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EXPLICIT DESCRIPTIONS OF BISYMMETRIC SUGENO INTEGRALS

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ABSTRACT. We provide sufficient conditions for a Sugeno integral to be bisymmetric. We explicitly describe bisymmetric Sugeno integrals over chains.

1. INTRODUCTION

Aggregation functions essentially model the process of merging a set of values into a single representative one. The need to aggregate values in a meaningful way has become more and more present in an increasing number of areas in mathematics and physics, and especially in applied fields such as engineering, computer science, and economical and social sciences. For recent references, see [2, 14, 15].

In this paper, we are interested in aggregation functions $f\colon A^n \to A$ satisfying the following identity

$$f\big(f(a_{11},\dots,a_{1n}),\dots,f(a_{n1},\dots,a_{nn})\big) = f\big(f(a_{11},\dots,a_{n1}),\dots,f(a_{1n},\dots,a_{nn})\big),$$

for all $a_{ij} \in A$ $(1 \leq i, j \leq n)$. The relevance of this property is made apparent in works pertinent to different areas of mathematical research. In functional equation theory, and in particular in aggregation function theory, this property is referred to as *bisymmetry* and it is naturally interpreted when reading off data provided by square matrices: essentially, it expresses the fact that aggregating the data by rows and then aggregating the resulting column outputs the same value as that of aggregating the data by columns and then aggregating the resulting row. For motivations and general background, see [1, 14]. In the algebraic setting, bisymmetry appears as the natural generalization of the notion of "mediality". It is also called self-commutation and it is tightly related to the notions of entropic algebras and centralizer clones.

Among noteworthy aggregation functions are the (discrete) Sugeno integrals, which were introduced by Sugeno [21, 22] as a way to compute the average of a function with respect to a nonadditive measure. Since their introduction, Sugeno integrals have been thoroughly investigated and are now considered as one of the most relevant families of aggregation functions in the qualitative setting of ordinal information (e.g., when the values to be aggregated are simply defined on a chain without further structure). This is partially due to the fact that, unlike other aggregation functions, Sugeno integrals can be defined over ordered structures where the usual arithmetic operations are not necessarily available. For general background, see [2, 14].

A convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. As it was observed in [17], Sugeno integrals can be regarded as polynomial functions $f\colon X^n \to X$ which are idempotent, that is, satisfying $f(x, \ldots, x) = x$.

In this paper, we address the question of characterizing those Sugeno integrals fulfilling the bisymmetry property. This question is answered for discrete Sugeno integrals over bounded chains. In Sect. 2, we recall basic notions in the universal-algebraic setting and settle the terminology used throughout the paper. Moreover, by showing that bisymmetry is preserved under several operations (e.g., permutation of variables, identification of variables and addition of dummy variables), we develop general tools for tackling the question of describing bisymmetric functions.

In Sect. 3, we survey well-known results concerning normal form representations of lattice functions which we then use to specify those Sugeno integrals on bounded chains which are bisymmetric. This explicit description is obtained by providing sufficient conditions for a Sugeno integral to be bisymmetric, and by showing that these conditions are also necessary in the particular case of Sugeno integrals over bounded chains. In Sect. 4 we point out problems which are left unsettled, and motivate directions of future research.

2. Preliminaries

In this section, we introduce some notions and terminology as well as establish some preliminary results that will be used in the sequel. For an integer $n \geq 1$, set $[n] := \{1, 2, \ldots, n\}$. With no danger of ambiguity, we denote the tuple $(x_1, \ldots, x_n)$ of any length by $\mathbf{x}$.

Let A be an arbitrary nonempty set. An *operation* on A (or *function*) is a map $f\colon A^n \to A$ for some integer $n \geq 1$, called the *arity* of f. We denote by $\mathcal{O}_A^{(n)}$ the set of all n-ary operations on A, and we denote by $\mathcal{O}_A$ the set of all finitary operations on A, i.e., $\mathcal{O}_A := \bigcup_{n \geq 1} \mathcal{O}_A^{(n)}$. We assume some familiarity with basic notions of universal algebra and lattice theory, and we refer the reader to [3, 4, 10, 11, 12, 16, 20] for general background.

2.1. Simple Minors.

Let $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$. We say that f is obtained from g by *simple variable substitution,* or f is a *simple minor* of g, if there is a mapping $\sigma\colon [m] \to [n]$ such that

$$f(x_1, \ldots, x_n) = g(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}).$$

If σ is not injective, then we speak of *identification of variables.* If σ is not surjective, then we speak of *addition of inessential variables.* If σ is bijective, then we speak of *permutation of variables.* For distinct indices $i, j \in [n]$, the function $f_{i \leftarrow j}\colon A^n \to A$ obtained from f by the simple variable substitution

$$f_{i \leftarrow j}(x_1, \ldots, x_n) := f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)$$

is called a *variable identification minor* of f, obtained by identifying x_i with x_j.

