MMSE Performance Analysis of Generalized Multibeam Satellite Channels

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Abstract—**Aggressive frequency reuse in the return link (RL) of multibeam satellite communications (SatComs) is crucial towards the implementation of next generation, interactive satellite services. In this direction, multiuser detection has shown great potential in mitigating the increased intrasystem interferences, induced by a tight spectrum reuse. Herein we present an analytic framework to describe the linear Minimum Mean Square Error (MMSE) performance of multiuser channels that exhibit full receive correlation: an inherent attribute of the RL of multibeam SatComs. Analytic, tight approximations on the MMSE performance are proposed for cases where closed form solutions are not available in the existing literature. The proposed framework is generic, thus providing a generalized solution straightforwardly extendable to various fading models over channels that exhibit full receive correlation. Simulation results are provided to show the tightness of the proposed approximation with respect to the available transmit power.**

Index Terms—**Multiuser detection, multibeam satellites, return link, linear minimum mean square error receivers.**

I. INTRODUCTION & RELATED WORK

TOWARDS next generation broadband, interactive, multi-beam satellite systems, higher throughput requirements especially for the return link (RL) are necessary. Multiuser detection (MUD) has already proven a key technique for the enhancement of the spectral efficiency of the RL [1] and the Forward Link (FL) [2] of multibeam satellite communications (SatComs). However, the inherent differences between the satellite and the terrestrial channel impose difficulties in the theoretical performance analysis of MUD for satellite systems. Added to that, linear MMSE receivers are more difficult to analyze compared to the optimal capacity achieving successive interference cancellation (SIC) techniques.

The basic characteristic of the RL multibeam satellite channel is the high receive correlation among the channels at the satellite side, resulting from the total lack of scatterers, as well as the practical collocation of the multibeam antenna

feeds, at the satellite side. As a result, all the receive antennas experience identical fading instances from a specific user. Specifically, each user's vector channel towards the satellite antennas is defined by the antenna radiation pattern. According to the multiplicative model and assuming block fading, the product between a single fading coefficient and the channel vector will characterize the fading instance. Subsequently, the channel matrix is given by the product of a full rank multibeam gain matrix with a diagonal matrix, composed of random fading coefficients, as originally proposed in [1] and extended in [3] and [4], for more realistic system models.

Linear multiuser receivers where extensively examined in [5]. More recently, numerous contributions examined the asymptotic performance of these architectures by employing random matrix theory methods described in [6] and the references therein. In the same field, the work of [7] is also noted.

Moreover, the mutual information of channels for which exact analytical formulas do not exist in literature, has been described via analytical, tight bounds. To this end, the Minkowski's inequality was initially proposed in [8] to provide a tight lower bound on the channel mutual information. This technique was extended in [9] to provide a new lower bound on the ergodic capacity of distributed MIMO systems. Furthermore, generic bounds on the sum rate of zero forcing (ZF) receivers over Rayleigh, Rice and correlated channels were given in [10].

Focusing on SatComs, the performance of linear MMSE receivers is not trivial to characterize analytically. In the existing literature, two different modeling approaches have been considered. The authors of [11] have modeled fading as the sum of two random matrices, each following a specific distribution, without however reporting results for linear receivers. The other approach, namely the multiplicative model, was introduced in [1], where the performance of optimal non-linear and linear receivers was given as a function of matrix arguments, with high computational complexity, for theoretical channel models. This model has been extended in [3], for composite, realistic channels and a novel, simple lower bound for the channel capacity has been deduced. Finally, the incorporation of rain fading has been tackled in [4], using an identical channel model, in which asymptotic closed form expressions for the system's capacity and the MMSE were derived.

The purpose of the present contribution is to extend the results of [3] and [4] for the case of linear MMSE receivers, thus

providing an analytic framework that covers the generalized type of channels with full receive correlation.

To this end, the information theoretic link between the channel mutual information and the MMSE given in [12], is exploited[1]. Results on the MMSE performance of the generalized satellite channel, for which the mutual information has been analytically described only through tight bounds, are proposed.

