On the connection between the categorical and the modal logic approaches to Quantum Mechanics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis aims at connecting the two research programs known as Categorical Quantum Mechanics and Dynamic Quantum Logic. This is achieved in three steps. First we define a procedure to extract a Modal Logic frame from a small category and a functor into the category of sets and relations. Second, we extend such methodology to locally small categories. Third, we apply it to the category of finite-dimensional Hilbert spaces to recover the semantics of Dynamic Quantum Logic.

This process prompts new lines of research. At a general level, we study some logics arising from wide classes of small categories. In the case of Hilbert spaces, we investigate how to obtain richer semantics, containing probabilistic information. We design a logic for this semantics and prove that, via translation, it preserves the validities of Dynamic Quantum Logic.

Contents

Introduction

The development of Quantum Computation and Information has caused a new wave of studies in Quantum Mechanics: the possibility of defining quantum algorithms, and the fact that some of them outperform their classical counterparts, has elicited both practical and theoretical questions.[1] On the practical side, for example, we are interested in the implementation of such algorithms. On the theoretical side, we seek to develop formal models to increase our understanding of quantum processes, with the hope of obtaining tools that will aid our work on quantum algorithms.

In this thesis we examine two research programs that belong to the second camp. They share a common goal: crafting a formalism that captures the features of quantum processes. The intended tool that we want to obtain from such a formalism is a formal system capable of proving the correctness of quantum algorithms.

Loosely speaking, an algorithm is correct when, for every suitable input, the output is the desired one, that is, the one expected from the intended behaviour of the algorithm. The correctness proofs of quantum algorithms are currently ad hoc, hence it would be useful to have a formal system wherein correctness proofs can be approached in a uniform way.

We thus have two approaches with the same theoretical aim and the same intended application. This constitutes a natural motivation to investigate the connection between the two. Let us now briefly introduce the two contestants.

The first research program, pioneered by Abramsky and Coecke, is a study of Quantum Mechanics through the lenses of Category Theory. The importance of Category Theory in the study of processes and transformations (in their broadest meaning) has now been established in Mathematics, Logic and Computer Science. Therefore it seems a suitable formal environment for the study of quantum algorithms.

This approach, called Categorical Quantum Mechanics, started from an analysis of the categorical structure of the category of finite-dimensional Hilbert spaces and linear maps.[2] In the last decade this research project has produced many results and a renewed interest in symmetric monoidal

[1] The standard reference on Quantum Computing and Information is [21].

[2] The first paper on Categorical Quantum Mechanics is [6]. See [7] for an extensive survey.

categories, the categories used to model compound systems.

The second approach, proposed by Baltag and Smets, exploits the formalism of Propositional Dynamic Logic, PDL henceforth, to represent quantum algorithms and to design a proof system able to prove their correctness. PDL is a complex modal logic whose modalities have an inbuilt algebraic structure. It was originally developed to model computer programs as labelled transition systems.

This line of research, named Dynamic Quantum logic, studies a quantum version of PDL called *LQP* (Logic of Quantum Programs) with the aim of capturing quantum algorithms.[3] It is connected with both the traditional logical studies of the foundations of Quantum Mechanics, the so-called *standard* Quantum Logic, and the "Dynamic Turn" in Logic, that is, the use of modal logics to reson about processes and information.

Contributions

From the short descriptions above it is immediately clear that a bridge between the two research projects can be found only by connecting the two underlying formalisms, Category Theory and Modal Logic. To this end, we describe a procedure to obtain a Modal Logic frame from a small category and a functor into the category of sets and relations. Although simple, the central idea of this procedure is, to the best of our knowledge, new to the literature.

For this reason a considerable part of this thesis is devoted to the analysis of the logics associated to the Modal Logic frames arising from certain classes of small categories. We prove the following results:

- The proof system of *DLT*, a dynamic logic with finitely many types designed to describe typed processes, is *sound and complete* with respect to the class of Modal Logic frames arising from small categories with finitely many objects. After lifting the cardinality restriction on the types, we prove soundness for a modified version of the proof system.
- The proof system of $S4$ is *sound and complete* with respect to the class of *all* Modal Logic frames arising from small categories, when the satisfaction of the diamond operator is adapted to the new setting.
- When we restrict our attention to a certain kind of functors, called singleton functors, we get a class of Modal Logic frames whose logic contains the validities of the Hybrid Logic of $S4$.

We then extend our procedure to cover locally small categories. This enables its application to the aforementioned category of Hilbert spaces.

[3] This line of research is developed in multiple papers, we refer to [11], and [10] in particular.

- We show that, with a particular choice of the functor, we can recover the class of Modal Logic frames for LQP. This constitutes the formal link between the two approaches.
- Choosing a second functor, which produces a richer semantics, we design a logic that captures all the features of LQP and more, namely: it can handle probabilities and encompasses both pure and strictly mixed states. We prove that, via translation, all the theorems of LQP are validities of this new logic.

Structure

The thesis is organized in five chapters. We divided the chapters in two groups to highlight the distinction between the literature review and our contributions.

In the first chapter we introduce Categorical Quantum Mechanics. We focus on the core aspect of this approach, that is, the categorical structure of the category of finite-dimensional Hilbert spaces. At the end of the chapter we explain how the notions from Quantum Mechanics are recovered in the categorical setting. As an example we give an informal illustration of the treatment of the Quantum Teleportation protocol.

Chapter 2 is dedicated to Dynamic Quantum Logic. We present the logic LQP and underline its main features: first, the idea of seeing the set of possible states of a physical system as the carrier of a Modal Mogic frame; second, the use of tests to model measurements; third, the use of programs to model quantum gates. We explain that LQP still lacks a pivotal element of quantum algorithms, namely the ability to handle compound systems. For this reason we introduce LQP^n, a system able to express locality and entanglement. We conclude the chapter by discussing the utility of LQP^n in proving the correctness of quantum protocols, exemplifying the representation of the Teleportation protocol.

In the third chapter we describe the procedure to obtain Modal Logic frames from small categories. We call a logics arising from these kinds of Modal Logic frames a Logic of Small Categories, LSC. We analyze three examples of LSC logics. The first one is DLT, a dynamic logic designed to handle typed processes. The second one is a logic for the language with only the diamond operator; we prove it is equivalent to $S4$. The third is a logic related to the Hybrid Logic of $S4$. We successively suggest how the categorical structure of a small category can be transferred to the corresponding Modal Logic frame and then captured by the logic. Finally, we extend this methodology to locally small categories.

In Chapter 4 we apply this procedure to the category of finite-dimensional Hilbert spaces, in two different ways. In the first part we analyze the Modal Logic frames given by the functor S, sending a Hilbert space to the set of its one-dimensional subspaces. We observe that the corresponding class of Modal Logic frames contains the semantics of LQP and LQP^n. In the second part we consider the functor F, sending a Hilbert space to the set of functions corresponding to its density operators. The Modal Logic frames generated by this functor are richer than those generated by S. To capture their additional features we design a new language and prove that the logic of these Modal Logic frames in this language contains all the theorems of LQP and LQP^n.

In the last chapter we expand on two questions:

1. Can we characterize the image of F independently from Hilbert spaces?
2. Can we define on this image a counterpart of the categorical structure of the category of Hilbert spaces?

The discussion connects the work of the previous chapter with existing results in the area, such as Gleason's Theorem, and with the research of Baltag and Smets on Correlation Models.

We include an Appendix with the relevant background from Category Theory, Quantum Mechanics and Modal Logic.

