This is a postprint version of an article published as *Aequationes Mathematicae* **81** (2011) 263–278. DOI: 10.1007/s00010-010-0058-6.
The final publication is available at www.springerlink.com.

SELF-COMMUTING LATTICE POLYNOMIAL FUNCTIONS ON CHAINS

MIGUEL COUCEIRO AND ERKKO LEHTONEN

Abstract. We provide sufficient conditions for a lattice polynomial function to be self-commuting. We explicitly describe self-commuting polynomial functions on chains.

1. Introduction

Two operations $f\colon A^n \to A$ and $g\colon A^m \to A$ are said to commute, and we write $f \perp g$, if for all $a_{ij} \in A$ ($1 \leq i \leq n$, $1 \leq j \leq m$), the following identity holds

$$f\bigl(g(a_{11}, a_{12}, \dots, a_{1m}), g(a_{21}, a_{22}, \dots, a_{2m}), \dots, g(a_{n1}, a_{n2}, \dots, a_{nm})\bigr)$$
$$= g\bigl(f(a_{11}, a_{21}, \dots, a_{n1}), f(a_{12}, a_{22}, \dots, a_{n2}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})\bigr).$$

For $n = m = 2$, the above condition stipulates that

$$f\bigl(g(a_{11}, a_{12}), g(a_{21}, a_{22})\bigr) = g\bigl(f(a_{11}, a_{21}), f(a_{12}, a_{22})\bigr).$$

The Eckmann-Hilton Theorem [11] asserts that if both f and g have an identity element and $f \perp g$, then in fact $f = g$ and $(A; f)$ is a commutative monoid on A.

The relevance of the notion of commutation is made apparent in works of several authors. In particular, commutation is the defining property of entropic algebras [21, 22, 26] (an algebra is entropic if its operations commute pairwise; idempotent entropic algebras are called modes) and centralizer clones [17, 18, 24, 27] (the centralizer of a set F of operations is the set of all operations that commute with every operation in F; the centralizer of F is a clone).

We are interested in functions f that commute with themselves. An algebra $(A; f)$ where f is a binary operation that satisfies the identity

$$f\bigl(f(a_{11}, a_{12}), f(a_{21}, a_{22})\bigr) = f\bigl(f(a_{11}, a_{21}), f(a_{12}, a_{22})\bigr)$$

is called a medial groupoid [15, 16]. Hence, self-commutation generalizes the notion of mediality, and it has been investigated by several authors (see, e.g., [1, 3, 19, 25]). In the realm of functional equation theory, self-commutation is also known as bisymmetry; for motivations and general background, see [2, 3, 13].

In this paper, we address the question of characterizing classes of self-commuting operations. In Section 2, we recall basic notions in the universal-algebraic setting and settle the terminology used throughout the paper. Moreover, we develop general tools for tackling the question of describing self-commuting operations.

1991 *Mathematics Subject Classification.* Primary 39B05; Secondary 08A40.

Key words and phrases. Self-commutation, bisymmetry, lattice polynomial function, functional equation.

This question is partially answered for lattice polynomial functions in Section 3. We start by surveying well-known results concerning normal form representations of these lattice functions which we then use to specify those polynomial functions on bounded chains which are self-commuting. This explicit description is obtained by providing sufficient conditions for a lattice polynomial function to be self-commuting, and by showing that these conditions are also necessary in the particular case of polynomial functions on bounded chains.

In Section 4 we point out problems which are left unsettled, and motivate directions of future research.

2. Preliminaries

In this section, we introduce some notions and terminology as well as establish some preliminary results that will be used in the sequel. For an integer $n \geq 1$, set $[n] := \{1, 2, \ldots, n\}$. With no danger of ambiguity, we denote the tuple $(x_1, \ldots, x_n)$ of any length by $\mathbf{x}$.

2.1. **Operations and clones.** Let A be an arbitrary nonempty set. An *operation* on A is a map $f \colon A^n \to A$ for some integer $n \geq 1$, called the *arity* of f. We denote by $\mathcal{O}_A^{(n)}$ the set of all n-ary operations on A, and we denote by $\mathcal{O}_A$ the set of all finitary operations on A, i.e., $\mathcal{O}_A := \bigcup_{n \geq 1} \mathcal{O}_A^{(n)}$.

For $1 \leq i \leq n$, the operation $(a_1, \ldots, a_n) \mapsto a_i$ is called the n-ary i-th *projection* on A. If $f \in \mathcal{O}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{O}_A^{(m)}$, then the *composition* of f with $g_1, \ldots, g_n$ is the operation $f(g_1, \ldots, g_n) \in \mathcal{O}_A^{(m)}$ given by

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f\big(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m)\big)$$

for all $a_1, \ldots, a_m \in A$. A *clone* on A is a set $\mathcal{C} \subseteq \mathcal{O}_A$ of operations on A that contains all projections on A and is closed under composition, i.e., $f(g_1, \ldots, g_n) \in \mathcal{C}$ whenever $f, g_1, \ldots, g_n \in \mathcal{C}$ and the composition is defined.

The clones on A constitute a complete lattice under inclusion order. Therefore, for each set $F \subseteq \mathcal{O}_A$ of operations on A, there exists a smallest clone on A that contains F, which will be denoted by $\langle F \rangle$ and called the *clone generated by* F.

We assume that the reader is familiar with basic notions of universal algebra and lattice theory, and we refer the reader to [4, 5, 8, 9, 10, 14, 23] for general background.

2.2. **Essential variables and variable identification minors.** We say that the i-th variable of $f \colon A^n \to A$ is *essential,* if there exist elements $a_1, \ldots, a_n, b \in A$ such that

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

If a variable is not essential in f, then we say that it is *inessential* in f.

