A SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM IN A POLYGONAL DOMAIN*

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Abstract. The semilinear reaction-diffusion equation $-\varepsilon^2\triangle u+b(x,u)=0$ with Dirichlet boundary conditions is considered in a convex polygonal domain. The singular perturbation parameter ε is arbitrarily small, and the "reduced equation" $b(x,u_0(x))=0$ may have multiple solutions. An asymptotic expansion for u is constructed that involves boundary and corner layer functions. By perturbing this asymptotic expansion, we obtain certain sub- and super-solutions and thus show the existence of a solution u that is close to the constructed asymptotic expansion. The polygonal boundary forces the study of the nonlinear autonomous elliptic equation $-\triangle z+f(z)=0$ posed in an infinite sector, and then well-posedness of the corresponding linearized problem.

1. Introduction. Consider the singularly perturbed semilinear reaction-diffusion boundary-value problem

$$Fu \equiv -\varepsilon^2\triangle u+b(x,u)=0, \qquad x=(x_1,x_2)\in\Omega\subset\mathbb{R}^2, \tag{1.1a}$$
$$u(x)=g(x), \qquad x\in\partial\Omega. \tag{1.1b}$$

Here Ω is a convex polygonal domain, $\triangle=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ is the Laplace operator, and ε is a small positive parameter.

The "reduced problem" associated with (1.1) is defined by formally setting $\varepsilon=0$ in (1.1a), i.e.

$$b(x,u_0(x))=0 \quad \text{for } x\in\bar{\Omega}. \tag{1.2}$$

It is assumed that (1.2) has a smooth solution u_0 that is stable in a sense to be described below. The hypotheses on b are such as to include the possibility of multiple solutions to (1.2) and therefore to (1.1). Since it may happen that $u_0\neq g$ on $\partial\Omega$, the solutions may exhibit boundary layer behavior near $\partial\Omega$. Problems such as (1.1) have been considered in 1 dimension [4] and in 2 dimensions in the case that the boundary $\partial\Omega$ is smooth [3, 6, 9]. In these papers it is shown that for ε sufficiently small, there is a solution of (1.1) that is close to u_0 in the interior of Ω. In addition, robust numerical methods for the solution of (1.1) have been presented and analysed in [7, and references therein] in dimension 1, and in [6] in dimension 2 in the case when $\partial\Omega$ is smooth.

In this paper we consider the problem (1.1) in a plane convex polygonal domain. The presence of vertices in $\partial\Omega$ causes some complications in the analysis. In addition to the boundary layer functions, some "corner layer functions" must be used in the construction of an asymptotic expansion. These corner layer functions are solutions to certain nonlinear boundary value problems in a convex sector, and the added complications come in studying these problems, for which mere solution existence is not straightforward. The construction exhibits the boundary and corner layer

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behavior of the solution, which will be used in a forthcoming numerical analysis of the problem.

We denote vertices of Ω by $\{P_j\}_1^M$ and the sides by $\{\Gamma_j\}_1^M$. The vertices are arranged in counterclockwise order with the vertex P_{j-1} being at the intersection of Γ_{j-1} and Γ_j, under the notation $\Gamma_{M+1} = \Gamma_1$. We assume that the function b is smooth and that g is smooth on each Γ_j and continuous at each vertex P_j. In addition we make the following assumptions.

A1 (*stable reduced solution*) There is a number $\gamma > 0$ such that

$$b_u(x, u_0(x)) > \gamma^2 > 0 \quad \text{for all } x \in \Omega.$$

A2 (*boundary condition*) The boundary data $g(x)$ from (1.1b) satisfy

$$\int_{u_0(x)}^{v} b(x,s)\, ds > 0 \qquad \text{for all } \; v \in \big(u_0(x), g(x)\big]', \qquad x \in \partial\Omega.$$

Here the notation $(a, b]'$ is defined to be $(a, b]$ when $a < b$ and $[b, a)$ when $a > b$, while $(a, b]' = \emptyset$ when $a = b$.

A3 (*corner condition*) For each vertex P_j, if $g(P_j) \neq u_0(P_j)$, then

$$\frac{b(P_j, g(P_j))}{g(P_j) - u_0(P_j)} > 0.$$

A4 Only to simplify our presentation, we make a further assumption that

$$u_0(x) < g(x) \qquad \text{for all } \; x \in \partial\Omega.$$

Using A4, we can simplify A3 to $b(P_j, g(P_j)) > 0$.

Note that if $g(x) \approx u_0(x)$, then A2 follows from A1 combined with (1.2), while if $g(x) = u_0(x)$ at some point $x \in \partial\Omega$, then A2 does not impose any restriction on g at this point. Similarly, if $g(P_j) \approx u_0(P_j)$, then A3 follows from A1 combined with (1.2), while if $g(P_j) = u_0(P_j)$ at some vertex P_j, then A3 does not impose any restriction on g at this point.

Assumption A1 is local and permits the construction of multiple solutions to (1.2) and therefore to (1.1). Assumption A2 is standardly made along the smooth boundaries [3, 6, 9]; it yields existence of boundary-layer ingredients of the asymptotic expansion.

We shall now discuss the corner assumption A3, which is necessitated by the presence of vertices in $\partial\Omega$. A key ingredient of our analysis is a study of certain solutions of the semilinear equation

$$-\triangle z + f(z) = 0 \tag{1.3}$$

posed in an unbounded sector. Our interest in (1.3) is induced by the observation that corner layer functions associated with the vertex P_j are related to a solution of equation (1.3) with $f(z) = b(P_j, z)$ subject to the boundary condition $z = g(P_j)$ (compare with problem (2.9)). Assumption A3 is not only sufficient for existence of a solution z. A result of [11] implies that A3 is necessary for existence of z if we want to exclude spike-type phenomena in the solution u of (1.1) at the corners of Ω (see Remark 3.5 below for details). Furthermore, invoking A3, we establish stability of solutions of (1.3) in the sense that the principal eigenvalue of the linearization of

(1.3) about its solution z is bounded away from zero (see Section 3.4). This analysis lies at the heart of the paper and may be of independent interest.

The main outcome of this paper is a construction of a first-order asymptotic expansion $u_{\rm as}$ to the problem (1.1) and the proof that there exists a solution $u(x)$ such that $|u-u_{\rm as}| \le C\varepsilon^2$. Furthermore, pointwise estimates of the derivatives of particular components of the asymptotic expansion are given. We shall use these estimates in a forthcoming paper to derive a robust numerical method and establish its ε-uniform accuracy. Similar results have been obtained by Fife [3] and, more recently, Nefedov [9] for smooth domains. Our result seems the first for a nonlinear problem in a polygonal domain. Furthermore, our analysis can be extended to piecewise smooth convex domains and higher-order asymptotic expansions. Following [4, 9], we invoke the theory of sub- and super-solutions to establish existence. The desired sub- and super-solutions are obtained by perturbing a formal asymptotic expansion and therefore give tight control on the solution.

The paper is organized as follows. Section 2 defines some boundary layer functions associated with each side of the polygon $\partial\Omega$ and some corner layer functions associated with each vertex of $\partial\Omega$. The boundary layer functions are defined as solutions of some ordinary differential equations in a stretched independent variable. The corner layer functions are solutions of some elliptic partial differential equations in stretched independent variables. The existence and properties of the corner layer functions are established in Section 3, and this section should probably be considered the main contribution of the paper. In Section 4 these boundary and corner functions are assembled into a super- and sub-solution to the problem. Using these functions the existence and properties of a solution to (1.1) are established. To shorten the paper we have placed some proofs that involve much computation in [5].

Notation. Throughout the paper we let C, $\bar{C}$, c, c' denote generic positive constants that may take different values in different formulas, but are always independent of ε ($\bar{C}$ is usually used for a sufficiently large constant). A subscripted C (e.g., C_1) denotes a positive constant that is independent of ε and takes a fixed value. For any two quantities w_1 and w_2, the notation $w_1 = O(w_2)$ means $|w_1| \le C|w_2|$.

2. Boundary and corner layer functions. This section defines some boundary layer functions associated with each side of the polygon $\partial\Omega$ and some corner layer functions associated with each vertex of $\partial\Omega$. The boundary layer functions are defined as solutions of some ordinary differential equations in a stretched independent variable. The corner layer functions are solutions of some elliptic partial differential equations in stretched independent variables. The existence and properties of the corner layer functions are established in Section 3.

We use the functions

$$B(x,t) = b(x, u_0(x)+t), \qquad \tilde{B}(x,t;p) = b(x,u_0(x)+t) - p\,t. \tag{2.1}$$

The perturbed version $\tilde{B}$ of the function B is used, with $|p|$ sufficiently small, in the construction of sub- and super-solutions. In the constructions that follow, a tilde will always denote a perturbed function. The perturbed functions always depend on the parameter p, but we will sometimes not show the explicit dependence. Thus, we will sometimes write $\tilde{B}(x,t)$ for $\tilde{B}(x,t;p)$. We need a notation for the derivatives of $\tilde{B}$. For derivatives with respect to the first argument, we write $\nabla_x\tilde{B}$, $\nabla_x^2\tilde{B}$, etc., for the vector, matrix of second derivatives, etc., with respect to x. We write $\tilde{B}_t$, $\tilde{B}_{tt}$, etc., for derivatives with respect to t. Note also that $\tilde{B}(x,0) = 0$, so $\nabla_x^k\tilde{B}(x,0) = 0$ for

$k = 1, 2, \cdots$, so

$$|\nabla_x^k \tilde{B}(x,t)| \le C|t| \qquad \text{for } k = 0, 1, 2, \cdots. \tag{2.2}$$

We will also use, for any function f, the notations

$$f\big|_a^b = f(b) - f(a), \qquad f\big|_{a;\,b}^{c} = f(c) - f(b) - f(a). \tag{2.3}$$

Since $f\big|_{a;\,b}^{a+b} + f(0) = abf''(t)$, we see that $f(0) = 0$ implies $f\big|_{a;\,b}^{a+b} = O(|ab|)$ and therefore $f\big|_{a;\,b}^{a+b+c} = O(|c| + |ab|)$. In view of (2.2), we thus have

$$\nabla_x^k \tilde{B}(x,\cdot)\Big|_{a;\,b}^{c+a+b} = O(|c| + |ab|). \tag{2.4}$$

In the following 2 subsections we define functions needed to assemble a perturbed first order asymptotic expansion for our problem. The 2 subsections deal respectively with a side of Ω, and with a vertex of Ω. The perturbed asymptotic expansions are defined in Section 4, where they are then used to obtain the existence of a solution to (1.1).

2.1. Solution near a side. In this subsection we construct boundary layer functions associated with a particular side Γ_j of $\partial\Omega$. Throughout the subsection, let Γ denote the line that extends Γ_j. Extend u_0 and b to smooth functions, also denoted u_0 and b, on $\mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}$, respectively, so that (1.2) and A1 hold true for all $x \in \mathbb{R}^2$. Furthermore, extend $g|_{\Gamma_j}$ to a smooth function, also denoted g, on Γ, which satisfies the extended form of A2 and A4 for all $x \in \Gamma$.

Let $\mathbf{e}_s$ denote the unit vector pointing in the direction of Γ and oriented so as to point from P_{j-1} to P_j. Let $\mathbf{e}_r$ be the unit vector perpendicular to $\mathbf{e}_s$ and oriented to point into Ω. Let s denote the signed distance along Γ with $s = 0$ at P_{j-1}. For $x \in \mathbb{R}^2$ write $x = P_{j-1} + s\mathbf{e}_s + r\mathbf{e}_r$. Then $\bar{x} = P_{j-1} + s\mathbf{e}_s$ is the point on Γ which is closest to x and r is the signed distance from $\bar{x}$ to x, with $r > 0$ if $x \in \Omega$ ($\mathbf{e}_s$, $\mathbf{e}_r$, x and $\bar{x}$ are shown in Figure 2.1).

Let $\tilde{v}_0(\xi, s; p)$ be the solution to the nonlinear autonomous two point boundary value problem

$$\begin{gathered} -\frac{\partial^2 \tilde{v}_0}{\partial \xi^2} + \tilde{B}(\bar{x}, \tilde{v}_0; p) = 0, \\ \tilde{v}_0(0, s; p) = g(\bar{x}) - u_0(\bar{x}), \qquad \tilde{v}_0(\infty, s; p) = 0. \end{gathered} \tag{2.5}$$

The geometric meaning of the variable ξ is given by the formula $\xi = r/\varepsilon$. The variables p and s appear as parameters in the problem (2.5). The parameter p satisfies $|p| < \gamma^2$ and in general will be close to zero. We sometimes omit the explicit dependence of $\tilde{v}_0$ on p and write $\tilde{v}_0(\xi, s) = \tilde{v}_0(\xi, s; p)$.

We set $v_0(\xi, s) = \tilde{v}_0(\xi, s; 0)$. The function v_0 appears in the asymptotic expansion of the solution near the side Γ_j. With v_0 defined, we define a function $v_1(\xi, s)$ to be the solution to the linear two point boundary value problem

$$\begin{gathered} -\frac{\partial^2 v_1}{\partial \xi^2} + v_1 B_t(\bar{x}, v_0) = -\xi\, \mathbf{e}_r \cdot \nabla_x B(\bar{x}, v_0), \\ v_1(0, s) = v_1(\infty, s) = 0. \end{gathered} \tag{2.6}$$

Note that v_1 is not a perturbed function as it does not depend on p. We also define

$$\begin{aligned}
&\mathring{\tilde{v}}_0(\xi;p)=\tilde{v}_0(\xi,0;p), \qquad \mathring{v}_0(\xi)=v_0(\xi,0), \qquad \mathring{v}_1(\xi)=v_1(\xi,0),\\
&\tilde{v}=\tilde{v}_0+\varepsilon v_1, \quad v=v_0+\varepsilon v_1, \quad \mathring{\tilde{v}}=\mathring{\tilde{v}}_0+\varepsilon\mathring{v}_1, \quad \mathring{v}=\mathring{v}_0+\varepsilon\mathring{v}_1.
\end{aligned} \tag{2.7}$$

In our notation, a small circle above a function name indicates that in the argument of the function we have set $s=0$.