Scope

Metaphorically speaking, this thesis resides in a meta-meta-level: if the ground level is that of Quantum Mechanics and Quantum Computation, and the meta-level pertains to theories describing and modelling Quantum Mechanics and Computation, such as Categorical Quantum Mechanics and Dynamic Quantum Logic, then the study of the connection between these theories lives in an even more abstract environment.

This level of generality has some practical consequences for the thesis. First, Quantum Computing remains in the background. Second, the reader has to cope with three different formalisms, coming from Category Theory, Modal Logic and of course Quantum Mechanics. Third, to limit the size of the thesis, in presenting the two approaches we only go as far as is needed for our purpose.

Part I

Background

Chapter 1

Categorical Quantum Mechanics

In their paper [6], Abramsky and Coecke initiated a study of foundations of Quantum Mechanics from a category-theoretic perspective. This approach recasts the concepts of Hilbert space Quantum Mechanics in the abstract language of Category Theory, allowing for a novel analysis of the notions employed in Quantum Computation and Information.

The target of this study is $\mathbf{FdHil}_{\mathbb{C}}$, the category having as objects finite-dimensional Hilbert spaces over the field of complex numbers and as morphisms linear maps.[1] This category can be thought of as the formal environment where Quantum Computing takes place. Throughout the thesis we will always assume that the Hilbert spaces under consideration are over $\mathbb{C}$, hence we drop the subscript and write just **FdHil**. The limitation to *finite-dimensional* Hilbert spaces is a rather strandard one in Quantum Computation, see for example [21].

This line of research produced a wealth of results that can hardly be summarized in one chapter. We will focus on the core aspect, the categorical structure of **FdHil**. We start by articulating the key observation in this respect: **FdHil** is a dagger compact closed category with byproducts.[2] Most of the chapter is devoted to the understanding of this finding: we introduce the appropriate definitions one by one, providing examples and explanations. The final part is dedicated to the implication in the modelling of Quantum Mechanics. We outline how the common ingredients of Quantum Mechanics can be recasted in the category-theoretic language and sketch the representation of the Quantum Teleportation protocol.

The reference for this chapter is the survey paper [7]; we occasionally borrow terminology and definitions from [24].

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[1] Observe that the preservation of the inner product is *not* required.

[2] We use the terminology of [24], calling dagger compact closed categories what in [7] are called strongly compact closed categories.

1.1 The categorical structure of FdHil

First of all, the fact that **FdHil** is a category hinges on the following facts: the identity map is linear and the composition of two linear maps is a linear map. From these we can easily prove that the identity axioms and associativity hold.

However, **FdHil** turns out to be much richer than just any category.

Theorem 1 ([7]). *The category* **FdHil** *is a dagger compact closed category with biproducts.*

This in particular means that **FdHil** is

1. a symmetric monoidal category
2. a compact closed category
3. a dagger category
4. a category with byproducts

To understand the Theorem and prove it we will follow this checklist, introducing the correponding notions step by step.

1.1.1 Symmetric monoidal categories

Definition 1. A *symmetric monoidal category* $\mathbf{C}$ is a category equipped with a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, a distinguished object I and natural isomorphisms

$$\begin{aligned} \alpha_{A,B,C} &: (A \otimes B) \otimes C \simeq A \otimes (B \otimes C) \\ \sigma_{A,B} &: A \otimes B \simeq B \otimes A \\ \lambda_A &: I \otimes A \simeq A \\ \rho_A &: A \otimes I \simeq A \end{aligned}$$

where A, B, C are objects of the category. They are required to satisfy the coherence conditions of a monoidal category listed in [20] pp. 158-9, namely the commutation of diagrams

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C\otimes D}} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha_{A\otimes B,C,D}} & ((A \otimes B) \otimes C) \otimes D \\ \downarrow{\scriptstyle 1_A \otimes \alpha_{B,C,D}} & & & \nearrow{\scriptstyle \alpha_{A,B,C} \otimes 1_D} & \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow[\alpha_{A,B\otimes C,D}]{} & (A \otimes (B \otimes C)) \otimes D & & \end{array}$$

and

$$\begin{array}{ccc} A\otimes(I\otimes C) & \xrightarrow{\alpha_{A,I,C}} & (A\otimes I)\otimes C \\ {\scriptstyle Id_A\otimes\lambda_C}\downarrow & & \downarrow{\scriptstyle \rho_A\otimes Id_C} \\ A\otimes C & = & A\otimes C \end{array} \qquad \begin{array}{ccc} I\otimes I & = & I\otimes I \\ & {\scriptstyle \rho_I}\searrow \quad \swarrow{\scriptstyle \lambda_I} & \\ & I & \end{array}$$

plus the requirements of symmetric monoidal category ([20] p.180), namely equations $\sigma_{A,B}\circ\sigma_{B,A} = Id_{B,A}d$ and $\rho_A\circ\sigma_{A,I} = \lambda_A : A\otimes I \simeq A$ and the commutation of diagram

$$\begin{array}{ccccc} A\otimes(B\otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A\otimes B)\otimes C & \xrightarrow{\sigma_{(A\otimes B),C}} & C\otimes(A\otimes B) \\ {\scriptstyle Id_A\otimes\sigma_{B,C}}\downarrow & & & & \downarrow{\scriptstyle \alpha_{C,A,B}} \\ A\otimes(C\otimes B) & \xrightarrow[\alpha_{A,C,B}]{} & (A\otimes C)\otimes B & \xrightarrow[\sigma_{A,C}\otimes Id_A]{} & (C\otimes A)\otimes B \end{array}$$

Proposition 1. **FdHil** *is a symmetric monoidal category.*

Proof. Consider **FdHil** equipped with the tensor product $\bigotimes$. The unit is $\mathbb{C}$ as a one-dimensional vector space over itself, with inner product defined as $\langle c|c'\rangle = c^*c'$. For $\alpha_{H,V,W}$, where H, V, W are Hilbert spaces, take the isomorphism between $(H\bigotimes V)\bigotimes W$ and $H\bigotimes(V\bigotimes W)$. Similarly, $\sigma_{H,V}$ is the isomorphism between $H\bigotimes V$ and $V\bigotimes H$. The isomorphism $\lambda_H : \mathbb{C}\bigotimes H \to H$ is defined as $\lambda_H(1, v_i) = 1v_i$, where v_i is a basis of H; likewise for ρ_H. The relevant naturality and coherence conditions and easily seen to hold. □

Another example of symmetric monoidal category is **Rel**, the category of sets and relations, where the bifunctor is the cartesian product and the unit is a singleton.

Note that the tensor is intrinsically different from a categorical product. The crucial aspect is that we cannot recover an element of the tensor from its components. In particular we don't have the diagonal $H \to H\bigotimes H$ and the projections $H_1\bigotimes H_2 \to H_i$. We can however model parallel processes in virtue of bifunctoriality:

$$\begin{array}{ccc}
A \otimes B & \xrightarrow{f \otimes Id_B} & A' \otimes B \\
{\scriptstyle Id_A \otimes g}\downarrow & & \downarrow{\scriptstyle Id_{A'} \otimes g} \\
A \otimes B' & \xrightarrow[f \otimes Id_{B'}]{} & A' \otimes B'
\end{array}$$

As remarked in [7], symmetric monoidal categories provide a "setting for describing processes in a resource sensitive way, closed under sequential and parallel composition".

Scalars

In any monoidal category (a symmetric monoidal category without the natural isomorphism σ) we can define a general notion of scalar as a morphism $I \to I$.

In the case of **FdHil** this ties in nicely with the usual notion of scalar. To see this, consider that the basis of $\mathbb{C}$ seen as a Hilbert space is the number 1. As every linear map is determined by the action on the basis, every linear map $\mathbb{C} \to \mathbb{C}$ will be determined by the image of 1. This in turn means that we have a bijectioon between linear maps $\mathbb{C} \to \mathbb{C}$ and elements of $\mathbb{C}$.

