WHEELED PROPS IN ALGEBRA, GEOMETRY AND QUANTIZATION

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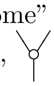
Abstract. Wheeled props is one the latest species found in the world of operads and props. We attempt to give an elementary introduction into the main ideas of the theory of wheeled props for beginners, and also a survey of its most recent major applications (ranging from algebra and geometry to deformation theory and Batalin-Vilkovisky quantization) which might be of interest to experts.

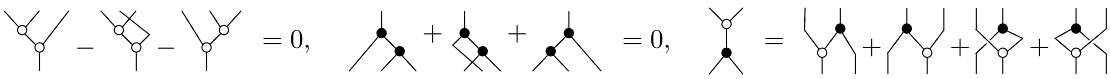
1. Introduction

Theory of operads and props undergoes a rapid development in recent years; its applications can be seen nowadays almost everywhere — in algebraic topology, in homological algebra, in differential geometry, in non-commutative geometry, in string topology, in deformation theory, in quantization theory etc. The theory demonstrates a remarkable unity of mathematics; for example, one and the same *operad of little 2-disks* solves the recognition problem for based 2-loop spaces in algebraic topology, describes homotopy Gerstenhaber structure on the Hochschild deformation complex in homological algebra, and also controls diffeomorphism invariant Hertling-Manin's integrability equations in differential geometry!

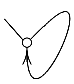
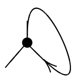
First examples of operads and props were constructed in 1960s in the classical papers of Gerstenhaber on deformation theory of algebras and of Stasheff on homotopy theory of loop spaces. The notion of prop was introduced by MacLane already in 1963 as a useful way to code axioms for operations with many inputs and outputs. The notion of operad was ultimately coined 10 years later by P.May through axiomatization of properties of earlier discovered *associahedra* polytopes and the associated A_∞*-spaces* by Stasheff and of the *little cubes operad* by Boardman and Fogt.

In this paper we attempt to explain the main ideas and constructions of the theory of wheeled operads and props and illustrate them with some of the most recent applications [Gr1, Gr2, Me1-Me7, MMS, MeVa, Mn, Str1, Str2] to geometry, deformation theory and Batalin-Vilkovisky quantization formalism of theoretical physics. In the heart of these applications lies a fact that some categories of local geometric and theoretical physics structures can be identified with the derived categories of surprisingly simple algebraic structures. The language of graphs is essential for the proof of this fact and permits us to reformulate it as follows: solution spaces of several important highly non-linear differential equations in geometry and physics are controlled by (wheeled) props which are resolutions of very compact graphical data, a kind of "genome"; for example, the "genome" of the species *local Poisson structures* is the prop of Lie 1-bialgebras built from two "genes", 

and , subject to the following engineering rules (see §2 for precise details),

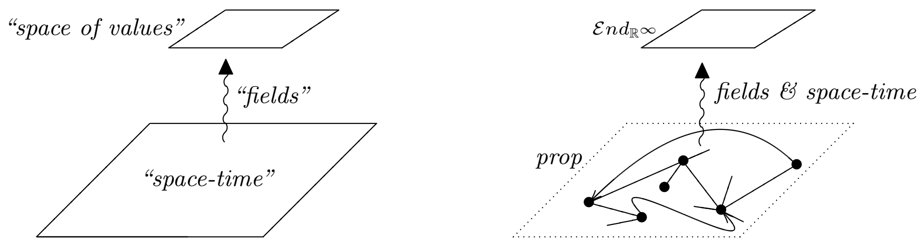


We shall explain how a slight modification of the above rules by addition of two extra conditions,

 = 0 and  = 0, , changes the resulting "species" dramatically: instead of the category

of local Poisson structures one gets the category of *quantum BV manifolds with split quasi-classical limit* which, for example, naturally emerges in the study of quantum master equations [BaVi, Sc] for BF-type quantum field theories (see §5 for precise details). Moreover, in the homotopy theory sense, this category is as perfect as, for example, the nowadays famous category of $\mathcal{L}ie_\infty$-algebras: quasi-isomorphisms of quantum BV manifolds turn out to be equivalence relations.

It is yet to see how non-trivial topology can be incorporated into the current pro(p)file of *local* differential geometry, but it is worth stressing already now that this approach to geometry and physics turns space-time — "the background of everything" — into an ordinary observable, a certain function (representation) on a prop and hence unveils a possibility for a new architecture:



Classical architecture:
a space-time is the fundamental
background for geometric structures

A new architecture of geometry and physics:
a prop is the fundamental background
for both a space-time and structures

In fact, some elements of this architecture have been envisaged long ago by Roger Penrose [Pe] in his "abstract index calculus".

The paper is organized as follows. In section 2 we give a short but self-contained introduction into the theory of (wheeled) operads and props. Sections 3 and 4 aim to give an account of most recent applications of that theory to geometry and, respectively, deformation theory. In Section 5 we explain some ideas of Koszul duality theory and its relation to the homotopy transfer formulae and Batalin-Vilkovisky formalism.

A few words about notations. The symbol $\mathbb{S}_n$ stands for the permutation group, i.e. for the group of all bijections, $[n] \to [n]$, where $[n]$ denotes (here and everywhere) the set $\{1, 2, \ldots, n\}$. Given a partition, $[n] = I_1 \sqcup \ldots \sqcup I_k$, the symbol $\sigma(I_1, \ldots, I_k)$ denotes the sign of the permutation $[n] \to \{I_1, \ldots, I_k\}$. If $V = \oplus_{i\in\mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$. We work throughout over a field $\mathbb{K}$ of characteristic 0.

2. An introduction to operads, dioperads, properads and props

2.1. **Directed graphs.** Let m and n be arbitrary non-negative integers. A *directed* (m,n)*-graph* is a triple (G, f_{in}, f_{out}), where G is a finite 1-dimensional CW complex whose every 1-dimensional cell ("edge") is oriented ("directed"), and

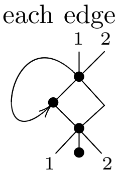
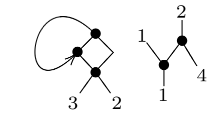
$$f_{in} : [m] \to \left\{\begin{array}{c}\text{the set of all 0-cells, } v\text{, of } G\\ \text{which have precisely one}\\ \text{adjacent edge directed from } v\end{array}\right\}, \quad f_{out} : [n] \to \left\{\begin{array}{c}\text{the set of all 0-cells, } v\text{, of } G\\ \text{which have precisely one}\\ \text{adjacent edge directed towards } v\end{array}\right\}$$

are injective maps of finite sets (called *labelling maps* or simply *labellings*) such that $\operatorname{Im} f_{in} \cap \operatorname{Im} f_{out} = \emptyset$. The set, $\mathfrak{G}^{\circlearrowright}(m,n)$, of all possible directed (m,n)-graphs carries an action, $(G, f_{in}, f_{out}) \to (G, f_{in} \circ \sigma^{-1}, f_{out} \circ \tau)$, of the group $\mathbb{S}_m \times \mathbb{S}_n$ (more precisely, the *right* action of $\mathbb{S}_m^{op} \times \mathbb{S}_n$ but we declare this detail implicit from now). We often abbreviate a triple (G, f_{in}, f_{out}) to G. For any $G \in \mathfrak{G}^{\circlearrowright}(m,n)$ the set,

$$V(G) := \{\text{all 0-cells of } G\} \setminus \{\operatorname{Im} f_{in} \cup \operatorname{Im} f_{out}\},$$

of all unlabelled 0-cells is called the set of *vertices* of G. The edges attached to labelled 0-cells, i.e. the ones lying in $\operatorname{Im} f_{in}$ or in $\operatorname{Im} f_{out}$ are called *incoming* or, respectively, *outgoing legs* of the graph G. The set

$$E(G) := \{\text{all 1-cells of } G\} \setminus \{\text{legs}\},$$

is called the set of *(internal) edges* of G. Legs and edges of G incident to a vertex $v \in V(G)$ are often called *half-edges* of v; the set of half-edges of v splits naturally into two disjoint sets, In_v and Out_v, consisting of incoming and, respectively, outgoing half-edges. In all our pictures the vertices of a graph will be denoted by bullets, the edges by intervals (or sometimes curves) connecting the vertices, and legs by intervals attached from one side to vertices. A choice of orientation on an edge or a leg will be visualized by as a choice of a particular direction (arrow) on the associated interval/curve; unless otherwise explicitly shown the direction of each edge in all our pictures is assumed to go *from bottom to the top*. For example, the graph  $\in \mathfrak{G}^{\circlearrowright}(2,2)$ has four vertices, four legs and five edges; the orientation of all legs and of four internal edges is *not* shown explicitly and hence, by default, flows *upwards*. Sometimes we skip showing explicitly labellings of legs (as in Table 1, for example). We set $\mathfrak{G}^{\circlearrowright} := \sqcup_{m,n\geq 0}\mathfrak{G}^{\circlearrowright}(m,n)$. Note that elements of $\mathfrak{G}^{\circlearrowright}$ are not necessarily connected, e.g.  $\in \mathfrak{G}^{\circlearrowright}(2,4)$.

2.2. **Decorated directed graphs.** Let E be an $\mathbb{S}$-*bimodule*, that is, a family, $\{E(p,q)\}_{p,q\geq 0}$, of vector spaces on which the group $\mathbb{S}_p$ acts on the left and the group $\mathbb{S}_q$ acts on the right, and both actions commute with each other. We shall use elements of E to decorate vertices of an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ as follows. First, for each vertex $v \in V(G)$ we construct a vector space

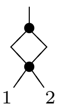
$$E(Out_v, In_v) := \langle Out_v\rangle \otimes_{\mathbb{S}_p} E(p,q) \otimes_{\mathbb{S}_q} \langle In_v\rangle,$$

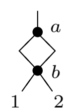
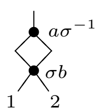
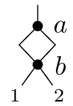
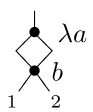
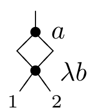
where $\langle Out_v\rangle$ (resp., $\langle In_v\rangle$) is the vector space spanned by all bijections $[\# Out_v] \to Out_v$ (resp., $In_v \to [\# In_v]$). It is (non-canonically) isomorphic to $E(p,q)$ as a vector space and carries natural actions of the automorphism groups of the sets Out_v and In_v. These actions make the following *unordered tensor product* over the set $V(G)$ (of cardinality, say, k),

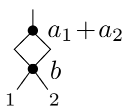
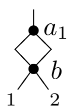
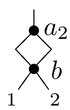
$$\bigotimes_{v\in V(G)} E(Out_v, In_v) := \left(\bigoplus_{i:[k]\to V(G)} E(Out_{i(1)}, In_{i(1)}) \otimes \ldots \otimes E(Out_{i(k)}, In_{i(k)})\right)_{\mathbb{S}_k},$$

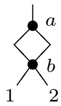
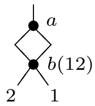
into a representation space of the automorphism group, $Aut(G)$, of the graph G which, by definition, is the subgroup of the symmetry group of the 1-dimensional CW-complex underlying the graph G which fixes its legs. Hence with an arbitrary graph $G \in \mathfrak{G}^{\circlearrowright}$ and an arbitrary $\mathbb{S}$-bimodule E one can associate a vector space,

$$G\langle E\rangle := \left(\otimes_{v\in V(G)} E(Out_v, In_v)\right)_{AutG},$$

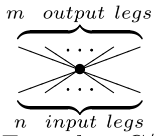
whose elements are called *decorated (by E) graphs*. For example, the automorphism group of the graph $G =$  is $\mathbb{Z}_2$ so that $G\langle E\rangle = E(1,2)\otimes_{\mathbb{Z}_2} E(2,2)$. It is useful to think of an element in $G\langle E\rangle$ as of the graph G whose vertices are literarily decorated by some elements $a \in E(1,2)$ and $b \in E(2,1)$ and subject to the following relations,

 $=$  $\sigma \in \mathbb{Z}_2$, $\lambda\Bigg($  $\Bigg) =$  $=$  $\forall \lambda \in \mathbb{K}$,

 $=$  $+$  and similarly for b.

It also follows from the definition that  $=$  , $(12) \in \mathbb{Z}_2$.

2.2.1. *Remark.* If $E = \{E(p,q)\}$ is a *differential graded* (dg, for short) $\mathbb{S}$-bimodule, i.e. if each vector space $E(p,q)$ is a complex equipped with an $\mathbb{S}_p \times \mathbb{S}_q$-equivariant differential δ, then, for any graph $G \in \mathfrak{G}^{\circlearrowright}(m,n)$, the associated graded vector space $G\langle E\rangle$ comes equipped with an induced $\mathbb{S}_m \times \mathbb{S}_n$-equivariant differential δ_G so that the collection, $\{\bigoplus_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} G\langle E\rangle\}_{m,n\geq 0}$, is again a *dg* $\mathbb{S}$-bimodule. We sometimes abbreviate δ_G to δ.