For the solvability and properties of problems (2.5) and (2.6) we have

Lemma 2.1. *There is $p_0\in(0,\gamma^2)$ such that for all $|p|\le p_0$ there exist functions $\tilde{v}_0$ and v_1 that satisfy* (2.5), (2.6). *For the function $\tilde{v}_0=\tilde{v}_0(\xi,s;p)$ we have*

$$\tilde{v}_0\ge 0, \qquad \frac{\partial\tilde{v}_0}{\partial p}\ge 0. \tag{2.8}$$

Furthermore, for any $k\ge 0$ and arbitrarily small but fixed δ, there is a $C>0$ such that for $0\le\xi<\infty$, $s\in\mathbb{R}$ and $k=0,1,\cdots$,

$$\Big|\frac{\partial^k\tilde{v}_0}{\partial\xi^k}\Big|+\Big|\frac{\partial^k\tilde{v}_0}{\partial s^k}\Big|+\Big|\frac{\partial^k v_1}{\partial\xi^k}\Big|+\Big|\frac{\partial^k v_1}{\partial s^k}\Big|+\Big|\frac{\partial\tilde{v}_0}{\partial p}\Big|+\Big|\frac{\partial^2\tilde{v}_0}{\partial p\,\partial s}\Big|\le Ce^{-(\gamma-\sqrt{|p|}-\delta)\xi}.$$

Proof. The existence and properties of $\tilde{v}_0$ follow from [7, Lemmas 2.2 and 2.3]. For v_1, we use a result presented in [3, Lemma 2.2] and [12, §2.3.1]. □

2.2. Solution near a vertex. In this subsection we construct corner layer functions associated with a particular vertex P_{j-1}. These corner layer functions will be used in the asymptotic expansion of the solution as well as the construction of a sub- and super-solution.

Some notation is required for the constructions. We place the vertex P_{j-1} at the origin O. Let S_j, or, when there is no ambiguity, simply S, be the infinite sector with angle ω at the apex, obtained by extending the two sides Γ_j and Γ_{j-1} in the direction away from O. The ray that extends the side Γ_j is denoted Γ, while the ray that extends the side Γ_{j-1} is denoted Γ^-. We extend $g|_{\Gamma_j}$ to a function on Γ; in this section the extended function is denoted g. Similarly, we extend $g|_{\Gamma_{j-1}}$ to a function on Γ^- that in this section is denoted g^-. These extensions are made in such a way that A2 and A4 hold. Let s denote the distance along Γ, measured from O, and let r denote the perpendicular distance to a point $x\in S$. Thus, $x\to(s,r)$ is a linear orthogonal map. We also let $\mathbf{e}_s$ and $\mathbf{e}_r$ denote the unit vectors along Γ and orthogonal to Γ respectively, so $x=r\mathbf{e}_r+s\mathbf{e}_s$. We denote by $\bar{x}=s\mathbf{e}_s$ the point of Γ that is closest to x. In a similar manner, we define variables (s^-,r^-), so $x=r^-\mathbf{e}_{r^-}+s^-\mathbf{e}_{s^-}$, and $\bar{x}^-=s^-\mathbf{e}_{s^-}$ associated with the side Γ^-. The variable s^- denotes the distance along Γ^-, measured from O. We will also need the stretched variables $\eta=x/\varepsilon$, $\xi=r/\varepsilon$, $\sigma=s/\varepsilon$, $\xi^-=r^-/\varepsilon$, $\sigma^-=s^-/\varepsilon$. These variables are shown in Figure 2.1.

Using these notations Section 2.1 gives functions $\tilde{v}_0(\xi,s;p)$ and $v_1(\xi,s)$ associated with the side Γ and functions $\tilde{v}_0^-(\xi^-,s^-;p)$ and $v_1^-(\xi^-,s^-)$ associated with the side Γ^-. We also recall the notations in (2.7) and use corresponding notations for the side Γ^-. The function $\tilde{v}$ matches the disparity between the boundary conditions of (1.1b) and the value of u_0 on Γ, but leaves a rapidly decaying boundary value on Γ^-. The function $\tilde{v}^-$ has a similar behavior, with a rapidly decaying boundary value on Γ. To deal with these rapidly decaying boundary values we construct functions $\tilde{z}_0(\eta;p)$ and $z_1(\eta)$, defined in terms of the stretched variable η.

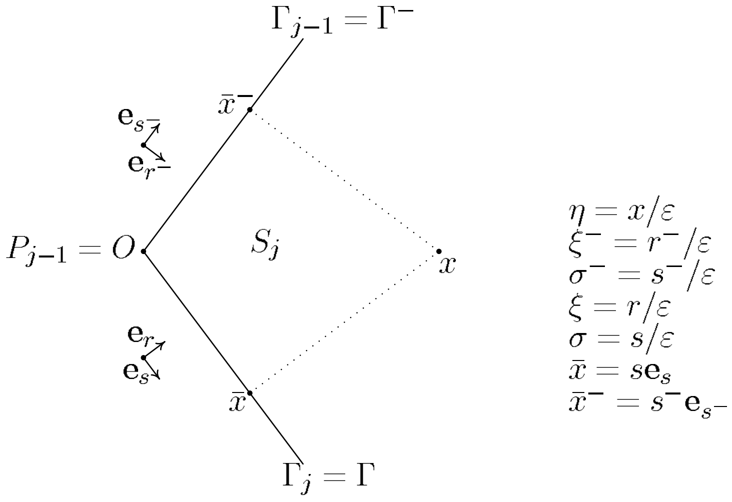

FIG. 2.1. *Geometry of a sector $S_j = S$*

The function $\tilde{z}_0$ is defined to be a bounded solution of the autonomous nonlinear elliptic boundary value problem

$$\begin{aligned} -\triangle_\eta \tilde{z}_0 + \tilde{B}(O, \tilde{z}_0; p) &= 0 \qquad && \text{in } S, \\ \tilde{z}_0 = A := g(O) - u_0(O) & && \text{on } \partial S. \end{aligned} \tag{2.9}$$

Here we have $A > 0$, by our assumption A4 at the point $P_{j-1} = O$. We also set $z_0(\eta) = \tilde{z}_0(\eta; 0)$. The existence and properties of $\tilde{z}_0$ are given in the following theorem; the proof is deferred to Section 3.

THEOREM 2.2. *There is a positive constant p^* such that if $|p| \le p^*$, the problem* (2.9) *has, for each p, a solution $\tilde{z}_0$ which satisfies $\tilde{z}_0 \le A$ and*

$$0 < \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\} \le \tilde{z}_0(\eta; p) \le \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\} + C|\eta|^{-1}, \tag{2.10}$$

and which is an increasing function of p. Also, $|\nabla \tilde{z}_0|$ is bounded in S. Finally there is a constant $C > 0$ such that

$$\tilde{z}_0(\eta) \le C\left(e^{-\gamma\xi} + e^{-\gamma\xi^-}\right). \tag{2.11}$$

We also consider a function $z_1(\eta)$ which satisfies the linear elliptic boundary value problem

$$\begin{gathered} -\triangle_\eta z_1 + z_1 B_t(O, z_0) = -\eta \cdot \nabla_x B(O, z_0) \quad \text{in } S, \\ z_1 = \sigma \frac{\partial}{\partial s}(g - u_0)\Big|_{x=O} \ \text{ on } \Gamma, \qquad z_1 = \sigma^- \frac{\partial}{\partial s^-}(g^- - u_0)\Big|_{x=O} \ \text{ on } \Gamma^-, \end{gathered} \tag{2.12}$$

The functions $\tilde{z}_0$ and z_1 form a correction $\tilde{z}_0 + \varepsilon z_1$ to the reduced solution u_0 in close proximity of the vertex O. To extend it further away from O, the corrections $\tilde{v}_0 + \varepsilon v_1$ and $\tilde{v}_0^- + \varepsilon v_1^-$ to u_0 near the sides Γ and Γ^- are to be invoked as follows. We use the corner functions $\tilde{z}_0$ and z_1 together with the boundary functions $\tilde{v}_0$, v_1, $\tilde{v}_0^-$, v_1^- to define a related pair of corner functions $\tilde{q}_0$ and q_1, which, rather than $\tilde{z}_0$ and z_1, will appear in a formal asymptotic expansion of the solution of (1.1) within an $O(1)$ distance to the vertex $O = P_{j-1}$; see Section 4.

We shall use the following notation. Pick a point $\eta \in S$. Having chosen η, the formulas

$$\eta = \xi \mathbf{e}_r + \sigma \mathbf{e}_s = \xi^- \mathbf{e}_{r^-} + \sigma^- \mathbf{e}_{s^-} \tag{2.13}$$

determine numbers $\xi, \sigma, \xi^-, \sigma^-$; see Figure 1. With this notation, and using the functions $\tilde{z}_0$, z_1 and $\mathring{\tilde{v}}_0$, $\mathring{\tilde{v}}_0^-$, $\mathring{v}_1$, $\mathring{v}_1^-$ of (2.5), (2.6),(2.7), we define

$$\tilde{q}_0(\eta; p) = \tilde{z}_0(\eta; p) - \mathring{\tilde{v}}_0(\xi; p) - \mathring{\tilde{v}}_0^-(\xi^-; p), \tag{2.14a}$$

$$q_1(\eta) = z_1(\eta) - [\mathring{v}_1(\xi) + \sigma \mathring{v}_{0,s}(\xi)] - [\mathring{v}_1^-(\xi^-) + \sigma^- \mathring{v}_{0,s^-}^-(\xi^-)], \tag{2.14b}$$

and furthermore,

$$\tilde{q}(\eta; p) = \tilde{q}_0(\eta; p) + \varepsilon q_1(\eta), \quad q_0(\eta) = \tilde{q}_0(\eta; 0), \quad q(\eta) = q_0(\eta) + \varepsilon q_1(\eta). \tag{2.14c}$$

In these formulas, following the notational conventions of (2.7), we mean

$$\mathring{v}_{0,s}(\xi) = \tfrac{\partial}{\partial s} v_0(\xi, s)\big|_{s=0}, \qquad \mathring{v}_{0,s^-}^- = \tfrac{\partial}{\partial s^-} v_0^-(\xi^-, s^-)\big|_{s^-=0}. \tag{2.14d}$$

Under this notation, the boundary conditions in (2.12) become $z_1 = \sigma\, \mathring{v}_{0,s}$ on Γ, and $z_1 = \sigma^- \mathring{v}_{0,s^-}^-$ on Γ^-.

From the above formulas and (2.5), (2.9), we derive a nonlinear boundary value problem satisfied by $\tilde{q}_0$:

$$\triangle_\eta \tilde{q}_0 = \tilde{B}(O, \tilde{q}_0 + \mathring{\tilde{v}}_0 + \mathring{\tilde{v}}_0^-) - \tilde{B}(O, \mathring{\tilde{v}}_0) - \tilde{B}(O, \mathring{\tilde{v}}_0^-), \tag{2.15a}$$

$$\tilde{q}_0 = -\mathring{\tilde{v}}_0^- \text{ on } \Gamma, \qquad \tilde{q}_0 = -\mathring{\tilde{v}}_0 \text{ on } \Gamma^-. \tag{2.15b}$$

Similarly, using (2.5), (2.6) and (2.12), in [5, Lemma 2.4] we also formally derive a linear boundary value problem satisfied by q_1:

$$\begin{aligned} -\triangle_\eta q_1 + q_1 B_t(O, z_0) &= -\eta \cdot \nabla_x B(O, \cdot)\Big|_{\mathring{v}_0;\ \mathring{v}_0^-}^{z_0} \\ &\quad -(\mathring{v}_1 + \sigma \mathring{v}_{0,s})\ B_t(O, \cdot)\Big|_{\mathring{v}_0}^{z_0} - (\mathring{v}_1^- + \sigma^- \mathring{v}_{0,s^-}^-)\ B_t(O, \cdot)\Big|_{\mathring{v}_0^-}^{z_0}, \\ q_1 = -(\mathring{v}_1^- + \sigma^- \mathring{v}_{0,s^-}^-) \text{ on } \Gamma,& \qquad q_1 = -(\mathring{v}_1 + \sigma \mathring{v}_{0,s}) \text{ on } \Gamma^-, \end{aligned} \tag{2.16}$$

where we used the notation (2.3). Finally, by formally differentiating relation (2.14a) and problem (2.9) (or the equivalent problem (2.15)) with respect to p and invoking (2.1), we formally derive a boundary value problem that is satisfied by $\tilde{q}_{0,p}$:

$$\begin{aligned} -\triangle_\eta \tilde{q}_{0,p} + \tilde{q}_{0,p} \tilde{B}_t(O, \tilde{z}_0) &= \tilde{q}_0 - \mathring{\tilde{v}}_{0,p}\ \tilde{B}_t(O, \cdot)\Big|_{\mathring{\tilde{v}}_0}^{\tilde{z}_0} - \mathring{\tilde{v}}_{0,p}^-\ \tilde{B}_t(O, \cdot)\Big|_{\mathring{\tilde{v}}_0^-}^{\tilde{z}_0}, \\ \tilde{q}_{0,p} = -\mathring{\tilde{v}}_{0,p}^- \text{ on } \Gamma,& \qquad \tilde{q}_{0,p} = -\mathring{\tilde{v}}_{0,p} \text{ on } \Gamma^-. \end{aligned} \tag{2.17}$$

The problem (2.15) will be used in Section 3.3 to show that the function $\tilde{q}_0$ is exponentially decaying as $|\eta| \to \infty$. Also, it will be seen that the data in the linear problems (2.16) and (2.17) are exponentially decaying. This will be used in Section 3.5 to show that each of these linear problems is well-posed and so has a solution depending continuously on the data and exponentially decaying. In view of (2.14b), the existence of q_1 immediately implies existence of z_1. Similarly, having proved the existence of the solution to (2.17), an integration is used to show that this solution is in fact the derivative of $\tilde{q}_0$ with respect to p.