Let $f \in \mathcal{O}_A^{(n)}$, $g \in \mathcal{O}_A^{(m)}$. We say that f is obtained from g by *simple variable substitution,* or f is a *simple minor* of g, if there is a mapping $\sigma \colon [m] \to [n]$ such that

$$f(x_1, \ldots, x_n) = g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}).$$

For distinct indices $i, j \in [n]$, the function $f_{i \leftarrow j} \colon A^n \to A$ obtained from f by the simple variable substitution

$$f_{i \leftarrow j}(x_1, \ldots, x_n) := f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n)$$

is called a *variable identification minor* of f, obtained by identifying x_i with x_j. Note that the i-th variable of $f_{i\leftarrow j}$ is always inessential.

For studies of classes of operations that are closed under taking simple minors, see, e.g., [6, 20].

2.3. **Self-commutation.** Let $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ be operations on A. We say that f *commutes* with g, denoted $f \perp g$, if for all a_{ij} ($i \in [n]$, $j \in [m]$), it holds that

$$\begin{aligned} &f\big(g(a_{11}, a_{12}, \dots, a_{1m}), g(a_{21}, a_{22}, \dots, a_{2m}), \dots, g(a_{n1}, a_{n2}, \dots, a_{nm})\big) \\ &\qquad = g\big(f(a_{11}, a_{21}, \dots, a_{n1}), f(a_{12}, a_{22}, \dots, a_{n2}), \dots, f(a_{1m}, a_{2m}, \dots, a_{nm})\big). \end{aligned}$$

We clearly have that $f \perp g$ if and only if $g \perp f$. If $f \perp f$, then we say that f is *self-commuting.*

For any set $F \subseteq \mathcal{O}_A$ of operations on A, the *centralizer* of F is the set

$$\bigcap_{f \in F} \{g \in \mathcal{O}_A : f \perp g\}.$$

It is a well-known fact that the centralizer of F is a clone.

Let $f\colon A^n \to A$ and $c \in A$. For $i \in [n]$, we define $f_c^i\colon A^{n-1} \to A$ to be the operation

$$f_c^i(a_1, \dots, a_{n-1}) = f(a_1, \dots, a_{i-1}, c, a_i, \dots, a_{n-1}).$$

The following lemma is an immediate consequence of the fact that the centralizer of a set of operations is a clone.

Lemma 2.1. *Assume that $f \in \mathcal{O}_A^{(n)}$ is self-commuting. Then the following assertions hold.*

(i) For every $i, j \in [n]$ $(i \neq j)$, $f_{i\leftarrow j}$ is self-commuting.
(ii) If f preserves $c \in A$, i.e., $f(c, \dots, c) = c$, then for every $i \in [n]$, f_c^i is self-commuting.

3. Self-commuting lattice polynomial functions

Let $(L; \wedge, \vee)$ be a lattice. With no danger of ambiguity, we denote lattices by their universes. In this section we study the self-commutation property on lattice polynomial functions, i.e., mappings $f\colon L^n \to L$ which can be obtained as compositions of the lattice operations and applied to variables (projections) and constants. As shown by Goodstein [12], lattice polynomial functions have neat normal form representations in the case when L is a bounded distributive lattice. Thus, unless stated otherwise, we assume in what follows that L is a bounded distributive lattice, and the least and the greatest elements of L will be denoted by 0 and 1, respectively.

We recall results concerning the representation of lattice polynomials, and we introduce some related concepts and terminology in Subsection 3.1. Then, we consider the property of self-commutation on these functions. We start by providing sufficient conditions for a lattice polynomial function to be self-commuting, namely, if a lattice polynomial function has so-called chain form, then it is self-commuting. Moreover, we will show that whenever the underlying lattice is a chain, the self-commuting polynomial functions are precisely the ones that have chain form.

3.1. **Preliminary results: representations of lattice polynomial functions.** The members of the clone generated by the lattice operations $\wedge$ and $\vee$ and all constant functions $x \mapsto c$, $c \in L$, are called (*lattice*) *polynomial functions* on L. Idempotent polynomial functions are also referred to as (*discrete*) *Sugeno integrals* [7, 13]. In the case of bounded distributive lattices, Goodstein [12] showed that polynomial functions are exactly those which allow representations in disjunctive normal form (see Proposition 3.1(i) below, first appearing in [12, Lemma 2.2]; see also Rudeanu [23, Chapter 3, §3] for a later reference).

Let $2^{[n]}$ denote the set of all subsets of $[n]$. For $I \subseteq [n]$, let $\mathbf{e}_I$ be the *characteristic vector* of I, i.e., the tuple in L^n whose i-th component is 1 if $i \in I$, and 0 otherwise. Note that the mapping $\alpha\colon 2^{[n]} \to \{0,1\}^n$ given by $\alpha(I) = \mathbf{e}_I$, for every $I \in 2^{[n]}$, is an order-isomorphism.

Proposition 3.1. *Let L be a bounded distributive lattice.*

(i) (Goodstein [12]). A function $f\colon L^n \to L$ is a polynomial function if and only if there exist $a_I \in L$, $I \subseteq [n]$ such that, for every $\mathbf{x} \in L^n$,

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \bigl(a_I \wedge \bigwedge_{i \in I} x_i\bigr). \tag{3.1}$$

(ii) (Goodstein [12]). A function $f\colon L^n \to L$ is a polynomial function if and only if for every $\mathbf{x} \in L^n$,

$$f(\mathbf{x}) = \bigvee_{I \subseteq [n]} \bigl(f(\mathbf{e}_I) \wedge \bigwedge_{i \in I} x_i\bigr). \tag{3.2}$$

(iii) Let $f\colon L^n \to L$ be a polynomial function on L given by (3.1). The following are equivalent:

- *$a_I = f(\mathbf{e}_I)$ for all $I \subseteq [n]$.*
- *$a_I \leq a_J$ whenever $I \subseteq J \subseteq [n]$.*

The expression given by (3.1) is usually referred to as a *disjunctive normal form* (DNF) representation of the polynomial function f. It is easy to see that the DNF representations of a polynomial function $f\colon L^n \to L$ are not necessarily unique. For instance, if for some $I \subseteq [n]$ we have $a_I \leq \bigvee_{J \subsetneq I} a_J$, then for every $\mathbf{x} \in L^n$,

$$f(\mathbf{x}) = \bigvee_{I \neq J \subseteq [n]} \bigl(a_J \wedge \bigwedge_{i \in J} x_i\bigr).$$

(For a discussion on the uniqueness of DNF representations of lattice polynomial functions see [7].)