2.2.2. *Remark.* The one vertex graph $\mathfrak{C}_{m,n}$:= 

$\in \mathfrak{G}^{\circlearrowright}(m,n)$ is often called *the* (m,n)*-corolla*. It is clear that for any $\mathbb{S}$-bimodule E one has $G\langle E\rangle = E(m,n)$.

2.3. **Wheeled props.** A *wheeled prop* is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps,

$$\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n)}\,, \quad m,n\geq 0,$$

parameterized by elements $G \in \mathfrak{G}^{\circlearrowright}$, which satisfy a "three-dimensional" associativity condition,

$$\mu_G = \mu_{G/H} \circ \mu'_H, \tag{1}$$

for any subgraph $H \subset G$. Here G/H is the graph obtained from G by shrinking the whole subgraph H into a single internal vertex, and $\mu'_H : G\langle E\rangle \to (G/H)\langle E\rangle$ stands for the map which equals μ_H on the decorated vertices lying in H and which is identity on all other vertices of G.

If the $\mathbb{S}$-bimodule $\mathcal{P}$ underlying a wheeled prop has a differential δ satisfying, for any $G \in \mathfrak{G}^{\circlearrowright}$, the condition $\delta \circ \mu_G = \mu_G \circ \delta_G$, then the wheeled prop $\mathcal{P}$ is called *differential*.

By Remark 2.2.2, the values of the maps μ_G can be identified with decorated corollas, and hence the maps themselves can be visually understood as *contraction* maps, $\mu_{G\in\mathfrak{G}^{\circlearrowright}(m,n)} : G\langle\mathcal{P}\rangle \to \mathfrak{C}_{m,n}\langle\mathcal{P}\rangle$, contracting all the edges and vertices of G into a single vertex.

2.3.1. *Remark.* Strictly speaking, the notion introduced in § 2.3 should be called a wheeled prop *without unit*. A wheeled prop *with unit* can be defined as in §2.1.1 provided one enlarges $\mathfrak{G}^{\circlearrowright}$ by adding a family of graphs, $\{\uparrow\uparrow\cdots\uparrow\circlearrowright\circlearrowright\cdots\circlearrowright\}$, *without vertices* [MMS].

2.4. **Props, properads, operads, etc. as $\mathfrak{G}$-algebras.** Let $\mathfrak{G} = \sqcup_{m,n}\mathfrak{G}(m,n)$ be a subset of the set $\mathfrak{G}^{\circlearrowright}$, say, one of the subsets defined in Table 1 below. A subgraph H of a graph $G \in \mathfrak{G}$ is called *admissible* if $H \in \mathfrak{G}$ and $G/H \in \mathfrak{G}$, i.e. a contraction of a graph from $\mathfrak{G}$ by a subgraph belonging to $\mathfrak{G}$ gives a graph which again belongs to $\mathfrak{G}$.

A $\mathfrak{G}$-*algebra* is, by definition (cf. § 2.3), an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m \times \mathbb{S}_n$-equivariant maps, $\{\mu_G : G\langle\mathcal{P}\rangle \to \mathcal{P}(m,n)\}_{G\in\mathfrak{G}^{\circlearrowright}(m,n)}$, parameterized by elements $G \in \mathfrak{G}$, which satisfy condition (1) for any admissible subgraph $H \subset G$. Applying this idea to the subfamilies $\mathfrak{G} \subset \mathfrak{G}^{\circlearrowright}$ from Table 1 gives us, in the chronological order, the notions of *prop, operad, dioperad, properad,* $\frac{1}{2}$*-prop* and their *wheeled* versions which have been introduced, respectively, in the papers [Mc, May, Ga, Va1, Ko1, Me5, MMS].

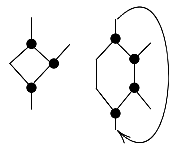
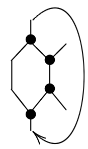
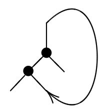
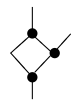
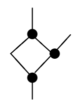
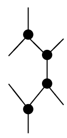
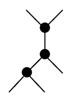
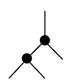
We leave it to the reader as an exercise to check that $\mathfrak{G}^{|}$-algebra structures on an $\mathbb{S}$-bimodule E with only $E(1,1)$ non-zero are precisely associative algebra structures on $E(1,1)$. This fact implies that, for any $\mathfrak{G}$-algebra $E = \{E(m,n)\}_{m,n\geq 0}$, the space $E(1,1)$ is an associative algebra.

2.5. **Basic examples of $\mathfrak{G}$-algebras.** (i) For any $\mathfrak{G}$ and any finite-dimensional vector space V the $\mathbb{S}$-bimodule $\mathcal{E}nd_V = \{\mathrm{Hom}(V^{\otimes n}, V^{\otimes m})\}$ is naturally a $\mathfrak{G}$-algebra with contraction maps $\mu_{G\in\mathfrak{G}}$ being ordinary compositions and, possibly, traces of linear maps; it is called the *endomorphism* $\mathfrak{G}$*-algebra of* V. In the cases $\mathfrak{G} \neq \mathfrak{G}^{\circlearrowright}, \mathfrak{G}^{\circlearrowright}_c$ the assumption on finite-dimensionality of V can be dropped (as the defining operations μ_G do not employ traces).

(ii) With any $\mathbb{S}$-bimodule, $E = \{E(m,n)\}$, there is associated another $\mathbb{S}$-bimodule, $\mathcal{F}^{\mathfrak{G}}\langle E\rangle = \{\mathcal{F}^{\mathfrak{G}}\langle E\rangle(m,n)\}$ with $\mathcal{F}^{\mathfrak{G}}\langle E\rangle(m,n) := \bigoplus_{G\in\mathfrak{G}(m,n)} G\langle E\rangle$, which has a natural $\mathfrak{G}$-algebra structure with the contraction maps μ_G being tautological. The $\mathfrak{G}$-algebra $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ is called *the free $\mathfrak{G}$-algebra generated by the $\mathbb{S}$-bimodule E*. We often abbreviate notations by replacing $\mathcal{F}^{\mathfrak{G}^{\circlearrowright}}$ by $\mathcal{F}^{\circlearrowright}$, $\mathcal{F}^{\mathfrak{G}^{\curlywedge}}$ by $\mathcal{F}^{\curlywedge}$, etc.

(*iii*) Definitions of $\mathfrak{G}$-*sub*algebras, $\mathcal{Q} \subset \mathcal{P}$, of $\mathfrak{G}$-algebras, of their ideals, $\mathcal{I} \subset \mathcal{P}$, and the associated quotient $\mathfrak{G}$-algebras, $\mathcal{P}/\mathcal{I}$, are straightforward. We omit the details.

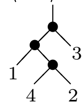
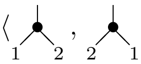
Table 1: A list of $\mathfrak{G}$-algebras

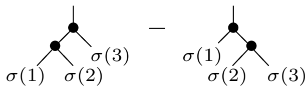
$\mathfrak{G}$	**Definition**	**$\mathfrak{G}$-algebra**	**Typical examples**
$\mathfrak{G}^{\circlearrowright}$	All possible directed graphs	Wheeled prop	
$\mathfrak{G}^{\circlearrowright}_c$	A subset $\mathfrak{G}^{\circlearrowright}_c \subset \mathfrak{G}^{\circlearrowright}$ consisting of all *connected* graphs	Wheeled properad	
$\mathfrak{G}^{\circlearrowright}_{oper}$	A subset $\mathfrak{G}^{\circlearrowright}_{oper} \subset \mathfrak{G}^{\circlearrowright}_c$ consisting of graphs whose vertices have at most one output leg	Wheeled operad	
$\mathfrak{G}^{\uparrow}$	A subset $\mathfrak{G}^{\uparrow} \subset \mathfrak{G}^{\circlearrowright}$ consisting of graphs with no *wheels*, i.e. with no directed closed paths of edges	Prop	
$\mathfrak{G}^{\uparrow}_c$	A subset $\mathfrak{G}^{\uparrow}_c \subset \mathfrak{G}^{\uparrow}$ consisting of all *connected* graphs	Properad	
$\mathfrak{G}^{\uparrow}_{c,0}$	A subset $\mathfrak{G}^{\uparrow}_{c,0} \subset \mathfrak{G}^{\uparrow}_c$ consisting of graphs of genus zero	Dioperad	
$\mathfrak{G}^{\frac{1}{2}}$	A subset $\mathfrak{G}^{\frac{1}{2}} \subset \mathfrak{G}^{\uparrow}_{c,0}$ consisting of all (m,n)-graphs with the number of directed paths from input legs to the output legs equal to mn	$\frac{1}{2}$-Prop	
$\mathfrak{G}^{\curlywedge}$	A subset $\mathfrak{G}^{\curlywedge} \subset \mathfrak{G}^{\uparrow}_{c,0}$ consisting of graphs whose vertices have precisely one output leg	Operad	
$\mathfrak{G}^{\mid}$	A subset $\mathfrak{G}^{\mid} \subset \mathfrak{G}^{\curlywedge}$ consisting of graphs whose vertices have precisely one input leg	Associative algebra	

2.6. **Morphisms and resolutions of $\mathfrak{G}$-algebras.** A morphisms of $\mathfrak{G}$-algebras, $\rho : \mathcal{P}_1 \to \mathcal{P}_2$, is a morphism of the underlying $\mathbb{S}$-bimodules such that, for any graph G, one has $\rho \circ \mu_G = \mu_G \circ (\rho^{\otimes G})$, where $\rho^{\otimes G}$ is a map, $G\langle\mathcal{P}_1\rangle \to G\langle\mathcal{P}_2\rangle$, which changes decorations of each vertex in G in accordance with ρ. A morphism of $\mathfrak{G}$-algebras, $\mathcal{P} \to \mathcal{E}nd\langle V\rangle$, is called a *representation* of the $\mathfrak{G}$-algebra $\mathcal{P}$ in a graded vector space V.

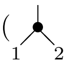
A *free resolution* of a dg $\mathfrak{G}$-algebra $\mathcal{P}$ is, by definition, a dg free $\mathfrak{G}$-algebra, $(\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta)$, together with a morphism, $\pi : (\mathcal{F}\langle E\rangle, \delta) \to \mathcal{P}$, which induces a cohomology isomorphism. If the differential δ in $\mathcal{F}\langle \mathcal{E}\rangle$ is decomposable with respect to compositions μ_G, then it is called a *minimal model* of $\mathcal{P}$ and is often denoted by $\mathcal{P}_\infty$.