3. Existence and properties of the corner layer functions . In this section the existence and properties of the functions $\tilde{z}_0$ and z_1 are established. The existence of a solution to (2.9) comes from the theory of sub- and super-solutions which is presented in Section 3.1. This theory is also used in Section 4 to show the existence of a solution to (1.1). The existence of a solution $\tilde{z}_0$ to (2.9) and its decay properties that are asserted in Theorem 2.2, are established in Section 3.2. Because (2.9) may have many solutions, we first construct specific sub- and super-solutions to (2.9); the function $\tilde{z}_0$ is then defined as the *unique* minimal solution that lies between these two constructed functions. In Section 3.3 we analyze the exponential-decay properties of the component $\tilde{q}_0$ of $\tilde{z}_0$.

To prepare for the existence and properties of z_1 and $\partial\tilde{z}_0/\partial p$, the linearization of (2.9) around the function $\tilde{z}_0$ must be analyzed. This is done in Section 3.4. It is shown there that the eigenvalue of the linearized problem is bounded away from zero, and as a consequence the linearized operator is invertible. The analysis in this section lies at the heart of the paper, and may have an independent interest. The existence and properties of z_1 and $\partial\tilde{z}_0/\partial p$ are then obtained in Section 3.5.

3.1. Sub- and super-solutions. The theory of sub- and super-solutions (also called lower and upper solutions) is presented, for example, in [1, 2, 10]. We state here the definitions and some important facts in this theory. These are stated for the problem (1.1), which was posed on the polygon Ω. In fact the theory of sub- and super-solutions is more generally applicable, in particular, when problem (1.1) is posed on the sector S. This observation enables us to use the results below both in the analysis of problems (1.1) and (2.9), the latter clearly being of type (1.1) with $\varepsilon = 1$.

A function β is a super-solution of the problem (1.1) if β is continuous and bounded in $\bar{\Omega}$, if $\beta \geq g$ on $\partial\Omega$, and if for each $\chi \in C_0^\infty(\Omega)$ with $\chi \geq 0$,

$$\iint_\Omega \big[-\beta\varepsilon^2 \triangle\chi + b(x,\beta)\chi \big] dx \geq 0. \tag{3.1}$$

Here $C_0^\infty(\Omega)$ denotes the set of infinitely differentiable functions with compact support in Ω. Similarly, α is a sub-solution of (1.1) if α is continuous and bounded in $\bar{\Omega}$, if $\alpha \leq g$ on $\partial\Omega$, and if the reverse inequality in (3.1) holds, with β replaced by α. The following lemma may be found in [1, 10].

Lemma 3.1. *If β_1 and β_2 are 2 super-solutions of* (1.1)*, which are in $C^2(\Omega)$, then $\min\{\beta_1, \beta_2\}$ is a super-solution of* (1.1)*. If α_1 and α_2 are 2 sub-solutions of* (1.1)*, which are in $C^2(\Omega)$, then $\max\{\alpha_1, \alpha_2\}$ is a sub-solution of* (1.1)*.*

The next lemma shows the reason for considering sub- and super-solutions; they provide a way of proving the existence of a solution. This result is stated in [2] and [10] with the assumption that the domain is bounded and has a smooth boundary, and in [1] with the nature of the boundary unspecified.

Lemma 3.2. *Let α and β be respectively a sub-solution and a super-solution of* (1.1) *with $\alpha \leq \beta$ in Ω. Then* (1.1) *has a solution u satisfying $\alpha \leq u \leq \beta$ in Ω. Furthermore* (1.1) *has a minimal solution u_m in the sense that if u is another solution with $\alpha \leq u \leq \beta$ in Ω, then $u_m \leq u$ in Ω.*

The following lemma will be useful in several places.

Lemma 3.3. *Let u_m be the unique minimal solution of* (1.1) *corresponding to a sub-solution α. Let $\hat{\Omega} \subset \Omega$. Let $\hat{u}_m$ be the unique minimal solution of the problem consisting of* (1.1) *in $\hat{\Omega}$ with the boundary condition $\hat{u}_m|_{\partial\hat{\Omega}} = u_m|_{\partial\hat{\Omega}}$ corresponding to the same sub-solution α restricted to $\hat{\Omega}$. Then $\hat{u}_m = u_m$ in $\hat{\Omega}$.*

Proof. Since $u_m \geq \alpha$ is a solution of the problem satisfied by $\hat{u}_m \geq \alpha$ and $\hat{u}_m$ is the minimal solution of this problem, $\hat{u}_m \leq u_m$ in $\hat{\Omega}$. Define β in Ω by $\beta = \hat{u}_m$ in $\hat{\Omega}$, $\beta = u_m$ in $\Omega \setminus \hat{\Omega}$. We claim that β is a super-solution of (1.1). For, letting Γ be the portion of $\partial\hat{\Omega}$ that lies inside Ω with n the unit normal on $\partial\hat{\Omega}$ pointing out of $\hat{\Omega}$, if $\chi \in C_0^\infty(\Omega)$ with $\chi \geq 0$, then since $\varepsilon^2 \triangle \hat{u}_m = b(x, \hat{u}_m)$ in $\hat{\Omega}$ and $\varepsilon^2 \triangle u_m = b(x, u_m)$ in $\Omega \setminus \hat{\Omega}$,

$$\begin{aligned}\iint_\Omega \big[- \beta\varepsilon^2 \triangle\chi + b(x,\beta)\chi\big]\, dx &= \iint_{\hat{\Omega}} \big[- \hat{u}_m\varepsilon^2 \triangle\chi + b(x,\hat{u}_m)\chi\big]\, dx \\ &\quad + \iint_{\Omega\setminus\hat{\Omega}} \big[- u_m\varepsilon^2 \triangle\chi + b(x,u_m)\chi\big]\, dx \\ &= \int_\Gamma \varepsilon^2 \chi \big(\frac{\partial \hat{u}_m}{\partial n_\Gamma} - \frac{\partial u_m}{\partial n_\Gamma}\big) \geq 0.\end{aligned}$$

Here the final assertion follows from $\hat{u}_m \leq u_m$ in $\hat{\Omega}$. Now β being a super-solution implies that $u_m \leq \beta$ in Ω. Therefore $u_m \leq \hat{u}_m$ in $\hat{\Omega}$, so $u_m = \hat{u}_m$ in $\hat{\Omega}$. □

LEMMA 3.4. *Let u_m be the unique minimal solution of* (1.1) *corresponding to a sub-solution α. Let $\hat{\Omega} \subset \Omega$. Let $\bar{u}_m$ be the unique minimal solution of the problem consisting of* (1.1) *in $\hat{\Omega}$ with a boundary condition $\bar{u}_m|_{\partial\hat{\Omega}} \geq u_m|_{\partial\hat{\Omega}}$ corresponding to the same sub-solution α restricted to $\hat{\Omega}$. Then $\bar{u}_m \geq u_m$ in $\hat{\Omega}$.*

Proof. Clearly $\hat{u}_m \leq \bar{u}_m$, where $\hat{u}_m$ is from Lemma 3.3. Combine this with $\hat{u}_m = u_m$ in $\hat{\Omega}$. □

3.2. Existence of $\tilde{z}_0$. In this section we prove Theorem 2.2. Let S be a convex sector with apex at O and with boundary $\Gamma \cup \Gamma^-$. Let $z = \tilde{z}_0$ and $f(z) = \tilde{B}(O, z)$. We are concerned with the boundary value problem (2.9), which in terms of z and f is

$$-\triangle z + f(z) = 0 \text{ in } S, \qquad z = A \text{ on } \partial S. \tag{3.2}$$

Here $A > 0$, by A4, and for sufficiently small $|p|$, by (2.1) combined with (1.2) and A1–A3, the function f satisfies

$$f(0) = 0, \quad f(A) > 0, \quad f'(0) > \gamma^2, \quad \int_0^s f(t)dt > 0 \text{ for } s \in (0, A]. \tag{3.3}$$

Note that (3.3) implies that 0 and A are sub- and super-solutions for (3.2). Therefore, there exists at least one solution $z \in [0, A]$ of problem (3.2). However, to establish the desired solution bounds, we shall invoke more precise sub- and super-solutions. By Lemma 3.1, the function $\alpha = \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\}$ gives a sub-solution of (3.2) such that $0 \leq \alpha \leq A$. We define $\tilde{z}_0$ to be the *unique minimal solution* corresponding to the sub-solution α and the super-solution A. Thus, $\tilde{z}_0 \leq A$ and, more generally, $\tilde{z}_0 \leq \beta$ for any super-solution β such that $\beta \geq \alpha$. In the following proof of Theorem 2.2 we construct a more precise upper bound for $\tilde{z}_0$.

Proof of Theorem 2.2:

Let $\eta_0 = \xi_0 \mathbf{e}_r + \sigma_0 \mathbf{e}_s$ be a point in S which is closer to Γ than to Γ^- (as the other case is similar). Let $\bar{\eta}_* = \sigma_0 \mathbf{e}_s$, $\bar{\eta}_*^- = \sigma_0 \mathbf{e}_{s^-}$ and let $\mathcal{O}$ denote the disc lying in S, tangent to Γ at $\bar{\eta}_*$, and tangent to Γ^- at $\bar{\eta}_*^-$. The radius of $\mathcal{O}$ is $\rho = \sigma_0 \tan(\omega/2)$ and is large if η_0 is far from O (thus $\mathcal{O}$ is centered at $\rho\mathbf{e}_r + \sigma_0\mathbf{e}_s$, while η_0 lies on the segment joining the center and $\bar{\eta}_*$). Consider problem (3.2) in $\mathcal{O}$ instead of S and denote its solution $z_{\mathcal{O}}$:

$$-\triangle z_{\mathcal{O}} + f(z_{\mathcal{O}}) = 0 \text{ in } \mathcal{O}, \qquad z_{\mathcal{O}} = A \text{ on } \partial\mathcal{O}. \tag{3.4}$$

Make the change of variable $\hat{\eta} = \eta/\rho$, which transforms $\mathcal{O}$ into the unit disc $\hat{\mathcal{O}}$. The problem (3.4) transforms into the problem for $\hat{z}_{\mathcal{O}}(\hat{\eta}) := z_{\mathcal{O}}(\eta)$:

$$-\rho^{-2}\triangle_{\hat{\eta}}\hat{z}_{\mathcal{O}} + f(\hat{z}_{\mathcal{O}}) = 0 \text{ in } \hat{\mathcal{O}}, \qquad \hat{z}_{\mathcal{O}} = A \text{ on } \partial\hat{\mathcal{O}}, \tag{3.5}$$

where ρ^{-1} is a small parameter for sufficiently large σ_0, i.e. we have a singularly perturbed problem of type (1.1) in a smooth unit-circle domain. Such problems were studied, e.g., in [6]; in particular, we invoke [6, Corollary 2.9]. From this result and some changes in notation, one concludes that there is a constant C such that $\chi(\hat{d})\,\mathring{\tilde{v}}_0(\rho\hat{d};p+\hat{p}) + b_2^{-1}\hat{p}$ with $\hat{p} = C\rho^{-1}$ is a super-solution of the problem (3.5). Here a constant $b_2 \ge |b_{uu}(O,t)|$ for $t \in [u_0(O), g(O)]$, $\hat{d} = \hat{d}(\hat{\eta})$ denotes the distance from a point $\hat{\eta} \in \hat{\mathcal{O}}$ to $\partial\hat{\mathcal{O}}$, and $\chi(\hat{d})$ is a smooth cut-off function equal to 1 for $\hat{d} > 1/2$ and 0 for $\hat{d} < 1/4$. (Note that $\mathring{\tilde{v}}_0(\rho\hat{d};p+\hat{p})$ replaces $v(\rho\hat{d}, l; p+\hat{p})$ in the notation of [6], which in the present situation, does not depend on l.)

Interpreting this for the original variable η with the distance $d = d(\eta) = \rho\hat{d}$ from a point $\eta \in \mathcal{O}$ to $\partial\mathcal{O}$, we see that $\chi(d/\rho)\,\mathring{\tilde{v}}_0(d;p+\hat{p}) + b_2^{-1}\hat{p}$ with $\hat{p} = C\rho^{-1}$ gives a super-solution for the problem (3.4). We also have the pair $\alpha = \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\}$ and A of sub- and super-solutions for the problem (3.4). Combining the two super-solutions by Lemma 3.1, we get a more precise super-solution: $\beta_{\mathcal{O}} := \min\{\chi(d/\rho)\,\mathring{\tilde{v}}_0(d;p+\hat{p}) + b_2^{-1}\hat{p},\, A\}$, where $\hat{p} = C\rho^{-1}$.