In **Rel** there can be only two relations between a singleton and itself, the empty and the total relation. So in this case there are only two scalars.

1.1.2 Compact closed categories

Definition 2. A *compact closed category* **C** is a symmetric monoidal category where to each object A is assigned a dual object A^*, a unit map $\eta_A : I \to A^* \bigotimes A$ and a counit map $\epsilon_A : A^* \bigotimes A \to I$ such that both

$$\begin{array}{ccccc}
A & \xrightarrow{\rho_A^{-1}} & A \otimes I & \xrightarrow{Id_A \otimes \eta_A} & A \otimes (A^* \otimes A) \\
{\scriptstyle Id_A}\downarrow & & & & \downarrow{\scriptstyle \alpha_{A,A^*,A}} \\
A & \xleftarrow[\lambda_A]{} & I \otimes A & \xleftarrow[\epsilon_A \otimes Id_A]{} & (A \otimes A^*) \otimes A
\end{array}$$

and the corresponding diagram for A^* commute.

Notice that in any compact closed category we can define a contravariant endofunctor $()^* : \mathbf{C} \to \mathbf{C}$ that sends objects to their duals and morphisms $f : A \to B$ to the morphisms $f^* : B^* \to A^*$ defined as

$$\begin{array}{ccccc} B^* & \xrightarrow{\lambda^{-1}_{B^*}} & I\otimes B^* & \xrightarrow{\eta_A\otimes Id_{B^*}} & A^*\otimes A\otimes B^* \\ {\scriptstyle f^*}\downarrow & & & & \downarrow{\scriptstyle Id_{A^*}\otimes f\otimes Id_{B^*}} \\ A^* & \xleftarrow[\rho_{A^*}]{} & A^*\otimes I & \xleftarrow[Id_{A^*}\otimes \epsilon_B]{} & A^*\otimes B\otimes B^* \end{array}$$

Proposition 2. **FdHil** *is a compact closed category.*

Proof. We have already seen that **FdHil** is a symmetric monoidal category. Given a Hilbert space H, take H^* to be the conjugate space of H, i.e., the space having the same set of vectors and the same addition operation but with scalar multiplication and inner product defined as

$$c \cdot_{H^*} v := c^* \cdot_H v \qquad \langle v_1|v_2\rangle_{H^*} := \langle v_2|v_1\rangle_H$$

where $v, v_1, v_2 \in H$ and c^* is the complex conjugate of c. We can then define the counit $\epsilon_H : H \otimes H^* \to I$ as $\epsilon_H(v_1, v_2) = \langle v_2|v_1\rangle_H$ and the unit $\eta_H : I \to H^* \otimes H$ by sending 1 to $\sum_{i=1}^{i=n} v_i \bigotimes v_i$, where v_i is a basis of H and n is its dimension.

For the commutation of the diagram in the definition, note that, starting from a vector v in a space H in the top-left of the diagram, we decompose it into the basis v_i, obtaining $\sum_{i=1}^{i=n} c_i v_i$. Following the diagram from left to right we obtain $(\sum_{i=1}^{i=n} c_i v_i) \bigotimes (\sum_{j=1}^{j=n} v_j \bigotimes v_j) = \sum_{i,j=1}^{i,j=n} c_i v_i \bigotimes v_j \bigotimes v_j$. Regrouping and taking the inner product (going down and left in the diagram) we can eliminate all terms containing orthogonal vectors, that is where $j \neq i$. We thus end up with the original vector $\sum_{i=1}^{i=n} c_i v_i$. We follow a similar procedure for the dual diagram. □

The category **Rel** is compact closed if we take $X^* = X$, for X object of **Rel**. Then the unit $\eta_X : \{*\} \to X \times X$ is defined as $\{(*, (xx))|x \in X\}$ and the counit $\epsilon_X : X \times X \to \{*\}$ as $\{((xx), *)|x \in X\}$. For R^* we take the converse of R, $R^{\cup}$.

Compact closed categories offer a categorical environment to treat dual objects, such as the dual vector space in the category of vector spaces. They also admit a two-dimensional graphical representation that can be used to model networks of quantum processes.[3]

1.1.3 Dagger categories

Definition 3. A *dagger category* **C** is a category where to each morphism $f : A \to B$ is associated a morphism $f^\dagger : B \to A$, called the *adjoint* of f, such that

[3]See [7] p.17 and followings for details.

- $Id_A^\dagger = Id_A$
- $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$
- $f^{\dagger\dagger} = f$

In other words, **C** is equipped with a functor $\dagger : \mathbf{C} \to \mathbf{C}$ that is identity on objects, contravariant and involutive.

Definition 4. Call *unitary* the isomorphisms f such that $f^{-1} = f^\dagger$. Call a morphism f *sel-adjoint* if $f = f^\dagger$.

Proposition 3. FdHil *is a dagger category.*

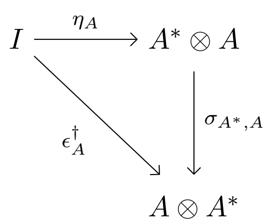
Proof. Given a linear map $L : H \to V$, take $L^\dagger$ to be the unique map such that $\langle v|L(w)\rangle = \langle L^\dagger(v)|w\rangle$ for all $v \in V$ and $w \in H$. □

Therefore the unitary and self-adjoint morphisms are in **FdHil** exactly the unitary and the self-adjoint linear maps.

Dagger compact closed categories

Definition 5. A *dagger compact closed category* **C** is both a dagger category and a compact closed category. Moreover, is satisfies the following coherence conditions:

1. the natural isomorphisms α, σ, λ and ρ are all unitary
2. $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
3. the following diagram commutes



Proposition 4. FdHil *is a dagger compact closed category.*

Proof. The isomorphisms in **FdHil** are all unitary maps. The second condition is given by the fact that $(L \otimes L')^\dagger = L^\dagger \otimes L'^\dagger$. The fact that $\sigma_{H^*,H} \circ \eta_H$

is the adjoint map of ϵ_H can be checked by observing that:

$$\begin{aligned}
\langle \sigma_{H^*,H} \circ \eta_H(c) | v_1 \bigotimes v_2 \rangle_{H \bigotimes H^*} &= \langle c \sum_i v_i \bigotimes v_i | v_1 \bigotimes v_2 \rangle_{H \bigotimes H^*} \\
&= c^* \sum_i \langle v_i | v_1 \rangle_H \langle v_i | v_2 \rangle_{H^*} \\
&= c^* \sum_i \langle v_i | v_1 \rangle_H \langle v_2 | v_i \rangle_H \\
&= c^* \langle v_2 | v_1 \rangle_H \\
&= \langle c | \langle v_2 | v_1 \rangle_H \rangle_I \\
&= \langle c | \epsilon_H (v_1 \bigotimes v_2) \rangle_I
\end{aligned}$$

where $c \in \mathbb{C}$ and $v_1, v_2 \in H$. □

Rel is trivially dagger compact closed: we take $R^\dagger = R^* = R^\cup$.

The dagger structure together with the functor $()^*$ can be used to describe operations on morphisms. In the case of **FdHil** the linear maps $L^\dagger$, L^* and $L^{\dagger *}$ correspond respectively to the conjugate-transpose, transpose and conjugate of the linear map L.

1.1.4 Categories with byproducts

Definition 6. A *zero object* $\mathbf{0}$ in a category $\mathbf{C}$ is an object that is initial and terminal, that is, for every other object A of the category there is a unique morphisms $A \to \mathbf{0}$ and a unique morphism $\mathbf{0} \to A$. If a category has a zero objects then there is a unique morphism $A \to \mathbf{0} \to B$, called $0_{A,B}$, between any two objects A and B.