The rest of this paper is structured as follows. In Sec. II the generalized multibeam satellite channel is introduced. The analytic framework proposed is described in detail in Sec. III. Simulation results are presented in Sec. IV, while Sec. V concludes the findings of this paper.

Notation: Throughout the paper, $\mathcal{E}\{\cdot\}$, $(\cdot)^\dagger$, denote the expectation, and the conjugate transpose of a matrix. Bold face lower case characters denote column vectors and upper case denote matrices, while $\mathbf{I}_n$ denotes an identity matrix of size n.

II. Signal & Channel Model

Let us consider a multiuser (MU) single input multiple output (SIMO) multiple access channel (MAC) with K single-antenna terminals transmitting towards a single receiver equipped with $N = K$ antennas. The commonly adopted assumption of symmetric systems (e.g. [9]) is a requisite for the analysis. As explained with more detail in [3], mathematical manipulations are facilitated by square channel matrices. Also, in multibeam SatComs, the assumption of a single user per beam scheduled in each time slot is usually adopted [1], [3], [14]–[16]. The input-output relationship reads as

$$\mathbf{y} = \sqrt{\gamma}\mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{s} \in \mathcal{C}^{K \times 1}$ is the transmitted baseband signal vector, such that $\mathcal{E}\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}_K$, $\mathbf{y} \in \mathcal{C}^{N \times 1}$ is the received signal vector, $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ is the complex noise vector and γ is the normalized transmit signal-to-noise ratio (SNR). Also, the matrix $\mathbf{H} \in \mathcal{C}^{N \times K}$ represents the complex-valued block fading channel and admits a number of expressions depending on the model.

A. Generalized Multibeam Channel: In the present contribution, the channel matrix is assumed to be modeled as the product of a full rank matrix with a random fading matrix. The assumption of full receive correlation, that will be hereafter considered, shall impose a diagonal fading matrix [1], [3] on the multiplicative model, thus yielding

$$\mathbf{H} = \mathbf{B}\mathbf{D}^{1/2}, \tag{2}$$

where $\mathbf{B}$ is a fixed, deterministic, full rank matrix with normalized power (i.e. $\text{trace}(\mathbf{B}^2) = K$) and $\mathbf{D} = \text{diag}(\mathbf{d})$ is a random diagonal matrix composed of the fading coefficients. The deterministic matrix $\mathbf{B}$ models the multibeam antenna radiation pattern, while the random elements of the diagonal matrix $\mathbf{D}$, are drawn from distributions that model various small or large scale effects. Subsequently, the adopted model, provides a generalized approach towards systems with full

receive side correlation. By changing the fading distribution different systems can be modeled. Two specific examples, relevant for multibeam SatComs will be studied hereafter.

1) Composite fading: This channel has been proposed in [3] to model the multibeam mobile satellite channel, by incorporating small scale Rician fading and large scale log normal shadowing. The resulting model reads as

$$\mathbf{H}_c = \mathbf{B}\mathbf{H}_d\sqrt{\mathbf{X}_d}, \tag{3}$$

where $\mathbf{H}_d$ and $\mathbf{X}_d$ are diagonal fading matrices with elements drawn from Rice and log-normal distributions respectively.

2) Rain fading: When fixed satellite services are assumed, then the high antenna directivity imposes an AWGN channel and allows for the utilization of higher frequency bands, where rain fading can dramatically deteriorate system performance. For this case the model proposed in [4] will yield

$$\mathbf{H}_r = \mathbf{B}\sqrt{\mathbf{L}_d}, \tag{4}$$

where the random elements of $\mathbf{L}_d$ are drawn from a log-normal distribution in dB scale[2].