To have a solution $z_{\mathcal{O}}(\eta)$ of (3.4) between α and $\beta_{\mathcal{O}}$, it is crucial to check that $\alpha \le \beta_{\mathcal{O}}$, which follows from $\alpha \le \chi(d/\rho)\,\mathring{\tilde{v}}_0(d;p+\hat{p}) + b_2^{-1}\hat{p}$. If a point η is closer to Γ, the latter assertion is equivalent to $\mathring{\tilde{v}}_0(\xi;p) \le \chi(d/\rho)\,\mathring{\tilde{v}}_0(d;p+\hat{p}) + b_2^{-1}\hat{p}$. In the region where $\chi = 1$, this follows from the inequalities $\mathring{\tilde{v}}_0(\xi;p) \le \mathring{\tilde{v}}_0(\xi;p+\hat{p}) \le \mathring{\tilde{v}}_0(d;p+\hat{p})$, where we used the monotonicity of $\mathring{\tilde{v}}$ in both of its arguments and the fact that $\xi \ge d(\eta)$ for any $\eta = (\xi,\sigma)$. Otherwise, if $\chi < 1$, then $\xi \ge d(\eta) \ge \rho/2$ and $\hat{p} = C\rho^{-1}$ imply $\mathring{\tilde{v}}_0(\xi;p) \le Ce^{-C\xi} \le b_2^{-1}\hat{p}$ provided that ρ is sufficiently large. Thus we showed that $\alpha \le \beta_{\mathcal{O}}$ if ρ is sufficiently large.

Let $z_{\mathcal{O}}(\eta)$ be the minimal solution of (3.4) between α and $\beta_{\mathcal{O}}$. By Lemma 3.4 applied to problems (3.2) and (3.4), we get $\tilde{z}_0(\eta) \le z_{\mathcal{O}}(\eta) \le \beta_{\mathcal{O}}$ in $\mathcal{O}$ and, in particular, $\tilde{z}_0(\eta_0) \le \beta_{\mathcal{O}}(\eta_0)$. Note that at $\eta = \eta_0$ we have $d = \xi_0$ and therefore $\beta_{\mathcal{O}}(\eta_0) \le \mathring{\tilde{v}}_0(\xi_0;p+\hat{p}) + b_2^{-1}\hat{p} \le \mathring{\tilde{v}}_0(\xi_0;p) + C\hat{p}$. Finally recall that $\hat{p} = C\rho^{-1} = C\sigma_0^{-1} \le C|\eta_0|^{-1}$. This proves the upper bound for $\tilde{z}_0$ in (2.10) for $\eta = \eta_0$ closer to Γ than to Γ^-. Thus (2.10) is established.

By (2.10) and (3.3), there is a sufficiently large number $\Xi > 0$ such that, setting

$$S_\Xi = \{\eta \in S : \min\{\xi, \xi^-\} > \Xi\}, \tag{3.6}$$

if $\eta \in S_\Xi$, then $\tilde{z}_0(\eta;p)$ is so small that $f'(\tilde{z}_0) > \gamma^2$. From the mean value theorem, $\tilde{z}_0$ satisfies the linear equation $-\triangle_\eta \tilde{z}_0 + a(\eta)\tilde{z}_0 = 0$ on S_Ξ with $a(\eta) > \gamma^2$. Let $W(\eta) = e^{-\gamma\xi} + e^{-\gamma\xi^-}$. Then we have $\triangle_\eta W = \gamma^2 W$ and hence $\big[-\triangle_\eta + a(\eta)\big]W = \big[a(\eta) - \gamma^2\big]W > 0$. The boundary ∂S_Ξ consists of a straight line segment at distance Ξ from Γ and a straight line segment at distance Ξ from Γ^-, where we have $W \ge e^{-\gamma\Xi}$. Hence there is a constant $C > 0$ such that $CW \ge A \ge \tilde{z}_0$ on ∂S_Ξ. From the maximum principle, $\tilde{z}_0(\eta) \le CW(\eta)$ in S_Ξ. This proves (2.11).

If $p < p'$, then $\tilde{z}_0(\cdot, p')$ is a super-solution for the problem satisfied by $\tilde{z}_0(\cdot,\, p)$, so $\tilde{z}_0(\eta,p) \le \tilde{z}_0(\eta,p')$, which gives the monotonicity assertion of the theorem. Finally, since $\tilde{z}_0$ satisfies the linear equation $\triangle_\eta \tilde{z}_0 = F$ with the bounded function $F(\eta) = \tilde{B}(O,\tilde{z}_0)$, standard first derivative bounds show that $|\nabla_\eta \tilde{z}_0| \le C$ in S. This finishes the proof of Theorem 2.2. □

Remark 3.5. In the context of problem (3.2), we have seen that that assumption A3 is equivalent to $f(A) > 0$, which appears in (3.3) and is used above to establish existence of a solution z such that $0 \le z \le A$. Furthermore, A3 is necessary for existence of such a solution in the following sense. A result of [11] implies that if in (3.3) we replace $f(A) > 0$ by $f(A) < 0$, then there is $\theta_f \in (\pi/2, \pi)$ such that for a sector S with the angle at the apex being less than θ_f, there exists no solution z to problem (3.2) such that $0 \le z \le A$. (Note that [11] deals with bounded domains, but an inspection of the arguments shows that similar results apply to the unbounded sector S.) Thus if we violate A3 and instead impose $f(A) < 0$, then problem (3.2) has no solution $0 \le z \le A$ even if it is posed in a quarter-plane.

It should be noted that imposing $0 \le z \le A$, we exclude spike-type phenomena in the solution u of (1.1) at the corners of Ω. Indeed, recalling that $z = \tilde{z}_0$ is a corner layer function to be used in an asymptotic expansion, we observe that z should be negligible away from $\Gamma \cup \Gamma^-$ and should also satisfy $z \approx \mathring{\tilde{v}}_0(\xi)$ and $z \approx \mathring{\tilde{v}}_0^-(\xi^-)$ near the boundaries Γ and Γ^-, respectively, away for the vertex. As $\mathring{\tilde{v}}, \mathring{\tilde{v}}^- \in [0, A]$, then $0 \le z \le A$ can be violated only for $|x - P_{j-1}| \ll 1$, and therefore will result in a spike in $u \approx z(\eta) + u_0(P_{j-1})$ in very close proximity of P_{j-1}.

3.3. Exponential decay of $\tilde{q}_0$. The function $\tilde{q}_0$ is defined by (2.14a), and it has been shown that this function satisfies the nonlinear boundary value problem (2.15). In this subsection we use an equivalent variant of this boundary value problem to establish the exponential decay of $\tilde{q}_0$.

Lemma 3.6. *There are constants C_1 and c_1 such that*

$$|\tilde{q}_0| + |\nabla \tilde{q}_0| \le C_1 e^{-c_1|\eta|} \quad \text{ in } S.$$

Proof. The boundary conditions (2.15b) are exponentially decaying, but it seems difficult to analyze the behavior of the right hand side of (2.15a). Instead, it is convenient to study the function $\tilde{q}_0 + \mathring{\tilde{v}}^-$. Define operators $\mathcal{N}$ and $\mathcal{L}$ by

$$\mathcal{N}[W] = -\triangle W + \tilde{B}(O, \cdot)\big|_{\mathring{\tilde{v}}_0}^{W + \mathring{\tilde{v}}_0}, \qquad \mathcal{L}W = -\triangle W + W \tilde{B}_t(O, \mathring{\tilde{v}}_0).$$

One has

$$\tilde{B}(O, \cdot)\big|_{\mathring{\tilde{v}}_0}^{W + \mathring{\tilde{v}}_0} = W \tilde{B}_t(O, \mathring{\tilde{v}}_0) + W^2 R,$$

where $|R| \le C_R = \frac{1}{2} \sup |\tilde{B}_{tt}|$. Hence

$$\mathcal{N}[W] = \mathcal{L}W + W^2 R. \tag{3.7}$$

The operator $\mathcal{N}$ will be used to obtain a boundary value problem for the function $\tilde{q}_0 + \mathring{\tilde{v}}_0^-$. From (2.15a) and (2.5) we obtain

$$\mathcal{N}[\tilde{q}_0 + \mathring{\tilde{v}}_0^-] = 0 \quad \text{ in } S. \tag{3.8a}$$

Next, from (2.15b), we immediately get

$$\tilde{q}_0 + \mathring{\tilde{v}}_0^- = 0 \quad \text{ on } \Gamma. \tag{3.8b}$$

Because $\tilde{q}_0 + \mathring{\tilde{v}}_0^-$ does not decay exponentially on the entire S, our boundary value problem is formulated on a subdomain of S. Let $\omega_1 \le \omega/2$, let Γ_{ω_1} be the ray in

S which makes angle ω_1 with Γ, and let $S_{\omega_1} \subset S$ be the sector with sides Γ and Γ_{ω_1}. One can see that $\mathring{\tilde{v}}_0^- \le \mathring{\tilde{v}}_0$ in S_{ω_1}. Hence, from (2.10), $0 < \mathring{\tilde{v}}_0 \le \tilde{z}_0$ in S_{ω_1} so $0 \le \tilde{z}_0 - \mathring{\tilde{v}}_0 = q_0 + \mathring{\tilde{v}}_0^- < \tilde{z}_0$ in S_{ω_1}. Therefore, using (2.11), one finds that given any $c_2 > 0$, no matter how small, there is a sufficiently large $\bar{\rho} = \bar{\rho}(c_2) > 0$ such that

$$|\tilde{q}_0 + \mathring{\tilde{v}}_0^-| \le Ce^{-\gamma\xi} = Ce^{-\gamma(\tan\omega_1)\sigma} \le c_2 e^{-\gamma(\tan\omega_1)\sigma/2} \text{ on } \Gamma_{\omega_1}, \text{ for } \sigma > \bar{\rho}. \tag{3.8c}$$

Let $S'_{\omega_1} = \{\eta \in S_{\omega_1} : \sigma > \bar{\rho}\}$, let Γ'_{ω_1} denote the portion of the ray Γ_{ω_1} that lies along S'_{ω_1}, and let $\Gamma_{\bar{\rho}}$ denote the portion of $\partial S'_{\omega_1}$ with $\sigma = \bar{\rho}$. Finally, by (2.10), one can make

$$|\tilde{q}_0 + \mathring{\tilde{v}}_0^-| = |\tilde{z}_0 - \mathring{\tilde{v}}_0| \le C|\eta|^{-1} \le C/\bar{\rho} \le c_2 \quad \text{on } \Gamma_{\bar{\rho}}, \tag{3.8d}$$

by making $\bar{\rho}$ sufficiently large.

Consider the nonlinear problem

$$\mathcal{N}[W] = 0 \text{ in } S'_{\omega_1}, \qquad W = \tilde{q}_0 + \mathring{\tilde{v}}_0^- \text{ on } \partial S'_{\omega_1}. \tag{3.9}$$

We construct an exponentially decaying super-solution $\bar{W}$ to (3.9). The super-solution has the form $\bar{W} = w(\xi)\phi(\sigma)$ where $\phi(\sigma) = c_\phi e^{-a\sigma}$ with a and c_ϕ chosen sufficiently small, and where w is a particular solution to the equation

$$-w'' + w\tilde{B}_t(O, \mathring{\tilde{v}}_0) = 1.$$

To construct w recall that function $\mathring{\tilde{v}}_0(\xi) \ge 0$ is monotonically decreasing, exponentially decaying and satisfies $-\mathring{\tilde{v}}_0'' + \tilde{B}(O, \mathring{\tilde{v}}_0) = 0$. Setting $\chi(\xi) = -\mathring{\tilde{v}}_{0,\xi} \ge 0$ we have (see [3, proof of Lemma 2.2])

$$w(\xi) = \chi(\xi) \int_0^\xi \chi(\eta)^{-2}\, \mathring{\tilde{v}}_0(\eta) d\eta + \chi(\xi). \tag{3.10}$$

Since $\underline{\gamma}\mathring{\tilde{v}}_0 \le \chi \le \bar{\gamma}\mathring{v}_0$ (see, e.g. [7, estimate (A.2)]), a calculation shows that for some c_w and C_w we have

$$0 < c_w \le w(\xi) \le C_w.$$

We now show that with a proper choice of a and c_ϕ, $\bar{W}$ is the desired super-solution. One has

$$\begin{aligned} \mathcal{L}[\bar{W}] &= -w''\phi - w\phi'' + [w\phi]\tilde{B}_t(O, \mathring{\tilde{v}}_0) = \big[- w'' + w\tilde{B}_t(O, \mathring{\tilde{v}}_0)\big]\phi - w\phi'' \\ &= \phi - w\phi'' = \phi(1 - a^2 w). \end{aligned}$$

Thus $\mathcal{L}[\bar{W}] \ge \frac{1}{2}\phi$ if $a^2 \le \frac{1}{2}C_w^{-1}$. Also, using (3.7), we get

$$\begin{aligned} \mathcal{N}[\bar{W}] &= \mathcal{L}\bar{W} + w^2\phi^2 R \\ &\ge \tfrac{1}{2}\phi - C_w^2\phi^2 C_R = \phi\,[\tfrac{1}{2} - C_w^2 C_R\phi] = \phi\,[\tfrac{1}{2} - C_w^2 C_R c_\phi e^{-a\sigma}] \\ &\ge 0 = \mathcal{N}[\tilde{q}_0 + \mathring{\tilde{v}}_0^-] \qquad \text{if } c_\phi = \frac{1}{2}C_w^{-2}C_R^{-1}. \end{aligned}$$

Since $\bar{W} > 0$ on Γ, from (3.8b) we have $\bar{W} \ge \tilde{q}_0 + \mathring{\tilde{v}}_0^-$ on Γ. We now show that $\bar{W} \ge \tilde{q}_0 + \mathring{\tilde{v}}_0^-$ on $\partial S'_{\omega_1}$. Using (3.8c), we have

$$|\tilde{q}_0 + \mathring{\tilde{v}}_0^-| \le c_2 e^{-\gamma(\tan\omega_1)\sigma/2} \le c_w \cdot c_\phi e^{-a\sigma} \le w\phi \quad \text{on } \Gamma'_{\omega_1}$$

if $c_2 < c_w c_\phi$ and $a \le \gamma(\tan\omega_1)/2$. Using (3.8c) we can pick c_2 small enough to satisfy this inequality by picking $\bar\rho$ sufficiently large. Finally, on $\Gamma_{\bar\rho}$, using (3.8d) we have

$$|\tilde q_0 + \mathring{\tilde v}_0^-| \le c_2 \le c_w \cdot c_\phi e^{-a\bar\rho} \le w\phi \quad \text{ on } \Gamma_{\bar\rho}$$

provided $c_2 < c_w c_\phi$ and a is sufficiently small. Again, this is achieved by making $\bar\rho = \bar\rho(c_2)$ sufficiently large independently of a and then choosing $a = a(\bar\rho)$ sufficiently small. In summary, if $\bar\rho$ is sufficiently large and a is sufficiently close to 0, the function $\bar W$ is a super-solution to the problem (3.9).