Definition 7. Suppose $\mathbf{C}$ has zero object, products and coproducts. Given objects $A_1, \ldots, A_n$, a *biproduct* is an object $A_1 \oplus \cdots \oplus A_n$ equipped with morphisms $p_j : A_1 \oplus \cdots \oplus A_n \to A_j$ and $q_j : A_j \to A_1 \oplus \cdots \oplus A_n$, for $j \in \{1, \ldots, n\}$, such that:

- $p_j \circ q_j = Id_{A_j}$
- $p_i \circ q_j = 0_{A_i,A_j}$ for $i \neq j$
- $A_1 \oplus \cdots \oplus A_n$ is both the product and the coproduct of $A_1, \ldots, A_n$

Byproducts stand for objects that are completely determined by their components.

Proposition 5. FdHil *has biproducts.*

Proof. The zero object is the 0-dimensional vector space, containing only the zero vector. The biproduct is the direct sum of Hilbert spaces, denoted by $\bigoplus$. Given H_1, H_2, the morphisms $p_1 : H_1 \bigoplus H_2 \to H_1$ and $q_1 : H_1 \to H_1 \bigoplus H_2$ are defined as $p_1(v_1, v_2) = v_1$ and $q_1(v_1) = (v_1, 0)$, respectively. □

In **Rel** the biproduct is the disjoint union, denoted by $\uplus$.

Dagger compact closed categories with byproducts

Definition 8. A *dagger compact closed category with byproducts* **C** is a dagger compact closed category in which there are biproducts and the following coherence condition holds: $p_i^\dagger = q_i : A_i \to A_1 \oplus A_2$ for all $i = 1, 2$ and A_1, A_2.

Proposition 6. FdHil *is a dagger compact closed category with byproducts.*

Proof. We can ascertain that $\langle q_1(v_1)|(v_1', v_2)\rangle = \langle (v_1, 0)|(v_1', v_2)\rangle = \langle v_1|v_1'\rangle + \langle 0|v_2\rangle = \langle v_1|v_1'\rangle = \langle v_1|p_1(v_1', v_2)\rangle$. $\square$

Finally, we remark that in a dagger compact closed category the tensor distributes over the biproduct ([7], Proposition 24 p.33).

The collection of the propositions proved in this section constitutes a proof of Theorem 1.

1.2 Back to Quantum Computing

We now highlight how the central ingredients of Quantum Computing can be recovered in this categorical framework. We have already see how unitary maps can be characterized using the dagger structure. The projections of type $H \to H$ are the self-adjoint morphisms P such that $P \circ P = P$. The compound systems are handled with the tensor operation, as expected.

A state s of a state space H is given by a morphism $\mathbb{C} \to H$, from the unit to the object. A state space H is n-dimensional if there is a unitary $\bigoplus_{i=1}^{i=n} I \to H$; each of such unitaries constitutes a basis for H.

Given two states, $\phi, \phi' : \mathbb{C} \to H$, we can define their inner product as $\phi^\dagger \circ \phi' : \mathbb{C} \to \mathbb{C}$. Recall that $\phi^\dagger$ is the transpose of ϕ, its bra in Dirac notation, and that morphisms $\mathbb{C} \to \mathbb{C}$ are the scalars; thus this is indeed the usual inner product.

We can also define abstractly a notion of partial trace: given a morphisms $L : H \bigotimes V \to W \bigotimes V$ its partial trace $tr_V(L) : H \to W$ is defined as

$$\begin{array}{ccccc}
H & \xrightarrow{\rho_H^{-1}} & H \bigotimes \mathbb{C} & \xrightarrow{Id_H \bigotimes \epsilon_V^\dagger} & H \bigotimes (V \bigotimes V^*) \\
\downarrow{\scriptstyle tr_V(L)} & & & & \downarrow{\scriptstyle L \bigotimes Id_V \bigotimes Id_{V^*}} \\
W & \xleftarrow[\rho_W]{} & W \bigotimes \mathbb{C} & \xleftarrow[Id_W \otimes \epsilon_V]{} & W \bigotimes (V \bigotimes V^*)
\end{array}$$

This corresponds to the usual notion of partial trace.

The biproduct structure allows to break a state spaces into orthogonal spaces, for example saying that there is a unitary $H \to \bigoplus_{i=1}^{i=n} H_i$. In this way we can account for the branching due to measurement and keep track of what happens in each branch. In this respect, the distribution of the tensor over the biproduct represent the propagation of classical information through the system.

1.2.1 Quantum Teleportation

To exemplify the way in which a quantum protocol is represented categorically, we have a closer look at the Teleportation protocol. The treatment will be partially informal, as we have not presented enough formal background for a thorough explanation. Since the specific treatment of this protocol will not play a significant role in the next chapters, our goal is to give the reader an intuitive sample of the categorical perspective on Quantum Mechanics and to allow for a comparison with the treatment of the same protocol offered by the modal logic approach.

The Teleportation protocol describes a technique to transfer a quantum state from one agent, called with the fictional name Alice, to another agent, called Bob. This procedure does not require the existence of a quantum communication channel between Alice and Bob, but a classic communication channel is needed. The Hilbert space describing the system is the tensor product of three 2-dimensional systems $H = H_1 \bigotimes H_2 \bigotimes H_3$, that is, it is a space consisting of three qubits. We suppose Alice and Bob possess one qubit each of a entangled Bell state $\beta_{00} \in H_2 \bigotimes H_3$. Alice also has a qubit q_1 given by a state of H_1.

After obtaining their part of the entangled Bell state, Alice and Bob become separated; we assume $H_1 \bigotimes H_2$ is the part of the system available to Alice and H_3 is the part available to Bob. The goal of Alice is to teleport her additional qubit to the location of Bob, i.e., to turn the state of H_3 into the initial state of H_1.

In order to do so, Alice performs a measurement in the Bell basis, that is, a measurement such that each projector projects into one of the vector of the Bell basis, on her two qubits. The result of this measurement is a pair of classical bits. The action that Bob has to perform on q_3 to obtain the initial q_1 depends on the measurement outcome obtained by Alice, so using the classical communication channel between them, she sends this pair of classical bits to Bob, who performs a quantum gate according to the following table:

$$00 \longrightarrow Id$$

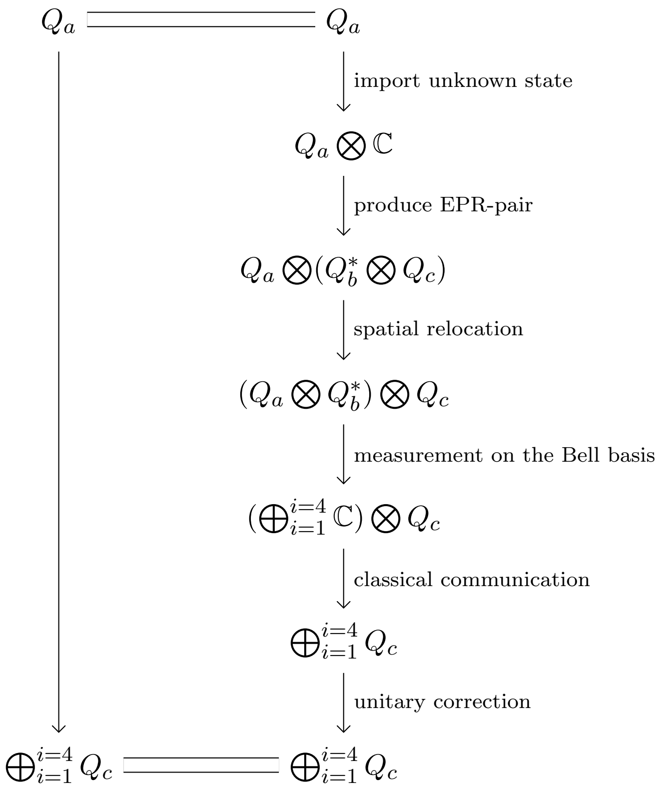
$$01 \longrightarrow Z$$

$$10 \longrightarrow X$$

$$11 \longrightarrow XZ$$

The final qubit q_3 of Bob is then equal to the initial q_1.