A DNF representation of a lattice polynomial function f of the form (3.2) is called a *canonical DNF* of f. By definition, there exists a unique canonical DNF for every lattice polynomial function.

We refer to the term $a_I \wedge \bigwedge_{i \in I} x_i$ in the canonical DNF of f as the *I-term* of f, or the *term* of f *associated with the set* I, and we say that $|I|$ is its *size*. We say that the I-term $a_I \wedge \bigwedge_{i \in I} x_i$ is *essential* if $a_I > \bigvee_{J \subsetneq I} a_J$; otherwise, we say that it is *inessential*.

Remark 3.2. Observe that we can omit inessential terms from the canonical DNF of a polynomial function f in order to obtain equivalent polynomial representations of f.

Remark 3.3. Every polynomial function is completely determined by its restriction to $\{0,1\}^n$. Moreover, every polynomial function is range-idempotent, i.e., for every $c \in \operatorname{Im} f$, $f(c,\dots,c)=c$.

Remark 3.4. Considering only bounded distributive lattices rather than arbitrary distributive lattices is not a serious restriction. Namely, let L be a distributive lattice, not necessarily bounded, and let L' be the lattice obtained from L by adjoining new top and bottom elements $\top$ and $\bot$, if necessary. Then, if f is a polynomial function on L induced by a polynomial p, then p induces a polynomial function f' on L', and it holds that the restriction of f' to L coincides with f. Similarly, if f' is a polynomial function on L' represented by the DNF

$$\bigvee_{I \subseteq [n]} \big(a_I \wedge \bigwedge_{i \in I} x_i\big),$$

then by omitting each term $a_I \wedge \bigwedge_{i \in I} x_i$ where $a_I = \bot$ and replacing each term $a_I \wedge \bigwedge_{i \in I} x_i$ where $a_I = \top$ by $\bigwedge_{i \in I} x_i$, we obtain an equivalent polynomial representation for f'. Unless f' is the constant function that takes value $\top$ or $\bot$ and this element is not in L, the function f on L induced by this new polynomial coincides with the restriction of f' to L. Moreover, self-commutation is preserved by this extension: if $f' \colon (L')^n \to L'$ is the extension of $f \colon L^n \to L$, then f is self-commuting if and only if f' is self-commuting.

3.2. **Self-commuting polynomial functions on chains.** In this subsection we provide explicit descriptions of self-commuting polynomial functions on chains. In order to simplify notation, if I is a singleton or a two-element set, then we write a_i and a_{ij} for $a_{\{i\}}$ and $a_{\{i,j\}}$, respectively. Also, to avoid cumbersome notation, we shall often denote $\wedge$ by juxtaposition.

We say that a lattice polynomial function $f \colon L^n \to L$ has *chain form* if

$$f(\mathbf{x}) = a_\emptyset \vee \bigvee_{i \in [n]} a_i x_i \vee \bigvee_{1 \le \ell \le r} \big(a_{S_\ell} \bigwedge_{i \in S_\ell} x_i\big), \tag{3.3}$$

for some subsets $S_1,\dots,S_r \subseteq [n]$ with at least two elements, $r \ge 0$, such that for $1 \le \ell \le r$, the S_ℓ-term is essential (in the canonical DNF of f) and the following conditions are satisfied:

(1) $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r$,
(2) if $r \ge 1$, then for all $i \in [n]$, there exists a $j \in S_1$ such that $a_i \le a_j$,
(3) $a_I \le a_J$ whenever $I \subseteq J$ and

$$I, J \in \{\emptyset\} \cup \{\{i\} : i \in [n]\} \cup \{S_\ell : 1 \le \ell \le r\}.$$

In the above definition, if $r = 0$, then f has the form

$$f(\mathbf{x}) = a_\emptyset \vee \bigvee_{i \in [n]} a_i x_i$$

for some elements $a_\emptyset$, a_i ($i \in [n]$) of L. In this case f is called a *weighted supremum.*

Theorem 3.5. *Let L be a bounded chain. A polynomial function $f \colon L^n \to L$ is self-commuting if and only if it has chain form.*

Theorem 3.5 will be a consequence of Lemmas 3.6 and 3.9 below. We start with the result that provides sufficient conditions for a polynomial to be self-commuting in the general case of distributive lattices.

Lemma 3.6. *Let L be a distributive lattice. Assume that a function $f\colon L^n \to L$ has chain form. Then f is self-commuting.*

Proof. Assume first that f is a weighted supremum, i.e., $r = 0$ in (3.3). We have that

$$\begin{aligned}
&f\bigl(f(x_{11}, x_{12}, \dots, x_{1n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})\bigr) \\
&\qquad = a_\emptyset \vee \bigvee_{i \in [n]} a_i \Bigl(a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij}\Bigr) = a_\emptyset \vee \bigvee_{i \in [n]} \bigvee_{j \in [n]} a_i a_j x_{ij} \\
&\qquad = a_\emptyset \vee \bigvee_{j \in [n]} \bigvee_{i \in [n]} a_j a_i x_{ij} = a_\emptyset \vee \bigvee_{j \in [n]} a_j \Bigl(a_\emptyset \vee \bigvee_{i \in [n]} a_i x_{ij}\Bigr) \\
&\qquad\qquad = f\bigl(f(x_{11}, x_{21}, \dots, x_{n1}), \dots, f(x_{1n}, x_{2n}, \dots, x_{nn})\bigr).
\end{aligned}$$

Thus, f is self-commuting.