As $\tilde z_0$ is the unique minimal solution corresponding to the sub-solution $\max\{\mathring{\tilde v}_0, \mathring{\tilde v}_0^-\}$, then, by Lemma 3.3, the function $W = \tilde q_0 + \mathring{\tilde v}_0^- = \tilde z_0 - \mathring{\tilde v}_0$ restricted to S'_{ω_1} is the unique minimal solution to (3.9) corresponding to the sub-solution $\underline{W} = \max\{\mathring{\tilde v}_0, \mathring{\tilde v}_0^-\} - \mathring{\tilde v}_0 = 0$. Since clearly $\underline{W} \le \bar W$, we conclude that $0 = \underline{W} \le W = \tilde q_0 + \mathring{\tilde v}_0^- \le \bar W \le Ce^{-a|\eta|}$ in S_{ω_1}.

Combining this with $|\mathring{\tilde v}_0^-| \le Ce^{-a|\eta|}$ in S_{ω_1}, yields $|\tilde q_0| \le Ce^{-a|\eta|}$ in S_{ω_1}. A similar argument shows that $|\tilde q_0| \le Ce^{-a|\eta|}$ in the sector $S^-_{\omega_1}$ of angle ω_1 and adjacent to the side Γ^- of S. The inequality (2.11) implies $|\tilde q_0| \le C_1 e^{-c_1|\eta|}$ in $S \setminus (S_{\omega_1} \cup S^-_{\omega_1})$, so $|\tilde q_0| \le C_1 e^{-c_1|\eta|}$ in S.

To bound first derivatives of $\tilde q_0$, let $\omega_2 \in (\frac{1}{2}\omega, \omega)$, let Γ_{ω_2} be the ray in S which makes an angle ω_2 with Γ, and let $S_{\omega_2} \subset S$ be the sector with sides Γ and Γ_{ω_2}. Using (3.7), (3.8a) and the exponential decay of $\tilde q_0$ and $\mathring{\tilde v}_0^-$ in S_{ω_2}, we find that $|\triangle(\tilde q_0 + \mathring{\tilde v}_0^-)| \le C|\tilde q_0 + \mathring{\tilde v}_0^-| \le Ce^{-c|\eta|}$ in S_{ω_2}. Now, applying the local Schauder-type estimate for first derivatives to pairs of concentric discs of radii 1 and 2, which can possibly intersect Γ, but not Γ_{ω_2}, one finds that $|\nabla(\tilde q_0 + \mathring{\tilde v}_0^-)| \le Ce^{-c|\eta|}$ and therefore $|\nabla \tilde q_0| \le Ce^{-c|\eta|}$ inside any admissible interior unit disc. Since such unit discs cover $\bar S_{\omega/2} \cap \{|\eta| \ge \bar C\}$, in view of Theorem 2.2 the desired exponential decay in the entire sector $\bar S_{\omega/2}$ follows. □

3.4. Well-posedness of the linearized problem in a sector. Let $\tilde z_0$ be the solution of (2.9) given by Theorem 2.2. Let $\tilde a(\eta) = \tilde B_t(O, \tilde z_0(\eta))$. In this section we establish the well-posedness of the linearized problem

$$\mathcal{M}W := -\triangle W + \tilde a W = F \ \text{ in } S, \qquad W = 0 \ \text{ on } \partial S. \tag{3.11}$$

As a consequence we obtain the existence of the functions z_1 and $\tilde z_{0,p}$ and exponential decay of their components q_1 and $\tilde q_{0,p}$.

Let S_R denote the truncated sector of radius R. We denote the 2 straight sides of S_R by Γ_R and Γ_R^-. We study the eigenvalue problem

$$\mathcal{M}\Phi_R = \lambda_R \Phi_R \ \text{ in } S_R, \qquad \Phi_R = 0 \ \text{ on } \partial S_R. \tag{3.12}$$

Applying the general eigenvalues/eigenfunctions theory [2, §6.5.1] to the operator $\mathcal{M} + C_a$, where $C_a > \max|\tilde a|$, we conclude that the problem (3.12) has a countable set of real eigenvalues $\lambda_{R,1} < \lambda_{R,2} \le \cdots$ and associated eigenfunctions. The principal eigenvalue $\lambda_{R,1}$ has only one eigenfunction, which we denote by ϕ. The eigenfunction $\phi > 0$ in S_R. Although $\tilde a$ is not necessarily non-negative, we will be able to show, see Theorem 3.13 below, that $\lambda_{R,1}$ is positive and bounded away from 0 uniformly in R. This implies, see Lemma 3.14 below, that the problem (3.11), but posed on the truncated sector, is well-posed and with a solution that is uniformly bounded in R. Taking the limit as $R \to \infty$ then gives the desired result, Theorem 3.15.

Let the direction of the $\bar\xi$-axis coincide with the ray $\omega/2$ and set

$$Z(\eta) := -\frac{\partial \tilde z_0}{\partial \bar\xi}\,. \tag{3.13}$$

Clearly, $\mathcal{M}Z = 0$ and $Z \geq 0$ on ∂S, since $\tilde{z}_0 = A$ on ∂S and $\tilde{z}_0 \leq A$ in S. We also note that since $\tilde{z}_0$ is constant on Γ, then $Z = -(\sin \frac{1}{2}\omega)\tilde{z}_{0,\xi}$ on Γ.

LEMMA 3.7. *We have $0 \leq Z \leq C$ in S. Furthermore, for each $R' > 0$ there is a $C_1 = C_1(R') > 0$ such that if $\eta \in \Gamma \cup \Gamma^-$ and $|\eta| > R'$, then $Z(\eta) \geq C_1$.*

Proof. The boundedness of Z follows from Theorem 2.2. To show that $Z \geq 0$, recall that $\tilde{z}_0$ was defined in Section 3.2 as the unique minimal solution of (2.9) corresponding to the sub-solution $\alpha = \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\}$. An inspection of the proof of [10, Theorem 7.1] shows that $\tilde{z}_0$ can be generated as the limit of an increasing sequence of sub-solutions $\{\alpha^{(k)}\}$, i.e. $\tilde{z}_0 = \lim_{k\to\infty} \alpha^{(k)}$, where $\alpha^{(0)} := \alpha$ and further $\alpha^{(k)}$ are defined inductively by

$$-\triangle\alpha^{(k)} + \bar{C}\alpha^{(k)} = \bar{C}\alpha^{(k-1)} - \tilde{B}(O, \alpha^{(k-1)}), \qquad \alpha^{(k)}\Big|_{\partial S} = A.$$

Here $\bar{C} \geq \tilde{B}_t(O,t)$ for all $t \in [0, A]$. Now a calculation shows that

$$[-\triangle + \bar{C}\,]\,\alpha^{(k)}_{\bar{\xi}} = [\bar{C} - \tilde{B}_t(O, \alpha^{(k-1)})]\,\alpha^{(k-1)}_{\bar{\xi}}, \qquad \alpha^{(k)}_{\bar{\xi}}\Big|_{\partial S} \leq 0,$$

where the relation on the boundary for each k follows from $\alpha^{(k)} \leq A$ in S. Note also that for $\alpha^{(0)} = \max\{\mathring{\tilde{v}}_0, \mathring{\tilde{v}}_0^-\}$ we get

$$\alpha^{(0)}_{\bar{\xi}} \leq 0 \qquad \text{in } \ S.$$

By the maximum principle and induction, these imply that

$$-\alpha^{(k)}_{\bar{\xi}} \geq 0 \qquad \text{in } \ S,$$

and hence, taking the limit as $k \to \infty$, we get the non-negativity of Z.

Now we shall show that for any $R' > 0$, there is a $C_1 > 0$ such that $Z \geq C_1 > 0$ for $\eta \in \Gamma \cup \Gamma^-$ and $|\eta| \geq R'$. Recall that $Z = -\partial\tilde{z}_0/\partial\bar{\xi} = -[\mathring{\tilde{v}}_0(\xi) + \mathring{\tilde{v}}_0^-(\xi^-)]_{\bar{\xi}} - \tilde{q}_{0,\bar{\xi}}$. Since the first term satisfies the desired estimate, and from Lemma 3.6, the term $\tilde{q}_{0,\bar{\xi}}$ can be made arbitrarily small by making $|\eta|$ large enough, there are positive constants R'' and C'' such that if $|\eta| > R''$ and $\eta \in \Gamma \cup \Gamma^-$ then $Z(\eta) > C''$.

It remains to show that for any $R' > 0$, there is a $C' > 0$ such that $Z \geq C'$ for $\eta \in \Gamma \cup \Gamma^-$ and $R' \leq |\eta| \leq R''$. This property immediately follows from $Z > 0$ on $\partial S\backslash O$. To show this, recall that $\tilde{z}_0 \leq A$ is a solution of problem (2.9), where $A > 0$ and $\tilde{B}(O, A) > 0$. Now, a calculation shows that

$$[-\triangle + \bar{C}](\tilde{z}_0 - A) < (\bar{C}t - \tilde{B}(O,t))\Big|_A^{\tilde{z}_0} \leq 0 \quad \text{in } S$$

Combining this with $\tilde{z}_0 - A = 0$ on ∂S and applying the maximum principle, yields $\tilde{z}_0 - A < 0$ in S. Furthermore, by Hopf's Lemma [2, §6.4.2], we have $\partial(\tilde{z}_0 - A)/\partial n > 0$ on $\partial S\backslash O$ (where O is excluded since it does not satisfy the interior ball condition). Finally, $Z = (\sin \frac{1}{2}\omega)\partial\tilde{z}_0/\partial n > 0$ on $\partial S\backslash O$. □

As the pair $(\lambda_{R,1}, \phi)$ solves (3.12), we get the equation

$$\lambda_{R,1}\phi\, Z = -(\triangle\phi)Z + \tilde{a}\phi Z = -(\triangle\phi)Z + \phi(\triangle Z),$$

integrating which over S_R and using the fact that $\phi = 0$ on ∂S_R we obtain

$$\lambda_{R,1} \iint_{S_R} \phi Z = -\int_{\partial S_R} Z\frac{\partial\phi}{\partial n} > 0. \tag{3.14}$$

Since ϕ and Z are positive in S_R, it follows that $\lambda_{R,1} > 0$. We now seek a lower bound for $\lambda_{R,1}$ that is independent of R.

In what follows we set $\rho(\eta) = \tilde{a}(\eta) - \lambda_{R,1}$, so $-\triangle\phi + \rho\phi = 0$. Recall that $\tilde{a}(\eta) = \tilde{B}_t(O, \tilde{z}_0(\eta))$ and, in view of (2.1), by the assumption A1, we have $\tilde{B}_t(O,0) > 0$. Using (2.11) pick a number $\Xi > 0$ such that $\tilde{a}(\eta) \geq \frac{3}{4}\tilde{B}_t(O,0)$ if $\min\{\xi, \xi^-\} \geq \Xi$. Set $S(\Xi) = \{\eta \in S_R : \min\{\xi, \xi^-\} \geq \Xi\}$. Thus,

$$\rho(\eta) \geq \tfrac{1}{4}\tilde{B}_t(O,0) \qquad \text{if} \quad \lambda_{R,1} \leq \tfrac{1}{2}\tilde{B}_t(O,0) \quad \text{and} \quad \eta \in S(\Xi). \tag{3.15}$$

Finally, for $R' < R$ let $\Gamma_{R',R}$ be the set of points $\eta \in \Gamma_R$ such that $|\eta| > R'$.