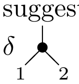
3. Applications to algebra and geometry

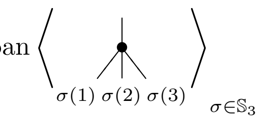
3.1. **Operad of associative algebras.** Let $A_0 = \{A_0(m,n)\}$ be an $\mathbb{S}$-bimodule with all $A_0(m,n) = 0$ except $A_0(1,2) := \mathbb{K}[\mathbb{S}_2]$. The associated free operad $\mathcal{F}^{\curlywedge}\langle A_0\rangle$ can be identified with the vector space spanned by all connected planar graphs of the form  . In particular, $\mathcal{F}^{\curlywedge}\langle A_0\rangle(1,2) \cong A_0(1,2) \cong$ span⟨  ⟩. Let I_0 be an ideal of $\mathcal{F}^{\curlywedge}\langle A_0\rangle$ generated by the following 6 planar graphs,

(2)  $\in \mathcal{F}^{\curlywedge}\langle A_0\rangle(1,3), \ \forall\, \sigma \in \mathbb{S}_3.$

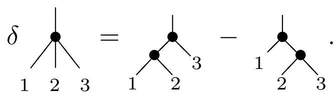
3.1.1. **Claim.** *There is a 1-1 correspondence between representations, $\rho : \mathcal{A}ss \to \mathcal{E}nd_V$, of the quotient operad, $\mathcal{A}ss := \mathcal{F}^{\curlywedge}\langle A_0\rangle/\langle I_0\rangle$, in a space V and associative algebra structures on V.*

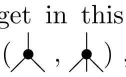
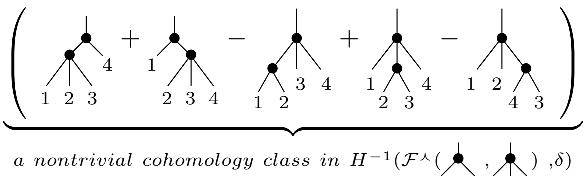
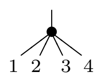
Proof. The values of ρ on arbitrary (equivalence classes of) planar graphs is uniquely determined by its value, $\rho($  $) \in \mathrm{Hom}(V^{\otimes 2}, V)$, on one of the two generators. Denote this value by μ. As ρ sends any of the graphs (2) to zero, the multiplication in V given by μ must be associative. □

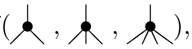
Thus the operad $\mathcal{A}ss$ can be called the *operad of associative algebras*. What could be a (minimal) free resolution of $\mathcal{A}ss$? By definition in § 2.6, this must be a *free* operad, $\mathcal{F}^{\curlywedge}\langle A\rangle$, generated by some $\mathbb{S}$-bimodule $A = \{A(1,n)\}_{n\geq 2}$ equipped with a differential δ and a projection $\pi : \mathcal{F}^{\curlywedge}\langle A\rangle \to \mathcal{A}ss$ inducing an isomorphism, $H(\mathcal{F}^{\curlywedge}\langle A\rangle, \delta) = \mathcal{A}ss$, at the cohomology level. The latter condition suggests that we can choose $A(1,2)$ to be identical to $A_0(1,2)$ and set a differential δ to satisfy δ  $= 0$. Then graphs (2) are cocycles in $\mathcal{F}^{\curlywedge}\langle A\rangle(1,3)$. In view of the cohomology isomorphism $\mathcal{F}^{\curlywedge}\langle A\rangle \to \mathcal{A}ss$, we have to make them coboundaries, and hence are forced to introduce an $\mathbb{S}_3$-module,

$$A(1,3) := \mathbb{K}[\mathbb{S}_3][1] = \mathrm{span}$$  ,

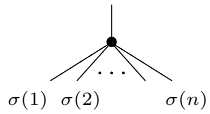
and set

(3) 

We get in this way a well-defined dg *free* operad together with a well-defined epimorphism, $\left(\mathcal{F}^{\curlywedge}\right.$  $\left., \delta\right) \longrightarrow (\mathcal{A}ss, 0)$, sending $(1,3)$-corolla to zero. However, this epimorphisms fails to be a quasi-isomorphism as δ  $= 0$. To kill this cohomology class we have to introduce a new generating $(1,4)$-corolla,  , of degree -2 and set the value of the differential on it to be equal to the underbraced expression above. Again we get a well-defined dg free operad together with a natural homomorphism,

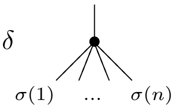
$\Big(\mathcal{F}($  $),\delta\Big) \to \mathcal{A}ss$, which, again, fails to be a quasi-isomorphism. To treat the new problem one has to introduce a new generating corolla of degree -3 with 5 input legs and so on.

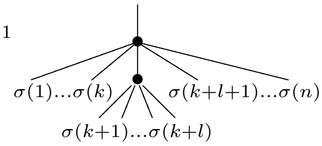
3.1.2. **Theorem** [Sta]**.** *The minimal resolution of $\mathcal{A}ss$ is a dg free operad, $\mathcal{A}ss_\infty := (\mathcal{F}^{\curlywedge}\langle A\rangle, \delta)$, generated by the $\mathbb{S}$-bimodule $A = \{A(1,n)\}$,*

(4) $$A(1,n) := \mathbb{K}[\mathbb{S}_n][n-2] = \text{span}\left\langle \right.$$ 

$$\left.\right\rangle_{\sigma\in\mathbb{S}_n},$$

and with the differential given on the generators as

(5) δ 

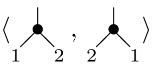
$$= \sum_{k=0}^{n-2}\sum_{l=2}^{n-k} (-1)^{k+l(n-k-l)+1}$$ 

.

3.1.3. **Definition.** Representations, $\mathcal{A}ss_\infty \to \mathcal{E}nd_V$, of the dg operad $(\mathcal{A}ss_\infty, \delta)$ in a dg vector space V are called *A_∞-structures* in V.

3.1.4. **Remark.** We now suggest the reader to re-read Stasheff's Theorem 3.1.2 from the end to the beginning: *given an infinite dimensional graph complex, $(\mathcal{A}ss_\infty, \delta)$, spanned by all possible planar graphs (without wheels) built from $(1,n)$-corollas with $n \geq 2$ and equipped with differential (5), then its cohomology, $H(\mathcal{A}ss_\infty, \delta)$, is generated by only $(1,2)$-corollas*, i.e. it is surprisingly small. It is often impossible to obtain such a result by a direct computation. One of the main theorem-proving technique in the theory of operads and props is called the *Koszul duality theory*, and a result of type (3.1.2) often requires a combination of ideas from homological algebra, algebraic topology, the theory of Cohen-Macaulay posets [Va2] and so on. Stasheff [Sta] proved Theorem 3.1.2 by constructing a remarkable family of polytopes called nowadays *associahedra*; in his approach the surprising smallness of $H(\mathcal{A}ss_\infty, \delta)$ gets nicely explained by the obvious contractibility of Stasheff's polytopes as topological spaces. We shall review some of theorem-proving techniques in §5 and continue this section with a list of examples which are most relevant to differential geometry.

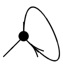
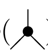
3.2. **Wheeled operad of finite-dimensional associative algebras.** Theorem 3.1.2 has been obtained in the category of algebras over the family of graphs, $\mathfrak{G}^{\curlywedge}$, which contain *no* closed directed paths of internal edges. What happens if we keep the *same* family of generators as in the case of $\mathcal{A}ss$,

$$A_0(m,n) = \begin{cases} \mathbb{K}[\mathbb{S}_2] = \text{span}\langle \ \cdots \ \rangle & \text{for } m=1, n=2 \\ 0 & \text{otherwise} \end{cases}$$ 

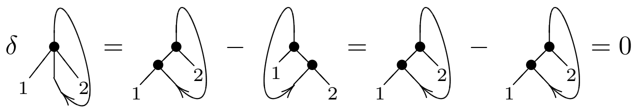
the *same* family of relations (1), but enlarge the family of graphs we work over from $\mathfrak{G}^{\curlywedge}$ to $\mathfrak{G}^{\circlearrowright}$? The associated quotient wheeled operad, $\mathcal{A}ss^{\circlearrowright} := \mathcal{F}^{\circlearrowright}\langle A_0\rangle/\langle I_0\rangle$, can be called the *operad of finite-dimensional associative algebras*. Indeed, one has the following

3.2.1. **Claim.** *There is a 1-1 correspondence between representations, $\rho : \mathcal{A}ss^{\circlearrowright} \to \mathcal{E}nd_V$, of $\mathcal{A}ss^{\circlearrowright}$ in a finite-dimensional vector space V and associative algebra structures on V.*

Proof. We need to explain only the subjective *finite-dimensional*, and that follows from the fact that representations of graphs $\in \mathcal{A}ss^{\circlearrowright}$ which have wheels involve traces. For example, the element

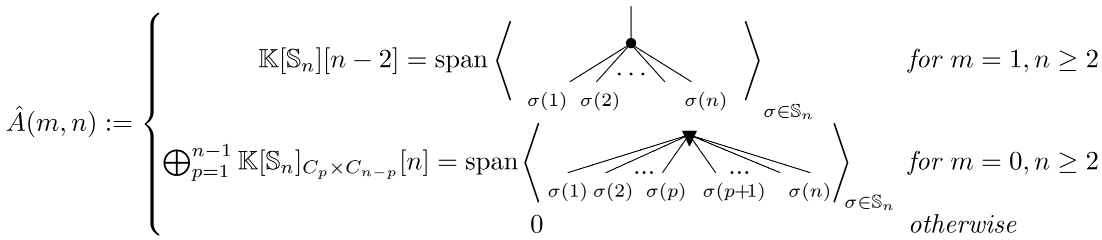
 $\in \mathcal{A}ss^{\circlearrowright}(0,1)$ gets represented in V as the image of the multiplication map $\rho($  $) \in \text{Hom}(V \otimes V, V)$ under a natural trace map $\text{Hom}(V \otimes V, V) \to \text{Hom}(V, \mathbb{K})$. □

It is easy to see that the straightforward analogue of Theorem 3.1.2 can *not* hold true for the operad of *finite-dimensional* associative algebras as, for example, formula (3) implies

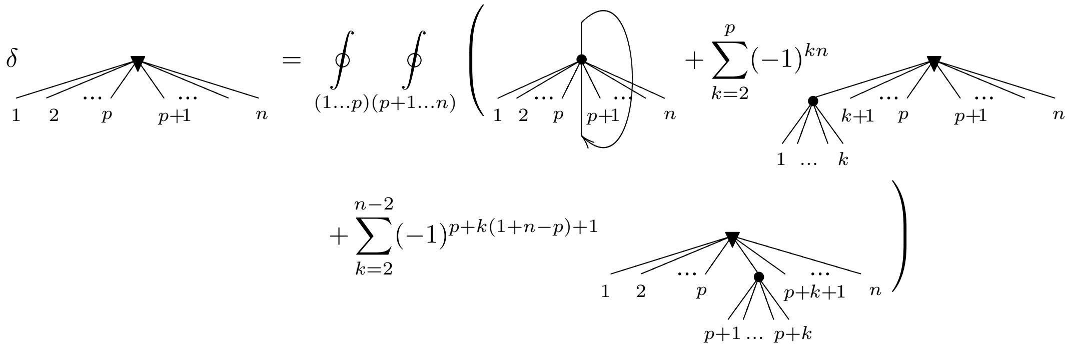


and whence provides us with a non-trivial cohomology class in $H^{-1}(\mathcal{F}^{\circlearrowright}\langle A\rangle, \delta)$ which maps under the natural projection $\mathcal{F}^{\circlearrowright}\langle A\rangle \to \mathcal{A}ss^{\circlearrowright}$ to zero. The correct analogue of Stasheff's result for finite-dimensional associative algebras was found in [MMS].

3.2.2. **Theorem.** *The minimal resolution of $\mathcal{A}ss^{\circlearrowright}$ is a dg free wheeled operad, $(\mathcal{A}ss^{\circlearrowright})_\infty := (\mathcal{F}^{\circlearrowright}\langle\hat{A}\rangle, \delta)$ generated by an $\mathbb{S}$-bimodule $\hat{A} = \{\hat{A}(m,n)\}$,*



where $C_p \times C_{n-p}$ is the subgroup of $\mathbb{S}_n$ generated by two commuting cyclic permutations $\zeta := (12\dots p)$ and $\xi := (p+1\dots n)$, and $k[\mathbb{S}_n]_{C_p\times C_{n-p}}$ stands for coinvariants.
The differential is given on the generators of $\hat{A}(1,n)$ by (5) and one the generators of $\hat{A}(0,n)$ by



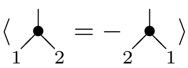
where the symbol $\oint_{(i_1\dots i_k)}$ stands for the cyclic skewsymmetrization of the indices $(i_1\dots i_k)$.

Thus the minimal resolution, $(\mathcal{A}ss^{\circlearrowright})_\infty$, of the operad of finite-dimensional associative algebras is different from the naive "wheelification", $(\mathcal{A}ss_\infty)^{\circlearrowright}$, of the Stasheff's minimal resolution of the operad, $\mathcal{A}ss$, of arbitrary associative algebras. A similar phenomenon occurs for the operad of commutative algebras [MMS]. In contrast, the operad, $\mathcal{L}ie$, of Lie algebras is rigid with respect to the wheelification:

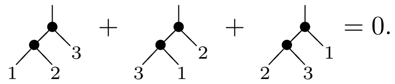
3.2.3. **Fact** [Me5]**.** $(\mathcal{L}ie^{\circlearrowright})_\infty = (\mathcal{L}ie_\infty)^{\circlearrowright}$, i.e. wheeled L_∞-algebras are exactly the same as ordinary finite-dimensional L_∞-algebras.

