-2)\quad y(k-1)\right]^T$$

$$\mathbf{s}(k)\equiv\left[Q(y(k-2))\quad Q(y(k-1))\right]^T$$

# Main results

- Formulation (based on Orla Feely's work)

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 2\cos\theta \end{bmatrix} \quad \mathbf{B} \equiv \begin{bmatrix} 0 & 0 \\ -1 & 2\cos\theta \end{bmatrix}$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}(k) - \mathbf{s}(k))$$

$$Q(y) \equiv \begin{cases} 1 & y \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

  - There are only finite numbers of possibilities of **s**(k). Hence, **s**(k) can be viewed as symbol and **s**(k) is called a symbolic sequence.

# Main results

- Formulation (based on Orla Feely's work)
  - This is a nonlinear state space dynamical model for second order marginally stable bandpass SDM.
  - The SDM is marginally stable because all the poles of **A** is on the unit circle.
  - This SDM is a bandpass system because we assume that $\theta \notin \{0, \pi, -\pi\}$.
  - The sum of the numerator and denominator polynomials of the loop filter is equal to one.
  - In this presentation, we only consider the step input, that is $\mathbf{u}(k) = \overline{\mathbf{u}}$ for $k \geq 0$ and one-bit quantizer.

# Main results

- Theorem 1: Necessary and sufficient conditions for the occurrence of limit cycles for this bandpass SDM

New

$$\mathbf{T} \equiv \begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix} \quad \hat{\mathbf{A}} \equiv \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{x}_0^* \equiv \left(\mathbf{I} - \mathbf{A}^M\right)^{-1}\left(\sum_{j=0}^{M-1} \mathbf{A}^{M-1-j}\mathbf{B}\left(\overline{\mathbf{u}} - \mathbf{s}(j)\right)\right)$$

$$\mathbf{x}_{i+1}^* \equiv \mathbf{A}\mathbf{x}_i^* + \mathbf{B}\left(\overline{\mathbf{u}} - \mathbf{s}(i)\right) \text{ for } \quad i = 0,1,\cdots,M-2$$

$$\hat{\mathbf{x}}_i(k) \equiv \mathbf{T}^{-1}\left(\mathbf{x}(kM+i) - \mathbf{x}_i^*\right) \text{ for } \quad i = 0,1,\cdots,M-1 \quad \text{and} \quad k \geq 0$$

# Main results

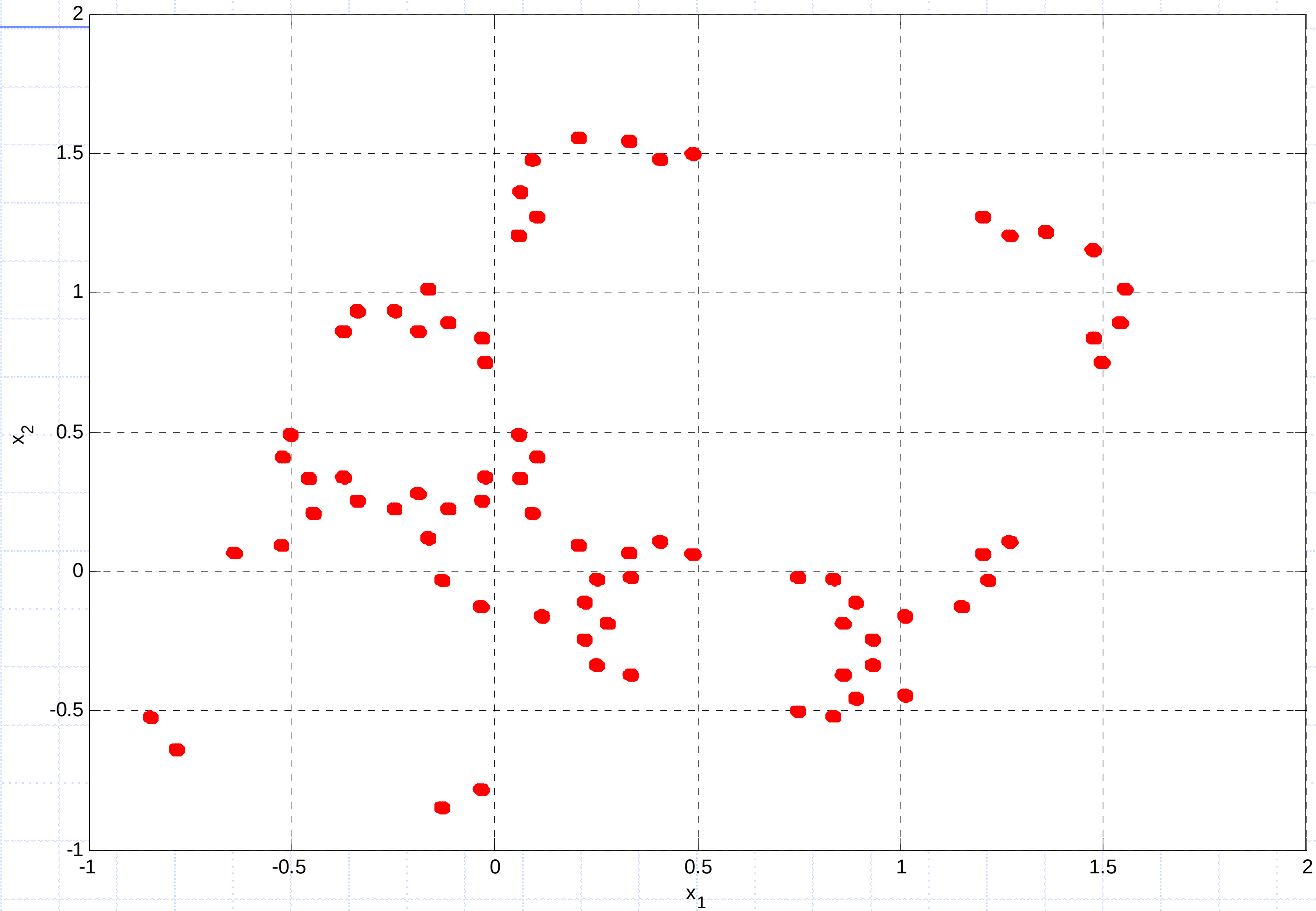
$$\exists M \in Z^{+} \text{ such that } \forall i \geq 0, \mathbf{s}(M+i) = \mathbf{s}(i)$$
$$\hat{\mathbf{x}}_i(k+1) = \hat{\mathbf{A}}^M \hat{\mathbf{x}}_i(k) \quad \text{for} \quad i = 0,1,\cdots,M-1 \quad \text{and} \quad k \geq 0$$
$$\mathbf{x}(0) \in \Sigma_M \equiv \left\{ \mathbf{x}(0) : \left\| \mathbf{T}^{-1}\left(\mathbf{x}(i) - \mathbf{x}_i^{*}\right) \right\| \leq \left\| \mathbf{x}_i^{*} \right\|_{\infty} \right\} \quad \text{for} \quad i = 0,1,\cdots,M-1$$

- The following three statements are equivalent:
  - The symbolic sequence is periodic with period M.
  - M ellipses would be exhibited on the phase plane.
  - The set of initial condition that exhibits limit cycle with period M consists of M ellipses.
- The size of ellipses depends on the initial condition, the orientation depends on the filter parameter θ, and the centres of these ellipses are at $\mathbf{x}_i^{*}$.

# Main results

- These ellipses cannot cut across principle axes.
- For M>1, all the centers of these ellipses cannot be in the same quadrant. By downsampling the output sequence by M, the downsampled output sequence becomes constant.
- When M=1, the ellipse is either on quadrant 1 or quadrant 3, but cannot in quadrant 2 or quadrant 4.
- Periodic symbolic sequence does not imply periodic state sequences. (However, this implication is true for lowpass SDMs.) The state vector is periodic only if $\theta$ is a rational multiple of $\pi$.
- Average value of s(k) does not equal to that of u(k). (This is approximately true for lowpass SDMs, and follows devil's staircase.)

# Main results



# Main results

- The importance of Theorem 1 is that it provides information for the choice of appropriate filter parameters and input step size so that limit cycles do not occur.
- Another importance of Theorem 1 is that the periodicity of the output sequence can be estimated by counting the number of ellipses on the phase plane or the number of elliptical sets of initial conditions.