For studies of classes of operations that are closed under taking simple minors, see, e.g., [5, 19].

2.2. **Bisymmetry.** Recall that $f \in \mathcal{O}_A^{(n)}$ is *bisymmetric* if

$$f\big(f(a_{11},\ldots,a_{1n}),\ldots,f(a_{n1},\ldots,a_{nn})\big) = f\big(f(a_{11},\ldots,a_{n1}),\ldots,f(a_{1n},\ldots,a_{nn})\big),$$

for all $a_{ij} \in A$ $(1 \leq i,j \leq n)$.

The following proposition asserts that the class of bisymmetric operations on A is minor closed.

Proposition 2.1. *If $f \in \mathcal{O}_A$ is bisymmetric, then every simple minor of f is bisymmetric.*

In the particular case when A is finite, Corollary 2.1 translates into saying that the class of bisymmetric operations on A is definable by functional equations in the sense of [6].

The set of bisymmetric functions is also closed under special type of substitutions of constants for variables, as described by the following lemma. Let $f\colon A^n \to A$ and $c \in A$. For $i \in [n]$, we define $f_c^i\colon A^{n-1} \to A$ to be the function

$$f_c^i(a_1,\ldots,a_{n-1}) = f(a_1,\ldots,a_{i-1},c,a_i,\ldots,a_{n-1}).$$

Lemma 2.2. *Assume that $f\colon A^n \to A$ preserves $c \in A$, i.e., $f(c,\ldots,c) = c$. If f is bisymmetric, then for every $i \in [n]$, f_c^i is bisymmetric.*

3. Bisymmetric Sugeno Integrals

Let $(L;\wedge,\vee)$ be a lattice. With no danger of ambiguity, we denote lattices by their universes. In this section we study bisymmetry on Sugeno integrals. As mentioned, Sugeno integrals can be regarded as certain lattice polynomial functions, i.e., mappings $f\colon L^n \to L$ which can be obtained as compositions of the lattice operations and applied to variables (projections) and constants. This view has several appealing aspects, in particular, concerning normal form representations of Sugeno integrals. Indeed, as shown by Goodstein [13], polynomial functions on bounded distributive lattices coincide exactly with those functions which are representable in disjunctive normal form (DNF). Thus, in what follows we assume that L is a bounded distributive lattice with least and greatest elements 0 and 1, respectively.

We recall the necessary results concerning the representation of lattice polynomials as well as introduce some related concepts and terminology in Subsect. 3.1. Then, we consider the property of bisymmetry on Sugeno integrals. In Subsect. 3.1, we present explicit descriptions of bisymmetric Sugeno integrals on chains as a corollary of Theorem 3.6 in [8].

3.1. **Preliminary Results: Representations of Lattice Polynomials.** An *n-ary* (*lattice*) *polynomial function* from L^n to L is defined recursively as follows:

(i) For each $i \in [n]$ and each $c \in L$, the projection $\mathbf{x} \mapsto x_i$ and the constant function $\mathbf{x} \mapsto c$ are polynomial functions from L^n to L.
(ii) If f and g are polynomial functions from L^n to L, then $f \vee g$ and $f \wedge g$ are polynomial functions from L^n to L.
(iii) Any polynomial function from L^n to L is obtained by finitely many applications of the rules (i) and (ii).

If rule (i) is only applied for projections, then the resulting polynomial functions are called (*lattice*) *term functions* [4, 16, 11]. Idempotent polynomial functions are referred to as (*discrete*) *Sugeno integrals* [9, 14]. In the case of bounded distributive lattices, Goodstein [13] showed that polynomial functions are exactly those which allow representations in disjunctive normal form (see Proposition 3.1 below, first appearing in [13, Lemma 2.2]; see also Rudeanu [20, Chapter 3, §3] for a later reference).

Proposition 3.1. *Let L be a bounded distributive lattice. A function $f\colon L^n \to L$ is a polynomial function if and only if there exist $a_I \in L$, $I \subseteq [n]$, such that, for every $\mathbf{x} \in L^n$,*

$$f(\mathbf{x}) = \bigvee_{I\subseteq[n]} (a_I \wedge \bigwedge_{i\in I} x_i). \tag{1}$$

In particular, a function $f\colon L^n \to L$ is a Sugeno integral if and only if it can be represented by a formula (1) where $\bigwedge_{I\subseteq[n]} a_I = 0$ and $\bigvee_{I\subseteq[n]} a_I = 1$.