Lemma 3.8. *If $\lambda_{R,1} < \frac{1}{2}\tilde{B}_t(O,0)$ then for each $R' \in (0,R)$ there is a positive constant $C_2(R')$, independent of R, such that*

$$\lambda_{R,1} \geq C_2 \frac{\int_{\Gamma_{R',R}} \partial\phi/\partial\xi}{\|\phi\|_{L_1(S_R \cap \{\xi \leq \Xi\})}}. \tag{3.16}$$

Proof. Since $\partial\phi/\partial n \leq 0$ on ∂S_R and $-\partial\phi/\partial n = \partial\phi/\partial\xi$ on Γ_R, relation (3.14) and Lemma 3.7 imply

$$\lambda_{R,1} = \frac{\int_{\partial S_R} |\partial\phi/\partial n| Z}{\iint_{S_R} \phi Z} \geq 2C_1 \frac{\int_{\Gamma_{R',R}} \phi_\xi}{\|Z\|_{L_\infty(S_R)}\|\phi\|_{L_1(S_R)}},$$

where $C_1 = C_1(R')$ is from Lemma 3.7. Using the bounds for Z given by Lemma 3.7, it remains to show that

$$\|\phi\|_{L_1(S_R)} \leq C\|\phi\|_{L_1(S_R \cap \{\xi \leq \Xi\})} \tag{3.17}$$

From (3.15), we have

$$\begin{aligned} 0 &\geq \int_{\partial S_R} \frac{\partial\phi}{\partial n} = \iint_{S_R} \triangle\phi = \iint_{S_R} \rho\phi = \iint_{S(\Xi)} \rho\phi + \iint_{S_R \setminus S(\Xi)} \rho\phi \\ &\geq \tfrac{1}{4}\tilde{B}_t(O,0)\|\phi\|_{L_1(S(\Xi))} - (\max_S |\rho|)\, \|\phi\|_{L_1(S_R \setminus S(\Xi))}. \end{aligned}$$

Recalling that $\tilde{B}_t(O,0) > 0$, we get

$$\|\phi\|_{L_1(S(\Xi))} \leq C\|\phi\|_{L_1(S_R \setminus S(\Xi))} \leq 2C\|\phi\|_{L_1(S:\xi \leq \Xi)}$$

where in the final assertion we used the symmetry of ϕ with respect to ξ and ξ^-. Adding $\|\phi\|_{L_1(S_R \setminus S(\Xi))}$ to both sides gives (3.17). ☐

Now we present an auxiliary lemma, which will enable us to get a lower bound for $\int_{\Gamma_{R',R}} \partial\phi/\partial\xi$ in the forthcoming Lemma 3.11.

Lemma 3.9. *Let positive numbers ρ_0, $\bar{\delta}$ and a be given. Let $0 < \delta \leq \bar{\delta}$. Let Ψ be a bounded positive function on $\mathbb{R}$ with $\Psi(x) = 0$ for $|x| > a$. Let a bounded function $\psi(x,y)$ be defined for $(x,y) \in (-\infty,\infty) \times [0,\delta]$ by*

$$-\triangle\psi + \rho_0\psi = 0, \qquad \psi(x,0) = 0, \qquad \psi(x,\delta) = \Psi(x)$$

Then $\psi \geq 0$ and there exist positive numbers $\bar{a} = \bar{a}(\bar{\delta})$ and $C_0(\bar{\delta})$ such that, setting $I = \int_{-a}^{a} \Psi(x)dx$, we have

$$\int_{-(a+\bar{a})}^{(a+\bar{a})} \psi_y(x,0)dx \geq \frac{C_0(\bar{\delta})}{\delta} I, \qquad \int_{|x|>|a+\bar{a}|} \psi_y(x,0)dx \leq \frac{C_0(\bar{\delta})}{2\delta} I.$$

Proof. We apply the Fourier transform in the variable x to ψ to obtain the solution formula $\psi(x,y) = \int_{-a}^{a} G(x-t,y)\Psi(t)dt$, where

$$G(x,y) = \frac{1}{\pi}\int_0^\infty \cos(xs)\ \frac{\sinh(y\sqrt{s^2+\rho_0})}{\sinh(\delta\sqrt{s^2+\rho_0})}ds.$$

By the maximum principle, $G(x,y) \geq 0$, while $G(x,0) = 0$, and hence

$$G_y(x,0) = \frac{1}{\pi}\int_0^\infty \cos(xs)\ \frac{\sqrt{s^2+\rho_0}}{\sinh(\delta\sqrt{s^2+\rho_0})}ds \geq 0.$$

Since $\psi_y(x,0) = \int_{-a}^{a} G_y(x-t,0)\Psi(t)\,dt$, we obtain

$$\int_{-a-\bar{a}}^{a+\bar{a}} \psi_y(x,0)\,dx = \int_{-a}^{a} I_1(t)\Psi(t)dt, \qquad I_1(t) := \int_{-a-\bar{a}-t}^{a+\bar{a}-t} G_y(x,0)\,dx.$$

If $t \in [-a,a]$, then $[-a-\bar{a}-t, a+\bar{a}-t] \supset [-\bar{a},\bar{a}] \supset [-\delta\tilde{a},\delta\tilde{a}]$, where $\tilde{a}$ is any positive number $\leq \bar{a}/\bar{\delta}$ and $\delta \leq \bar{\delta}$. Combining this with $G_y(x,0) \geq 0$, we get

$$I_1(t) \geq \frac{1}{\pi}\int_{-\delta\tilde{a}}^{\delta\tilde{a}} dx \int_0^\infty \cos(xs)\ \frac{\sqrt{s^2+\rho_0}}{\sinh(\delta\sqrt{s^2+\rho_0})}ds.$$

Changing variables to $\hat{x} = x/\delta$ and $\hat{s} = s\delta$ so that $xs = \hat{x}\hat{s}$ and $dx\,ds = d\hat{x}\,d\hat{s}$ we arrive at

$$I_1(t) \geq \frac{1}{\delta\,\pi}\int_{-\tilde{a}}^{\tilde{a}} d\hat{x} \int_0^\infty \cos(\hat{x}\hat{s})\ \frac{\sqrt{\hat{s}^2+\delta^2\rho_0}}{\sinh(\sqrt{\hat{s}^2+\delta^2\rho_0})}d\hat{s}.$$

Note that here $\delta^2\rho_0 \in (0, \bar{\delta}^2\rho_0]$, while $e^{-s} \leq s/\sinh s \leq 2e^{-s/2}$ for any positive s. Therefore

$$e^{-\bar{\delta}\sqrt{\rho_0}}e^{-\hat{s}} \leq \frac{\sqrt{\hat{s}^2+\delta^2\rho_0}}{\sinh(\sqrt{\hat{s}^2+\delta^2\rho_0})} \leq 2e^{-\hat{s}/2}.$$

Combining this with $\cos(\hat{x}\hat{s}) > \frac{1}{2}$ for $\hat{s} \leq 1/|\hat{x}|$ and $|\cos(\hat{x}\hat{s})| \leq 1$ otherwise, we get

$$\begin{aligned} I_1(t) &\geq \frac{1}{\delta\,\pi}\int_{-\tilde{a}}^{\tilde{a}} d\hat{x}\Big[\tfrac{1}{2}\,e^{-\bar{\delta}\sqrt{\rho_0}}\int_0^{1/|\hat{x}|} e^{-\hat{s}}d\hat{s} - 2\int_{1/|\hat{x}|}^\infty e^{-\hat{s}/2}d\hat{s}\Big] \\ &= \frac{C}{\delta}\int_{-\tilde{a}}^{\tilde{a}} d\hat{x}\Big[1 - e^{-1/|\hat{x}|} - C'e^{-1/(2|\hat{x}|)}\Big], \end{aligned}$$

where $C = C(\bar{\delta})$ and $C' = C'(\bar{\delta})$. If $\tilde{a} > 0$ is chosen sufficiently small, depending only on $\bar{\delta}$, the integrand is $\geq C > 0$ and hence $I_1(t) \geq C_0(\bar{\delta})/\delta$.

Now consider

$$\int_{a+\bar{a}}^\infty \psi_y(x,0)\,dx = \int_{-a}^{a} I_2(t)\Psi(t)dt, \qquad I_2(t) := \int_{a+\bar{a}-t}^\infty G_y(x,0)\,dx.$$

Since $t \in [-a,a]$, we see that $a+\bar{a}-t \geq \bar{a} \geq \delta\bar{a}/\bar{\delta}$. Combining this with $G_y(x,0) \geq 0$ we get

$$0 \leq I_2(t) \leq \int_{\delta\bar{a}/\bar{\delta}}^\infty G_y(x,0)\,dx = \frac{1}{2\pi}\int_{\delta\bar{a}/\bar{\delta}}^\infty dx \int_{-\infty}^\infty e^{ixs}\ \frac{\sqrt{s^2+\rho_0}}{\sinh(\delta\sqrt{s^2+\rho_0})}ds.$$

Again using $\hat{x} = x/\delta$ and $\hat{s} = s\delta$, we get

$$I_2(t) \le \frac{\delta^{-1}}{2\pi} \int_{\bar{a}/\bar{\delta}}^{\infty} d\hat{x} \int_{-\infty}^{\infty} e^{i\hat{x}\hat{s}}\; g(\hat{s}) d\hat{s} = \frac{\delta^{-1}}{2\pi} \int_{\bar{a}/\bar{\delta}}^{\infty} \frac{d\hat{x}}{(i\hat{x})^2} \int_{-\infty}^{\infty} e^{i\hat{x}\hat{s}}\; g''(\hat{s}) d\,\hat{s},$$

where

$$g(\hat{s}) = \frac{\sqrt{\hat{s}^2 + \delta^2\rho_0}}{\sinh(\sqrt{\hat{s}^2 + \delta^2\rho_0})}$$

and we used integration by parts twice. A calculation shows that $g(\hat{s})$ and its derivatives are well-defined (e.g. they are bounded at 0) and

$$|g(\hat{s})| + |g'(\hat{s})| + |g''(\hat{s})| \le Ce^{-C|\hat{s}|}$$

Taking absolute values,

$$I_2(t) \le \frac{C}{\delta} \int_{\bar{a}/\bar{\delta}}^{\infty} \frac{d\hat{x}}{\hat{x}^2} = \frac{C}{\delta \bar{a}/\bar{\delta}} = \frac{C_0(\bar{\delta})}{4\delta}$$

if $\bar{a} = \bar{a}(\bar{\delta})$ is chosen sufficiently large. The integral over $(-\infty, -(a+\bar{a}))$ is estimated in the same way. □

COROLLARY 3.10. *Under the conditions of Lemma 3.9, there exists a function $\tilde{\psi}$ that satisfies*

$$-\triangle\tilde{\psi} + \rho_0\tilde{\psi} = 0, \quad \tilde{\psi}(x,0) = 0, \quad \tilde{\psi}(x,\delta) = \Psi(x), \quad \tilde{\psi}(x, \pm(a+\bar{a})) \le 0$$

in the domain $[-(a+\bar{a}), a+\bar{a}] \times [0,\delta]$, and

$$\int_{-(a+\bar{a})}^{(a+\bar{a})} \tilde{\psi}_y(x,0)dx \ge C_0'(\bar{\delta}) \int_{-a}^{a} \Psi(x)dx.$$

Proof. Let ψ be the function given by Lemma 3.9 and let

$$\tilde{\psi}(x,y) := \psi(x,y) - \psi(2(a+\bar{a}) - x, y) - \psi(-2(a+\bar{a}) - x, y).$$

Then $\tilde{\psi}(\pm(a+\bar{a})) = -\psi(\mp 3(a+\bar{a})) \le 0$; and $\tilde{\psi}$ satisfies the same equation as ψ. We obtain the asserted inequality with $C_0'(\bar{\delta}) = \frac{1}{2}\bar{\delta}^{-1}C_0(\bar{\delta})$. □

LEMMA 3.11. *There are positive numbers $\bar{a}$ and $\bar{C}$, independent of R, such that for any σ_1, σ_2 with $0 < \sigma_1 < \sigma_2$, if $[0,\Xi] \times [\sigma_1 - \bar{a}, \sigma_2 + \bar{a}] \subset S_R$, where η is interpreted as $\eta = (\xi, \sigma)$, then*

$$\int_{\sigma_1 - \bar{a}}^{\sigma_2 + \bar{a}} \phi_\xi(0,\sigma)d\sigma \ge \bar{C}\|\phi\|_{L_1([0,\Xi]\times[\sigma_1,\sigma_2])}.$$

Proof. Since $\phi \ge 0$, the mean value theorem applied to the positive function $\Phi(\xi) := \int_{\sigma_1}^{\sigma_2} \phi(\xi,\sigma)d\sigma$ gives

$$\|\phi\|_{L_1([0,\Xi]\times[\sigma_1,\sigma_2])} = \Xi \int_{\sigma_1}^{\sigma_2} \phi(\delta,\sigma)d\sigma$$

for some $\delta \in (0,\Xi)$.