# Main results

- Lemma 2: Conditions for occurrence of elliptical fractal patterns for this bandpass SDM

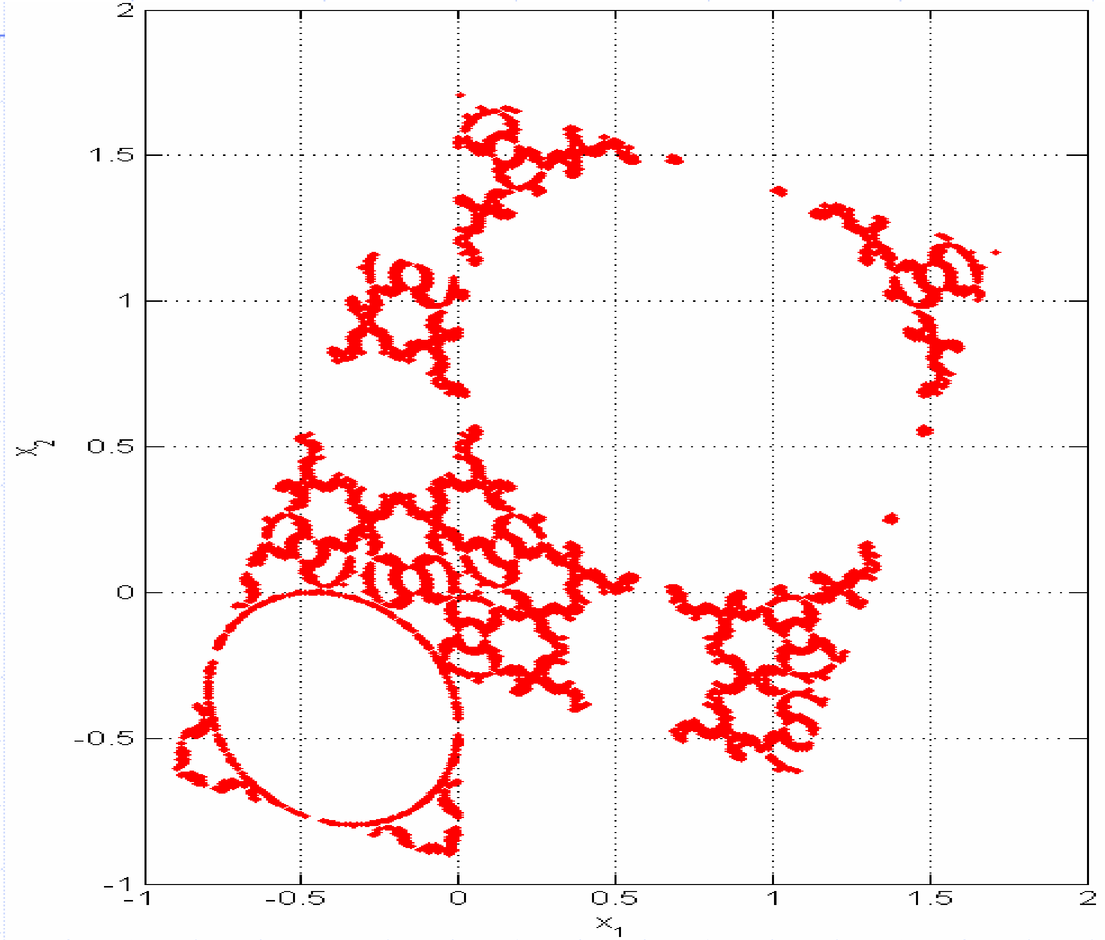


If $\left|\cos\theta\right| < \frac{1}{2}$, $\left|\bar{u}\right| < \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$, $\mathbf{s}(k)$ is aperiodic and $\mathbf{x}(0) \in \Gamma \setminus \bigcup_{M\geq 1} \Sigma_M$, where $\Gamma$ is the trapezoid, then $\mathbf{x}(k) \in \Gamma \setminus \bigcup_{M\geq 1} \Sigma_M$ for $k \geq 0$

# Main results

- Since ellipses are smooth objects that cannot cover by the whole phase plane, there must exist some initial conditions which not exhibited limit cycles.
- Lemma 2 accounts for the occurrence of elliptic fractal pattern. If the initial conditions are inside the fractal regions confined in the two trapezoids, the trajectories will stay inside these two trapezoids and will neither enter the ellipses inside the trapezoids nor leave the trapezoidal regions. Since $\Gamma$ consists of two trapezoids, $\bigcup_{M \geq 1} \Sigma_M$ representing the set of all periodic state sequences, $\Gamma$ will look like elliptic fractal regions confined within the trapezoids.

# Main results



# Main results

- The importance of Lemma 2 is that it provides information for the SDM designers to operate the SDM under the elliptic fractal behavior so that limit cycle behavior is avoided.
- We can see from Lemma 2 that in order to have elliptic fractal behavior, the filter parameter should be selected within a region, the input step size should be kept below certain bound and the initial condition should be in the elliptical fractal region.

# Main results

- We will show later that the trajectory will eventually trap inside the trapezoid if the initial condition are outside the trapezoid, since the trajectory may exhibit either limit cycle or fractal behavior inside the trapezoid, under what condition will the trajectory converge to limit cycle behavior?

# Main results

- Corollary 3: Necessary and sufficient conditions for convergence of limit cycles for this bandpass SDM

New

$$\tilde{\mathbf{x}}_i(k) \equiv \mathbf{T}^{-1}\left(\mathbf{x}(kM + k_0 + i) - \mathbf{x}_i^{\ *}\right) \text{for} \quad i = 0,1,\cdots,M-1 \quad \text{and} \quad k \geq 0$$

$$\tilde{\mathbf{x}}_0^* \equiv \left(\mathbf{I} - \mathbf{A}^M\right)^{-1}\left(\sum_{j=0}^{M-1} \mathbf{A}^{M-1-j}\mathbf{B}\left(\bar{\mathbf{u}} - \mathbf{s}(k_0 + j)\right)\right)$$

$$\tilde{\mathbf{x}}_{i+1}^* \equiv \mathbf{A}\tilde{\mathbf{x}}_i^* + \mathbf{B}\left(\bar{\mathbf{u}} - \mathbf{s}(k_0 + i)\right) \text{ for} \quad i = 0,1,\cdots,M-2$$

$$\exists M \in Z^+ \text{ such that } \forall i \geq 0, \mathbf{s}(k_0 + M + i) = \mathbf{s}(k_0 + i)$$

$$\tilde{\mathbf{x}}_i(k+1) = \hat{\mathbf{A}}^M \tilde{\mathbf{x}}_i(k) \text{ for} \quad i = 0,1,\cdots,M-1 \quad \text{and} \quad k \geq 0$$

$$\mathbf{x}(0) \in \Sigma'_M \equiv \left\{\mathbf{x}(0) : \left\|\mathbf{T}^{-1}\left(\mathbf{x}(k_0 + i) - \mathbf{x}_i^{\ *}\right)\right\| \leq \left\|\mathbf{x}_i^{\ *}\right\|_\infty\right\} \text{for} \quad i = 0,1,\cdots,M-1$$

# Main results

- Corollary 4: Conditions for global convergence of limit cycles for this bandpass SDM (effects of input step size and filter parameter)

New

If $\left|\cos\theta\right| > \frac{1}{2}$ and $\left|\bar{u}\right| > \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$, then $\forall \mathbf{x}(0) \in \Re^2$, $\bigcup_{M \geq 1} \Sigma'_M = \Re^2$

# Main results

- Corollary 5: Conditions for global convergence of limit cycles for this bandpass SDM (effect of filter parameter, independent of input step size)

New

If $\theta = q\pi$, where Q denotes the set of rational number, then $\forall \mathbf{x}(0) \in \Re^2$ and $\forall \bar{u} \in \Re$, $\exists M \geq 1$ and $\exists k_0 \geq 1$ such that $\mathbf{x}(k + k_0 + M) = \mathbf{x}(k + k_0)$, $\forall k \geq 0$

# Main results

- If the filter parameter are selected outside the range and the input step size is larger than certain bound, then conditions in Lemma 2 do not satisfied. This implies that the elliptic fractal pattern will not occur and limit cycles would occur no matter where the initial condition is.
- On the contrary, the trajectory of the lowpass SDM may diverge and unstable behavior may result if the input step size is increased.
- If the natural frequency of the loop filter is a rational multiple of $\pi$, then elliptic fractal pattern will not occur and limit cycles would occur no matter where the initial condition is and what input step size is.