Assume then that $r \geq 1$. The assumption that for every $i \in [n]$ there is a $j \in S_1$ such that $a_i \leq a_j$ implies that $a_i \leq a_{S_\ell}$ (and hence $a_i a_{S_\ell} = a_i$) for all $i \in [n]$ and for all $\ell \in [r]$. Using this observation and distributivity we get

$$\begin{aligned}
&f\bigl(f(x_{11}, x_{12}, \dots, x_{1n}), f(x_{21}, x_{22}, \dots, x_{2n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})\bigr) \\
&= a_\emptyset \vee \bigvee_{i \in [n]} a_i \Bigl[a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq r} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij}\Bigr] \\
&\qquad \vee \bigvee_{1 \leq t \leq r} a_{S_t} \bigwedge_{i \in S_t} \Bigl[a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq r} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij}\Bigr] \\
&= a_\emptyset \vee \underbrace{\bigvee_{i \in [n]} \bigvee_{j \in [n]} a_i a_j x_{ij}}_{\text{(I)}} \vee \underbrace{\bigvee_{i \in [n]} \bigvee_{1 \leq \ell \leq r} a_i \bigwedge_{j \in S_\ell} x_{ij}}_{\text{(II)}} \\
&\qquad \vee \underbrace{\bigvee_{1 \leq t \leq r} \bigwedge_{i \in S_t} \Bigl[a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq r} a_{S_t} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij}\Bigr]}_{\text{(III)}}.
\end{aligned}$$

Every term in (II) is absorbed by a term in (I): for every $i \in [n]$, there is a $k \in S_1$ such that $a_i \leq a_k$, and hence for any $\ell \in [r]$, the term $a_i \bigwedge_{j \in S_\ell} x_{ij} = a_i a_k x_{ik} \bigwedge_{j \in S_\ell \setminus \{k\}} x_{ij}$ in (II) is absorbed by the term $a_i a_k x_{ik}$ in (I).

In (III), for a fixed t, if $\ell > t$, then the term $a_{S_t} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij} = a_{S_t} \bigwedge_{j \in S_\ell} x_{ij}$ is absorbed by $a_{S_t} \bigwedge_{j \in S_t} x_{ij} = a_{S_t} a_{S_t} \bigwedge_{j \in S_t} x_{ij}$, and hence (III) simplifies to

$$\bigvee_{1 \leq t \leq r} \underbrace{\bigwedge_{i \in S_t} \Bigl[a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq t} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij}\Bigr]}_{\text{(IV)}}. \tag{3.4}$$

For a fixed t, (IV) expands to the disjunction of all possible conjunctions $\bigwedge_{i \in S_t} \phi_i$ of $|S_t|$ terms, where each ϕ_i is one of $a_\emptyset$, $a_j x_{ij}$ for some $j \in [n]$, or $a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij}$ for some $1 \leq \ell \leq t$. If $\phi_i = a_\emptyset$ for some $i \in S_t$, then the conjunction is absorbed by $a_\emptyset$. If $\phi_i = a_i x_{ii}$ for some $i \in S_t$, then the conjunction is absorbed by the term $a_i a_i x_{ii} = a_i x_{ii}$ in (I).

Consider then such a conjunction $\bigwedge_{i \in S_t} \phi_i$ where for all $i \in S_t$, ϕ_i is not equal to $a_\emptyset$ nor to $a_i x_{ii}$, but for some $i \in S_t$, $\phi_i = a_j x_{ij}$ for some $j \neq i$. By our assumption, there is a $k \in S_1$ such that $a_j \leq a_k$ and hence $a_j = a_j a_k$. We have that ϕ_k equals either $a_\ell x_{k\ell}$ for some $\ell \neq k$ or $a_{S_\ell} \bigwedge_{m \in S_\ell} x_{km}$ for some $1 \leq \ell \leq t$. In the former case, $\phi_i \phi_k = a_j a_k x_{ij} a_\ell x_{k\ell}$, and hence the conjunction $\bigwedge_{i \in S_t} \phi_i$ is absorbed by the term $a_k a_\ell x_{k\ell}$ in (I). In the latter case, $\phi_i \phi_k = a_j a_k x_{ij} a_{S_\ell} \bigwedge_{m \in S_\ell} x_{km}$, and hence the conjunction $\bigwedge_{i \in S_t} \phi_i$ is absorbed by the term $a_k a_k x_{kk} = a_k x_{kk}$ in (I).

The remaining conjunctions that arise from the expansion of (IV) are of the form

$$\bigwedge_{i \in S_t} a_{S_{\ell_i}} \bigwedge_{j \in S_{\ell_i}} x_{ij}$$

where $1 \leq \ell_i \leq t$ ($i \in S_t$). Let $\ell' = \min_{i \in S_t} \ell_i$. If $\ell' < t$, then this conjunction is absorbed by $a_{S_{\ell'}} \bigwedge_{i \in S_{\ell'}} \bigwedge_{j \in S_{\ell'}} x_{ij}$, which arises from the expansion of

$$\bigwedge_{i \in S_{\ell'}} \Big[a_\emptyset \vee \bigvee_{j \in [n]} a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq \ell'} a_{S_\ell} \bigwedge_{j \in S_\ell} x_{ij} \Big]$$

in (3.4). Thus, the only remaining conjunction that arises from the expansion of (IV) is $a_{S_t} \bigwedge_{i \in S_t} \bigwedge_{j \in S_t} x_{ij}$.