The Teleportation protocol is represented in the diagram below, borrowed from [7]. The symbols Q_a, Q_b, Q_c stand for the state spaces of the three qubits (the first qubit lives in Q_a, etcetera). The arrow on the left of the diagram encode the expected effect of the protocol, namely the fact that the qubit in Q_a is trasferred to the system Q_c in all four possible evolutions of the system (the number 4 is given by the number of possible outputs of the measurement on the Bell basis). The arrow on the right side depicts the protocol itself. The commutation of this diagram expresses the correctness of the algorithm.



Starting from the top-right corner of the diagram, the steps of the algorithm are the following. We first import a generic qubit from the system Q_a. Successively we generate an EPR-pair. In the third step the two parties, Alice and Bob, divide the two qubits of the entangled pair between themselves and become spatially separated. These steps constitute the preparation of the Teleportation protocol. Then Alice performs a measurement on the Bell basis. This is seen as a morphism $(Q_a \bigotimes Q_b^*) \to (\bigoplus_{i=1}^{i=4} \mathbb{C})$: each projector sends the state space into the corresponding ray, so applied together they collapse the space into one of the four copies of $\mathbb{C}$). Communicating the result of the measurement means informing Bob of which branch is actual. This is modelled with the distribution of the tensor over the direct sum. Finally, Bob applies a different unitary gate depending on the branch.

The commutation of this diagram can be proved in this categorical framework, see Theorem 30 in [7] p.41.

Chapter 2

Dynamic Quantum Logic

In this chapter we present a logic, the *Logic of Quantum Programs LQP*, specifically designed by Baltag and Smets to express quantum algorithms and prove their correctness. The core ideas behind this logic are two. First, we can see the states of a physical system as states of a Modal Logic frame. Second, the dynamics of the system can be captured by means of a PDL-style formalism, that is, a modal logic formalism containing constructors for modalities. In particular, the intuition is that measurements can be seen as tests and the evolutions of the system as programs.

How do we prove the correctness of an algorithm in this setting? Essentially, by proving that it is a validity of the logic. More precisely, if we are able to represent the systems we want to study as Modal Logic frames, we can express the correctness of an algorithm by proving that the formula encoding the algorithm is true at all states in all systems, i.e., is a validity of the corresponding class of Modal Logic frames. Thus the key result that we need to apply the above line of reasoning is Soundness: we need to show that if a formula is provable in the logic (from some premises) then it is true in all states in all Modal Logic frames (satisfying the premises).

We start presenting an implementation of this ideas, the logic LQP. This logic is however not enough for our purpose: to express quantum algorithms we need a way to refer to subsystems of a given system. For this reason in the second section we strengthen the language and the proof system of LQP, obtaining the logic LQP^n.

It turns out that LQP^n is powerful enough to prove the correctness of some important quantum protocols. Its language, however, is not able to account for probabilities, and this constitutes a limit to the usefulness of LQP^n.

The logic and its semantics have been developed in the articles [11] and [10]. The main reference for this chapter is [11]. In what follows we use the terms "program"and "action"as synonyms.

2.1 The logic LQP

We begin introducing the logic LQP, a formalism able to capture the dynamic of a quantum system.

2.1.1 Syntax

Given a set of atomic propositions At and a set of atomic actions $AtAct$, the set of formulas $\mathcal{F}_{LQP}$ is built by mutual recursion as follows:

$$\psi ::= p \mid \neg\psi \mid \psi \wedge \phi \mid [\pi]\psi$$

where $p \in At$ and the action π is defined as

$$\pi ::= U \mid \pi^{\dagger} \mid \pi \cup \pi' \mid \pi; \pi' \mid \psi?$$

where $U \in AtAct$. We will use as symbols the names of the unitary maps, e.g., we will use H as a symbol to denote the Hadamard gate. The distinction between the symbol and the object, that is, the linear map, will be clear by the context.

Definition 9. Call a program *deterministic* if it not built using the constructor $\cup$.

Abbreviations

The classical disjunction and implication are defined as usual. We write $\top$ for a generic classical tautology, like $p \vee \neg p$. Define

- $\langle\pi\rangle\psi := \neg[\pi]\neg\psi$, the dual of $[\pi]$
- $\Diamond\psi := \langle\psi?\rangle\top$, the *measurement modality*, expressing the possibility to perform the measurement of the property represented by ψ
- $\Box\psi := \neg\Diamond\neg\psi$
- $\sim\psi := \Box\neg\psi$, the *orthocomplement*
- $\psi \leq \phi := \Box\Box(\psi \to \phi)$, ψ is *logically weaker* than ϕ
- $\psi \perp \phi := \psi \leq \sim \phi$
- $T(\psi) := \sim\sim \psi \leq \psi$, ψ is *testable*

2.1.2 Semantics

Definition 10. Given a Hilbert space H, a *concrete quantum dynamic frame* is a tuple $\langle \Sigma_H, \{\xrightarrow{P_a?}\}_{a \in L_H}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ such that

1. Σ_H is the set of all one-dimensional linear subspaces of H
2. $\{\xrightarrow{P_a?}\}_{a \in L_H}$ is a family of *quantum tests*, partial maps from Σ_H into Σ_H associated to the projectors of the Hilbert space H. Given $\overline{v} \in \Sigma_H$, the partial map $\xrightarrow{P_a?}$ is defined as $\xrightarrow{P_a?} (\overline{v}) = \overline{P_a(v)}$. The map is undefined if $P_a(v)$ is the zero vector.
3. $\{\xrightarrow{U}\}_{U \in \mathcal{U}}$ is a collection of partial maps from Σ_H into Σ_H associated to the unitary maps from H into H. As for projectors, the map $\xrightarrow{U}$ is defined as $\xrightarrow{U} (\overline{v}) = \overline{U(v)}$.

Call Γ_{CQDF} the class of all concrete quantum dynamic frames.

Definition 11. Given a concrete quantum dynamic frame, a set $T \subseteq \Sigma_H$ and a relation $R \subseteq \Sigma_H \times \Sigma_H$, define the following operations

- $T^{\perp} = \{s | \forall t \in T \; s \perp t\}$
- $[R]T = \{s | \forall t \in T \; (s,t) \in R \Rightarrow t \in T\}$
- $R^{\dagger} = \{(s,t) | t \in ([R]\{s\}^{\perp})^{\perp}\}$

where $s \perp t$ is the orthogonality relation between one-dimensional linear subspaces in Σ_H.

Definition 12. An *LQP-model* M consists of a concrete quantum dynamic frame $\langle \Sigma_H, \{\xrightarrow{P_a?}\}_{a \in L_H}, \{\xrightarrow{U}\}_{U \in \mathcal{U}} \rangle$ and a valuation function $V : At \to \wp(\Sigma_H)$.

Given an *LQP*-model, we define an interpretation of the actions and the satisfaction relation by mutual recursion.