# Main results

- Hence, to avoid the occurrence of limit cycle, the filter parameter should be selected within a region, the input step size should be kept below certain bound, the initial condition should be in the elliptical fractal region, and the natural frequency of the loop filter should not be selected as a rational multiple of $\pi$.

# Main results

- Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not equal to one

$$F(z) = \frac{c\,z^{-1} + d\,z^{-2}}{1 - 2\cos\theta\, z^{-1} + z^{-2}}$$

# Main results

- Corollary 6: Fixed point behavior for this SDM

If $\mathbf{x}(0) = \frac{(c+d)(\bar{u}-\bar{s})}{2(1-\cos\theta)}\begin{bmatrix}1\\1\end{bmatrix}$, where $\bar{s} = Q(x_1(0))$, then

$\mathbf{x}(k) = \mathbf{x}(0)$ for $k \geq 0$

New

  - If $|\bar{u}|<1$ and $d+c>0$, there is no fixed point no matter where the initial condition is.
  - The importance of this corollary is that it provides information for the choice of appropriate filter parameters and input step size so that the fixed point behavior can be avoided no matter where the initial condition is.

# Main results

- Corollary 7: Necessary and sufficient condition for occurrence of limit cycles for this SDM

New

Theorem 1 is still applied by putting $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ d & c \end{bmatrix}$

# Main results

- Lemma 8: Admissible set of periodic sequence for this SDM

New

$$\zeta \equiv [s_1(0), \quad s_2(0), \quad \cdots \quad s_1(M-1), \quad s_2(M-1)]^T$$

$$\mathbf{D}_j \equiv \begin{bmatrix} d \sin j\theta & c \sin j\theta \\ d \sin(j+1)\theta & c \sin(j+1)\theta \end{bmatrix}$$

$$\mathbf{D} \equiv \begin{bmatrix} \mathbf{D}_{M-1} & \mathbf{D}_{M-2} & \cdots & \mathbf{D}_1 & \mathbf{D}_0 \\ \mathbf{D}_0 & \mathbf{D}_{M-1} & \ddots & & \mathbf{D}_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathbf{D}_{M-1} & \mathbf{D}_{M-2} \\ \mathbf{D}_{M-2} & \cdots & \cdots & \mathbf{D}_0 & \mathbf{D}_{M-1} \end{bmatrix}$$

# Main results

$$\mathbf{K} \equiv diag\left(\left(\mathbf{I}-\mathbf{A}^{M}\right)^{-1}, \quad \cdots, \quad \left(\mathbf{I}-\mathbf{A}^{M}\right)^{-1}\right)$$

$$\boldsymbol{\upsilon} \equiv \left[\frac{(d+c)\overline{u}}{\sin\theta}\sum_{j=0}^{M-1}\sin j\theta, \quad \frac{(d+c)\overline{u}}{\sin\theta}\sum_{j=1}^{M}\sin j\theta\right]$$

$$\boldsymbol{\tau} \equiv [\boldsymbol{\upsilon}, \quad \cdots, \quad \boldsymbol{\upsilon}]^{T}$$

$$\mathbf{s} \equiv (\mathbf{s}(0), \mathbf{s}(1), \cdots, \mathbf{s}(M-1))$$

$\Sigma$ be the admissible set of periodic output sequences with period M

$$\Sigma = \{\mathbf{s} : \mathbf{Q}(\mathbf{K}(\mathbf{D}\zeta + \boldsymbol{\tau})) = \zeta\}$$

# Main results

- This lemma provides information on characterizing the admissible set of periodic output sequences and can be used to check whether a given periodic output bit pattern is admissible or not.
- Note that not all of the periodic output sequences are admissible. For example, when $|\bar{u}|<1$ and $d+c>0$, it is impossible to have a constant output sequence, as discussed in Corollary 6.
- However, if an initial condition satisfies the condition in Corollary 7, then the corresponding output sequence will be in the admissible set.

# Main results

- Corollary 9: Conditions for occurrence of elliptical fractal patterns for this SDM

New

If $|\cos\theta| < \frac{1}{2}$, $c = -2d\cos\theta$, $d < 0$, $|\bar{u}| < \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$, $\mathbf{s}(k)$ is aperiodic and $\mathbf{x}(0) \in \Gamma \setminus \bigcup_{M\geq 1} \Sigma_M$, then $\mathbf{x}(k) \in \Gamma \setminus \bigcup_{M\geq 1} \Sigma_M$ for $k \geq 0$

# Main results

- This corollary stated that the conditions for the occurrence of trapezoids not only depend on $\theta$, but also on the input step size and other filter parameters as well.
- This corollary provides information for SDM designers to choose appropriate filter parameters and input step size so that elliptic fractal behaviors may occur. In this case, limit cycle behaviors can be avoided.
- To have elliptic fractal behaviors, the input step size should not exceed the bound. The denominator coefficient $a$ ($2\cos\theta$) should be selected within a range, and the numerator coefficients should be selected in proportion. The leading coefficient of the numerator should also be negative.

# Main results

- It is not enough to conclude that the SDM would exhibit fractal behavior only from filter parameters and input step size ($\left|\cos\theta\right| < \frac{1}{2}$, $c = -2d\cos\theta$, $d < 0$ and $\left|\bar{u}\right| < \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$), the SDM may exhibit limit cycle behavior for some initial conditions.

# Main results

- Corollary 10: Conservation of noise output gain and input output gain

New

The noise output gain and input output gain at the natural frequency of the loop filter of this SDM are the same as that of the SDM with sum of the numerator and denominator polynomials of the loop filter equal to one.

# Main results

- Second order strictly stable bandpass SDMs
  - In the practical situation, there are leakages on the integrators. This originates from the internal resistances of the components.
  - Even though the leakages may sometimes be negligible, engineers and circuit designers may impose leakage on the integrators so as to improve the stability of the overall systems.
  - Therefore, the eigenvalues of the system matrices are strictly inside the unit circle, and the system matrices are actually strictly stable.

# Main results

- Corollary 11: Necessary and sufficient condition for occurrence of limit cycles for this SDM

New

$$\mathbf{D} \equiv \begin{bmatrix} re^{j\theta} & 0 \\ 0 & re^{-j\theta} \end{bmatrix} \mathbf{T} \equiv \begin{bmatrix} \frac{1}{\sqrt{r}} e^{-\left(\frac{j\theta}{2}\right)} & \frac{1}{\sqrt{r}} e^{\frac{j\theta}{2}} \\ \sqrt{r} e^{\frac{j\theta}{2}} & \sqrt{r} e^{-\left(\frac{j\theta}{2}\right)} \end{bmatrix}$$

$$\mathbf{x}_0^* \equiv \sum_{j=0}^{M-1} \mathbf{T}\mathbf{D}^{M-1-j} \left( \lim_{p \to +\infty} \sum_{m=0}^{p-1} \mathbf{D}^{mM} \right) \mathbf{T}^{-1} \mathbf{B}\left(\bar{\mathbf{u}} - \mathbf{s}\left(k_0 + j\right)\right)$$

$$\mathbf{x}_{i+1}^* \equiv \mathbf{A}^i \mathbf{x}_0^* + \sum_{m=0}^{i-1} \mathbf{A}^{i-1-m} \mathbf{B}\left(\bar{\mathbf{u}} - \mathbf{s}\left(k_0 + m\right)\right) \text{for} \quad i = 1,2,\cdots, M-1$$

# Main results

$$\mathbf{s}(k_0 + M + i) = \mathbf{s}(k_0 + i)\ \forall i \geq 0$$

$$\lim_{k\to+\infty} \mathbf{x}(kM + k_0 + i) = \mathbf{x}_i^* \text{ for } \quad i = 0,1,\cdots,M-1$$

$$\mathbf{x}(0) \in \Sigma''_M \equiv \left\{ \begin{array}{l} \mathbf{x}(0) : \exists k_0 \in Z^+ \bigcup \{0\} \text{such that } \forall k \geq 0 \text{ and} \\ i = 0,1,\cdots,M-1, Q(\mathbf{x}(kM + k_0 + i)) = Q(\mathbf{x}_i^*) \end{array} \right\}$$

# Main results

- Second order strictly stable bandpass SDMs
  - The leakage of the system depends on the values of r. If r is closer to 0, then the poles are closer to the origin and the leakage is more serious. If r is closer to 1, then the poles are closer to the unit circle and the leakage is less significant. If r=1, it reduces to the marginally stable case.
  - The trajectories will converge to the set of fixed points $\left\{\mathbf{x}_0^*, \mathbf{x}_1^*, \cdots, \mathbf{x}_{M-1}^*\right\}$, and the periodicity of the steady states of the output sequence is equal to the number of fixed points on the phase plane.