3.2.4. **Reminder on L_∞-algebras and their homotopy classification.** For future reference we recall here a few useful facts about Lie and L_∞-algebras [Ko2]. The operad, $\mathcal{L}ie$, of Lie algebras is the quotient operad, $\mathcal{L}ie := \mathcal{F}^{\curlywedge}\langle L_0\rangle/I$, of the free operad generated by an $\mathbb{S}$-bimodule $L_0 = \{L_0(m,n)\}$,

(6) $L_0(m,n) = \begin{cases} \mathrm{sgn}_2 = \mathrm{span}\langle \text{(see diagram)} \rangle & \text{for } m = 1, n = 2 \\ 0 & \text{otherwise} \end{cases}$

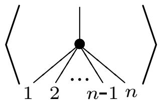


modulo the ideal I generated by the following relations

(7) 

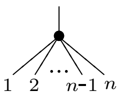
Its minimal resolution, $\mathcal{L}ie_\infty$, is a dg free operad, $\mathcal{F}^{\curlywedge}\langle L\rangle$ generated by an $\mathbb{S}_n$-bimodule

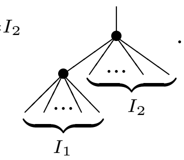
$$L(m,n) := \begin{cases} \mathrm{sgn}_n[n-2] = \mathrm{span}\left\langle \text{(corolla)} \right\rangle & \text{for } m=1, n\geq 2, \\ 0 & \text{otherwise} \end{cases}$$



with the differential given by

(8) $$\delta \;(\text{corolla}) = \sum_{\substack{[n]=I_1\sqcup I_2\\ \#I_1\geq 2, \#I_2\geq 1}} (-1)^{\sigma(I_1,I_2)+(\#I_1+1)\#I_2} \;(\text{tree}) .$$





Here (and elsewhere) sgn_n stands for the 1-dimensional sign representation of $\mathbb{S}_n$.

With an arbitrary graded vector space V one can associate a formal graded manifold, $\mathcal{M}_V$, whose structure sheaf, $\mathcal{O}_{\mathcal{M}_V}$, is, by definition, the completed graded cocommutative coalgebra $\widehat{\odot}(V[1])$; if V is finite dimensional, then one can equivalently view $\mathcal{M}_V$ as a small neighbourhood of zero in the space $V[1]$ equipped with the algebra (rather than coalgebra), $\widehat{\odot}(V^*[-1])$, of ordinary smooth formal functions. It is well known (see, e.g., [Ko2]) that L_∞-structures in a dg space V, that is, representations $\mathcal{L}_\infty \to \mathcal{E}nd_V$, are in one-to-one correspondence with degree 1 vector fields, $\eth$, on $\mathcal{M}_V$ which vanish at the distinguished point, $\eth\,|_{0\in\mathcal{M}_V}=0$, and satisfy the condition $[\eth,\eth]=0$ (such vector fields are called *cohomological*). The pairs $(\mathcal{M}_V,\eth)$ are often called *dg manifolds*. This interpretation of L_∞-structures permits us to use simple and concise geometric instruments to describe notions which, in the pure algebraic translation, look awkwardly large. For example, a *morphism* of L_∞-algebras $V\to W$ is nothing but a smooth map, $f:\mathcal{M}_V\to\mathcal{M}_W$, of the associated formal manifolds such that $f_*(\eth_V)=\eth_W$.

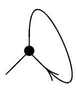
A L_∞ algebra $(\mathcal{M}_V,\eth)$, is called *minimal* if the first Taylor coefficient, $\eth_{(1)}$, of the homological vector field $\eth$ at the distinguished point $0\in\mathcal{M}_V$ vanishes. It is called *linear contractible* if the higher Taylor coefficients $\eth_{(\geq 2)}$ vanish and the first one $\eth_{(1)}$ has trivial cohomology when viewed as a differential in V. According to Kontsevich [Ko2], *any L_∞-algebra (or, better, the associated dg manifold) is isomorphic to the direct product of a minimal and of a linear contractible ones.* This fact implies that quasi-isomorphisms in the category of L_∞-algebras are equivalence relations. A dg manifold is called *contractible* if it is isomorphic to a linear contractible one.

3.3. **Unimodular Lie algebras.**

Many important Lie algebras $\mathfrak{g}$ (e.g., all semisimple Lie algebras) have an additional property that, for any $g\in\mathfrak{g}$, the trace of the associated adjoint action

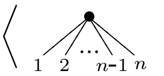
$$\begin{array}{rccc} \mathrm{Ad}_g: & \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ & e & \longrightarrow & [g,e] \end{array}$$

vanishes. Lie algebras with this property are called *unimodular*. The wheeled operad, $\mathcal{UL}ie$, controlling unimodular Lie algebras is the quotient of the free wheeled operad, $\mathcal{F}^{\circlearrowright}\langle L_0\rangle$, generated by the $\mathbb{S}$-bimodule (6) modulo the ideal generated by the Jacobi relations (7) and the unimodularity

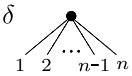
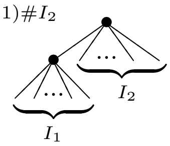
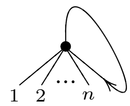
ones,  $=0$. Its minimal resolution has been found in [Gr1]:

3.3.1. **Theorem.** *The operad $\mathcal{ULie}_\infty$ is a dg free operad, $\mathcal{F}^{\circlearrowright}\langle\hat{L}\rangle$ generated by the $\mathbb{S}$-bimodule,*

$$\hat{L}(m,n) := \begin{cases} \mathrm{sgn}_n[n-2] & \text{for } m=1, n\geq 2,\\ \mathrm{sgn}_n[n] = \mathrm{span}\langle \ \ \rangle & \text{for } m=0, n\geq 1,\\ 0 & \text{otherwise}\end{cases}$$

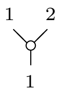
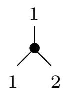


with the differential on the generators of $\hat{L}(1,n)$ given by (8) and on the generators of $\hat{L}(1,n)$ by

δ  $= \sum_{\substack{[n]=I_1\sqcup I_2\\ \#I_1\geq 2, \#I_2\geq 0}} (-1)^{\sigma(I_1,I_2)+(\#I_1+1)\#I_2}$  $+$  .

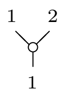
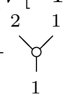
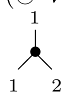
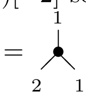
Geometrically, unimodular L_∞-structures in V can be interpreted as pairs, $(\eth,\omega)$, where $\eth$ is a cohomological vector field and ω a Q-invariant section of the Berezinian bundle on $V^*[1]$ (see [Gr1]).

3.4. **Lie 1-bialgebras and Poisson geometry.** A *Lie n-bialgebra* on a graded vector space V is a pair of linear maps,

$\Delta \simeq$  $: V \to V\wedge V, \quad [\bullet] \simeq$  $: \wedge^2(V[-n]) \to V[-n]$

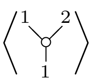
making the space V into a Lie coalgebra and the space $V[-n]$ into a Lie algebra and satisfying, for any $a,b\in V$, the compatibility condition

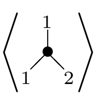
$$\Delta[a\bullet b] = \sum a_1\otimes[a_2\bullet b] + [a\bullet b_1]\otimes b_2 + (-1)^{|a||b|+n|a|+n|b|}([b\bullet a_1]\otimes a_2 + b_1\otimes[b_2\bullet a]),$$

Here $\Delta a =: \sum a_1\otimes a_2$ and $\Delta b =: \sum b_1\otimes b_2$. The case $n=0$ gives the notion of Lie bialgebra which was introduced by Drinfeld [Dr] in the context of quantum groups. The case $n=1$, as we shall see below, is relevant to Poisson geometry. In this case one has $\wedge^2(V[-1]) = (\odot^2 V)[-2]$ so that the basic binary operations have the following symmetries,  $= -$  and  $=$  .

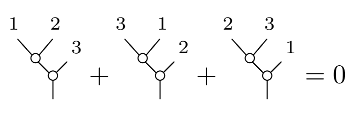
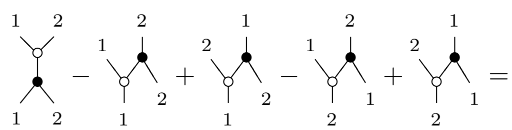
Thus the prop of Lie 1-bialgebras, $\mathcal{L}ie^1\mathcal{B}$, is the quotient of the free prop, $\mathcal{F}^{\uparrow}\langle B\rangle$, generated by an $\mathbb{S}$-bimodule,

(9)
$$B(m,n) := \begin{cases} \mathrm{sgn}_2\otimes \mathbb{1}_1 = \mathrm{span}\langle\ \ \rangle & \text{if } m=2, n=1,\\ \mathbb{1}_1\otimes\mathbb{1}_2[-1] \equiv \mathrm{span}\langle\ \ \rangle & \text{if } m=1, n=2,\\ 0 & \text{otherwise}\end{cases}$$





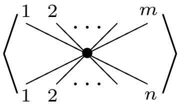
modulo the ideal generated by Jacobi relations (7) and the following ones

(10)  $= 0,$  $= 0.$

Its minimal resolution, $\mathcal{L}ie^1\mathcal{B}_\infty$, has been computed in [Me3].

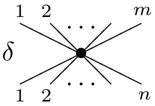
3.4.1. **Theorem.** (i) *$\mathcal{L}ie^1\mathcal{B}_\infty$ is a dg free prop, $\mathcal{F}^{\uparrow}\langle X\rangle$, generated by an $\mathbb{S}$-bimodule,*

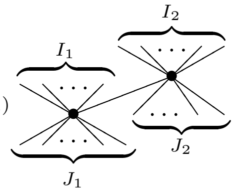
(11) $$X(m,n)[-1] = \mathrm{sgn}_m \otimes 1\!\!1_n[m-2] = \mathrm{span}\left\langle \right\rangle_{m\geq 1, n\geq 1, m+n\geq 3}$$



and with the differential given on the generators as follows,

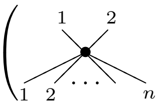
$$\delta \quad = \sum_{\substack{[m]=I_1\sqcup I_2, [n]=J_1\sqcup J_2\\ |I_1|\geq 0, |I_2|\geq 1, |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$$





(ii) *For any $d\in\mathbb{N}$, there is a one-to-one correspondence between representations of the dg prop $\mathcal{L}ie^1\mathcal{B}_\infty$ in $\mathbb{R}^d$ and formal Poisson structures, π, on $\mathbb{R}^d$ vanishing at the origin.*

Proof. The proof of (i) is straightforward (see, e.g., [MeVa, Me5, Va1]) once one uses rather non-straightforward Koszul duality theory for for dioperads, [GiKa, Ga], and Kontsevich's ideas of $\frac{1}{2}$-props and path filtrations [Ko1, MaVo]. We shall discuss some of these ideas in §5 and show here now only the proof of (ii). Since $\mathbb{R}^p$ is concentrated in degree zero, an arbitrary representation $\rho: \mathcal{L}ie^1\mathcal{B}_\infty \to \mathcal{E}nd_{\mathbb{R}^p}$ can have non-zero values only on (m,n)-corollas with $m=2$. Denote these values, $\rho\left(\quad \right) \in \mathrm{Hom}(\odot^n\mathbb{R}^p, \wedge^2\mathbb{R}^p)$, by π_n. As the tangent space, $\mathcal{T}_0$, to $\mathbb{R}^p$ at zero



can be identified with $\mathbb{R}^p$ itself, we can identify the total sum $\pi := \sum_{n\geq 1}\pi_n \in \mathrm{Hom}(\odot^{\geq 1}\mathbb{R}^p, \wedge^2\mathcal{T}_0)$ with a formal bi-vector field on $\mathbb{R}^p$. Then the equation $\rho\circ\delta = \delta\circ\rho$ becomes precisely the Poisson equation, $[\pi,\pi]_S = 0$, where $[\ ,\ ]_S$ are the Schouten brackets. □

It is worth pointing out that the vanishing condition $\pi|_{0\in\mathbb{R}^p} = 0$ in Theorem 3.4.1(ii) is no serious restriction: given an arbitrary formal or analytic Poisson structure π on $\mathbb{R}^p$ (not necessary vanishing at $0\in\mathbb{R}^p$), then, for any parameter λ viewed as a coordinate on $\mathbb{R}$, the product $\lambda\pi$ is a Poisson structure on $\mathbb{R}^{p+1} = \mathbb{R}^p\times\mathbb{R}$ vanishing at zero $0\in\mathbb{R}^{n+1}$ and hence is a representation of the prop $\mathcal{L}ie^1\mathcal{B}_\infty$.

3.4.2. **Bi-Hamiltonian geometry.** The prop profile of a *pair of compatible Poisson structures* (which is an important concept in the theory of integrable systems) has been computed by Strohmayer in [Str2] with the help of an earlier result of Dotsenko and Khoroshkin [DoKh].