Definition 13. An *interpretation of the actions* in an *LQP*-model is a function $i : At \to \{\xrightarrow{P_a?}\}_{a \in L_H} \cup \{\xrightarrow{U}\}_{U \in \mathcal{U}}$ such that

- $i(U) \in \{\xrightarrow{U}\}_{U \in \mathcal{U}}$
- $i(\pi \cup \pi') = i(\pi) \cup i(\pi')$
- $i(\pi^{\dagger}) = i(\pi)^{\dagger}$
- $i(\pi; \pi') = i(\pi); i(\pi')$
- $i(\psi?) = \xrightarrow{P_a?}$ where a is the span of the set $\{s \in \Sigma_H | M, s \vDash_{LQP} \psi\}$

Where ; on the right-end side is the composition of relations (partial functions in this case), $\cup$ is the union of relations and $\dagger$ is defined as above.

Definition 14. Given a model M, a state s in the model and a formula $\psi \in \mathcal{F}_{LQP}$, the satisfaction relation $\vDash_{LQP}$ is defined as

- $M, s \vDash_{LQP} p$ iff $s \in V(p)$
- $M, s \vDash_{LQP} \neg\psi$ iff $M, s \nvDash_{LQP} \psi$
- $M, s \vDash_{LQP} \psi \wedge \phi$ iff $M, s \vDash_{LQP} \psi$ and $M, s \vDash_{LQP} \phi$
- $M, s \vDash_{LQP} [\pi]\psi$ iff for all $(s, s') \in i(\pi)$ we have $M, s' \vDash_{LQP} \psi$

2.1.3 Proof System

Definition 15. The proof system for LQP consists of the following axioms:

1. the tautologies of classical propositional logic
2. $[\pi](p \rightarrow q) \rightarrow ([\pi]p \rightarrow [\pi]q)$
3. $[\pi]p \leftrightarrow \neg\langle\pi\rangle\neg p$
4. $\langle\pi\rangle\langle\pi'\rangle p \leftrightarrow \langle\pi;\pi'\rangle p$
5. $\langle\pi \cup \pi'\rangle p \leftrightarrow \langle\pi\rangle p \vee \langle\pi'\rangle p$
6. $\langle q?\rangle p \rightarrow \Diamond p$, **Testability**
7. $\langle p?\rangle q \rightarrow [p?]q$, **Partial Functionality**
8. $p \wedge q \rightarrow \langle p?\rangle q$, **Adequacy**
9. $T(p) \rightarrow [p?]p$, **Repeatability**
10. $\langle\pi\rangle\Box\Box p \rightarrow [\pi']p$, **Proper Superposition**
11. $\neg[U]q \leftrightarrow [a]\neg q$, **Unitary Functionality**
12. $p \leftrightarrow [U;U^{\dagger}]p$, **Unitary Bijectivity 1**
13. $p \leftrightarrow [U^{\dagger};U]p$, **Unitary Bijectivity 2**
14. $p \rightarrow [\pi]\Box\langle\pi^{\dagger}\rangle\Diamond p$, **Adjointness**

We briefly explain the meaning of the axioms. Axioms 1 asserts the fact that the proof system contains all the classical tautologies. Axioms 2 and 3 are the normality condition for a modal logic. Axioms 4 and 5 dictate the behaviour of the constructor ; and $\cup$, asserting that the composition of actions is the sequential application of them and that the union of actions is the non-deterministic choice of either of them. Testability ensures that if a property p can be actualized by a measurement then we can succesfully perform a measurement of p. Axiom 7 forces the partial functionality of the quantum tests. Axiom 8 states that testing a true property does not change the state. Axiom 9 guarantees that a testable property holds after it has been tested successfully. Axiom 10 ensures that states can be superposed. Axiom 11, 12 and 13 entail the functionality and bijectivity of the basic actions. Axiom 14 regulates the behaviour of the adjoint action.[1]

The inference rules of this proof system are:

- Modus Ponens: from $\vdash_{LQP} p, p \to q$ infer $\vdash_{LQP} q$
- Uniform Substitution: from $\vdash_{LQP} \psi$ infer $\vdash_{LQP} \psi[p/\phi]$[2]
- Generalization Rule: from $\vdash_{LQP} p$ infer $\vdash_{LQP} [\pi]p$
- Test Generalization Rule: if q does not occur in ψ or ϕ then from $\vdash_{LQP} \psi \to [q?]\phi$ infer $\vdash_{LQP} \psi \to \Box\phi$

Theorem 2 (Theorem 3 in [11], p. 21)**.** *The proof system of LQP is sound with respect to the class* Γ_{CQDF}*.*

It is worth noting that, when the proof system is enriched with two additional axioms, called Piron's Covering Law and Mayet's Condition, we obtain a proof system wich is sound and *complete* with respect to the subclass of Γ_{CQDF} generated by infinite-dimensional Hilbert spaces. This result is illustrated in [10].

2.2 The logic LQP^n

Nevertheless, the formalism of LQP is not enough to express quantum protocols. We need to express *locality*, that is, we need to express the fact that some quantum gates or measurements are performed locally, on certain subsystem. For this reason we develop an enhanced version of LQP, called LQP^n.

2.2.1 Syntax

Suppose given a natural number n. Set $N = \{1, \ldots, n\}$. Given a set of atomic propositions At and a set of atomic actions $AtAct$, the set of formulas $\mathcal{F}_{LQP^n}$ is built by mutual recursion as follows:

$$\psi ::= \top_I \,|\, p \,|\, 1 \,|\, + \,|\, \neg\psi \,|\, \psi \wedge \phi \,|\, [\pi]\psi$$

[1]See Theorem 2 in [11] for the link with the property of the adjoint map in the Hilbert Space setting.

[2]This notation means: replace every instance of p in ψ with the formula ϕ.

where $p \in At$, $I \subseteq N$ and the action π is defined as

$$\pi ::= triv_I \,|\, U \,|\, \pi^\dagger \,|\, \pi \cup \pi' \,|\, \pi; \pi' \,|\, \psi?$$

where $a \in AtAct$ and again $I \subseteq N$. Essentially the new symbols for the formulas are the constants $\top_I$, 1 and $+$, and the new symbols for actions are $triv_I$.

Abbreviations

We assume all the abbreviations of LQP. Moreover, we now take $\top := \top_N$ and postulate the following additional abbreviations:

- $\bot_N := \neg\top_N$
- $\psi = \phi := \Box\Box(\psi \leftrightarrow \phi)$, ψ is *logically equivalent* to ϕ
- $\psi_I := \top_I \wedge \langle\top_{N\setminus I}\rangle\psi$
- $\psi =_I \phi := \psi \leq \top_I \wedge \phi \leq \top_I \wedge \psi_I = \phi_I$
- $I(\psi) := \psi = \psi_I$

2.2.2 Semantics

Definition 16. Let H' be a Hilbert space of dimension 2 with basis $\{|1\rangle, |0\rangle\}$. Given the Hilbert space $H := \bigotimes_{i=1}^{i=n} H'$ consisting of n copies of H', call *n-partite quantum dynamic frame* the concrete quantum dynamic frame associated to H.

Set $N = \{1, \ldots, n\}$. We write H_I to indicate the tensor product of the Hilbert spaces indexed by the indeces in I. Thus in particular $H_N = H$.

Call Γ_{CQDF^n} the class of n-partite quantum dynamic frames.

Definition 17. Given an n-partite quantum dynamic frame and a set of indices $I \subseteq N$, a partial map $R \subseteq \Sigma_H \times \Sigma_H$ is called *I-local* if it correspond to a linear map $L : H \to H$ such that $L = L'_I \bigotimes Id_{N\setminus I}$. Call Loc_I the set of I-local maps. An action is I-local if it is the union of I-local maps. Call $\top_I^{\Sigma_H \times \Sigma_H} = \bigcup Loc_I$ the union of all the local maps.