III. Performance Analysis

The performance of MUD is evaluated via the achievable SINR_k after MMSE detection at the kth user, given by [18]

$$\gamma_{\text{mmse},k} = \frac{1}{\left[\left(\mathbf{I}_K + \gamma\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\right]_{kk}} - 1. \tag{5}$$

Averaged over the users and all channel instances, the system spectral efficiency, will be given by [18]

$$C_{\text{mmse}}(\gamma) = \mathcal{E}_\mathbf{H}\left\{I_{\text{mmse}}\right\} = \mathcal{E}_\mathbf{H}\left\{\frac{1}{K}\sum_{i=1}^{K}\log_2\left(1 + \gamma_{\text{mmse},k}\right)\right\}, \tag{6}$$

in bits/sec/Hz. By combining (5) and (6) and directly applying the Jensen's inequality the following stands

$$C_{\text{mmse}} \geq \mathcal{E}_\mathbf{H}\left\{-\log_2\left(\frac{1}{K}\text{trace}\left(\left(\mathbf{I}_K + \gamma\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\right)\right)\right\}. \tag{7}$$

Thus, in common practice, another measure, namely the instantaneous, per user, minimum mean square error, is also adopted [6], [12]:

$$\epsilon^2 = \frac{1}{K}\text{trace}\left(\left(\mathbf{I}_K + \gamma\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\right). \tag{8}$$

The presence of the inverse of a matrix sum in (7) hinders the computation of ϵ^2 when the eigenvalue distribution of $\mathbf{H}^\dagger\mathbf{H}$ cannot be computed analytically. To solve this problem, we propose to use the approximation that follows.

Lemma 1. *The instantaneous per user MMSE of an uplink MU SIMO system can be approximated by*

$$\hat{\epsilon}^2 = \left(1 + \gamma\exp\left(\frac{1}{K}\ln\det\left(\mathbf{H}^\dagger\mathbf{H}\right)\right)\right)^{-1}, \tag{9}$$

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[1] For completeness we also note the work of [13] that provides an analytic framework for linear MMSE receivers, which however does not cover the case studied herein.

[2] In natural units, these elements are produced by exponentiated log-normal elements [17] and the relevant distribution is also referred to as log-log-normal.

Proof: In [12], an explicit relationship between the channel mutual information $I_e = \log_2 \det \left(I + \gamma H^\dagger H \right)$ in bits/sec/Hz and ϵ^2 was derived by differentiating with respect to the SNR. This result is easily extended for vector channels yielding [6]

$$\gamma \frac{\partial}{\partial \gamma} I_e(\gamma) = K - \text{trace} \left(\left(I_K + \gamma H^\dagger H \right)^{-1} \right) \quad (10)$$

$$= K(1 - \epsilon^2). \quad (11)$$

Furthermore, with the use of Minkowski's inequality, a bound on I_e, tight over the whole SNR region and exact for the high SNRs, has been derived in [8]:

$$I_e \geq I_{lb} = K \log_2 \left(1 + \gamma \det \left(\left(H^\dagger H \right)^{1/K} \right) \right). \quad (12)$$

By differentiation with respect to γ we get the following:

$$\gamma \frac{\partial}{\partial \gamma} I_{lb} = \frac{\gamma K \det \left(\left(H^\dagger H \right)^{1/K} \right)}{1 + \gamma \det \left(\left(H^\dagger H \right)^{1/K} \right)} \overset{(10)}{\Longrightarrow}$$

$$\hat{\epsilon}^2 = \left(1 + \gamma \det \left(\left(H^\dagger H \right)^{1/K} \right) \right)^{-1}. \quad (13)$$

Finally (13) can be rewritten as in (9). ∎

Since the differentiation does not preserve the direction of the bound, the characteristics of this approximation will be studied in more detail in the following lemma.

Lemma 2. *Let $\hat{\epsilon}^2$ be the derived approximation of ϵ^2, then $\hat{\epsilon}^2$ and ϵ^2 as functions of γ, will have a crossing point.*

Proof: Denoting as λ_i the ordered eigenvalues of $H^\dagger H$, let us define the function $D(\gamma) \doteq I_e(\gamma) - I_{lb}(\gamma)$, where $I_e(\gamma) = \sum_i \log(1 + \gamma \lambda_i)$ and $I_{lb}(\gamma) = \log \left(1 + \gamma \left(\prod_i \lambda_i \right)^{1/K} \right)$, with the following properties: 1) $D(0) = 0$. 2) For γ_o sufficiently large, but finite, we have that $D(\gamma_o) = 0$. From properties 1 and 2, a straightforward application of Cauchy's mean-value theorem yields that there will be at least one point $\gamma^* \in (0, \gamma_o)$, with zero derivative and subsequently a crossing point between the approximation and the actual function. ∎