# Main results

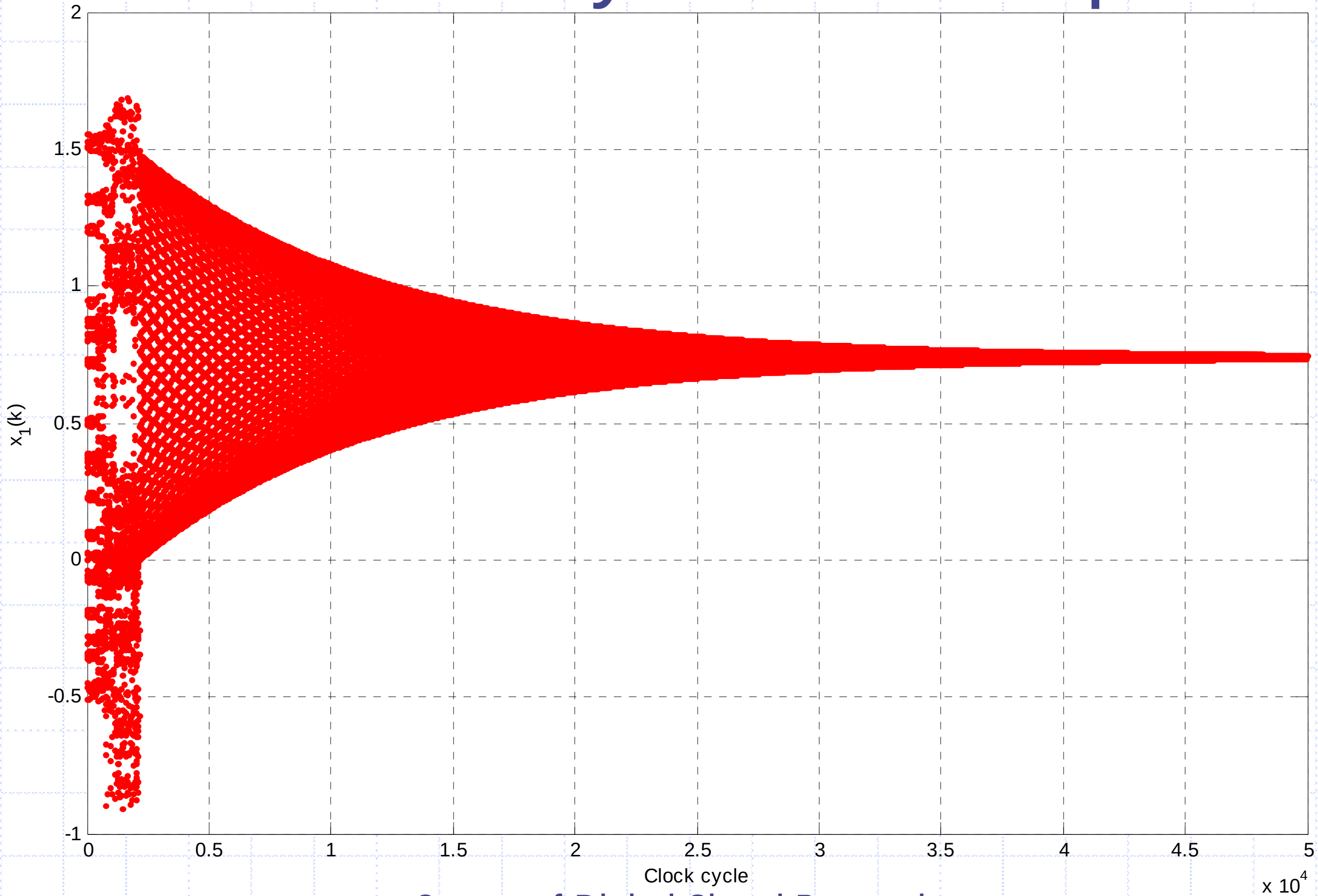
- Second order strictly stable bandpass SDMs
  - Although the state vector is converging to a periodic orbit, it never reaches these periodic points. That means, the state vector is aperiodic even though the output sequence is eventually periodic. This result is different from the case when r=1 and $\theta$ is a rational multiple of $\pi$.
  - Although $\mathbf{x}_i^*$, for $i = 0,1,\cdots,M-1$, depends on $\mathbf{s}(i)$, for $i = 0,1,\cdots,M-1$, it does not depend on $\mathbf{x}(0)$ directly. That is, the fixed points leading to a given symbol sequence are not directly depended on the initial conditions.

# Main results

- Second order strictly stable bandpass SDMs
  - The state vectors of the corresponding linear system will converge to $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\bar{\mathbf{u}}$, which is not the same as that of $\mathbf{x}_0^*$ for M=1. Comparing these two values, there are DC shifts and the DC shifts are exactly dropped at the output sequences.
  - For M=1, the rate of convergence only depends on r when the output sequence becomes steady. This is because the DC terms do not affect the rate of convergence. However, if we look at the transient response of the system, that is, the time duration when the output sequence is not constant, the system dynamics could be very complex.

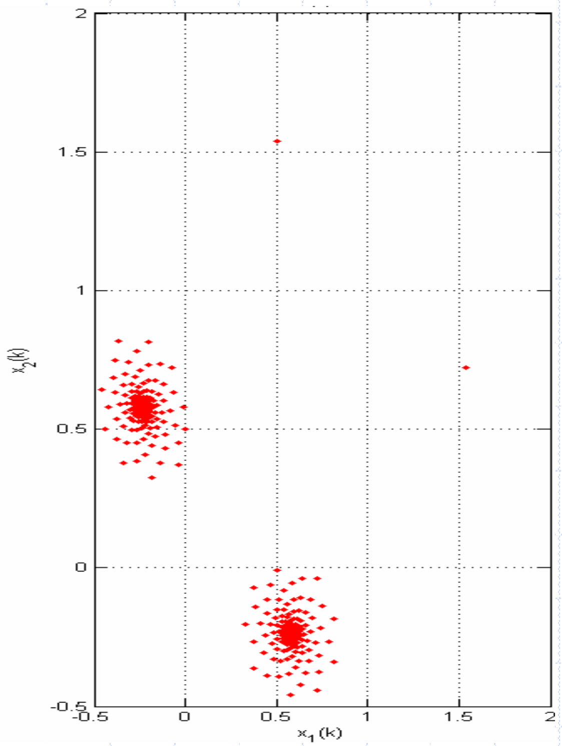
# Main results

- Second order strictly stable bandpass SDMs



# Main results

- Second order strictly stable bandpass SDMs



# Main results

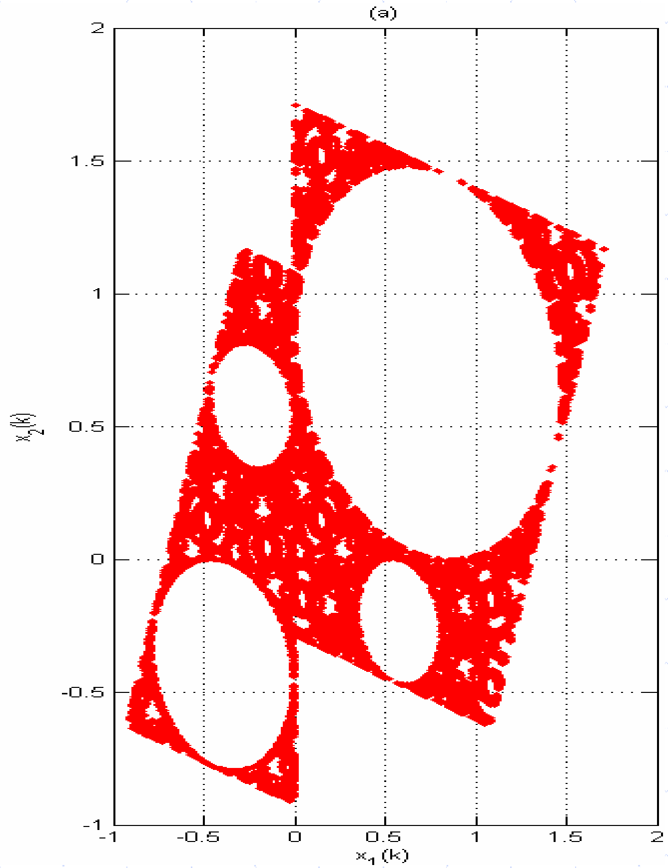
- Second order strictly stable bandpass SDMs
  - Intuitively, fractals would not be exhibited on the phase plane when the system matrices of the bandpass SDM are strictly stable. However, if the sets of initial conditions corresponding to convergent or limit cycle behavior do not cover the whole phase plane, then fractal patterns may also be exhibited in the phase plane.
  - In this case, the output sequence is aperiodic.