3.4.3. **Wheeled Poisson structures?** Theorem 3.4.1 says that the minimal resolution,

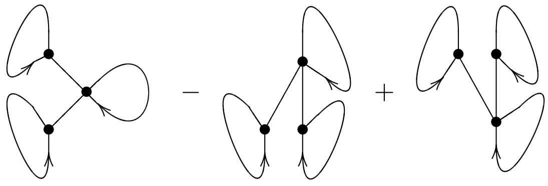
$$\mathcal{L}ie^1\mathcal{B}_\infty = (\mathcal{F}^{\uparrow}\langle X\rangle, \delta),$$

of the prop, $\mathcal{L}ie^1\mathcal{B}$, of arbitrary Lie 1-bialgebras controls the category of local (formal) smooth Poisson structures. What can be said about a minimal resolution, $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$, of the wheeled prop, $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$, of *finite dimensional* Lie 1-bialgebras whose representations can, in view of Theorem 3.4.1(ii), be called *wheeled Poisson structures*? Note that $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ has the same generators and relations as $\mathcal{L}ie^1\mathcal{B}$, the only difference being that graphs now might have wheels. As in the case of associative algebra, the naive wheelification,

$$(\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright} := (\mathcal{F}^{\circlearrowright}\langle X\rangle, \delta),$$

creates new *non-trivial* cohomology classes, as e.g. this one [Me5]

(12)



$$\in (\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright}$$

which map under the natural projection $(\mathcal{L}ie^1\mathcal{B}_\infty)^{\circlearrowright} \to \mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ to zero. Thus the set of generators of a minimal resolution, $(\mathcal{L}ie^1\mathcal{B}^{\circlearrowright})_\infty$, of $\mathcal{L}ie^1\mathcal{B}^{\circlearrowright}$ must be larger than the set (11), and at present its computation is beyond reach. All we can say now about mysterious *wheeled* Poisson structures on a graded formal manifold M is that (i) they are Maurer-Cartan elements of a certain L_∞-algebra extension of the ordinary Schouten bracket on M which involves divergence operators (in fact, graph (12) gives us a glimpse of the μ_3 composition in that L_∞-algebra), and (ii) they can be deformation quantized in exactly the same sense as ordinary Poisson structures; moreover, it is proven in [Me6] with the help of the theory of wheeled props that there exists universal formulae for deformation quantization of wheeled Poisson structures which involve only rational numbers $\mathbb{Q}$.

3.5. **Pre-Lie algebras, Nijenhuis geometry and contractible dg manifolds.** A *pre-Lie* algebra is a vector space together with a binary operation, $\circ : V^{\otimes 2} \to V$, satisfying the condition

$$(a\circ b)\circ c - a\circ(b\circ c) - (-1)^{|b||c|}(a\circ c)\circ b + (-1)^{|b||c|}a\circ(c\circ b) = 0$$

for any $a,b,c \in V$. Any pre-Lie algebra is naturally a Lie algebra with the bracket, $[a,b] := a\circ b - (-1)^{|a||b|}b\circ a$. Let us consider the following extension of this notion: a *pre-Lie*^{2} algebra is a pre-Lie algebra $(V,\circ)$ equipped with a compatible Lie bracket in degree 1, i.e. with a linear map[1] $[\,\bullet\,] : \wedge^2(V[-1]) \to V[-1]$ satisfying the Jacobi identities and the following compatibility condition,

$$[a\bullet b]\circ c + (-1)^{|b|}a\circ[b\bullet c] + (-1)^{|b||a|+|b|}b\circ[a\bullet c] = \\ (-1)^{|b||c|+|c|}[(a\circ c)\bullet b] + (-1)^{(|a|+1)(|b|+|c|)+|a|}[(b\circ c)\bullet a], \quad \forall\, a,b,c\in V$$

This compatibility condition can be understood as follows. The vector space $V\oplus V[-1]$ is naturally a complex with trivial cohomology. If we write elements of $V\oplus V[-1]$ as $a+\Pi b$, where $a,b\in V$ and Π is a formal symbol of degree 1, then the natural differential in $V\oplus V[-1]$ is given by $d(a+\Pi b) = 0+\Pi a$. Given two arbitrary binary operations,

$$\circ : V\otimes V \to V, \quad [\,\bullet\,] : \odot^2 V \to V[1],$$

define a degree zero map, $[\ ,\ ] : \wedge^2(V\oplus V[-1]) \to V\oplus V[-1]$ by setting,

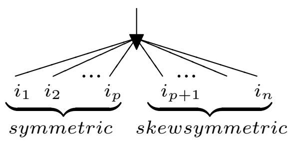
$$[a,b] := a\circ b - (-1)^{|a||b|}b\circ a, \quad [\Pi a, b] := -(-1)^{|a|}[a\bullet b] + \Pi a\circ b, \quad [\Pi a, \Pi b] := \Pi\,[a\bullet b]\,.$$

3.5.1. **Proposition** [Me4]**.** *The data* $(V\oplus V[-1], d, [\ ,\ ])$ *is a (contractible) dg Lie algebra if and only if* $(V,\circ,[\,\bullet\,])$ *is a pre-Lie*^{2} *algebra.*

Rather surprisingly, the minimal resolution, *pre-*$\mathcal{L}ie^2_\infty$, of the operad of pre-Lie^2-algebras has much to do with the famous Nijenhuis integrability condition in differential geometry. The following result is based on the works [ChLi, Me4, Str2].

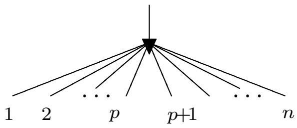
3.5.2. **Theorem.** (i) *The operad pre-*$\mathcal{L}ie^2_\infty$ *is a free operad,* $\mathcal{F}^{\curlywedge}\langle N\rangle$*, generated by an* $\mathbb{S}$*-bimodule* N *with all* $N(m,n)=0$ *except the following ones,*

$$N(1,n) := \bigoplus_{p=1}^{n} \mathrm{Ind}^{\mathbb{S}_n}_{\mathbb{S}_p\times\mathbb{S}_{n-p}} \mathbb{1}_p \otimes \mathrm{sgn}_{n-p}[n-p-1] = \mathrm{span}\left\langle \quad \right\rangle, \quad n\geq 2,$$

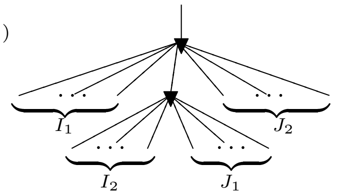


and equipped with a differential given on the generators by

$$d$$

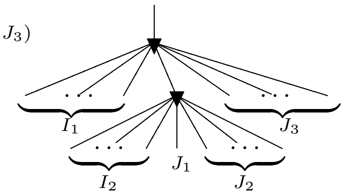


$$= \sum_{\substack{I_1\sqcup I_2=(1,\dots,p)\\ J_1\sqcup J_2=(p+1,\dots,n)\\ \#I_2\geq 1, \#I_1+\#J_2\geq 1\\ \#I_2+\#J_1\geq 2}} (-1)^{\#J_2+\sigma(J_1,J_2)}$$



[1]Equivalently, a linear map $[\,\bullet\,] : \odot^2 V \to V[1]$.

$$- \sum_{\substack{I_1\sqcup I_2=(1,\ldots,n-p)\\ J_1\sqcup J_2\sqcup J_3=(n-p+1,\ldots,n)\\ \#I_1\geq 1, \#I_2\geq 1\\ \#I_1+\#J_3\geq 1, \#I_2+\#J_2\geq 1}} (-1)^{\#J_2+\#J_3+\sigma(J_1,J_2,J_3)}$$



(ii) *For any $d\in\mathbb{N}$, there is a one-to-one correspondence between representations of pre-$\mathcal{L}ie^2_\infty$ in $\mathbb{R}^d$ and endomorphisms, $J:\mathcal{T}_{\mathbb{R}^d}\to\mathcal{T}_{\mathbb{R}^d}$, of the tangent bundle on the affine space $\mathbb{R}^d$ satisfying the Nijenhuis integrability condition, $N_J=0$, and the vanishing condition $J\mid_{0\in\mathbb{R}^d}=0$.*

We recall that the Nijenhuis tensor of an endomorphism, $J:\mathcal{T}_M\to\mathcal{T}_M$, of the tangent bundle of an arbitrary smooth manifold M (in particular, of $\mathbb{R}^m$) can be defined as a map

$$\begin{array}{rccc} N_J: & \wedge^2\mathcal{T}_{\mathbb{R}^m} & \longrightarrow & \mathcal{T}_{\mathbb{R}^m}\\ & X\otimes Y & \longrightarrow & N_J(X,Y):=[JX,JY]+J^2[X,Y]-J[X,JY]-J[JX,Y], \end{array}$$

and that its beauty is hidden in the far from being obvious fact that it is linear not only over $\mathbb{R}$ but also over arbitrary smooth functions, $f\in\mathcal{O}_M$, on M, that is, $N_J(fX,Y)=N_J(X,fY)=fN_J(X,Y)$.

A representation of this dg operad in an arbitrary *graded* vector space V might be called a *graded* or *extended* Nijenhuis structure on V (viewed as a formal manifold). Interestingly, the category of these *extended Nijenhuis manifolds* is almost identical (see [Me4]) to the category of *contractible dg manifolds* which we first met in §3.2.4 when discussing Kontsevich's homotopy classification of dg manifolds. Proposition 3.5.1 above is in fact one of the simplest manifestations of this more general phenomenon.

3.6. **Gerstenhaber algebras and Hertling-Manin geometry.** We conclude this section with an example which was actually the first one to reveal strong interconnections between derived (via minimal resolutions) categories of rather simple algebraic structures and solution sets of highly non-linear *diffeomorphism covariant* differential equations on ordinary smooth manifolds.

A *Gerstenhaber* algebra is, by definition, a graded vector space V together with two linear maps, $\circ:\odot^2V\to V$ and $[\,\bullet\,]:\odot^2V\to V[1]$ such that $(V,\circ)$ is a graded commutative algebra, $(V[-1],[\,\bullet\,])$ is a graded Lie algebra, and the compatibility equation,

$$[(a\circ b)\bullet c]=a\circ[b\bullet c]+(-1)^{|b|(|c|+1)}[a\bullet c]\circ b,\quad \forall\ a,b,c\in V,$$

holds. The operad of Gerstenhaber algebras is often denoted by $\mathcal{G}$. Its minimal resolution, $\mathcal{G}_\infty$, has been computed in [GeJo]; it is one of the most important operads in mathematics which found many applications in homological algebra, algebraic topology and deformation quantization. It was shown in [Me2] that $\mathcal{G}_\infty$ has also a differential geometric dimension:

3.6.1. **Theorem.** *For any $d\in\mathbb{N}$, there is a one-to-one correspondence between representations of the dg operad $\mathcal{G}_\infty$ in $\mathbb{R}^d$ (concentrated in degree 0) and morphisms of sheaves, $\mu:\odot^2\mathcal{T}_{\mathbb{R}^d}\to\mathcal{T}_{\mathbb{R}^d}$, making the tangent sheaf, $\mathcal{T}_{\mathbb{R}^d}$, into a commutative and associative algebra and satisfying the Hertling-Manin integrability condition, $R_\mu=0$, and the vanishing condition $\mu\mid_{0\in\mathbb{R}^d}=0$.*

We recall that the Hertling-Manin tensor, R_μ, of an arbitrary commutative and associative product, $\mu:\mathcal{T}_M\odot\mathcal{T}_M\to\mathcal{T}_M$, on the tangent sheaf of an arbitrary smooth manifold M is a map [HeMa]

$$\begin{array}{rccc} R_\mu: & \otimes^4\mathcal{T}_M & \longrightarrow & \mathcal{T}_M\\ & X\otimes Y\otimes Z\otimes W & \longrightarrow & R_\mu(X,Y,Z,W) \end{array}$$

where

$$\begin{aligned} R_\mu(X,Y,Z,W) \;=\;& [\mu(X,Y),\mu(Z,W)]-\mu([\mu(X,Y),Z],W)-\mu(Z,[\mu(X,Y),W])\\ &-\mu(X,[Y,\mu(Z,W)])-\mu[X,\mu(Z,W)],Y)+\mu(X,\mu(Z,[Y,W]))\\ &+\mu(X,\mu([Y,Z],W))+\mu([X,Z],\mu(Y,W))+\mu([X,W],\mu(Y,Z)). \end{aligned}$$

A remarkable fact is that this map is linear not only over $\mathbb{R}$ but also over arbitrary smooth functions, $f \in \mathcal{O}_M$, on M, that is, $R_\mu(fX,Y,Z,W) = fR_\mu(X,Y,Z,W)$, $R_\mu(X,fY,Z,W) = fR_\mu(X,Y,Z,W)$, etc. One can view the Hertling-Manin integrability equation as a diffeomorphism covariant version of the WDVV equation [HeMa, HMT].