Thus, we have that

$$f\big(f(x_{11}, x_{12}, \ldots, x_{1n}), f(x_{21}, x_{22}, \ldots, x_{2n}), \ldots, f(x_{n1}, x_{n2}, \ldots, x_{nn})\big) = a_\emptyset \vee \bigvee_{i \in [n]} \bigvee_{j \in [n]} a_i a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq r} a_{S_\ell} \bigwedge_{i \in S_\ell} \bigwedge_{j \in S_\ell} x_{ij}. \tag{3.5}$$

In a similar way, we can deduce that

$$f\big(f(x_{11}, x_{21}, \ldots, x_{n1}), f(x_{12}, x_{22}, \ldots, x_{n2}), \ldots, f(x_{1n}, x_{2n}, \ldots, x_{nn})\big) = a_\emptyset \vee \bigvee_{j \in [n]} \bigvee_{i \in [n]} a_i a_j x_{ij} \vee \bigvee_{1 \leq \ell \leq r} a_{S_\ell} \bigwedge_{j \in S_\ell} \bigwedge_{i \in S_\ell} x_{ij}. \tag{3.6}$$

The right-hand sides of (3.5) and (3.6) are clearly equal, and we conclude that f is self-commuting. □

The necessity of the conditions in Theorem 3.5 follows from Lemma 3.9. In its proof, we will need the following two auxiliary results.

Lemma 3.7. *Let $f \colon L^n \to L$ be a polynomial function. Assume that the I-term of f is essential.*

(i) If $i, j \notin I$, then the I-term of $f_{i \leftarrow j}$ is essential.
(ii) If $i, j \in I$, then the I'-term of $f_{i \leftarrow j}$ is essential, where $I' := I \setminus \{i\}$.

Proof. (i) Since $i, j \notin I$, for every $K \subseteq I$ we have that $f_{i \leftarrow j}(\mathbf{e}_K) = f(\mathbf{e}_K)$. Thus,

$$f_{i \leftarrow j}(\mathbf{e}_I) = f(\mathbf{e}_I) > \bigvee_{K \subsetneq I} f(\mathbf{e}_K) = \bigvee_{K \subsetneq I} f_{i \leftarrow j}(\mathbf{e}_K),$$

where the inequality holds because the I-term of f is essential. Hence the I-term of $f_{i \leftarrow j}$ is essential.

(ii) Since $i, j \in I$, for every $K \subseteq I$ we have that $f_{i \leftarrow j}(\mathbf{e}_K) = f(\mathbf{e}_{\nabla K})$, where

$$\nabla K = \begin{cases} K & \text{if } j \notin K, \\ K \cup \{i\} & \text{if } j \in K. \end{cases}$$

Since $\nabla I' = I$, we have

$$f_{i \leftarrow j}(\mathbf{e}_{I'}) = f(\mathbf{e}_{\nabla I'}) = f(\mathbf{e}_I) > \bigvee_{T \subsetneq I} f(\mathbf{e}_T) \geq \bigvee_{K \subsetneq I'} f(\mathbf{e}_{\nabla K}) = \bigvee_{K \subsetneq I'} f_{i \leftarrow j}(\mathbf{e}_K),$$

where the first inequality holds because the I-term of f is essential. The second inequality holds because every joinand on the right-hand side appears on the left-hand side. Thus the I'-term of $f_{i \leftarrow j}$ is essential. □

Lemma 3.8. *Let $f\colon L^n \to L$ be a polynomial function, and assume that the i-th variable of f is inessential.*

(i) For every $I \subseteq [n]$ such that $i \in I$, we have that the I-term of f is inessential.
(ii) If the I-term of f is essential (and hence $i \notin I$ by (i)), then the I'-term of f^i_c is essential, where $c \in L$, $I' := \sigma[I]$ and $\sigma\colon [n] \setminus \{i\} \to [n-1]$ is the bijection given by

$$\sigma(a) = \begin{cases} a & \text{if } a < i, \\ a - 1 & \text{if } a > i. \end{cases}$$

(iii) f has chain form if and only if f^i_c $(c \in L)$ has chain form.

Proof. (i) Let $I \subseteq [n]$ be such that $i \in I$. Since the i-th variable of f is inessential, we have $f(\mathbf{e}_I) = f(\mathbf{e}_{I \setminus \{i\}})$. Thus, the I-term of f is inessential.

(ii) By the definition of f^i_c and by the assumption that the i-th variable of f is inessential, we have that $f^i_c(\mathbf{e}_{I'}) = f(\mathbf{e}_I)$. The following calculation shows that the I'-term of f^i_1 is essential:

$$f^i_c(\mathbf{e}_{I'}) = f(\mathbf{e}_I) > \bigvee_{T \subsetneq I} f(\mathbf{e}_T) \geq \bigvee_{K \subsetneq I'} f(\mathbf{e}_{\sigma^{-1}(K)}) = \bigvee_{K \subsetneq I'} f^i_c(\mathbf{e}_K).$$

Here the first inequality holds since the I-term of f is essential, and the second inequality holds since every joinand on the right-hand side occurs on the left-hand side.

(iii) Assume first that f has chain form, i.e., f is of the form (3.3) and satisfies conditions (1)–(3) of the definition of chain form. Since the i-th variable is inessential in f, then by (i) none of the essential terms of f is associated with a subset $S \subseteq [n]$ that contains i. Therefore, we may assume that f is of the form

$$f(\mathbf{x}) = a_\emptyset \vee \bigvee_{j \in [n] \setminus \{i\}} a_j x_j \vee \bigvee_{1 \leq \ell \leq r} \bigl(a_{S_\ell} \bigwedge_{j \in S_\ell} x_j\bigr),$$

where $i \notin S_r \supseteq \cdots \supseteq S_1$. Then

$$f^i_c(\mathbf{x}) = a_\emptyset \vee \bigvee_{j \in [n] \setminus \{i\}} a_{\sigma(j)} x_{\sigma(j)} \vee \bigvee_{1 \leq \ell \leq r} \bigl(a_{\sigma(S_\ell)} \bigwedge_{j \in \sigma(S_\ell)} x_j\bigr), \tag{3.7}$$

which has chain form, since σ is bijective.