# Main results

- Second order strictly stable bandpass SDMs
  - The importance of this corollary is that it is not necessary to place unstable poles in the system matrices of bandpass SDMs to generate signals with rich frequency spectra in order to suppress unwanted tones from quantizers. In fact, fractals can be generated via system matrices with strictly stable poles. Since the output sequences are aperiodic, which consist of rich frequency spectra, the unwanted tones could be suppressed using these aperiodic signals without the tradeoff of the stability of the systems.

# Main results

- Second order strictly stable bandpass SDMs



# Main results

- High order lowpass SDMs

$$\mathbf{x}(k) \equiv \left[y(k-N), \quad \cdots, \quad y(k-1)\right]^T$$

$$\mathbf{u}(k) \equiv \left[u(k-N), \quad \cdots, \quad u(k-1)\right]^T$$

$$\mathbf{s}(k) \equiv \left[Q(y(k-N)), \quad \cdots, \quad Q(y(k-1))\right]^T$$

$$F(z) \equiv \frac{\sum_{j=1}^{N} b_j z^{-j}}{\sum_{i=0}^{N} a_i z^{-i}}$$

# Main results

- High order lowpass SDMs

$$\mathbf{A} \equiv \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_N}{a_0} & \cdots & \cdots & \cdots & -\frac{a_1}{a_0} \end{bmatrix} \quad \mathbf{B} \equiv \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \frac{b_N}{a_0} & \cdots & \cdots & \cdots & \frac{b_1}{a_0} \end{bmatrix}$$

  - Suppose there are $n_d$ eigenvalues of **A** located at 1, where $1 \le n_d \le N$, and the remaining $N-n_d$ eigenvalues are distinct.

# Main results

- High order lowpass SDMs
  - Corollary 12:

Denote $L_i$, for i=1,2,…,N, be the *i th* element of the vector $\sum_{j=0}^{P-1} \mathbf{A}^{P-1-j}\mathbf{B}\left(\mathbf{u}\left(k_0+j\right)-\mathbf{s}\left(k_0+j\right)\right)$, in which $P\in Z^{+}$ and $k_0\geq 0$. Let $\mathbf{r}_j$, for j=1,2,…,N, be the *j th* row of $\mathbf{I}-\mathbf{A}^P$, where **I** is an NxN identity matrix. Then there exist real constants, $c_{i,n}\in\Re$, for $i=1,2,\ldots,N-n_d$ and $n=1,2,\ldots,n_d$, such that

$$\sum_{i=1}^{N-n_d} c_{i,n}\mathbf{r}_i = \mathbf{r}_{N-n_d+n}$$

New

# Main results

- High order lowpass SDMs

Denote $\Psi_P \equiv \left\{ \mathbf{x}(0) : \mathbf{r}_i \mathbf{x}(0) = L_i \ \text{ for } \ i = 1,2,\cdots, N - n_d \right\}$. Then the SDM will exhibit limit cycle behavior for $k \geq k_0$ with period P if $\sum_{i=1}^{N-n_d} c_{i,n} L_i = L_{N-n_d+n}$ for all $n=1,2,\ldots,n_d$. In this case, $\Psi_p$ is the non-empty set of initial conditions such that the SDM exhibits limit cycle behavior with period P.

If there exist some $n \in \{1,2,\ldots,n_d\}$ such that $\sum_{i=1}^{N-n_d} c_{i,n} L_i \neq L_{N-n_d+n}$, then there will not exist any fixed point or periodic state sequence.

# Main results

- High order lowpass SDMs
  - Since the analysis is based on the feedforward structure, any minimal realization of the feedforward structure can be transformed to the direct form structure via simple transformation.
  - In general, an arbitrary SDM may not give fixed point or limit cycle behavior. The importance of Corollary 12 is that it gives the condition for the occurrence of fixed points or limit cycles, and characterizes the corresponding set of initial conditions so that the SDM designers can avoid such unwanted behavior.

# Main results

- High order lowpass SDMs
  - The existence of an invariant set is not a necessary and sufficient condition for the occurrence of the fixed points or limit cycles. For example, the trajectory of the SDM may exhibit fractal or irregular chaotic pattern confined in an invariant set. In this case, even though there exists an invariant set, the trajectory does not exhibit fixed point or limit cycle behavior.

# Main results

- High order lowpass SDMs
  - Even if the trajectory exhibits fixed point or limit cycle behaviors for some initial conditions, it is not sufficient to conclude that the trajectories with other initial conditions will be eventually converged to the fixed points or limit cycles.

# Main results

- High order lowpass SDMs
  - To check whether the trajectory within an invariant set would exhibit fixed point or limit cycle behavior, the first step is to characterize the invariant set. The second step is to determine whether this invariant set is the set of initial conditions that corresponds to fixed point or limit cycle behaviors. If the invariant set satisfies the condition stated in Corollary 12, then this invariant set is the set of initial condition that eventually gives fixed point or limit cycle behaviors.

# Main results

- Stability analysis of SDMs
  - Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter equal to one:
    - Local stability
    - Global attractor

# Main results

- Stability analysis of SDMs
  - Second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
    - Global stability
    - Conditions for divergence
    - Rate of divergence

# Main results

- Stability analysis of SDMs
  - High order lowpass SDMs
    - Global stability

# Main results

- Local stability of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter equal to one
  - For the stability of the elliptical trajectory, if the initial condition is strictly inside an elliptical set, a small perturbation of it will give rise to limit cycle with the same period as that of the original initial condition. Hence, the corresponding trajectory is regarded as locally stable.

# Main results

- Local stability of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter equal to one)
  - However, if the initial condition is on the boundary of an elliptical set, a small perturbation of it may give rise to a very different dynamical behavior. In this case, the trajectory is regarded as locally unstable.

# Main results

- Global attractor of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter equal to one)
  - Peter Ashwin stated in 2003 that the trapezoid is the global attractor of this SDM.
  - However, he had just stated this fact and did not provide an analytical proof.

# Main results

- Global attractor of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter equal to one)
  - The points on one side of the boundaries will converge to the trapezoids, while the points on the other side of the boundaries exhibit fractal behaviors.
  - This kind of stability is quite different from the classical one, in which all the points in both sides of boundaries either converge to the boundaries or diverge from the boundaries.

# Main results

- Global attractor of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter equal to one)
  - We prove by the following approach: the mapping from the whole state space to itself is not injective, but surjective.
  - That means, if one or more than one ellipses or elliptic fractal pattern exist, then there will always exist some points outside the ellipses or the elliptic fractal region that are mapped to the ellipses or the elliptic fractal region.

# Main results

- Global attractor of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter equal to one)
  - However, the mapping from the ellipses or the elliptic fractal region to themselves is bijective, so the trajectory will stay inside the ellipses or the elliptic fractal region.
  - As a result, the ellipses or the elliptic fractal region are global attractors of the SDM.

# Main results

New

Suppose $\Gamma$ exists. Define $G:\Gamma \rightarrow \Gamma$ such that $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}))$, then $G$ is bijective.