A state $s \in \Sigma_H$ is *I-separated* if its corresponding unitary vector v can be written as a pure tensor $v = x_I \bigotimes y_{N\setminus I}$, where $x_I \in H_I$ and $y_{N\setminus I} \in H_{N\setminus I}$. If s is I-separated, call s_I the ray $\overline{x_I}$.

A property $T \subseteq \Sigma_H$ is a *I-local state* if, for all $s \in T$, s is I-separated and $s_I = t$, where t is a state (that is, one-dimensional linear subspace) of H_I. A property $T \subseteq \Sigma_H$ is a *I-local* if it is a union of I-local states. Equivalently, there is $S' \subseteq \Sigma_{H_I}$ such that $S = \{s \in \Sigma_H | s_I \in S'\}$. Call $\top_I^\Sigma$ the union of all I-local properties.

Definition 18. We extend the interpretation of the actions i from At to $\{\xrightarrow{P_a?}\}_{a\in L_H} \cup \{\xrightarrow{U}\}_{U\in\mathcal{U}}$ to the new symbols by putting $i(triv_I) = \top_I^{\Sigma_H\times\Sigma_H}$.

Definition 19. The satisfaction relation $\models_{LQP^n}$ contains that of LQP and moreover is defined on the new formulas as

- $M, s \models_{LQP^n} 1$ iff $s = \overline{\bigotimes_{i=1}^{i=n} |1\rangle_i}$
- $M, s \models_{LQP^n} +$ iff $s = \overline{\bigotimes_{i=1}^{i=n} |+\rangle_i}$
- $M, s \models_{LQP^n} \top_I$ iff $s \in \top_I^\Sigma$

Note that the last condition means that $\top_I$ is true at a state iff that state is I-separated.

2.2.3 Expressing locality and entanglement

In order to introduce the new axioms we need to construct some additional abbreviations. Due to the enhanced language, we are now able to express the fact that, say, the subsystem i is in state $|1\rangle$ by the formula 1_i (which is short for $1_{\{i\}}$). We can also define the new propositional constants $0_i :=\sim 1_i$ and $-_i :=\sim +_i$.

Definition 20. First define $\pi[\psi] =\sim [\pi^\dagger] \sim \psi$. This represent the strongest testable post-condition ensured by applying program π to any state satisfying precondition ψ. Second, given a vector $\vec{c} = (c(i))_{i\in I} \in \{0, 1, +\}^{|I|}$, put

$$\vec{c}_I := \bigwedge_{i\in I} c(i)_i$$

Then define the formula $I(\pi)$, meaning that the program π is I-local, as

$$I(\pi) := \bigwedge_{\vec{c},\vec{d},\vec{d'}} (\vec{d}_{N\setminus I} =_{N\setminus I} \pi[\vec{c}_I \wedge \vec{d}_{N\setminus I}] =_I \pi[\vec{c}_I \wedge \vec{d'}_{N\setminus I}])$$

where $\vec{c} \in \{0, 1, +\}^{|I|}$ and $\vec{d}, \vec{d'} \in \{0, 1, +\}^{n-|I|}$. This formula expresses the fact that the action π acts only on the subsystem I: the first equality inside the parenthesis states that the subsystem $N\setminus I$ is left unchanged by the program, while second one asserts that the action of the program is independent from the state of the subsystem $N\setminus I$.

Relying on the isomorphism between the tensor space $H_I \bigotimes H_J$ and the space of linear maps $H_I \to H_J$, we now want to express the fact that a state is entangled according to a program π.

Definition 21. For equicardinal and disjoint $I, J \subseteq N$ and deterministic π, define

$$\overline{\pi}_{I,J} := \top_{I\cup J} \wedge \bigwedge_{c\in\{0,1,+\}} ([c_I?](\pi[c_I])_J \wedge (\sim c_I \rightarrow \pi(c_I)) =\perp)$$

This formula encodes the fact that the state is entangled according to π, that is, any measurement of the subsystem I resulting in a state c_I collapses the subsystem J into $\pi[c_I]_J$.

2.2.4 Proof System

Definition 22. The proof system for LQP^n contains all the axioms and inference rules of LQP. In addition it contains the following axioms:

1. $\top_N$, **Separation 1**
2. $\top_I \wedge \top_J \rightarrow \top_{N\setminus I} \wedge \top_{I\cup J} \wedge \top_{I\cap J}$, **Separation 2**
3. $I(triv_I)$, **Axiom for** $triv_I$ **1**
4. $I(\pi) \rightarrow \langle\pi\rangle p \leq \langle triv_I\rangle p$, **Axiom for** $triv_I$ **2**
5. $T(p) \wedge I(p) \wedge I(q) \wedge \perp \neq p \leq \psi \rightarrow (p = \psi)$, **Local States**
6. For equicardinal and disjoint $I, J \subseteq N$, $c \in \{0, 1, +, -\}$ and deterministic π :
   $T(c_I) \wedge T(\overline{\pi}_{I,J})$, **Basic State Testability**
7. $+_i \rightarrow \Diamond 0_i \wedge \Diamond 1_i$, **Proper Superposition 1**
8. $-_i \rightarrow \Diamond 0_i \wedge \Diamond 1_i$, **Proper Superposition 2**
9. For deterministic programs π, π':
   $\bigwedge_{\vec{c}\in\{0,1,+\}^n} (\pi[\vec{c}_N] = \pi'[\vec{c}_N] \rightarrow \pi[p] = \pi'[p])$, **Deterministic Programs**
10. For π deterministic and I, J disjoint:
    $T(p_i) \rightarrow p_i?[\overline{\pi}_{I,J}] =_J \pi[p_i]$, **Entanglement**

The first two Axioms state that every state is N-separated and if a state is both I-separated and J-separated then it is also $N\setminus I$-separated, $I \cup J$-separated and $I \cap J$-separated. Axioms 3 and 4 express the fact that $triv_I$ is the weakest I-local program. Axiom 5 ensures that testable local properties are minimal among non-trivial local properties. The Basic State Testability Axioms guarantees that the properties represented by c_I and $\overline{\pi}_{I,J}$, being a certain state and being entangled with respect to some program, are testable. Axiom 7 and 8 postulate the existence of proper superpositions. Axiom 9 forces two deterministic programs that agree on the basis to agree on all properties. Axiom 10 ensures that the property of being entangled according to π holds for all testable I-local properties and not just on states satisfying c_I, as in the definition.

The proof system is also enriched with an additional inference rule called Local Atomicity Rule: if $I \neq N$ and p does not occur in ψ, ϕ, θ then from

$$\vdash_{LQP^n} \psi \wedge T(p_I) \wedge p_I \leq \phi \rightarrow p_I \leq \theta$$

infer

$$\vdash_{LQP^n} \psi \wedge I(\phi) \rightarrow \phi \leq \theta$$

To represent quantum protocol we need to introduce a few more abbreviations and axioms. For example, we need to know that the action X indeed behaves like the corresponding quantum gate. To this end we introduce a group of axioms describing the effect of the action X on the basis:

- $0_i \rightarrow [X_i]1_i$
- $1_i \rightarrow [X_i]0_i$
- $+_i \rightarrow [X_i]+_i$

where i is the subsystem on which X_i acts. We introduce similar groups of axioms for $CNOT, H$ and Z. The Bell states are characterized by formulas $\beta^{ij}_{xy} := (\overline{Z^x_1; X^y_1})_{ij}$.