4. Applications to deformation theory

4.1. **From minimal resolutions to L_∞-algebras.** One of the advantages of knowing a dg free resolution, $\mathcal{P}_\infty$, of a $\mathfrak{G}$-algebra controlling a mathematical structure $\mathcal{P}$ is that $\mathcal{P}_\infty$ paves a direct way to the deformation theory of $\mathcal{P}$-structures. In the heart of this approach to the deformation theory of many algebraic and geometric structures is observation 4.1.2 (see below) which was proven in [MeVa] in several ways. For its precise formulation we need the following notion.

4.1.1. **Definitions.** A L_∞-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n\mathfrak{g} \to \mathfrak{g}[2-n]\}_{n\geq 1})$ is called *filtered* if $\mathfrak{g}$ admits a non-negative decreasing Hausdorff filtration,

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \ldots \supseteq \mathfrak{g}_i \supseteq \ldots,$$

such that $\operatorname{Im}\mu_n \subset \mathfrak{g}_n$ for all $n \geq n_0$ beginning with some $n_0 \in \mathbb{N}$. In this case it make sense to define the associated set, $\mathcal{MC}(\mathfrak{g})$, of *Maurer-Cartan elements* as a subset of $\mathfrak{g}$ consisting of degree 1 elements Γ satisfying the equation $\sum_{n\geq 1}\frac{1}{n!}\mu_n(\Gamma,\ldots,\Gamma) = 0$.

A very useful fact is that to every Maurer-Cartan element, $\Gamma \in \mathcal{MC}(\mathfrak{g})$, of a filtered L_∞-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n\mathfrak{g} \to \mathfrak{g}\}_{n\geq 1})$ there corresponds a Γ-*twisted* L_∞-algebra structure, $\{\mu_n^\Gamma : \wedge^n\mathfrak{g} \to \mathfrak{g}\}_{n\geq 1}$, on $\mathfrak{g}$. If one thinks of the original L_∞ algebra as of a dg manifold $(\mathcal{M}_\mathfrak{g}, \eth)$ (see §4.1.2), then the set $\mathcal{MC}(\mathfrak{g})$ can be identified with the zero set of the homological vector field $\eth$, and the Γ-twisted $\mathcal{L}_\infty$-algebra structure on $\mathfrak{g}$ corresponds to that homological vector field $\eth^\Gamma$ on $\mathcal{M}_\mathfrak{g}$ which is obtained from $\eth$ by the translation diffeomorphism, $x \to x + \Gamma$, $\forall x \in \mathcal{M}_\mathfrak{g}$.

4.1.2. **Theorem** [MeVa]. *Let $(\mathcal{F}^\mathfrak{G}\langle E\rangle, \delta)$ be a dg free $\mathfrak{G}$-algebra (see Table 1) generated by an $\mathbb{S}$-bimodule E, and let $(\mathcal{Q}, \delta_\mathcal{Q})$ be an arbitrary dg $\mathfrak{G}$-algebra. Then*

- (i) *the graded vector space, $\mathfrak{g} := \operatorname{Hom}_\mathbb{S}(E, \mathcal{Q})[-1]$, is canonically a filtered L_∞-algebra;*
- (ii) *the set of all morphisms, $\{\mathcal{F}^\mathfrak{G}\langle E\rangle \to \mathcal{Q}\}$, of dg $\mathfrak{G}$-algebras is canonically isomorphic to the Maurer-Cartan set, $\mathcal{MC}(\mathfrak{g})$, of the L_∞-algebra in (i).*

Proof. As an illustration we show an elementary proof of the theorem in the simplest case $\mathfrak{G} = \mathfrak{G}^|$ (see Table 1), i.e. in the case when $\mathcal{F}^\mathfrak{G}\langle E\rangle$ is the free associative algebra, $\otimes^\bullet E$, generated by a graded vector space E and $(\mathcal{Q}, \delta_\mathcal{Q})$ is an arbitrary dg associative algebra (we refer the reader to [MeVa] for all other cases from Table 1 except $\mathfrak{G}^\circlearrowright$, and to [Gr2] for the case $\mathfrak{G}^\circlearrowright$). With these data we shall associate a cohomological vector field, $\eth$, on the space $\mathfrak{g}[1] = \operatorname{Hom}(E, \mathcal{Q}) = \mathcal{Q}\otimes E^*$, and we shall do it in local coordinates by assuming further (only for simplicity of signs factors in formulae) that the graded vector spaces E and $\mathcal{Q}$ are free modules over some graded commutative ring, $R = \oplus_{i\in\mathbb{Z}}R^i$, with *degree 0* generators $\{e_a\}_{a\in I}$ and, respectively, $\{e_\alpha\}_{\alpha\in J}$. Then the differentials in $\otimes^\bullet E$ and $\mathcal{Q}$, as well as multiplication $\circ$ in $\mathcal{Q}$, have, respectively, the following coordinate representations,

$$\delta e_a = \sum_{\substack{k\geq 1\\ a_1\ldots a_k\in I}} \delta_a^{a_1\ldots a_k} e_{a_1}\otimes\ldots\otimes e_{a_k}, \qquad \delta_\mathcal{Q} e_\alpha = \sum_{\beta\in J} Q_\alpha^\beta e_\beta, \qquad e_\alpha \circ a_\beta = \sum_{\gamma\in J}\mu_{\alpha\beta}^\gamma e_\gamma,$$

for some coefficients $\delta_a^{a_1\ldots a_k} \in R^1$, $Q_\alpha^\beta \in R^1$ and $\mu_{\alpha\beta}^\gamma \in R^0$. The vector space of all $\mathcal{R}$-linear maps, $\operatorname{Hom}(E, \mathcal{Q})$, is naturally graded, $\operatorname{Hom}(E, \mathcal{Q}) = \oplus_{i\in\mathbb{Z}}\operatorname{Hom}^i(E, \mathcal{Q})$, with $\operatorname{Hom}^i(E, \mathcal{Q})$ denoting the space of all homogeneous linear maps of degree i. In the chosen bases a generic element $\gamma \in \operatorname{Hom}^i(E, \mathcal{Q})$ gets a coordinate representation, $\gamma(e_a) = \sum_{\alpha\in J}\gamma_{a(i)}^\alpha e_\alpha$, for some coefficients $\gamma_{a(i)}^\alpha \in R^i$. The family of parameters $\{\gamma_{a(i)}^\alpha\}_{a\in I,\alpha\in J,i\in\mathbb{Z}}$ provides us with a coordinate system on the formal manifold $\mathcal{M}_{\mathfrak{g}} \simeq \mathrm{Hom}(E, \mathcal{Q})$. In these coordinates the required homological vector field on $\mathcal{M}_{\mathfrak{g}}$, that is, a $\mathcal{L}_\infty$-structure on $\mathrm{Hom}(E, \mathcal{Q})[-1]$, is given explicitly by

$$\eth = \left(\sum_{\alpha,\beta,a,i} Q^{\alpha}_{\beta} \gamma^{\beta}_{a(i)} - \sum_{a,a_\bullet,\alpha,i} (-1)^i \delta^{a_1\ldots a_k}_{a} \gamma^{\alpha}_{a_1\ldots a_k(i)} \right) \frac{\partial}{\partial \gamma^{\alpha}_{a(i)}}$$

where, for $k \geq 2$,

$$\gamma^{\alpha}_{a_1 a_2 \ldots a_k(i)} = \sum_{\substack{\beta_\bullet, \gamma_\bullet \in J \\ i_1+\ldots+i_k=i}} \mu^{\alpha}_{\beta_1\gamma_1} \mu^{\gamma_1}_{\beta_2\gamma_2} \cdots \mu^{\gamma_{k-2}}_{\beta_{k-1}\beta_k} \gamma^{\beta_1}_{a_1(i_1)} \gamma^{\beta_2}_{a_2(i_2)} \cdots \gamma^{\beta_k}_{a_k(i_k)}.$$

The equation $[\eth, \eth] = 0$ follows straightforwardly from the assumptions that $\delta^2 = 0$, $\delta^2_{\mathcal{Q}} = 0$, as well as from the associativity of the product $\circ$ and its compatibility with $\delta_{\mathcal{Q}}$. This proves (i).
The Maurer-Cartan set $\mathcal{MC}(\mathfrak{g})$ is precisely the set $\{\gamma \in \mathrm{Hom}^0(E, \mathcal{Q}) : \eth|_\gamma = 0\}$ and, therefore, consists of all points in $\mathrm{Hom}(E, \mathcal{Q})$ which have all the coordinates $\{\gamma^{\alpha}_{a(i)}\}_{i\neq 0}$ vanishing, and the coordinate $\gamma^{\alpha}_{a(0)}$ satisfying the equations,

$$\sum_{\beta\in J} Q^{\alpha}_{\beta} \gamma^{\beta}_{a(0)} - \sum_{a_1,\ldots,a_k\in I} \delta^{a_1\ldots a_k}_{a} \gamma^{\alpha}_{a_1\ldots a_k(0)} = 0.$$

Which just say that the associated to $\gamma^{\alpha}_{a(0)}$ map of associative algebras, $\odot^\bullet E \to \mathcal{Q}$, commutes with the differentials δ and $\delta_{\mathcal{Q}}$ defining thereby a morphism of *dg* algebras. This proves claim (ii). □

4.2. **Deformation theory.** The theory of operads and props gives a universal approach to the deformation theory of many algebraic and geometric structures and provides us with a conceptual explanation of the well-known "experimental" observation that a deformation theory is controlled by a differential graded (dg, for short) Lie or, more generally, a L_∞-algebra. What happens is the following [KoSo, MeVa, vdL]:

(I) an algebraic or a (germ of) geometric structure, $\mathfrak{s}$, in a vector space V (which is an *object* in the corresponding category, $\mathfrak{S}$, of algebraic or geometric structures) can often be interpreted as a *representation*, $\alpha_{\mathfrak{s}} : \mathcal{S} \to \mathcal{E}nd_V$, of a $\mathfrak{G}$-algebra $\mathcal{S}$ uniquely associated to the category of $\mathfrak{s}$-structures;

(II) a dg resolution, $\pi : \mathcal{S}_\infty = (\mathcal{F}^{\mathfrak{G}}\langle E\rangle, \delta) \to \mathcal{S}$, of the $\mathfrak{G}$-algebra $\mathcal{S}$ gives rise, by Theorem 4.1.2, to a filtered L_∞-algebra on the vector space $\mathfrak{g} = \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1])$ whose Maurer-Cartan elements correspond to all possible representations, $\mathcal{S}_\infty \to \mathcal{E}nd_V$; in particular, our original algebraic or geometric structure $\mathfrak{s}$ defines a Maurer-Cartan element $\Gamma_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \circ \pi$ in $\mathcal{MC}(\mathfrak{g})$;

(III) the $\Gamma_{\mathfrak{s}}$-twisted L_∞-algebra structure on $\mathfrak{g}$ is precisely the one which controls, in Deligne's sense, the deformation theory of $\mathfrak{s}$.

For example, if $\mathfrak{s}$ is a structure of associative algebra on a vector space V, then,

(i) there is an operad, $\mathcal{A}ss$, uniquely associated to the category of associative algebras such that $\mathfrak{s}$ corresponds to a morphism, $\alpha_{\mathfrak{s}} : \mathcal{A}ss \to \mathcal{E}nd_V$, of operads (see §2.1);

(ii) there is a unique minimal resolution (see Theorem 3.1.2), $\mathcal{A}ss_\infty$, of $\mathcal{A}ss$ which is generated by the $\mathbb{S}$-module $E = \{\mathbb{K}[\mathbb{S}_n][n-2]\}$ and whose representations, $\pi : \mathcal{A}ss_\infty \to \mathcal{E}nd_V$, in a dg space V are in one-to-one correspondence with Maurer-Cartan elements in the Lie algebra,

$$\left(\mathcal{G} := \mathrm{Hom}_{\mathbb{S}}(E, \mathcal{E}nd_V)[-1] = \oplus_{n\geq 1} \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)[1-n], [\ ,\ ]_G\right),$$

where $[\ ,\ ]_G$ are Gerstenhaber brackets.