The expression given in (1) is usually referred to as the *disjunctive normal form* (DNF) representation of the polynomial function f.

The following corollaries belong to the folklore of lattice theory and are immediate consequences of Theorems D and E in [13].

Corollary 3.2. *Every polynomial function is completely determined by its restriction to $\{0,1\}^n$.*

Corollary 3.3. *A function $g\colon \{0,1\}^n \to L$ can be extended to a polynomial function $f\colon L^n \to L$ if and only if it is nondecreasing. In this case, the extension is unique.*

It is easy to see that the DNF representations of a polynomial function $f\colon L^n \to L$ are not necessarily unique. For instance, in Proposition 3.1, if for some $I \subseteq [n]$ we have $a_I = \bigvee_{J\subset I} a_J$, then for every $\mathbf{x} \in L^n$,

$$f(\mathbf{x}) = \bigvee_{I\neq J\subseteq[n]} (a_J \wedge \bigwedge_{i\in J} x_i).$$

However, using Corollaries 3.2 and 3.3, one can easily set canonical ways of constructing these normal form representations of polynomial functions. (For a discussion on the uniqueness of DNF representations of lattice polynomial functions see [9].)

Let $2^{[n]}$ denote the set of all subsets of $[n]$. For $I \subseteq [n]$, let $\mathbf{e}_I$ be the *characteristic vector* of I, i.e., the n-tuple in L^n whose i-th component is 1 if $i \in I$, and 0 otherwise. Note that the mapping $\alpha\colon 2^{[n]} \to \{0,1\}^n$ given by $\alpha(I) = \mathbf{e}_I$, for every $I \in 2^{[n]}$, is an order-isomorphism.

Proposition 3.4 (Goodstein [13])**.** *Let L be a bounded distributive lattice. A function $f\colon L^n \to L$ is a polynomial function if and only if for every $\mathbf{x} \in L^n$,*

$$f(\mathbf{x}) = \bigvee_{I\subseteq[n]} \big(f(\mathbf{e}_I) \wedge \bigwedge_{i\in I} x_i\big).$$

Moreover, if f is a Sugeno integral, then $f(\mathbf{e}_\emptyset) = 0$ and $f(\mathbf{e}_{[n]}) = 1$.

It is noteworthy that Proposition 3.4 leads to the following characterization of the essential arguments of polynomial functions in terms of necessary and sufficient conditions [7]. Here, x_i is said to be *essential* in $f\colon L^n \to L$ if there are $a_1, \ldots, a_n, b_i \in L$, $a_i \neq b_i$, such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n).$$

Proposition 3.5. *Let L be a bounded distributive lattice and let $f\colon L^n \to L$ be a polynomial function. Then for each $j \in [n]$, x_j is essential in f if and only if there exists a set $J \subseteq [n] \setminus \{j\}$ such that $f(\mathbf{e}_J) < f(\mathbf{e}_{J\cup\{j\}})$.*

Remark 3.6. The assumption that the lattice L is bounded is not very crucial. Let L' be the lattice obtained from L by adjoining new top and bottom elements $\top$ and $\bot$, if necessary. Then, if f is a polynomial function over L induced by a polynomial p, then p induces a polynomial function f' on L', and it holds that the restriction of f' to L coincides with f. Similarly, if L' is a distributive lattice and f' is a polynomial function on L' represented by the DNF

$$\bigvee_{I\subseteq[n]} (a_I \wedge \bigwedge_{i\in I} x_i),$$

then by omitting each term $a_I \wedge \bigwedge_{i\in I} x_i$ where $a_I = \bot$ and replacing each term $a_I \wedge \bigwedge_{i\in I} x_i$ where $a_I = \top$ by $\bigwedge_{i\in I} x_i$, we obtain an equivalent polynomial representation for f'. Unless f' is a constant function that takes value $\top$ or $\bot$ and this element is not in L, the function f on L induced by this new polynomial coincides with the restriction of f' to L.

3.2. **Bisymmetric Sugeno Integrals on Chains.** A Sugeno integral $f\colon L^n \to L$ is said to be a *weighted disjunction* if it is of the form

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{i\in[n]} (a_i \wedge x_i) \tag{2}$$

for some elements a_i ($i \in [n]$) of L such that $\bigvee_{i\in[n]} a_i = 1$. Observe that every weighted disjunction (2) is idempotent since for every $x \in L$,

$$f(x, \ldots, x) = \bigvee_{i\in[n]} (a_i \wedge x) = (\bigvee_{i\in[n]} a_i) \wedge x = 1 \wedge x = x.$$

Thus every weighted disjunction is a Sugeno integral.