New

Suppose $\Gamma$ exists. Define $H:\Re^2 \rightarrow \Re^2$ such that $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{B}(\bar{\mathbf{u}} - Q(\mathbf{x}))$, then $H$ is surjective, but not injective. $\forall \mathbf{y} \in \Gamma$, $\exists \mathbf{x}_1 \in \Gamma$ and $\exists \mathbf{x}_2 \in R^2 \setminus \Gamma$ such that $H(\mathbf{x}_1) = H(\mathbf{x}_2) = \mathbf{y}$

If $\Gamma$ exists, $\forall \mathbf{x}(0) \in \Re^2 \setminus \Gamma, \exists k_0 \in Z^+$ such that $\mathbf{x}(k) \in \Gamma$, $k \geq k_0$

# Main results

- Global stability of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may not be equal to one)
  - Corollary 13: Conditions for global stability of this SDM

New

If $\left|\cos\theta\right| < \frac{1}{2}$, $c = -2d\cos\theta$, $d < 0$ and $\left|\bar{u}\right| < \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$, $\forall \mathbf{x}(0) \in \Re^2 \setminus \Gamma$, $\exists k_0 \in Z^+$, $\mathbf{x}(k) \in \Gamma$ for $k \geq k_0$

# Main results

- Unlike the second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter equal to one, this SDM is not necessarily globally stable.
- That is, the trajectory may diverge. The global convergence is guaranteed only when

$$\left|\cos\theta\right| < \frac{1}{2}, c = -2d\cos\theta, d < 0 \text{ and } \left|\bar{u}\right| < \min\left(1, \frac{1+2\cos\theta}{1-2\cos\theta}\right)$$

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may not be equal to one)
  - Corollary 14:

If $d > 0$, then $H$ is bijective.

New

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may *not* be equal to one)
  - In order to have a global attractor region, a bounded subset should exist in the state space such that all the states outside it will eventually move to it, and all the states inside it will stay inside it. That means, the mapping from this bounded subset to itself is bijective, but the mapping from the whole state space to this bounded subset is not injective.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may *not* be equal to one)
  - If the mapping from the whole state space to itself is bijective, such as the case in Corollary 14, then the states outside this bounded subset will not move to it even though it exists.
  - Hence, there does not exist a global attractor region and the trajectory of the SDM is not guaranteed to be bounded for those initial conditions that are not in the bounded subset.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may *not* be equal to one)
  - Since there does not exist a global attractor region if Corollary 14 is satisfied and the trajectory of the SDM is not guaranteed to be bounded for those initial conditions that are not in the bounded subset, the importance of this corollary is that it provides information for SDM designers to choose appropriate filter parameters such that Corollary 14 is *not* satisfied.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may *not* be equal to one)
  - If both Corollary 6 and Corollary 14 are satisfied, then the bounded subset will consist of at least one single element. Similarly, if both Corollary 7 and Corollary 14 are satisfied, then the bounded subset will consist of elliptical sets.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may *not* be equal to one
  - Hence, if both Corollary 6 and Corollary 14 or both Corollary 7 and Corollary 14 are satisfied, then there will exist some initial conditions exhibiting fixed point or limit cycle behavior. However, the other initial conditions will not result in fixed point or limit cycle behavior. So the fixed point or limit cycles are unstable.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Corollary 15:

New

If $d > 0$ and $\mathbf{x}(0)$ is not in that bounded subset, then $x_i(k) \to \infty$ for $i = 1,2$

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - The importance of this theorem is to suggest how to design the filter parameters so that the divergent behavior is avoided.
  - The global stability of the SDM depends on the numerator coefficients of the loop filter transfer function, that is, the matrix **B**.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Corollary 16:

Denote $S_i(\omega)$ for $i=1,2$ as the discrete-time Fourier transform of $s_i(k)$. If $d>0$ and $\mathbf{x}(0)$ is not in the bounded subset, then there does not exist $B_0 \in \Re$ such that $\left|S_i(\omega=\pm\theta)\right| < B_0$

New

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - The importance of this corollary is that the divergent behavior can be distinguished from the elliptic fractal pattern confined in the trapezoids or irregular chaotic pattern by examining the existence of a pole located at the natural frequency of the loop filter in the spectrum of the output sequence.

# Main results

- Conditions for divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - If the trajectory diverges, then there will not exist $B_0 \in \Re$ such that $\left|S_i(\omega = \pm\theta)\right| < B_0$ for $i = 1,2$. On the other hand, for the cases when an elliptic fractal pattern is confined in the trapezoids or an irregular chaotic pattern is exhibited in the phase plane, $\exists B_0 \in \Re$ such that $\left|S_i(\omega = \pm\theta)\right| < B_0$ for $i = 1,2$, and the trajectory is bounded.

# Main results

- Rate of divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - Corollary 17:

New

$\exists C_1, C_2, c_1, c_2 \in \Re$ such that $C_1 k + c_1 \leq |x_i(k)| \leq C_2 k + c_2$

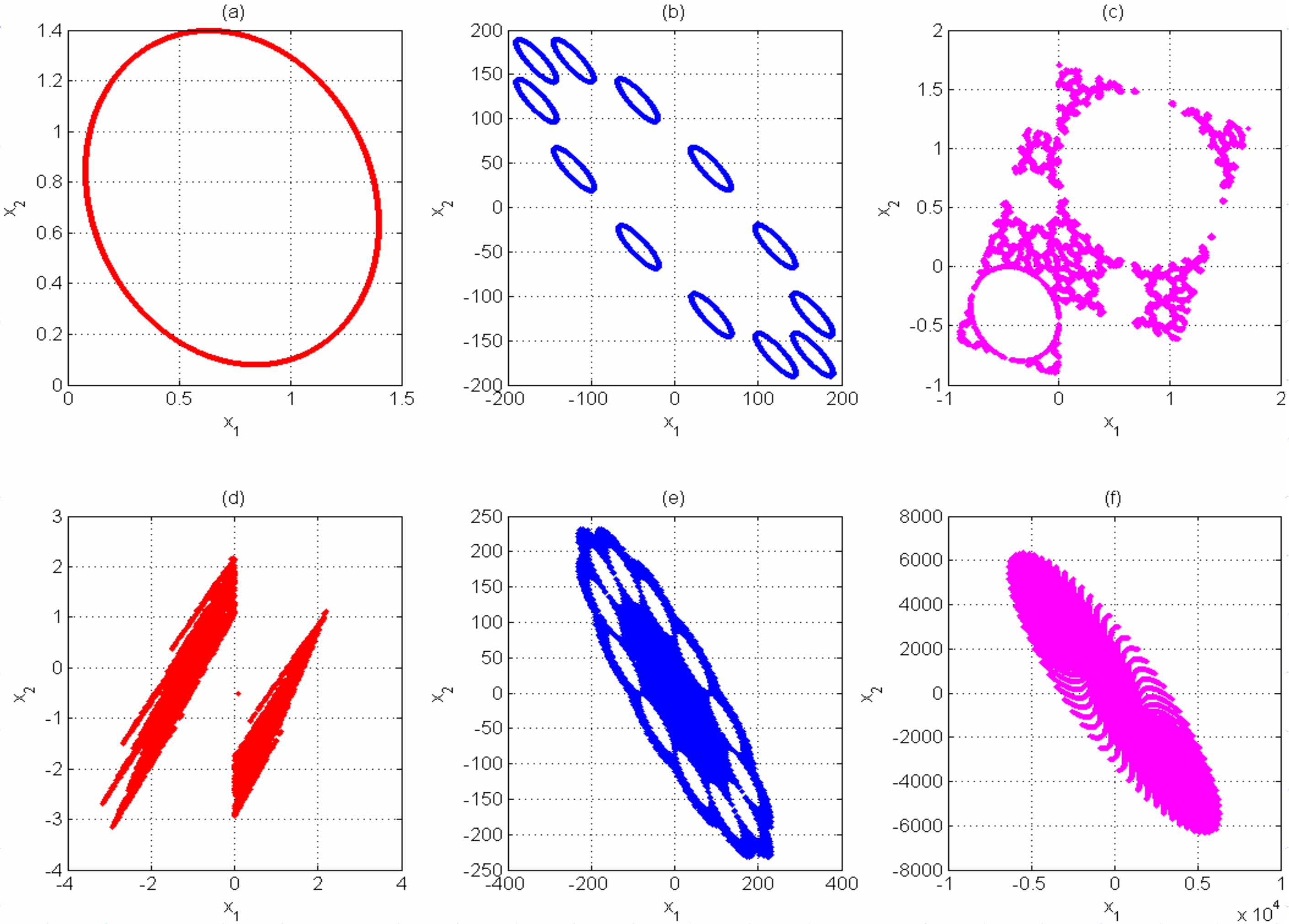
# Main results

- Rate of divergence of second order marginally stable bandpass SDMs (with sum of the numerator and denominator polynomials of the loop filter may not be equal to one)
  - Corollary 17 provides information regarding how the trajectory diverges. The divergence is not exponential. Instead, the state variables are bounded by two linear growth functions.
  - Even when the trajectory diverges, an elliptic fractal pattern may be exhibited on the phase plane. In this case, the elliptic fractal pattern is not confined in any particular region.