We apply Corollary 3.10 with $\rho_0 > 0$ such that $\rho_0 > \max_S \rho(\eta)$, $\bar{\delta} = \Xi$ and the function Ψ defined by $\Psi(\sigma) = \phi(\delta, \sigma)$ for $\sigma \in [\sigma_1, \sigma_2]$, $\Psi(\sigma) = 0$ otherwise. Thus there exists a function $\tilde{\psi}$ such that $-\triangle\tilde{\psi} + \rho_0\tilde{\psi} = 0$ and

$$\int_{\sigma_1-\bar{a}}^{\sigma_2+\bar{a}} \tilde{\psi}_\xi(0,\sigma)d\sigma \geq C_0'(\Xi)\int_{\sigma_1}^{\sigma_2}\phi(\delta,\sigma)d\sigma$$

Note that the choice of ρ_0 implies $-\triangle\phi + \rho_0\phi \geq 0$. Now by the maximum principle, $\phi \geq \tilde{\psi}(\xi, \sigma)$ and hence $\phi_\xi(0, \sigma) \geq \tilde{\psi}_\xi(0, \sigma)$, which yields

$$\int_{\sigma_1-\bar{a}}^{\sigma_2+\bar{a}} \phi_\xi(0,\sigma)d\sigma \geq C_0'(\Xi)\int_{\sigma_1}^{\sigma_2}\phi(\delta,\sigma)d\sigma = \Xi^{-1}C_0'(\Xi)\|\phi\|_{L_1([0,\Xi]\times[\sigma_1,\sigma_2])}.$$

The result is therefore obtained with $\bar{C} = \Xi^{-1}C_0'(\Xi)$. □

The next lemma gives another lower bound for $\int_{\Gamma_{R',R}}\phi_\xi$ in (3.16).

Lemma 3.12. *There exist positive numbers R' and C^*, independent of R, such that if $\lambda_{R,1} < \frac{1}{2}\tilde{B}_t(O, 0)$, we have $R > 2R'$ and*

$$\int_{\Gamma_{R',R}} \phi_\xi \geq C^* \max_{S_R} \phi.$$

Proof. It suffices to prove the desired estimate for ϕ scaled so that $\max_{S_R}\phi = 1$. Let the maximum be attained at (ξ^*, σ^*); clearly $\xi^* = \xi^*(R)$ and $\sigma^* = \sigma^*(R)$. Since ϕ is symmetric with respect to Γ_R and Γ_R^-, there is such a point (ξ^*, σ^*) closer to Γ. Note that for this point we have $\xi^* \leq \Xi$; indeed, by the maximum principle, ϕ cannot attain its positive maximum in $S(\Xi)$, since in this subdomain $-\triangle\phi + \rho\phi = 0$ with $\rho > 0$. Next, combining $\triangle\phi = \rho\phi$ with $\phi \leq 1$ in S_R, we get $|\nabla\phi| \leq C$ in Ω_R, where C is independent of R. Hence with $\delta = 1/(4C)$ we have $\phi(\xi^*, \sigma) \geq \frac{1}{2}$ for $\sigma \in [\sigma^*, \sigma^*+2\delta]$. Therefore $R > \sigma^* + 2\delta \geq 2\delta$, the rectangle $(0, \xi^*) \times (\sigma^*, \sigma^* + 2\delta)$ is in S_R and on its boundary $\phi(\xi, \sigma)$ satisfies

$$\phi(\xi^*,\sigma) \geq \frac{1}{4}\big[\cos\big([\sigma-(\sigma^*+\delta)]\pi/\delta\big)+1\big], \quad \phi(\xi,(\sigma^*+\delta)\pm\delta) \geq 0, \quad \phi(0,\sigma) = 0$$

We claim that this implies

$$\int_\delta^R \phi_\xi(0,\sigma)d\sigma \geq \int_{\sigma^*+\delta}^{\sigma^*+2\delta}\phi_\xi(0,\sigma)d\sigma \geq C^*, \tag{3.18}$$

which yields the assertion of the lemma with $R' = \delta$.

To prove (3.18) set $\sigma' = \sigma - (\sigma^* + \delta)$, let $\rho_0 > 0$ satisfy $\rho_0 > \max_S \rho(\eta)$, and set $\kappa = \sqrt{\rho_0 + (\pi/\delta)^2}$. Consider the barrier function

$$\psi(\xi,\sigma') := [\cos(\sigma'\pi/\delta)+1]\frac{\sinh(\kappa\xi)}{\sinh(\kappa\xi^*)} \text{ for } (\xi,\sigma') \in [0,\xi^*]\times[-\delta,\delta].$$

Clearly

$$\psi \geq 0, \qquad \psi(\xi,\pm\delta) = \psi(0,\sigma') = 0, \qquad \psi(\xi^*,\sigma') = \cos(\sigma'\pi/\delta)+1,$$

and furthermore

$$-\triangle\psi + \rho_0\psi = -(\pi/\delta)^2\frac{\sinh(\kappa\xi)}{\sinh(\kappa\xi^*)} \leq 0.$$

Finally note that since $\xi^* \le \Xi$,

$$\frac{\partial\psi}{\partial\xi}\Big|_{\xi=0} = [\cos(\sigma'\pi/\delta)+1]\frac{\kappa}{\sinh(\kappa\xi^*)} \ge [\cos(\sigma'\pi/\delta)+1]\frac{\kappa}{\sinh(\kappa\Xi)},$$

so

$$\frac{\partial\psi}{\partial\xi}\Big|_{\xi=0,|\sigma'|\le\delta/2} \ge \frac{\kappa}{\sinh(\kappa\Xi)}.$$

Now for $(\xi,\sigma') \in [0,\xi^*]\times[-\delta,\delta]$ we have $\phi \ge \frac{1}{4}\kappa\psi$. This indeed follows from $-\triangle\phi+\rho_0\phi \ge -\triangle\phi+\rho\phi = 0$, the boundary conditions on ϕ, and the maximum principle applied in the domain $(\xi,\sigma') \in [0,\xi^*]\times[-\delta,\delta]$. Thus we arrive at the bound $\phi_\xi\big|_{\xi=0,|\sigma'|\le\delta/2} \ge \frac{1}{4}\kappa/\sinh(\kappa\Xi)$, which yields

$$\int_{\sigma^*+\delta}^{\sigma^*+2\delta}\phi_\xi(0,\sigma)d\sigma \ge \int_0^{\delta/2}\phi_\xi(0,\sigma')d\sigma' \ge \frac{\kappa\delta}{8\sinh(\kappa\Xi)} = C^* > 0,$$

i.e. we have obtained (3.18). □

THEOREM 3.13. *The principal eigenvalue of S_R satisfies $\lambda_{R,1} \ge C > 0$, where C is independent of R.*

Proof. If $\lambda_{1,R} \ge \frac{1}{2}\tilde{B}_t(O,0)$, our assertion follows. Therefore we suppose that $\lambda_{1,R} < \frac{1}{2}\tilde{B}_t(O,0)$ and let R' be given by Lemma 3.12. Consider the largest domain $\Omega_0 = \{(\xi,\sigma) \in [0,\Xi]\times[\sigma_1,\sigma_2]\}$ such that $\Omega_0' = \{(\xi,\sigma) \in [0,\Xi]\times[\sigma_1-\bar{a},\sigma_2+\bar{a}]\} \subset S_R \setminus S_{R'}$. Then by Lemma 3.11, we have

$$\int_{\Gamma_{R',R}}\phi_\xi \ge \int_{\sigma_1-\bar{a}}^{\sigma_2+\bar{a}}\phi_\xi(0,\sigma)d\sigma \ge \bar{C}\|\phi\|_{L_1(\Omega_0)}.$$

Combining this with Lemma 3.12, we get

$$\int_{\Gamma_{R',R}}\phi_\xi \ge C\left[\|\phi\|_{L_1(\Omega_0)} + \max_{S_R}\phi\right]. \tag{3.19}$$

Note also that $S_R \cap \{\xi \le \Xi\}\backslash\Omega_0$ is of size $O(1)$ and therefore

$$\|\phi\|_{L_1,S_R\cap\{\xi\le\Xi\}} \le \|\phi\|_{L_1(\Omega_0)} + C\max_{S_R}\phi. \tag{3.20}$$

Combining (3.16), (3.19) and (3.20), we get the assertion of the theorem. □

LEMMA 3.14. *There is a constant $C > 0$, independent of R, such that if $R > 0$ and $F \in L_2(S_R)$ then the problem*

$$\mathcal{M}W = F \ \text{ in } S_R, \qquad W = 0 \ \text{ on } \partial S_R \tag{3.21}$$

has a solution W which satisfies

$$\|W\|_{L_\infty(S_R)} + \|W\|_{H^2(S_R)} \le C\|F\|_{L_2(S_R)}. \tag{3.22}$$

If $|F(\eta)| \le Ce^{-c|\eta|}$, then $|W(\eta)| \le C'e^{-c'|\eta|}$.

Proof. Since $\mathcal{M}$ is a self-adjoint operator on $L_2(S_R)$, the well-posedness of the boundary value problem and the inequality $\|W\|_{L_2(S_R)} \le C\|F\|_{L_2(S_R)}$ follows from Theorem 3.13 and an eigenfunction expansion. Write the differential equation as

$-\triangle W = F_1 := -\tilde{a}W + F$. Then $F_1 \in L_2(R)$ and has L_2 norm bounded uniformly in R. From the "second fundamental inequality" of Ladyzhenskaya ([8], Lemma 8.1) and the convexity of the sector S_R, one obtains the inequality $\|W\|_{H^2(S_R)} \leq C\|F_1\|_{L_2(S_R)} \leq C\|F\|_{L_2(S_R)}$, where C is independent of R, which is one of the inequalities in (3.22). Sobolev's inequality implies the other inequality in (3.22).

For the exponential decay, note first that the expansion of W into the eigenfunctions of $\mathcal{M}$ gives

$$\lambda_{R,1}\|W\|^2_{L_2(S_R)} \leq (\mathcal{M}W, W),$$

$$\|\nabla W\|^2_{L_2(S_R)} = (\mathcal{M}W, W) - (\tilde{a}W, W) \leq C(\mathcal{M}W, W).$$

Therefore we get the "strict Garding inequality"

$$\|W\|^2_{H^1(S_R)} \leq C_*(\mathcal{M}W, W) \text{ for } W \in H^1_0(S_R) \tag{3.23}$$

with a constant C_* that is independent of R.

Note that $|\eta|\cos(\omega/2) \leq \bar{\xi} \leq |\eta|$, where the variable $\bar{\xi}$ has the same meaning as in (3.13). Therefore, it suffices to establish the final exponential-decay assertion of the lemma with $|\eta|$ replaced by $\bar{\xi}$. Now suppose F satisfies $|F(\eta)| \leq Ce^{-c\bar{\xi}}$. Let $\kappa \in (0, c)$ and set $\tilde{W} = e^{\kappa\bar{\xi}}W$, $\tilde{F} = e^{\kappa\bar{\xi}}F$. Thus $|\tilde{F}(\eta)| \leq Ce^{-c'\bar{\xi}}$ with $c' = c - \kappa > 0$. The function $\tilde{W}$ satisfies

$$\mathcal{M}\tilde{W} + 2\kappa\tilde{W}_{\bar{\xi}} - \kappa^2\tilde{W} = \tilde{F}. \tag{3.24}$$

Applying (3.23) we get

$$C_*^{-1}\|\tilde{W}\|^2_{H^1(S_R)} \leq (\tilde{F} - 2\kappa\tilde{W}_{\bar{\xi}} + \kappa^2\tilde{W}, \tilde{W}) \leq \|\tilde{W}\|_{L_2(S_R)}\|\tilde{F}\|_{L_2(S_R)} + \kappa^2\|\tilde{W}\|^2_{L_2(S_R)},$$

where we used $(\tilde{W}, \tilde{W}_{\bar{\xi}}) = 0$. Choosing κ sufficiently small and using the arithmetic-geometric mean inequality we get

$$\|\tilde{W}\|_{H^1(S_R)} \leq C\|\tilde{F}\|_{L_2(S_R)}.$$

Setting $\tilde{F}_1 = \tilde{F} - 2\kappa\tilde{W}_{\bar{\xi}} + \kappa^2\tilde{W}$, this implies $\|\tilde{F}_1\|_{L_2(S_R)} \leq C\|\tilde{F}\|_{L_2(S_R)}$. Equation (3.24) becomes $\mathcal{M}\tilde{W} = \tilde{F}_1$ and the inequality (3.22) applied to this equation gives $|\tilde{W}(\eta)| \leq C\|\tilde{F}\|_{L_2(S_R)}$ for $\eta \in S_R$. Therefore $|W(\eta)| \leq Ce^{-\kappa\bar{\xi}}\|\tilde{F}\|_{L_2(S_R)} \leq Ce^{-\kappa\bar{\xi}}$ for $\eta \in S_r$. The constants in these inequalities are all independent of η. □

THEOREM 3.15. *There is a constant $C > 0$ such that if $F \in L_2(S)$ then the problem* (3.11) *has a solution W which satisfies*

$$\|W\|_{L_\infty(S)} + \|W\|_{H^2(S)} \leq C\|F\|_{L_2(S)}. \tag{3.25}$$

If $|F(\eta)| \leq Ce^{-c|\eta|}$, then $|W(\eta)| \leq C'e^{-c'|\eta|}$ for $c' < c$.