We say that f has *chain form* if

$$f(x_1, x_2, \ldots, x_n) = \bigvee_{i\in[n]} (a_i \wedge x_i) \vee \bigvee_{1\leq\ell\leq r} (a_{S_\ell} \wedge \bigwedge_{i\in S_\ell} x_i), \tag{3}$$

for a chain of subsets $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r \subseteq [n]$, $r \geq 1$, $|S_1| \geq 2$, and some elements a_i ($i \in [n]$), a_{S_ℓ} ($1 \leq \ell \leq r$) of L such that $a_{S_l} \leq a_{S_t}$ whenever $l \leq t$, $\bigvee_{i\in[n]} a_i \vee \bigvee_{\ell\in[r]} a_{S_\ell} = 1$, and for all $i \notin S_1$, there is a $j \in S_1$ such that $a_i \leq a_j$. As in the case of weighted disjunctions, it is easy to verify that every function which has chain form is a Sugeno integral.

Theorem 3.7. *Let L be a bounded chain. A Sugeno integral $f\colon L^n \to L$ is bisymmetric if and only if it is a weighted disjunction or it has chain form.*

Theorem 3.7 is a consequence of the following two results proved in [8]. The first provides sufficient conditions for a Sugeno integral to be bisymmetric in the general case of distributive lattices. Its proof was achieved by case analysis.

Lemma 3.8. *Let L be a distributive lattice. Assume that a function $f\colon L^n \to L$ is a weighted disjunction or has chain form. Then f is bisymmetric.*

The necessity of the conditions in Theorem 3.7 followed from the lemma below, which can be verified by induction on the arity of functions.

Lemma 3.9. *Let L be a bounded chain. If a Sugeno integral $f\colon L^n \to L$ is bisymmetric, then it is a weighted disjunction or it has chain form.*

Observe that every binary Sugeno integral is bisymmetric. This is not the case for $n \geq 3$. For instance, let $f\colon [0,1]^3 \to [0,1]$ be given by $f(x_1,x_2,x_3) = (0.5 \wedge x_3) \vee (x_1 \wedge x_2)$. Then

$$f\big(f(1,0,1), f(1,1,0), f(0,0,0)\big) = 0.5 \neq 0 = f\big(f(1,1,0), f(0,1,0), f(1,0,0)\big).$$

Clearly, this function is not a weighted disjunction and it does not have chain form because the condition "for all $i \notin S_1$, there is a $j \in S_1$ such that $a_i \leq a_j$" is not fulfilled. However, a minor modification of f yields a function that has chain form (and is hence bisymmetric): $f'(x_1,x_2,x_3) = (0.5 \wedge x_1) \vee (0.5 \wedge x_3) \vee (x_1 \wedge x_2)$.

4. Concluding Remarks and Future Work

We have obtained an explicit form of bisymmetric Sugeno integrals on chains. However, we do not know whether Theorem 3.7 still holds in the general case of Sugeno integrals over distributive lattices. This constitutes a topic of ongoing research.

Another problem which was not addressed concerns the following generalization of bisymmetry. Two operations $f\colon A^n \to A$ and $g\colon A^m \to A$ are said to commute, denoted $f \perp g$, if for all $a_{ij} \in A$ ($1 \leq i \leq n$, $1 \leq j \leq m$), the following identity holds

$$f\big(g(a_{11}, a_{12}, \dots, a_{1m}), g(a_{21}, a_{22}, \dots, a_{2m}), \dots, g(a_{n1}, a_{n2}, \dots, a_{nm})\big) = g\big(f(a_{11}, a_{21}, \dots, a_{n1}), f(a_{12}, a_{22}, \dots, a_{n2}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})\big).$$

Commutation has been considered in the realm of aggregation function theory. In this context, functions are often regarded as mappings $f\colon \bigcup_{n\geq 1} A^n \to A$, and bisymmetry is naturally generalized to what is referred to as strong bisymmetry. Denoting by f_n the restriction of f to A^n, the map f is said to be *strongly bisymmetric* if for any $n, m \geq 1$, we have $f_n \perp f_m$. This generalization is both natural and useful from the application point of view. To illustrate this, suppose one is given data in tabular form, say an $n \times m$ matrix, to be meaningfully fused into a single representative value. One could first aggregate the data by rows and then aggregate the resulting column; or one could first aggregate the columns and then the resulting row. What is expressed by the property of strong bisymmetry is that the final outcome is the same under both procedures. Extending the notion of Sugeno integral (as a polynomial function) to such families, we are thus left with the problem of describing those families of Sugeno integrals which are strongly bisymmetric.

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