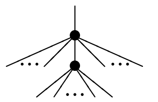
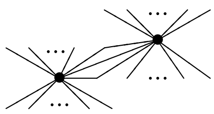
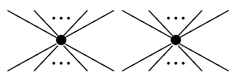
(iii) the particular associative algebra structure $\mathfrak{s}$ on V gives, therefore, rise to the associated Maurer-Cartan element $\gamma_{\mathfrak{s}} := \alpha_{\mathfrak{s}} \circ \pi$ in $\mathcal{G}$; twisting $\mathcal{G}$ by $\gamma_{\mathfrak{s}}$ gives the Hochschild dg Lie algebra, $\mathcal{G}_{\mathfrak{s}} = (\oplus_n \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, V)[1-n], [\ ,\ ]_G, d_H := [\gamma_{\mathfrak{s}},\ ]_G)$ which indeed controls the deformation theory of $\mathfrak{s}$.

This is a classical example illustrating how the machine works. For some new applications of this approach to deformation theory (e.g. to the proof of Deligne's conjecture or to the deformation theory of associative bialgebras) we refer to [KoSo, MeVa] and to many references cited there.

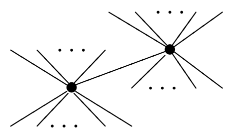
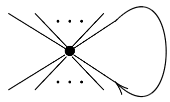
5. Koszul duality theory, quantum BV manifolds and effective BF-actions

5.1. **Quadratic $\mathfrak{G}$-algebras and their Koszul duals.** Koszul duality theory of quadratic $\mathfrak{G}$-algebras is one of the most powerful theorem-proving technique in the theory of (wheeled) operads and properads and their applications.

What is a *quadratic* $\mathfrak{G}$-algebra? Every family of graphs, $\mathfrak{G}$, from Table 1 has a uniquely defined subfamily, $\mathfrak{G}_{gen}$, of *generating* graphs, which, by definition, is the smallest subset of $\mathfrak{G}$ with the defining property such that, for every $G \in \mathfrak{G}$ and any $\mathfrak{G}$-algebra $\mathcal{P}$, the associated "contraction" composition $\mu_{\mathfrak{G}} : G\langle\mathcal{P}\rangle \to \mathcal{P}$ can be represented as an iteration (in the sense of (1)) of compositions μ_{G_i} for some $G_i \in \mathfrak{G}_{gen}$, $i \in I$. For example,

$\mathfrak{G}^{\curlywedge}_{gen} = \{$  $\}$, $\mathfrak{G}^{\uparrow}_{gen} = \{$  ,  $\}$,

and

$\mathfrak{G}^{\circlearrowright}_{c,gen} = \{$  ,  $\}$.

5.1.1. **Weight gradation.** let $\mathfrak{G}$ be a family of graphs from Table 1. For any graph $G \in \mathfrak{G}$ with p vertices and q wheels (that is, closed paths of directed internal edges) we set $||G|| := p + q$ and call it the *weight* of G. Thus $\mathfrak{G}_{gen} \subset \mathfrak{G}$ consists precisely of graphs of weight 2.

For an $\mathbb{S}$-bimodule E set $\mathcal{F}^{\mathfrak{G}}_{(\lambda)}\langle E\rangle$ to be an $\mathbb{S}$-subbimodule of the free $\mathfrak{G}$-algebra $\mathcal{F}^{\mathfrak{G}}\langle E\rangle$ spanned by decorated graphs of weight λ.

5.1.2. **Definition.** A $\mathfrak{G}$-algebra, $\mathcal{P}$, is called *quadratic* if it is the quotient, $\mathcal{F}^{\mathfrak{G}}\langle E\rangle/\langle\mathcal{R}\rangle$, of a free $\mathfrak{G}$-algebra (generated by an $\mathbb{S}$-bimodule E) modulo the ideal generated by a subspace $\mathcal{R} \subset \mathcal{F}^{\mathfrak{G}}_{(2)}\langle E\rangle = \bigoplus_{G\in\mathfrak{G}_{gen}} G\langle E\rangle$. It comes equipped with an induced weight gradation, $\mathcal{P} = \oplus_{\lambda\geq 1}\mathcal{P}_{(\lambda)}$, where $\mathcal{P}_{(\lambda)} = \mathcal{F}^{\mathfrak{G}}_{(\lambda)}\langle E\rangle/\langle\mathcal{R}\rangle$. In particular, $\mathcal{P}_{(1)} = E$ and $\mathcal{P}_{(2)} = \mathcal{F}^{\mathfrak{G}}_{(2)}\langle E\rangle/\mathcal{R}$.

5.2. **Koszul duality.** Let $\mathfrak{G}_c$ be any family of *connected* graphs from Table 1. In this case one can associate to any quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$ its Koszul dual $\mathfrak{G}_c$-coalgebra $\mathcal{P}^{\mathrm{i}}$. We omit technical details (referring to [GiKa, GeJo, Ga, Va1, MMS, Me7]) and explain just the working scheme:

(i) The notion of $\mathfrak{G}_c$-*coproperad* is obtained by an obvious dualization of the notion of $\mathfrak{G}_c$-algebra (see §2.3): this is an $\mathbb{S}$-bimodule $\mathcal{P} = \{\mathcal{P}(m,n)\}$ together with a family of linear $\mathbb{S}_m\times\mathbb{S}_n$-equivariant maps,

$$\{\Delta_G : \mathcal{P}(m,n) \to G\langle\mathcal{P}\rangle\}_{G\in\mathfrak{G}_c(m,n), m,n\geq 0}\,,$$

which satisfy the coassociativity condition, $\Delta_G = \Delta'_H \circ \Delta_{G/H}$, for any subgraph $H \subset G$ which belongs to the family $\mathfrak{G}$. Here $\Delta'_H : (G/H)\langle E\rangle \to G\langle E\rangle$ is the map which equals Δ_H on the distinguished vertex of G/H and which is identity on all other vertices of G.

(ii) There exists a pair of adjoint *exact* functors

$$\begin{array}{cccc} B: & \text{the category of dg } \mathfrak{G}_c\text{-algebras} & \rightleftarrows & \text{the category of dg } \mathfrak{G}_c\text{-coalgebras}: \quad B^c \\ & \mathcal{P} & \longrightarrow & (B(\mathcal{P}), \partial_{\mathcal{P}}) \\ & (B^c(\mathcal{Q}), \partial_{\mathcal{Q}}) & \longleftarrow & \mathcal{Q} \end{array}$$

such that, for any dg $\mathfrak{G}_c$-algebra $\mathcal{P}$ the composition $B^c(B(\mathcal{P}))$ is a dg free resolution of $\mathcal{P}$. The differential $\partial_{\mathcal{P}}$ in $B(\mathcal{P})$ encodes both the differential and all the generating contraction compositions, $\{\mu_{\mathfrak{G}} : G\langle\mathcal{P}\rangle \to \mathcal{P}\}_{G\in\mathfrak{G}_{gen}}$, in the $\mathfrak{G}_c$-algebra $\mathcal{P}$ (and similarly for $\partial_{\mathcal{Q}}$).

(iii) As a vector space, $B(\mathcal{P})$, is isomorphic to the free $\mathfrak{G}_c$-algebra, $\mathcal{F}^{\mathfrak{G}_c}\langle\hat{\mathcal{P}}\rangle$, generated by an $\mathbb{S}$-bimodule $\hat{\mathcal{P}}$ which is linearly isomorphic to $\mathcal{P}$. Hence it comes equipped with a double weight gradation, $B(\mathcal{P}) = \oplus_{\lambda,\mu\geq 1} B^{(\lambda)}_{(\mu)}(\mathcal{P})$, one of which (say, λ) is induced from the weight gradation of $\mathcal{P}$ and the other (say, μ), is given by the standard weight gradation, $\oplus_{\mu\geq 1}\mathcal{F}^{\mathfrak{G}_c}_{(\mu)}\langle\hat{\mathcal{P}}\rangle$, of a free $\mathfrak{G}_c$-algebra. The differential $\partial_{\mathcal{P}}$ preserves the λ-gradation and decreases by one the μ-gradation.

5.2.1. **Definition.** Given a quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$, the weight graded $\mathfrak{G}_c$-coalgebra $\mathcal{P}^{\mathsf{i}} = \oplus_{\lambda\geq 1}\mathcal{P}^{\mathsf{i}}_{(\lambda)}$ with $\mathcal{P}^{\mathsf{i}}_{(\lambda)} := B^{(\lambda)}_{(\lambda)}(\mathcal{P}) \cap \mathsf{Ker}\,\partial_{\mathcal{P}} \subset B(\mathcal{P})$ is called *Koszul dual* to $\mathcal{P}$.

The beauty of this notion is that $\mathcal{P}^{\mathsf{i}}$ is again quadratic and, moreover, can often be easily computed directly from generators and relations, E and $\mathcal{R}$, of $\mathcal{P}$.

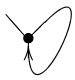
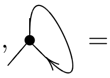
5.2.2. **Definition.** . A quadratic $\mathfrak{G}_c$-algebra $\mathcal{P}$ is called *Koszul*, if the associated inclusion of dg coproperads, $\imath : (\mathcal{P}^{\mathsf{i}}, 0) \longrightarrow (B(\mathcal{P}), \partial_{\mathcal{P}})$, is a quasi-isomorphism.

As the cobar construction functor B^c preserves quasi-isomorphisms between connected $\mathfrak{G}$-coalgebras, the composition

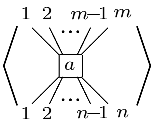
$$\pi : \mathcal{P}_\infty := B^c(\mathcal{P}^{\mathsf{i}}) \stackrel{B^c(\imath)}{\longrightarrow} B^c(B(\mathcal{P})) \stackrel{\text{natural projection}}{\longrightarrow} \mathcal{P}$$

is a quasi-isomorphism if and only if $\mathcal{P}$ is Koszul; in this case the dg free $\mathfrak{G}_c$-algebra $\mathcal{P}_\infty$ gives us a minimal resolution of the quadratic algebra $\mathcal{P}$. Almost all minimal resolutions listed in §2 have been obtained in this way.

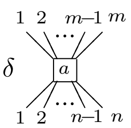
5.3. **Homotopy transfer formulae.** If $\mathcal{P}_\infty$ is a minimal resolution of some $\mathfrak{G}$-algebra $\mathcal{P}$, and (V, d) is a complex carrying a $\mathcal{P}$-structure, then one might expect that the associated cohomology space, $H(V, d)$, carries an induced structure of $\mathcal{P}_\infty$-algebra. In the case when $\mathcal{P}$ is an operad of associative algebras, existence of such induced $\mathcal{A}ss_\infty$-structures was proven by Kadeishvili in [Ka] and the first explicit formulae have been shown in [Me1]. Later Kontsevich and Soibelman [KoSo] have nicely rewritten these homotopy transfer formulae in terms of sums of decorated graphs. In fact, it is a general phenomenon that the homotopy transfer formulae can be represented as sums of graphs. The required graphs are precisely the ones which describe the image of the natural inclusion $\imath : (\mathcal{P}^{\mathsf{i}}, 0) \longrightarrow (B(\mathcal{P}), \partial_{\mathcal{P}})$, and apply to any quadratic $\mathfrak{G}$-algebra, not necessarily the Koszul one [Me7].

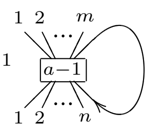
5.4. **Example: unimodular Lie 1-bialgebras versus quantum BV manifolds.** The wheeled prop, $\mathcal{ULie}^1\mathcal{B}$, of *unimodular* Lie 1-bialgebras was defined in [Me7] (cf. §3.4) as the quotient, $\mathcal{F}_c^\circlearrowright\langle B\rangle/\langle\mathcal{R}\rangle$, of the free wheeled properad generated by $\mathbb{S}$-bimodule (9) modulo the ideal generated by relations (7), (10) and the following ones,  $= 0$,  $= 0$, expressing unimodularity of both binary operations. This is a quadratic wheeled properad so that one can apply the above general machinery to compute its Koszul dual coproperad, $\mathcal{ULie}^1\mathcal{B}^{\mathsf{i}}$, and then the dg properad $\mathcal{P}_\infty := B^c(\mathcal{ULie}^1\mathcal{B}^{\mathsf{i}})$ which turns out to be a free wheeled properad, $\mathcal{F}_c^\circlearrowright\langle Z\rangle$, generated by an $\mathbb{S}$-bimodule,

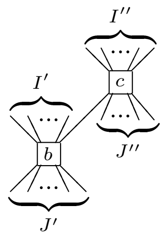
$$Z(m,n) := \bigoplus_{a\geq 0}^{\infty} \mathrm{sgn}_m \otimes \mathbb{1}_n[m-2-2a] = \mathrm{span}\left\langle \quad \right\rangle_{\substack{m+n+2a\geq 3\\ m+a\geq 1, n+a\geq 1}} .$$



and equipped with the following differential

δ 

$= (-1)^{m-1}$ 

$+ \sum_{\substack{a=b+c\\ b,c\geq 0}} \sum_{\substack{m=I'\sqcup I''\\ [n]=J'\sqcup J''}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1|(|I_2|+1)}$ 

It is not known at present whether or not $\mathcal{ULie}^1\mathcal{B}$ is Koszul, i.e. whether or not the above free properad is a (minimal) resolution of the latter. In any case, $\mathcal{ULie}^1\mathcal{B}_\infty$ gives us an approximation to that minimal resolution, and has, in fact, a geometrically meaningful set, $\{\mathcal{ULie}^1\mathcal{B}_\infty \to \mathcal{E}nd_V\}$, of all possible representations. To describe this set let us recall a few notions from the Schwarz model [Sc] of the Batalin-Vilkovisky quantization formalism [BaVi].