Lemma 3. *The average over the channel realizations, per user MMSE, $\mathcal{E}_H \{ \hat{\epsilon}^2 \}$, can be bounded by*

$$\left(1 + \gamma \exp \left(\frac{1}{K} \mathcal{E}_H \left\{ \ln \det \left(H^\dagger H \right) \right\} \right) \right)^{-1} \quad (14)$$

Proof: Let us consider the function $\phi(x) = \left(\left(1 + \exp(x) \right) \right)^{-1}$ which is convex for $x < 0$ and concave for $x > 0$. By applying Jensen's inequality over the two regions we get that

$$\mathcal{E}_H \{ \hat{\epsilon}^2 \} = \begin{cases} \geq a, & x \leq 0 \\ \leq a, & x > 0 \end{cases} \quad (15)$$

for $a = \left(1 + \gamma \exp \left(\frac{1}{K} \mathcal{E}_H \left\{ \ln \det \left(H^\dagger H \right) \right\} \right) \right)^{-1}$ and $x = 1/K \cdot \ln \det \left(\left(H^\dagger H \right) \right) + \ln \gamma$. ∎

Lemma 4. *The average per user MMSE approximation, for the composite multibeam satellite channel (as given by (3)), is analytically described by (18), where μ_m (dB) is the mean of*
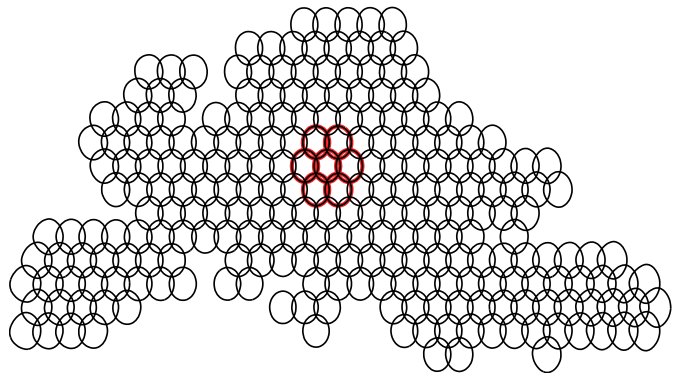

Fig. 1.   Multibeam antenna beam pattern for European coverage provided by the ESA [19]. A specific cluster of 7 beams (in red), is chosen.

the normal distribution, K_r the Rician factor, the g_2 function is given as $g_2(s^2) = \log s^2 + E_i(s^2)$ (as defined in [3]), E_i is the exponential integral and $s^2 = K_r$ is the non-centrality parameter of the associated χ^2-distribution.

Proof: The expectation of the logarithm of the determinant of the specific channel matrix has been derived in [3]. Direct application of these calculations on (14) concludes the proof. ∎

Lemma 5. *The per user average MMSE spectral efficiency of a multibeam satellite system under rain fading (as modeled by (4)) can be approximated by the closed form formula*

$$\mathcal{E}_H \left\{ \hat{\epsilon}_{rain}^2 \right\} = \left(1 + \gamma \exp \left(\frac{1}{K} \ln \left(\det B^2 \right) + \mu_l \right) \right)^{-1}, \quad (16)$$

where μ_l is the mean of the equivalent log-normal distribution.

Proof: Following the proof of Lemma 3, an analytical bound on $\mathcal{E}_H \left\{ \hat{\epsilon}_{rain}^2 \right\}$ will read as

$$\left(1 + \gamma \exp \left(\frac{1}{K} \mathcal{E}_H \left\{ \ln \det \left(B^2 L_d \right) \right\} \right) \right)^{-1} \quad (17)$$

where $\ln \det(B^2 L_d) = \ln \left(\det B^2 \right) + \ln \det L_d$ since the matrices are square and $\mathcal{E} \{ \ln \left(\det L_d \right) \} = \sum_{i=1}^{K} \mathcal{E} \{ \ln l_i \} = K \cdot \mu_l$. The parameter μ_l is the mean of the related log-normal distribution given by $\mu_l = \exp(\mu_m + \sigma^2/2)$. As before, μ_m and σ are the mean and variance of the associated normal distribution. Since B is a deterministic matrix, from the above analysis, (16) is deduced. ∎