For the converse, assume that f does not have chain form. Then

(a) f has essential terms associated with incomparable subsets $I, J \subseteq [n]$ of size at least two; or
(b) f has the form (3.3) and there is a $j \in [n] \setminus S_1$ such that $a_j > a_k$ for every $k \in S_1$.

If (a) holds, then by (ii) the $\sigma(I)$-term and $\sigma(J)$-term of f^i_c are essential but $\sigma(I)$ and $\sigma(J)$ remain incomparable. Hence f^i_c does not have chain form.

If (b) holds, then $j \neq i$, because the i-th variable is inessential in f. Moreover, f_c^i is given by (3.7) as above. From the fact that σ is bijective it follows that $a_{\sigma(j)} > a_{\sigma(k)}$ for every $k \in S_1$. Since $\sigma(j) \notin \sigma(S_1)$, f_c^i does not have chain form. □

Lemma 3.9. *Let L be a bounded chain. If a polynomial function $f\colon L^n \to L$ is self-commuting, then it has chain form.*

Proof. The statement clearly holds for $n = 1$ and $n = 2$, since every unary or binary polynomial function has chain form.

Suppose $n = 3$. Then, by Proposition 3.1, the canonical DNF of f has the form

$$f = a_\emptyset \vee a_1x_1 \vee a_2x_2 \vee a_3x_3 \vee a_{12}x_1x_2 \vee a_{13}x_1x_3 \vee a_{23}x_2x_3 \vee a_{123}x_1x_2x_3, \tag{3.8}$$

where $a_I \leq a_J$ whenever $I \subseteq J$.

We have that

$$\begin{aligned}
&f\bigl(f(1,1,0), f(0,1,1), f(0,0,0)\bigr)\\
&\quad= f(a_\emptyset \vee a_1 \vee a_2 \vee a_{12},\ a_\emptyset \vee a_2 \vee a_3 \vee a_{23},\ a_\emptyset)\\
&\quad= f(a_{12}, a_{23}, a_\emptyset)\\
&\quad= a_\emptyset \vee a_1a_{12} \vee a_2a_{23} \vee a_3a_\emptyset \vee\\
&\qquad\qquad a_{12}a_{12}a_{23} \vee a_{13}a_{12}a_\emptyset \vee a_{23}a_{23}a_\emptyset \vee a_{123}a_{12}a_{23}a_\emptyset\\
&\quad= a_1 \vee a_2 \vee a_{12}a_{23},\\
&f\bigl(f(1,0,0), f(1,1,0), f(0,1,0)\bigr)\\
&\quad= f(a_\emptyset \vee a_1,\ a_\emptyset \vee a_1 \vee a_2 \vee a_{12},\ a_\emptyset \vee a_2)\\
&\quad= f(a_1, a_{12}, a_2)\\
&\quad= a_\emptyset \vee a_1a_1 \vee a_2a_{12} \vee a_3a_2 \vee\\
&\qquad\qquad a_{12}a_1a_{12} \vee a_{13}a_1a_2 \vee a_{23}a_{12}a_2 \vee a_{123}a_1a_{12}a_2\\
&\quad= a_1 \vee a_2,
\end{aligned} \tag{3.9}$$

and since f is self-commuting, we have $a_1 \vee a_2 \vee a_{12}a_{23} = a_1 \vee a_2$. This equality translates into $a_{12}a_{23} \leq a_1 \vee a_2$. In a similar way, after suitably permuting the rows and columns of the 3×3 matrix $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ used in (3.9), we can deduce that

$$a_{ij}a_{jk} \leq a_i \vee a_j \leq a_{ij} \tag{3.10}$$

for $\{i, j, k\} = \{1, 2, 3\}$.

Since L is a chain, we have for some choice of $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$ that $a_{\alpha\beta} \leq a_{\beta\gamma} \leq a_{\alpha\gamma}$. Inequalities (3.10) then imply

$$a_\alpha \vee a_\beta = a_{\alpha\beta} \quad \text{and} \quad a_\beta \vee a_\gamma = a_{\beta\gamma},$$

i.e., the terms associated with the sets $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are inessential. Thus, f has at most one essential term of size 2. If f has no essential term of size 2, then it has chain form. Otherwise f has precisely one essential term of size 2, namely, the one associated with $S_1 = \{\alpha, \gamma\}$. Without loss of generality, assume that $\{\alpha, \gamma\} = \{1, 2\}$. Then $a_{12} > a_1 \vee a_2$ and

$$a_3 \leq a_{13} = a_{13}a_{12} \leq a_1 \vee a_2.$$

Since L is a chain, this implies that $a_3 \leq a_1$ or $a_3 \leq a_2$, and we conclude that f has chain form.

We now proceed by induction on n. Assume that the claim holds for $n < \ell$ for some $\ell \geq 4$. We show that it holds for $n = \ell$. In the sequel, we will write "$X \parallel Y$" for "$X \not\subseteq Y$ and $Y \not\subseteq X$".

Let $f = \bigvee_{I \subseteq [\ell]} a_I \bigwedge_{i \in I} x_i$ be self-commuting, and in view of Proposition 3.1, assume that $a_I \leq a_J$ whenever $I \subseteq J$. If f has no essential terms of size at least 2, then f has chain form. Thus, we suppose that f has an essential term of size at least 2. First we show that the essential terms of f of size at least 2 are associated with a chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r$. For a contradiction, suppose that there are $I, J \subseteq [k]$ such that $|I| \geq 2$, $|J| \geq 2$, $I \parallel J$ and the I-term and the J-term of f are essential. Fix such I and J so that $|I \cap J|$ is the largest possible, $|I| \leq |J|$ and $|J|$ is the largest among such pairs. We need to consider several cases.