Proof. Pick a sequence $R_j \to \infty$ and let W_j be the corresponding solution of (3.21). Using compactness and a diagonalization argument one obtains a subsequence of the W_j, which we again call W_j, and a function $W \in H^2(S)$, such that for each $R > 0$, $W_j \to W$ in $H^1(S_R)$ and W_j converges weakly to W in $H^2(S)$. Letting $\chi \in C_0^\infty(S)$ and taking the limit in the equation

$$\iint_{S_{R_j}} \nabla W_j \cdot \nabla\chi + \tilde{a}W_j\chi = \iint_{S_{R_j}} F\chi,$$

we conclude that W solves (3.11). For each $R > 0$ one has

$$\|W\|_{H^2(S_R)} \leq \liminf \|W_j\|_{H^2(S_R)} \leq C\|F\|_{L_2(S)}.$$

Letting $R \to \infty$ we therefore get the second inequality of (3.25). The first inequality follows from Sobolev's inequality. The assertion regarding exponential decay is proved in the same way as in the proof of Lemma 3.14. ◻

3.5. The existence and exponential decay of q_1 and $\tilde{q}_{0,p}$. The formula (2.14b) presumes to give a definition of the function q_1, the presumption being that the boundary value problem (2.12) has a solution. In fact, we will turn the matter around: from (2.14b) we have already obtained the boundary value problem (2.16) for the function q_1. The data of this problem is exponentially decaying. This allows us to establish the existence of q_1 and then, from (2.14b), to define z_1.

LEMMA 3.16. *The solution to the problem* (2.16) *exists and defines a function q_1 which is exponentially decaying in S, i.e. $|q_1| \leq Ce^{-c|\eta|}$ for some C, $c > 0$.*

Proof. We apply Theorem 3.15 to a version of the problem (2.16) for an auxiliary function $\hat{q}_1$, which is obtained from q_1 by subtracting a smooth exponentially decaying function that satisfies the boundary conditions in (2.16) so that $\hat{q}_1\big|_{\partial S} = 0$. For this we must show that q_1 is exponentially decaying on Γ and Γ^-, and $\mathcal{M}q_1$ is exponentially decaying on S. The exponential decay of the boundary conditions in (2.16) and their derivatives follows from the inequalities in Lemma 2.1 combined with the observation that $\xi = |\eta| \sin\omega$ on Γ^- and $\xi^- = |\eta|\sin\omega$ on Γ. We now show that there are positive constants c_1 and C_1 such that

$$|\mathcal{M}q_1| \leq C_1 e^{-c_1|\eta|}. \tag{3.26}$$

Using (2.16), denote the 3 terms on the right-hand side of $\mathcal{M}q_1$ by I, II, and II^-. Recalling that $z_0 = q_0 + \mathring{v}_0 + \mathring{v}_0^-$ and then applying (2.4) to I yields the bound $|I| \leq C|\eta|(|q_0| + \mathring{v}_0\mathring{v}_0^-)$. For II and II^- we readily get $|II| \leq |\mathring{v}_1 + \sigma\mathring{v}_{0,s}|(|q_0| + \mathring{v}_0^-)$ and $|II^-| \leq |\mathring{v}_1^- + \sigma^-\mathring{v}_{0,s^-}^-|(|q_0| + \mathring{v}_0)$. Combining these three estimates with Lemma 3.6 and Lemma 2.1, we get the exponential decay of each of the terms I, II, and II^- and therefore the desired estimate (3.26). ◻

By formally differentiating both sides of (2.9) with respect to p and invoking (2.1), one obtains a linear boundary value problem satisfied by $\tilde{z}_{0,p}$. However the data for this equation are not square integrable and so Theorem 3.15 cannot be used. It is better to instead to work with $\tilde{q}_0$. The boundary value problem satisfied by $\tilde{q}_{0,p}$ is given by (2.17).

LEMMA 3.17. *The function $\tilde{q}_{0,p}$ exists and is exponentially decaying in S.*

Proof. We apply Theorem 3.15 to the problem (2.17). For this we must show that $\tilde{q}_{0,p}$ is exponentially decaying on Γ and Γ^-, and we must show that the right-hand side is exponentially decaying on S. The proof of this exponential decay of the data is similar to that given in the preceding lemma. ◻

4. The perturbed asymptotic expansion; sub- and super-solutions; existence proof. In Section 2.1 we have defined boundary layer functions $\tilde{v} = \tilde{v}_0 + \varepsilon v_1$ associated with a side Γ of Ω, and in Section 2.2 we have defined corner layer functions $\tilde{q} = \tilde{q}_0 + \varepsilon\tilde{q}_1$ associated with a vertex P_{j-1} on Ω. We now define perturbed asymptotic expansions β_{S_j} associated with a vertex P_{j-1} of the polygon Ω. These functions are then assembled into a perturbed asymptotic expansion β_Ω of the original problem (1.1). When the perturbation parameter p vanishes, we obtain an asymptotic expansion associated with the problem that is eventually shown to be of second order. The perturbed asymptotic expansions are used to construct sub- and super-solutions for the problem, and these sub- and super-solutions are then used to establish the existence of a solution to (1.1).

Recalling the formulas (2.7) and (2.14), we define a perturbed asymptotic expansion β_{S_j} and an asymptotic expansion u_{as,S_j} associated with the vertex P_{j-1} as follows:

$$\begin{aligned} \beta_{S_j}(x;p) &= u_0(x) + \tilde{v}(\xi, s; p) + \tilde{v}^-(\xi^-, s^-; p) + \tilde{q}(\eta; p) + \theta p, \\ u_{\mathrm{as},S_j}(x) &= \beta_{S_j}(x;0) = u_0(x) + v(\xi, s) + v^-(\xi^-, s^-) + q(\eta), \end{aligned} \tag{4.1}$$

where $x \in S_j$, and the variables ξ, ξ^-, s, s^-, η are all associated with the sector S_j having apex at P_{j-1} (see Figure 2.1). A value for the positive parameter θ and a range of values for p will be chosen shortly. The functions u_{as,S_j} will be used to build an asymptotic approximation to the solution. The proof of the following lemma is given in [5, Lemma 3.1].

LEMMA 4.1. *We have* $Fu_{\mathrm{as},S_j} = O(\varepsilon^2)$ *for all* $x \in S_j$. *Furthermore, we have* $u_{\mathrm{as},S_j}(x) = g(x) + O(\varepsilon^2)$ *on* ∂S_j, *where* $g(x)$ *is extended from* $\Gamma_{j-1} \cup \Gamma_j$ *onto* ∂S_j *as described in Section 2.2.*

LEMMA 4.2. *There are positive numbers* θ, ε^*, p^*, c_1 *and* c_2 *such that for* $\varepsilon \le \varepsilon^*$ *and* $|p| \le p^*$ *one has* $|\beta_{S_j}(x;p) - u_{\mathrm{as},S_j}| \le C|p|$, *and*

$$\beta_{S_j}(x;-p) \le \beta_{S_j}(x;p) \qquad \text{for } p > 0, \tag{4.2a}$$

$$F\beta_{S_j} \ge \tfrac{1}{2}\theta\gamma^2 p - c_1\varepsilon^2 \qquad \text{for } p > 0, \tag{4.2b}$$

$$F\beta_{S_j} \le -\tfrac{1}{2}\theta\gamma^2|p| + c_1\varepsilon^2 \qquad \text{for } p < 0, \tag{4.2c}$$

$$(\mathrm{sgn} p)\,\beta_{S_j} \ge (\mathrm{sgn} p)\,g + \tfrac{1}{2}\theta|p| - c_2\varepsilon^2 \text{ on } \partial S_j \qquad \text{for } p \ne 0. \tag{4.2d}$$

Proof. The inequalities (4.2b), (4.2c) are established in [5, Lemma 4.4]. Note that it is crucial in the proof that the positive parameter θ in the definition (4.1) of β_{S_j} is chosen sufficiently small so that $0 < \theta \le |\lambda(x)|^{-1}$, where for some $\vartheta = \vartheta(x) \in (0,1)$ we have $\lambda(x) = b_{uu}(x, u_0(x) + \vartheta[v_0 + v_0^- + q_0])$.

Next, we invoke [5, Lemma 4.1], which gives another desired assertion $\beta_{S_j} = u_{\mathrm{as},S_j} + O(p)$, and also states that for some sufficiently small $\varepsilon^* > 0$, if $\varepsilon \le \varepsilon^*$ and $p \ge 0$, then

$$\beta_{S_j}(x;-p) \le u_{\mathrm{as},S_j}(x) - \tfrac{1}{2}\theta p \le u_{\mathrm{as},S_j}(x) + \tfrac{1}{2}\theta p \le \beta_{S_j}(x;p). \tag{4.3}$$

This immediately implies (4.2a). Furthermore, for $x \in \partial S_j$, combining (4.3) with the estimate $u_{\mathrm{as},S_j}(x) = g(x) + O(\varepsilon^2)$ of Lemma 4.1, we get $\beta_{S_j}(x;p) \ge g(x) - c_2\varepsilon^2 + \frac{1}{2}\theta p$ for $p > 0$, and $\beta_{S_j}(x;p) \le g(x) + c_2\varepsilon^2 - \frac{1}{2}\theta|p|$ for $p < 0$. Thus we have obtained the remaining assertion (4.2d). $\square$

To define corresponding perturbed asymptotic expansions for the whole domain Ω we require a suitable partition of unity. Let functions $\{\chi_j\}_{j=1}^M$ be non-negative smooth functions $\bar{\Omega} \to [0,1]$ which satisfy

$$\chi_j(P_{j-1}) = 1, \qquad \chi_j(x) + \chi_{j+1}(x) = 1 \text{ on } \bar{\Gamma}_j, \qquad \sum_{j=1}^{M} \chi_j(x) = 1 \text{ on } \bar{\Omega}.$$

We define the perturbed asymptotic expansion β_Ω associated with the problem (1.1) by

$$\beta_\Omega(x;p) = \sum_{j=1}^{M} \chi_j(x)\,\beta_{S_j}(x;p),$$
$$u_{\mathrm{as},\Omega}(x) = \beta_\Omega(x;0).$$

One has

LEMMA 4.3. *There are positive numbers θ, ε^*, p^*, c_1 and c_2 such that for $\varepsilon \le \varepsilon^*$ and $|p| \le p^*$ one has $|\beta_\Omega(x;p) - u_{\mathrm{as},\Omega}| \le C|p|$, and*

$$\begin{aligned}
\beta_\Omega(x;-p) &\le \beta_\Omega(x;p) && \text{for } p>0, && (4.4a)\\
F\beta_\Omega &\ge \tfrac12\theta\gamma^2\,p - c_1\varepsilon^2 && \text{for } p>0, && (4.4b)\\
F\beta_\Omega &\le -\tfrac12\theta\gamma^2|p| + c_1\varepsilon^2 && \text{for } p<0, && (4.4c)\\
(\mathrm{sgn}p)\,\beta_\Omega &\ge (\mathrm{sgn}p)\,g + \tfrac12\theta|p| - c_2\varepsilon^2 \text{ on } \partial\Omega && \text{for } p\ne 0. && (4.4d)
\end{aligned}$$

Proof. The proof of each inequality follows from the non-negativity of the partition of unity and the corresponding inequality in Lemma 4.2.

We shall dwell on getting (4.4b) and (4.4c) bearing in mind that F is nonlinear. If $\chi_j(x) = 1$ for some j, then $\beta_\Omega(x,p) = \beta_{S_j}(x,p)$ and (4.4b), (4.4c) are straightforward. Otherwise, x has to be a positive distance away from any vertex of Ω. Now, if we have $\chi_j(x) + \chi_{j+1}(x) = 1$ for some j, then the exponential decay of $\tilde{q}$ implies that $\beta_{S_{j+1}}(x,p) = \beta_{S_j}(x,p) + O(e^{-c/\varepsilon})$; thus we have $\beta_\Omega(x,p) = \beta_{S_j}(x,p) + O(e^{-c/\varepsilon})$ and $F\beta_\Omega(x,p) = F\beta_{S_j}(x,p) + O(e^{-c/\varepsilon})$, i.e. (4.4b), (4.4c) hold true in this case with possibly a larger constant c_1 than in (4.2b), (4.2c). Finally if $\chi_j(x) + \chi_{j+1}(x) \ne 1$ for all j, then x is a positive distance away from $\partial\Omega$, and now the exponential decay of $\tilde{q}$ and $\tilde{v}$ implies that $\beta_{S_j}(x,p) = u_0(x) + \theta p + O(e^{-c/\varepsilon})$ for all j; hence we have $\beta_\Omega(x,p) = \beta_{S_1}(x,p) + O(e^{-c/\varepsilon})$, and we again get (4.4b), (4.4c). □

We now present our main result.

THEOREM 4.4. *Let b satisfy assumptions A1–A4. Then there is a positive constant ε^* such that for all $\varepsilon \in (0,\varepsilon^*]$ the problem* (1.1) *has a solution $u(x)$ such that $|u(x) - u_{\mathrm{as},\Omega}(x)| \le C\varepsilon^2$.*

Proof. First set ε^* in Lemma 4.3 sufficiently small so that $c_1(\varepsilon^*)^2 \le \frac12\theta\gamma^2 p^*$ and $c_2(\varepsilon^*)^2 \le \frac12\theta p^*$. For any $\varepsilon \le \varepsilon^*$, choose $\bar{p} = \max\{c_1\varepsilon^2/(\frac12\theta\gamma^2),\, c_2\varepsilon^2/(\frac12\theta)\}$. Now, by Lemma 4.3, the functions $\beta_\Omega(x;\bar{p})$ and $\beta_\Omega(x;-\bar{p})$ are ordered super- and subsolutions, respectively, of problem (1.1). Applying Lemma 3.2 we obtain a solution u of (1.1) lying between $\beta_\Omega(x;\bar{p})$ and $\beta_\Omega(x;-\bar{p})$. Since, by Lemma 4.3, we have $|\beta_\Omega(x;\pm\bar{p}) - u_{\mathrm{as},\Omega}(x)| \le C\bar{p} \le C\varepsilon^2$, the desired estimate follows. □

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