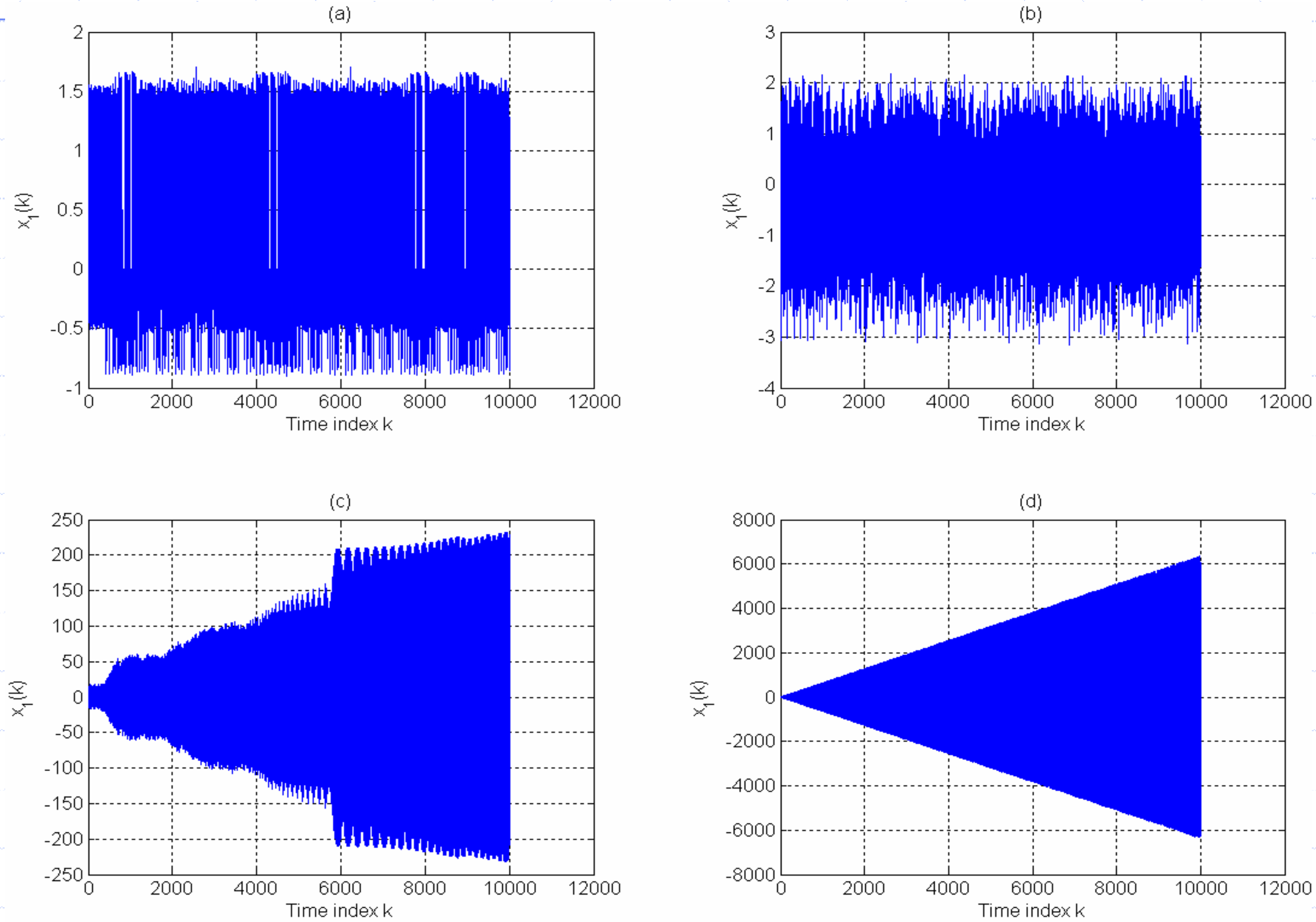
# Main results

- Rate of divergence of second order marginally stable bandpass SDMs with sum of the numerator and denominator polynomials of the loop filter may not be equal to one
  - When a limit cycle occurs, or an elliptic fractal pattern confined within two trapezoids, or an irregular chaotic pattern is exhibited in the phase plane, the trajectory will be bounded and the response does not grow linearly with respect to k. In these cases, the state variables will be bounded by the two horizontal straight lines.

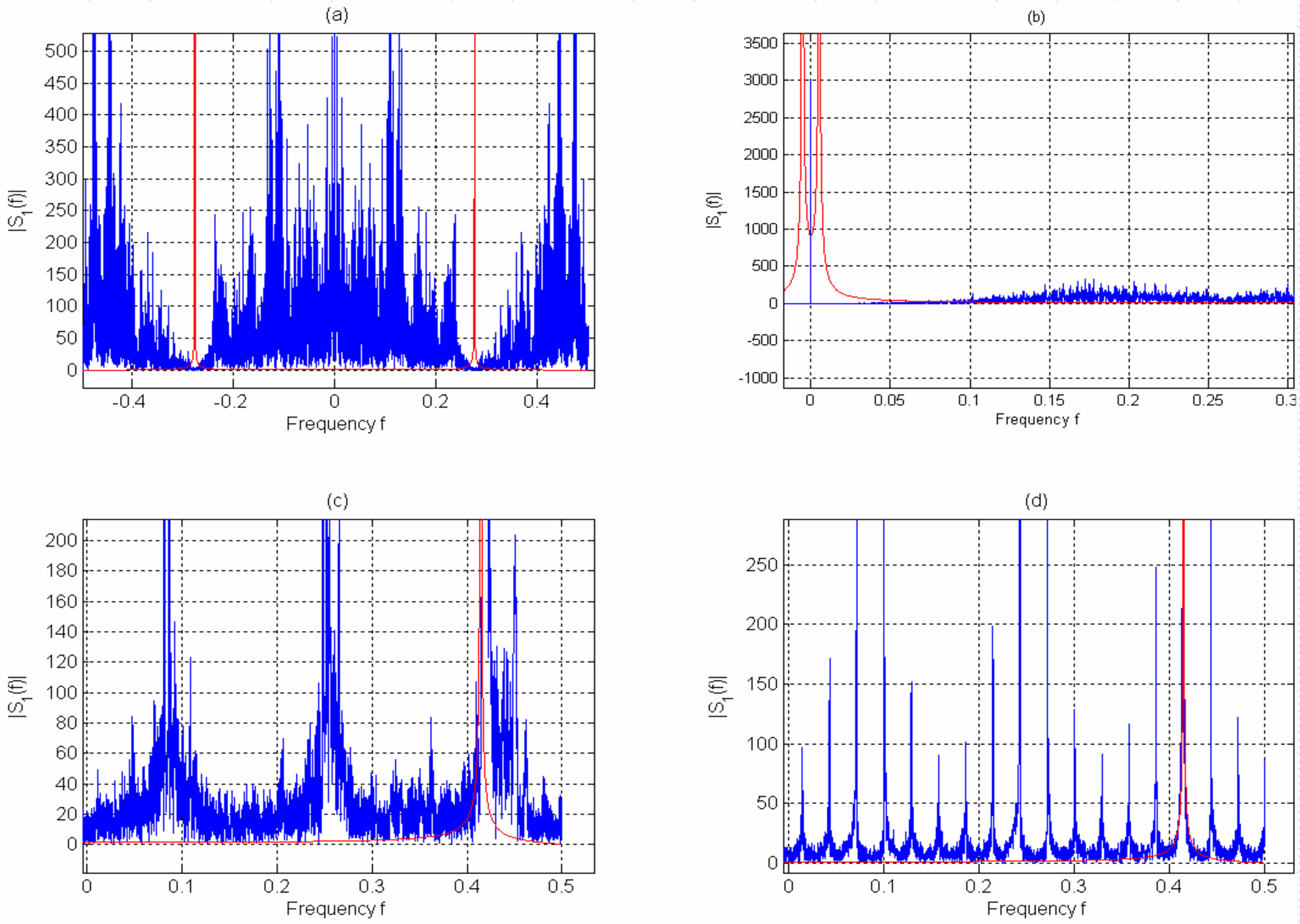
# Main results



# Main results



# Main results



# Main results

- Global stability of high order lowpass SDMs

  Denote the forward dynamics of the system as $\aleph_f : \Re^N \to \Re^N$. Denote the backward dynamics of the SDMs as $\aleph_b : \Re^N \to \Re^N$. Define

  $$x'(k) \equiv b_N \bar{u} + \sum_{i=1}^{N-1} b_{N-i}\left(\bar{u} - Q(x_i(k))\right) - \sum_{i=1}^{N} a_{N-i} x_i(k)$$

  - Corollary 18:

    If $|x'(k)| > |b_N|$ and $Q(a_N b_N) = 1$, or $Q(a_N b_N) = -1$, then the backward dynamics of the SDMs can be defined as $\mathbf{x}(k-1) = \left[ \frac{x'(k) - Q(x'(k) a_N) b_N}{a_N} \quad x_1(k) \quad \cdots \quad x_{N-1}(k) \right]^T$

# Main results

- Global stability of high order lowpass SDMs
  - Corollary 19:

New

For $Q(a_N b_N) = 1$, if $\exists k_0 \in Z$ such that $|x'(k)| < |b_N|$, then $\aleph_{\mathrm{b}}(\mathbf{x}(k))$ is not defined for $k \le k_0$

# Main results

- Global stability of high order lowpass SDMs
  - The importance of Corollary 18 and Corollary 19 is that it characterizes the conditions for the existence of the backward dynamics of the SDMs. This information is useful for computing the invariant set.