5.5. **Formal quantum BV manifolds.** Let $\{x^a,\psi_a,\hbar\}_{1\leq a\leq n}$, $n\in\mathbb{N}$, be a set of formal homogeneous variables of degrees $|x^a|+|\psi_a|=1$ and $|\hbar|=2$, and let $\mathcal{O}^\hbar_{x,\psi}:=\mathbb{K}[[x^a,\psi_a,\hbar]]$ be the associated free graded commutative ring which we view from now on as a $\mathbb{K}[[\hbar]]$-algebra. The degree -1 Lie bracket,

$$\{f\bullet g\}:=(-1)^{|f|}\Delta(fg)-(-1)^{|f|}\Delta(f)g-f\Delta(g),\ \forall\ f,g\in\mathcal{O}^\hbar_{x,\psi}$$

make $\mathcal{O}^\hbar_{x,\psi}$ into a Gerstenhaber $\mathbb{K}[[\hbar]]$-algebra (see §3.6). Here and elsewhere $\Delta := \sum_{a=1}^n(-1)^{|x^a|}\frac{\partial^2}{\partial x^a\partial\psi_a}$. A *quantum master function* is, by definition, a degree 2 element $\Gamma\in\mathcal{O}^\hbar_{x,\psi}$ satisfying a so called *quantum master equation*

$$\hbar\Delta\Gamma+\frac{1}{2}\{\Gamma\bullet\Gamma\}=0. \tag{13}$$

Such an element makes the $\mathbb{K}[[\hbar]]$-module $\mathcal{O}^\hbar_{x,\psi}$ *differential* with the differential $\Delta_\Gamma:=\hbar\Delta+\{\Gamma\bullet\ \}$. Note that this differential does *not* respect the algebra structure in $\mathcal{O}^\hbar_{x,\psi}$ but respects the Poisson brackets.

Consider a group of $\mathbb{K}[[\hbar]]$-algebra automorphisms, $F:\mathcal{O}^\hbar_{x,\psi}\to\mathcal{O}^\hbar_{x,\psi}$, preserving the Lie brackets, $F(\{f\bullet g\})=\{F(f)\bullet F(g)\}$ (but not necessarily the operator Δ); this group is uniquely determined by a collection, $\mathcal{N}:=\{|x^a|,|\psi_a|\}_{1\leq a\leq n}$, of $2n$ integers and is denoted by $Symp_{\mathcal{N}}$. It is often called a group of *symplectomorphsims* of the Gerstenhaber algebra $(\mathcal{O}^\hbar_{x,\psi},\{\ \bullet\ \})$. A remarkable fact [Kh] is that $Symp_{\mathcal{N}}$ acts on the set of quantum master functions by the formula,

$$e^{\frac{F(\Gamma)}{\hbar}}:=\left[\mathrm{Ber}\begin{pmatrix}\frac{\partial F(x^a)}{\partial x^b} & \frac{\partial F(x^a)}{\partial\psi_b}\\ \frac{\partial F(\psi_a)}{\partial x^b} & \frac{\partial F(\psi_a)}{\partial\psi_b}\end{pmatrix}\right]^{-\frac{1}{2}}e^{\frac{\Gamma(x,\psi,\hbar)}{\hbar}}. \tag{14}$$

5.5.1. **Definition.** An equivalence class of pairs, $(\mathcal{O}^\hbar_{x,\psi},\Gamma)$, under the action of the group $Symp_{\mathcal{N}}$ is called a formal *quantum BV manifold* $\mathcal{M}$ of dimension $\mathcal{N}$. A particular representative, $(\mathcal{O}^\hbar_{x,\psi},\Gamma)$, of $\mathcal{M}$ is called a *Darboux coordinate chart* on $\mathcal{M}$.

In geometric terms, $\mathcal{M}$ is a formal odd symplectic manifold equipped with a special type semidensity [Kh, Sc]. We need an extra structure on $\mathcal{M}$ which we again define with the help of a Darboux coordinate chart. Notice that the ideals, I_x and I_ψ, in the $\mathbb{K}[[\hbar]]$-algebra $\mathcal{O}^\hbar_{x,\psi}$ generated, respectively, by $\{x^a\}_{1\leq a\leq n}$ and $\{\psi_a\}_{1\leq a\leq n}$, are also Lie ideals; geometrically, they define a pair of transversally intersecting Lagrangian submanifolds of $\mathcal{M}$. A quantum BV manifold $\mathcal{M}$ is said to have *split quasi-classical limit* (or, slightly shorter, $\mathcal{M}$ is *quasi-classically split*) if it admits a Darboux coordinate chart in which the master function, $\Gamma(x,\psi,\hbar)=\sum_{n\geq 0}\Gamma_n(x,\psi)\hbar^n$, satisfies the following two boundary conditions,

$$\Gamma_0\in I_xI_y,\quad \Gamma_1\in I_x+I_y.$$

In plain terms, these conditions mean that $\Gamma(x,\psi,\hbar)$ is given by a formal power series of the form,

$$\Gamma(x,\psi,\hbar)=\underbrace{\sum_{a,b}\Gamma_{(0)}{}^a_b x^b\psi_a}_{\Gamma_0}+\underbrace{\sum_{\substack{p+q+2n\geq 3\\ p+n\geq 1, q+n\geq 0}}\frac{1}{p!q!}\Gamma_{(n)}{}^{b_1\dots b_q}_{a_1\dots a_p}x^{a_1}\dots x^{a_p}\psi_{b_1}\dots\psi_{b_q}\hbar^n}_{\mathbf{\Gamma}} \tag{15}$$

for some $\Gamma_{(n)}{}^{b_1\dots b_q}_{a_1\dots a_p}\in\mathbb{K}$. Quantum master equation (13) immediately implies that $\{\Gamma_0,\Gamma_0\}=0$ so that $\eth:=\{\Gamma_0\bullet\ \}$ is a differential in the Gerstenhaber algebra $\mathcal{O}^\hbar_{x,\psi}$. Then master equation (13) for a quasi-classically split quantum master function can be equivalently rewritten in the form,

$$\eth\mathbf{\Gamma}+\hbar\Delta\mathbf{\Gamma}+\frac{1}{2}\{\mathbf{\Gamma}\bullet\mathbf{\Gamma}\}=0,$$

where $\mathbf{\Gamma}$ is an element of $\mathcal{O}^\hbar_{x,\psi}$ of polynomial order at least three (here we set, by definition, the polynomial order of generators x and ψ to be equal to 1 and the polynomial order of $\hbar$ to be equal to 2). The differential $\eth$ induces a differential on the tangent space, $\mathcal{T}_*\mathcal{M}$, to $\mathcal{M}$ at the distinguished point; we denote it by the same letter $\eth$. Such a quantum BV manifold is called

minimal if $\eth = 0$ and *contractible* if there exists a Darboux coordinate chart in which $\mathbf{\Gamma} = 0$ (i.e. $\Gamma = \eth$) and the tangent complex $(\mathcal{T}_*\mathcal{M}, \eth)$ is acyclic (cf. §3.2.4).
An important class of so called BF field theories (see, e.g., [CaRo] and references cited there) have associated quantum BV manifolds which do satisfy the split quasi-classical limit condition.

5.5.2. **Proposition.** *For any dg vector space V, there is a one-to-one correspondence between representations, $\mathcal{ULie}^1\mathcal{B}_\infty \to \mathcal{E}nd_V$, and structures of formal quasi-classically split quantum BV manifold on $\mathcal{M}_{V\oplus V^*[1]}$, the formal manifold associated to $V \oplus V^*[1]$.*

5.6. **Morphisms of quantum BV manifolds** [Me7]**.** The above Proposition together with the Koszul duality theory approach to the homotopy transfer outlined in §5.3 provide us with highly non-trivial formulae for constructing quantum BV manifold structures out of dg unimodular Lie 1-bialgebras. We would like to have a *category* of quantum BV manifolds in which such homotopy transfer formulae can be interpreted as morphisms. This can be achieved via the following

5.6.1. **Definitions.** (i) A *morphism* of quantum BV manifolds, $F : \mathcal{M} \to \hat{\mathcal{M}}$, is, by definition, a morphism of dg $\mathbb{K}[[\hbar]]$-modules,

$$F : \left(\mathcal{O}_{\hat{\mathcal{M}}} \simeq \mathcal{O}^{\hbar}_{\hat{x},\hat{\psi}}, \Delta_{\hat{\Gamma}}\right) \longrightarrow \left(\mathcal{O}_{\mathcal{M}} \simeq \mathcal{O}^{\hbar}_{x,\psi}, \Delta_{\Gamma}\right),$$

reducing in the classical limit $\hbar \to 0$ to a morphism of algebras, $F|_{\hbar=0} : \mathcal{O}_{\hat{x},\hat{\psi}} \to \mathcal{O}_{x,\psi}$ preserving the ideals, $F|_{\hbar=0}(\langle\hat{x}\rangle) \subset \langle x\rangle$ and $F|_{\hbar=0}(\langle\hat{\psi}\rangle) \subset \langle\psi\rangle$, of the distinguished Lagrangian submanifolds in $\hat{\mathcal{M}}|_{\hbar=0}$ and $\mathcal{M}|_{\hbar=0}$.
(ii) If $F : \mathcal{M} \to \hat{\mathcal{M}}$ is a morphism of quantum BV manifolds, then $dF|_{\hbar=0}$ induces in fact a morphism of dg vector spaces, $(\mathcal{T}_*\mathcal{M}, \eth) \to (\mathcal{T}_*\hat{\mathcal{M}}, \hat{\eth})$; F is called a *quasi-isomorphism* if the latter map induces an isomorphism of the associated cohomology groups.

5.6.2. **Theorem** [Me7]**.** *Every quantum quasi-classically split BV manifold is isomorphic to a product of a minimal one and of a contractible one. In particular, every such a manifold is quasi-isomorphic to a minimal one.*

5.7. **Homotopy transfer of quantum BV-structures via Feynman integral.** Homotopy transfer formulae of $\mathcal{P}_\infty$-structures given by Koszul duality theory are given by sums of decorated graphs which resemble Feynman diagrams in quantum field theory. This resemblance was made a rigorous fact in [Mn] for the case of the wheeled operad of unimodular Lie algebras (see §3.3).

Given any complex V and a dg Lie 1-bialgebra structure on V with degree 0 Lie cobrackets $\Delta^{CoLie} : V \to \wedge^2 V$ and degree 1 Lie brackets $[\bullet] : \odot^2 V \to V[1]$, the associated by Koszul duality theory homotopy formulae transfer this rather trivial quantum BV manifold structure on V to a highly non-trivial quantum master function on its cohomology $H(V)$; the same formulae can also be described [Me7] by a standard Batalin-Vilkovisky quantization [BaVi] of a BF-type field theory on the space $V \oplus V^*[1]$ with the action given by,

$$\begin{array}{rccc} S: & V \oplus V^*[1] & \longrightarrow & \mathbb{K} \\ & p \oplus \omega & \longrightarrow & S(p,\omega) := \langle p, d\omega\rangle \ + \ \frac{1}{2}\langle p, [\omega,\omega]\rangle + \frac{1}{2}\langle [p\bullet p], \omega\rangle, \end{array}$$

where $\langle\ ,\ \rangle$ stands for the natural pairing, and $[\ ,\ ] : \odot^2(V^*[1]) \to V^*[2]$ for the dualization of Δ^{CoLie}. Thus at least in some cases the Koszul duality technique for homotopy transfer of ∞-structures is identical to the Feynman diagram technique in theoretical physics. The beauty of the latter lies in its combinatorial simplicity (due to the Wick theorem), while the power of the former lies in its generality: the Koszul duality theory applies to any (non-commutative case including) quadratic $\mathfrak{G}$-algebras.

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