Case 1: $|I \cap J| \geq 2$. Take distinct $i, j \in I \cap J$. By Lemma 3.7, the I'-th and the J'-th terms of $f_{i \leftarrow j}$ are essential, where $I' = I \setminus \{i\}$, $J' = J \setminus \{i\}$. Since $I' \parallel J'$, $f_{i \leftarrow j}$ does not have chain form. Choose $c \in \operatorname{Im} f_{i \leftarrow j}$. By Lemma 3.8, $(f_{i \leftarrow j})^i_c$ does not have chain form either. By our induction hypothesis, $(f_{i \leftarrow j})^i_c$ is not self-commuting, which contradicts Lemma 2.1 which asserts that self-commutation is preserved by the two operations that we just performed.

Case 2: $|I \cap J| \leq 1$ and $|J| \geq 3$. Take distinct $i, j \in J \setminus I$. As in Case 1, we derive a contradiction by taking $I' = I$, $J' = J \setminus \{i\}$.

Case 3: $|I \cap J| = 1$, $|J| = 2$ and $\ell \geq 5$. Take distinct $i, j \in [\ell] \setminus (I \cup J)$. As in Case 1, we derive a contradiction by taking $I' = I$, $J' = J$.

Case 4: $|I \cap J| = 1$, $|J| = 2$ and $\ell = 4$. Then $[4] \setminus (I \cup J) = \{i\}$ for some $i \in [4]$. Since the I-term and the J-term of $f^i_{a_\emptyset}$ are clearly essential, $f^i_{a_\emptyset}$ does not have chain form, and by the induction hypothesis it is not self-commuting. This is a contradiction to Lemma 2.1, since f is assumed to be self-commuting.

Case 5: $|I \cap J| = 0$ and $|J| = 2$. Note that in this case the essential terms of f of size at least 2 are associated with pairwise disjoint sets and have size exactly 2. Let $I = \{i, j\}$, $J = \{k, t\}$. If f has an essential term of size (at least) 2 associated with a set K distinct (and hence disjoint) from I and J, then by Lemma 3.7, the J-term and the K-term of $f_{i \leftarrow j}$ are essential. Since $J \parallel K$, $f_{i \leftarrow j}$ does not have chain form.

Otherwise I and J are the only essential terms of size at least 2. Since L is a chain, $a_I \leq a_J$ or $a_J \leq a_I$. Assume without loss of generality that $a_J \leq a_I$. Then f is of the form

$$f = a_\emptyset \vee a_i x_i \vee a_j x_j \vee a_{ij} x_i x_j \vee a_k x_k \vee a_t x_t \vee a_{kt} x_k x_t \vee \cdots$$

where the remaining terms are associated with singletons distinct from $\{i\}$, $\{j\}$, $\{k\}$, $\{t\}$. Consider then

$$f_{i \leftarrow j} = (a_i \vee a_j \vee a_{ij}) x_j \vee a_k x_k \vee a_t x_t \vee a_{kt} x_k x_t \vee \cdots .$$

But $f_{i \leftarrow j}$ does not have chain form, because condition (2) in the definition of chain form is not satisfied by $f_{i \leftarrow j}$ as the following computation shows:

$$a_i \vee a_j \vee a_{ij} = a_{ij} \geq a_{kt} > a_k \vee a_t,$$

where the first inequality holds by our assumption that $a_J \leq a_I$, and the second inequality holds by the assumption that the J-term of f is essential.

Thus, for a $c \in \operatorname{Im} f_{i \leftarrow j}$, $(f_{i \leftarrow j})^i_c$ does not have chain form by Lemma 3.8. The induction hypothesis implies that $(f_{i \leftarrow j})^i_c$ is not self-commuting, which contradicts Lemma 2.1.

Thus, the essential terms of f of size at least 2 are associated with a chain $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_r$. To complete the proof, we need to show that for every $i \notin S_1$, there is a $j \in S_1$ such that $a_i \leq a_j$. For a contradiction, suppose that there is an $i \in [n]$ such that $a_i > a_j$ for every $j \in S_1$. We consider two cases.

Case 1: $|S_1| \geq 3$. Take distinct $k, m \in S_1$, and consider $f_{k \leftarrow m}$. Using the polynomial expression obtained by removing all inessential terms from the canonical DNF of f, and applying Lemma 3.7, one can see that the essential terms of $f_{k \leftarrow m}$ of size at least 2 are associated with a chain $S'_1 \subseteq S'_2 \subseteq \cdots \subseteq S'_r$, where $S'_i := S_i \setminus \{k\}$ for $1 \leq i \leq r$, and the $\{m\}$-term of f is $(a_k \vee a_m)x_m$. Since $a_i > a_j$ for every $j \in S_1$, the induction hypothesis implies that $(f_{k \leftarrow m})^k_c$ is not self-commuting for any $c \in \operatorname{Im} f_{k \leftarrow m}$. As above, this contradicts Lemma 2.1.

Case 2: $|S_1| = 2$. Then there is a $t \in [\ell] \setminus (S_1 \cup \{i\})$. Consider $f_{t \leftarrow i}$. The essential terms of $f_{t \leftarrow i}$ of size at least 2 are associated with a chain whose least element is S_1, and the $\{i\}$-term of this function is $(a_i \vee a_t)x_i$. Since for every $j \in S_1$, $a_i > a_j$, we also have $a_i \vee a_t > a_j$. Hence $f_{t \leftarrow i}$ does not have chain form, and thus $(f_{t \leftarrow i})^t_c$, $c \in \operatorname{Im} f_{t \leftarrow i}$, is not self-commuting. As above, this contradicts Lemma 2.1. □

Proof of Theorem 3.5. Lemma 3.6, when restricted to chains, shows that the condition is sufficient. Necessity follows from Lemma 3.9. □

4. Concluding remarks and future work

We have obtained an explicit form of self-commuting polynomial functions on chains (in fact, unique up to addition of inessential terms). As Lemma 3.6 asserts, our condition is sufficient in the general case of polynomial functions over distributive lattices. Whether it is also a necessary condition in the general case constitutes a topic of ongoing research.