# Main results

- Global stability of high order lowpass SDMs
  - Corollary 20:

    If $\aleph_b(\mathbf{x}(k))$ is defined for all $k \le 0$, then the invariant set $\wp$ can be expresses as

$$\wp = \left\{ \mathbf{x}(0) : \aleph_f(\mathbf{x}(k)) \in \wp \quad \text{for } k \ge 0 \quad \text{and} \quad \aleph_b(\mathbf{x}(k)) \in \wp \quad \text{for } k \le 0 \right\}$$

  - Corollary 21:

    For $Q(a_N b_N) = 1$ if $\exists k_0 \in Z$ such that $|x'(k)| < |b_N|$, then $\mathbf{x}(0) \notin \wp$

New

New

# Main results

- Global stability of high order lowpass SDMs
  - The importance of Corollary 20 and Corollary 21 is that it can be used for computing the invariant set via the dynamics of SDMs. This is essential for defining the domain and the range of the system mapping.

# Main results

- Global stability of high order lowpass SDMs
  - Corollary 22:

    If $Q(a_N b_N) = -1$, then $\Im$ is not injective. Hence, it is not bijective.

  - The importance of Corollary 22 is that it reveals that the system mapping from the invariant set to itself is not necessarily bijective.

New

# Main results

- Global stability of high order lowpass SDMs
  - Corollary 23:

New

If $Q(a_N b_N) = -1$, then the invariant $\wp'$ can also be expressed as:

$$\wp' = \left\{ \begin{array}{l} \mathbf{x}(0) : \aleph_{\mathrm{f}}(\mathbf{x}(k)) \in \wp' \quad \text{for } k \geq 0, \quad \aleph_{\mathrm{b}}(\mathbf{x}(k)) \in \wp' \quad \text{for } k \leq 0, \quad \text{and} \\ \mathbf{x}(k) + \begin{bmatrix} \dfrac{2b_N Q(x_1(k))}{a_N} & 0 & \cdots & 0 \end{bmatrix}^T \notin \wp' \quad \text{if } \mathbf{x}(k) \in \wp' \quad \forall k \in Z \end{array} \right\}$$

If $Q(a_N b_N) = 1$, suppose $\aleph_{\mathrm{b}}(\mathbf{x}(k))$ is defined for all $k \leq 0$, then the invariant set $\wp'$ can also be expressed as :

$$\wp' = \{ \mathbf{x}(0) : \aleph_{\mathrm{f}}(\mathbf{x}(k)) \in \wp' \quad \text{for } k \geq 0, \quad \aleph_{\mathrm{b}}(\mathbf{x}(k)) \in \wp' \quad \text{for } k \leq 0 \}$$

# Main results

- Global stability of high order lowpass SDMs
  - Corollary 24:

    Suppose $\aleph_b(\mathbf{x}(k))$ is defined for all $k \le 0$ and the invariant set $\wp'$ exists. Define $\mathfrak{I}' : \wp' \to \wp'$. Then $\mathfrak{I}'$ is bijective.
  - The importance of Corollary 23 and Corollary 24 is that it characterizes the invariant set in the state space in which the system map is bijective from $\wp'$ to $\wp'$. Compared to the set $\wp$, the system map is not bijective from $\wp$ to $\wp$. This result is important for deriving the global stability of SDM.

New

# Main results

- Global stability of high order lowpass SDMs
  - Theorem 25:

New

For $Q(a_N b_N) = -1$, suppose the invariant set $\wp'$ exists, then $\forall \mathbf{x}(0) \in \Re^N \setminus \wp', \exists k_0 > 0$ such that $\mathbf{x}(k) \in \wp'$ for all $k \geq k_0$. If $Q(a_N b_N) = 1$, then the SDMs are *not* globally stable.

# Main results

- Global stability of high order lowpass SDMs
  - The importance of Theorem 25 is that the state trajectory of the SDM will eventually fall inside $\wp'$ when $Q(a_N b_N) = -1$ if $\wp'$ exists. Once the trajectory falls inside $\wp'$, the trajectory will stay inside $\wp'$ and will never leave $\wp'$. Hence, the trajectory can be regarded as globally stable because the trajectory is confined in $\wp'$ no matter where the initial condition is.

# Main results

- Global stability of high order lowpass SDMs
  - The dynamical behavior of the trajectory inside $\wp'$ could be very complex. Fixed point, limit cycle, fractal or irregular chaotic behavior could be exhibited.

# Main results

- Global stability of high order lowpass SDMs
  - The bounded behavior depends not only on the existence of the invariant set or whether the range of the system mapping forming a partition on the invariant set, but also on the size of the hyperplane formed from the projection of the invariant set onto its first co-ordinate (based on the direct form realization), and the sign of the product of the leading filter coefficients in the numerator and that in the denominator as well.

# Main results

- Global stability of high order lowpass SDMs
  - The condition in Corollary 12 is not a sub-condition of Theorem 25. Therefore, there exists some initial conditions such that Corollary 12 is satisfied, while Theorem 25 is not satisfied. Hence, the SDM could exhibit fixed point or limit cycle behavior for some initial conditions, while the trajectory does not converge to the fixed point or limit cycle for other initial conditions.

# Part 3:

## Conclusions

# Conclusions

- The necessary and sufficient conditions for the occurrence of fixed point or limit cycle of bandpass SDMs are derived.
- The periodicity of the output sequences, the stability of these elliptical trajectories and the admissible set of periodic output sequences of bandpass SDMs are discussed.

# Conclusions

- The phenomenon that the ellipses or elliptic fractal regions are the global attractors of the second order marginally stable bandpass SDM with sum of the numerator and denominator polynomials of the loop filter equal to one has been analytically explained and proved.

# Conclusions

- The trajectory of the second order marginally stable bandpass SDM with sum of the numerator and denominator polynomials of the loop filter equal to one will always converge to limit cycle and fractal pattern will not occur if the input step size is larger than a certain value or the natural frequency of the loop filter is a rational multiple of $\pi$.

# Conclusions

- The conditions for the occurrence of the trapezoids in bandpass SDMs are also given.
- Fractal patterns may also be exhibited in the phase plane when the system matrices of bandpass SDMs are strictly stable. This occurs when the sets of initial conditions corresponding to convergent or limit cycle behavior do not cover the whole phase plane.

# Conclusions

- When the leading coefficient of the numerator of the loop filter of bandpass SDMs is positive, the limit cycle (if it exists) will become unstable, and the global attractor region does not exist. Hence, the trajectory will diverge for some initial conditions.
- The growth is linear and the spectrum of the output sequence has a pole at the natural frequency of the loop filter. This result can be used to distinguish the spectra of elliptic fractal patterns confined in the trapezoids or irregular chaotic patterns from that of the divergent patterns.

# Conclusions

- The global stability criterion of high order lowpass SDMs is investigated. The global stability of SDMs does not only depend on the existence of an invariant set or whether the range of the system mapping form a partition on the invariant set or not, but it also depends on the size of the hyperplane formed from the projection of the invariant set onto its first co-ordinate (based on the direct form realization), and the sign of the product of the leading filter coefficients in the numerator and that in the denominator as well.

# Part 4:

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# Part 5:

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# Part 6:

## Questions and Answers

# Q & A