IV. SIMULATION RESULTS

Realistic beam patterns from multibeam coverage over Europe are chosen to provide accurate results for the system performance. In Fig. 1 the employed 245-beam pattern is visualised. This beam pattern is designed for European coverage by the European Space Agency (ESA) and provided in the framework of [19]. In multibeam systems, multiple GWs - each only serving a fraction of the total number of beams- are deployed due to limitations in the feeder link bandwidth. In order to focus on a specific GW, only a part of the total beam pattern (i.e. 7 beams) has been considered in the analysis as presented in Fig. 1. These beams correspond to the beams served by a specific gateway (GW) over which MUD will

$$\mathcal{E}_{\mathbf{H}}\left\{\hat{\epsilon}_{\text{comp}}^2\right\} = \left(1 + \gamma \exp\left(\frac{1}{K}\log\left(\det \mathbf{B}^2\right) + \mu_m + g_2\left(s^2\right) - \log\left(K_r + 1\right)\right)\right)^{-1}. \tag{18}$$
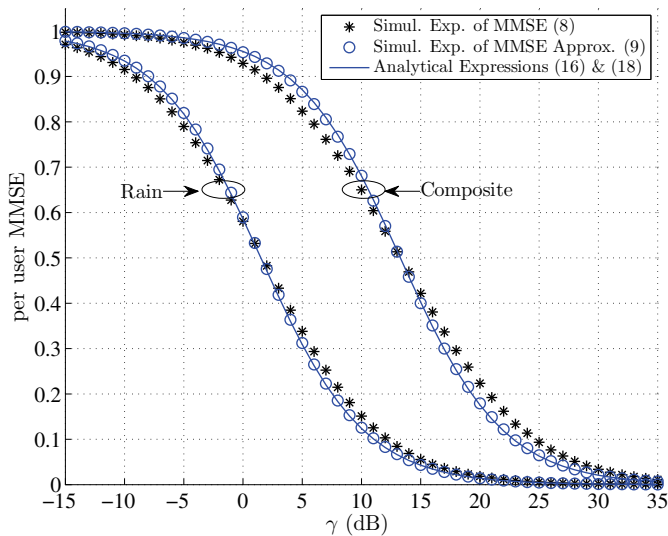

Fig. 2. MMSE for seven normalized ESA beams with $K_r = 10$dB, $\mu_m =$ -2.63, σ=0.5.

be performed in a distributed GW scenario. Monte Carlo (MC) simulations have been performed to calculate the exact performance over the random channel. The analytic results of Sec. III provide deterministic approximations on the system performance with respect to γ and are compared to the MC results to illustrate the tightness of the deduced formulas.

In Fig. 2, the simulated expectation of the MMSE given by (8) over 1000 channel instances for the composite and the rain faded channels, is presented with star markers. In the same figure, circle markers represent the simulated expectation of the proposed approximation, given by (9). The analytic formulas presented in (18) and (16), are plotted with continuous lines. It is clear that the analytical expressions precisely fit the expected values of the proposed approximation, thus providing a strong analytic performance evaluation tool. Over the entire SNR region, the maximum deviation from the actual MMSE value is no more than 1.5dB for the composite fading case, while for the rain fading, the deviation is less than 1dB. Especially for the SNR regions around 12.5 dB and 2dB for the two cases respectively, the approximation becomes almost exact. In practical systems, these regions are of main interest. Consequently, the proposed expressions are very tight for systems with finite users and conventional link budgets, in contrast to asymptotic results based on large system dimensions or high SNR approaches.

V. CONCLUSIONS

A new, accurate approximation for the MMSE of a MU SIMO channel that exhibits full receive correlation has been derived, under an analytic framework that generalizes to various fading distributions. In particular, two cases of generalized channels, namely the mobile composite and the fixed satellite

channels, have been evaluated in terms of MMSE performance. The proposed analytic expressions provide accurate approximations of the expected value of the system MMSE, valid for finite system dimensions and over a large SNR region.

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