Another problem which was not addressed concerns commutation. As mentioned, self-commutation appears within the scope of functional equation theory under the name of bisymmetry. Also, in the context of aggregation function theory, functions are often regarded as mappings $f\colon \bigcup_{n \geq 1} A^n \to A$. In this framework, bisymmetry is naturally generalized to what is referred to as strong bisymmetry. Denoting by f_n the restriction of f to A^n, the map f is said to be *strongly bisymmetric* if for any $n, m \geq 1$, we have $f_n \perp f_m$. This generalization is both natural and useful from the application point of view. To illustrate this, suppose one is given data in tabular form, say an $n \times m$ matrix, to be meaningfully fused into a single representative value. One could first aggregate the data by rows and then aggregate the resulting column; or one could first aggregate the columns and then the resulting row. What is expressed by the property of strong bisymmetry is that the final outcome is the same under both procedures. Extending the notion of polynomial functions to such families, we are thus left with the problem of describing those families of polynomial functions which are strongly bisymmetric.

Acknowledgments

We would like to thank Jean-Luc Marichal for introducing us to the topic and for helpful discussions. Also, we are grateful to the reviewers for their useful comments which led into improvement of the current manuscript.

References

[1] J. Aczél, *On mean values,* Bull. Amer. Math. Soc. *54* (1948), 392–400.
[2] J. Aczél, *Lectures on Functional Equations and Their Applications,* Academic Press, New York, 1966; Dover Publications, Mineola, 2006.
[3] J. Aczél and J. Dhombres, *Functional Equations in Several Variables,* Encyclopedia of Mathematics and Its Applications, vol. 31, Cambridge University Press, Cambridge, 1989.
[4] G. Birkhoff, *Lattice Theory,* 3rd edition, Coll. Publ., XXV, American Mathematical Society, 1967.
[5] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra,* Springer-Verlag, 1981.
[6] M. Couceiro and S. Foldes, *On closed sets of relational constraints and classes of functions closed under variable substitution,* Algebra Universalis *54* (2005), 149–165.
[7] M. Couceiro and J.-L. Marichal, *Characterizations of discrete Sugeno integrals as polynomial functions over distributive lattices,* Fuzzy Sets and Systems *161* (2010), 694–707.
[8] B. Davey and H. A. Priestley, *Introduction to Lattices and Order,* 2nd edition, Cambridge University Press, Cambridge, 2002.
[9] K. Denecke and S. L. Wismath, *Universal Algebra and Applications in Theoretical Computer Science,* Chapman & Hall/CRC, Boca Raton, 2002.
[10] K. Denecke and S. L. Wismath, *Universal Algebra and Coalgebra,* World Scientific, 2009.
[11] B. Eckmann and P. J. Hilton, *Group-like structures in general categories I—multiplications and comultiplications,* Math. Ann. *145* (1962), 227–255.
[12] R. L. Goodstein, *The solution of equations in a lattice,* Proc. Roy. Soc. Edinburgh Sect. A *67* (1965/1967), 231–242.
[13] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation Functions,* Encyclopedia of Mathematics and Its Applications, vol. 127, Cambridge University Press, Cambridge, 2009.
[14] G. Grätzer, *Universal Algebra,* 2nd edition, Springer-Verlag, 1979.
[15] J. Ježek and T. Kepka, *Equational theories of medial groupoids,* Algebra Universalis *17* (1983), 174–190.
[16] J. Ježek and T. Kepka, *Medial groupoids,* Rozpravy Československé Akad. Věd, Řada Mat. Přírod. Věd *93* (1983), 93 pp.
[17] B. Larose, *On the centralizer of the join operation of a finite lattice,* Algebra Universalis *34* (1995), 304–313.
[18] H. Machida and I. G. Rosenberg, *On the centralizers of monoids in clone theory,* 33rd IEEE International Symposium on Multiple-Valued Logic (ISMVL 2003), 16–19 May 2003, Tokyo, Japan, IEEE Computer Society, 2003, pp. 303–308.
[19] J.-L. Marichal, P. Mathonet and E. Tousset, *Characterization of some aggregation functions stable for positive linear transformations,* Fuzzy Sets and Systems *102* (1999), 293–314.
[20] N. Pippenger, *Galois theory for minors of finite functions,* Discrete Math. *254* (2002), 405–419.
[21] G. Polák and Á. Szendrei, *Independent basis for the identities of entropic groupoids,* Comment. Math. Univ. Carolin. *22* (1981), 71–85.
[22] A. Romanowska and J. D. Smith, *Modes,* World Scientific, Singapore, 2002.
[23] S. Rudeanu, *Lattice Functions and Equations,* Discrete Mathematics and Theoretical Computer Science Series, Springer-Verlag, London, 2001.
[24] J. Sichler and V. Trnková, *Essential operations in centralizer clones,* Algebra Universalis *59* (2008), 277–301.
[25] J.-P. Soublin, *Étude algébrique de la notion de moyenne,* J. Math. Pure Appl. *50* (1971), 53–264.
[26] M. M. Stronkowski, *Cancellation in entropic algebras,* Algebra Universalis *60* (2009), 439–468.
[27] V. Trnková and J. Sichler, *All clones are centralizer clones,* Algebra Universalis *61* (2009), 77–95.

(M. Couceiro) University of Luxembourg, Mathematics Research Unit, 6, rue Richard Coudenhove-Kalergi, L–1359 Luxembourg, Luxembourg
E-mail address: `miguel.couceiro@uni.lu`

(E. Lehtonen) University of Luxembourg, Computer Science and Communications Research Unit, 6, rue Richard Coudenhove-Kalergi, L–1359 Luxembourg, Luxembourg
E-mail address: `erkko.lehtonen@uni.lu`