Theorem 3 (Theorem 7 in [11], p.28)**.** *The proof system of LQP^n is sound with respect to the class Γ_{CQDF^n}.*

2.3 Expressing Quantum Algorithms

We now turn to the representation of quantum algorithms. The main result concerning the correctness proofs of known quantum algorithms is

Theorem 4 ([11] and [8])**.** *In the logic LQP^n we can give a formal correctness proof of the following algorithms: Teleportation, Quantum Secret Sharing, Superdense Coding, Entanglement Swapping and Logic Gate Teleportation.*

We exemplify the case of the Teleportation protocol, describing how it is coded in the logic. We use an alternative formulation of the same protocol. After the preparation of the protocol, instead of measuring in the Bell basis Alice performs the following actions.

The first move of Alice is the entanglement of the two qubits in her possession: she first perform a CNOT gate on the two qubits and successively a Hadamard gate on the first one (the one in H_1). She then measures the system of both qubits in the standard base. The rest of the protocol remains unchanged: she communicates the result of the measurement to Bob, who performs a unitary correction on his qubits.

The program in LQP^n corresponding to the Teleportation protocol can be written as

$$Tel := \bigcup_{x,y \in \{0,1\}} (CNOT_{1,2}; H_1; (x_1 \wedge y_2)?; X_3^y; Z_3^x)$$

where X^0 and Z^0 are the identity. Notice that the program follows closely the sequential application of the quantum gates described above. The big union in the front captures the indeterminacy of the measurement outcome, and at the same time divides the program into 4 branches which are dependent on such outcome. The validity expressing the correctness of the overall procedure is

$$Tel[q_1 \wedge \beta_{00}^{23}] =_3 id_{13}[q_1]$$

which reads as follows: the state obtained after performing the protocol Tel to a state prepared in $q_1 \wedge \beta_{00}^{23}$ is, with respect to the subsystem indexed by 3, equivalent to the state having subsystem 3 in state q_1. For the proof of this formula from the axioms of LQP^n we refer to [11] p. 30.

Despite these encouraging results, we can see that LQP^n falls short in one respect: its formalism cannot express probabilities. Hence this logic will not be able to represent any algorithm in which probabilities play a significant role. This suggest the need for a further improvement of LQP^n.

Part II

Drawing the connection

Chapter 3

Modal Logic for Small Categories

In this chapter we present a procedure to obtain a Modal Logic frame from a small category. The primary aim is to have a formal tool to connect the approaches we presented in the previous two chapters. Our methodology however touches a very general issue, namely the relation between Category Theory and Modal Logic. In particular, the combination of the types given by the categorical setting with a dynamic logic formalism seems to us especially interesting, as it can be used to describe typed processes. We spend a considerable part of this chapter analyzing such Dynamic Logic with Types, abbreviated DLT.

The chapter is structured as follows. First we introduce such a procedure in full generality. Second, we devote some pages to the study of different logics that can be used to describe Modal Logic frames arising from small categories: the dynamic logic with types DLT, the logic $S4$ and Hybrid Logic. Third, we elaborate on the possibility to connect the categorical structure to the logic. We conclude the chapter extending the approach to cover locally small categories.

3.1 Frames and models

Definition 23. A *small category* is a category such that the collection of objects and the collection of maps are both sets. A *locally small category* is a category such that, for each pair of objects I, J, the collection of morphisms from I to J is a set.

In what follows we will only consider small categories, if not indicated otherwise. We will use the notation $\mathbf{C}_0$ and $\mathbf{C}_1$ to indicate the collection of objects and arrows of a category $\mathbf{C}$, respectively.

Definition 24. Given a small category $\mathbf{C}$ and a functor $U : \mathbf{C} \to \mathbf{Rel}$, a $(\mathbf{C}, U)$-frame is a pair $\langle W, Rel \rangle$ such that

- The set W is defined as

$$W := \bigcup \{U(I) | I \in \mathbf{C}_0\}$$

- The set Rel of relations on W is defined as

$$Rel := \{U(f) | f \in \mathbf{C}_1\}$$

Notice that if $\mathbf{C}$ is small then W is the union of set-many sets, and thus is a set. Similarly, as there are set-many morphisms in $\mathbf{C}$, Rel will be a set.

Definition 25. Call Γ the class of all $(\mathbf{C}, U)$-frames, that is, the class of Modal Logic frames arising from any such pair $(\mathbf{C}, U)$.

Definition 26. Fixed a set At of atomic propositions, a $(\mathbf{C}, U)$-model is a triple $\langle W, Rel, V \rangle$ such that

- $\langle W, Rel \rangle$ is a $(\mathbf{C}, U)$-frame
- V is a function $V : At \to \wp(W)$

Definition 27. Call $\mathcal{M}(\Psi)$ the class of models over the Modal Logic frames in the class Ψ.

3.2 Logics for small categories

Definition 28. Suppose given a satisfaction relation $\vDash$ over a class of models $\mathcal{M}(\Psi)$ and a set of formulas $\mathcal{F}$. We say that a formula ψ is *globally true* in a model M if $M, s \vDash \psi$ for every $s \in W$. A formula is *valid in a frame* if it is satisfied in every state of every model based on that Modal Logic frame. Given a class of Modal Logic frames Ψ, a formula is *valid* in Ψ if it is valid in every frame belonging to Ψ. The set of all formulas in $\mathcal{F}$ that are valid in a class of Modal Logic frames Φ is called the *logic* of Φ, and we denote it with $\Lambda^{\mathcal{F}}_{\Psi}$.

Definition 29. Call *LSC*-logic, Logic of Small Categories, a logic given by a set of formulas $\mathcal{F}$, a class of Modal Logic frames $\Gamma' \subseteq \Gamma$ and a satisfaction relation $\vDash$ suitably defined.

3.3 Examples of LSC logics

Varying the category $\mathbf{C}$, the functor $U : \mathbf{C} \to \mathbf{Rel}$, the syntax or the satisfaction relation we can obtain different Modal Logic frames and different logics. In these subsection we show how some interesting logics can be recovered by particular choices of these ingredients.

Although not directly relevant for the rest of the thesis, we decided to include these results to highlight that our procedure to obtain Modal Logic frames from small categories is interesting per se.

3.3.1 Dynamic logic with types

Categories are used in Computer Science to model datatypes with associated operations[1], and PDL has been developed to represent computer programs.[2] In light of the connection that we established between small categories and Modal Logic frames, it is natural to ask: can we combine the typing system with the modal logic formalism in order to have a *logic for typed processes*?

In what follows we propose a candidate for such a logic, a dynamic logic with types called DLT. Its syntax is similar to the syntax of PDL but with a substantial difference: we have constants for types of objects and the actions are typed.

We define the set $\mathcal{F}_{DLT}$ of formulas of the logic as

$$\psi := t \,|\, p \,|\, \neg\psi \,|\, \psi \wedge \phi \,|\, [\pi_{t,t'}]\psi$$

where t's come from a set of types Typ of cardinality n, and p's belong to a set of atomic propositions At. The actions $\pi_{I,J}$ are taken from a set $Act_{I,J}$ defined recursively as

$$\pi_{t,t'} := Id_t \,|\, a_{t,t'} \,|\, \pi_{t,t''}; \pi'_{t'',t'}$$

where $a_{t,t'}$ are atomic actions belonging to a set $AtAct_{t,t'}$, and Id_t's are constants defined only in the case $t = t'$. Set $Act := \bigcup_{t,t' \in Typ} Act_{t,t'}$. The connectives $\langle \pi_{t,t'} \rangle$, $\to$, $\vee$ and $\leftrightarrow$ are defined as usual from the basic ones.

Definition 30. Call Γ_n the class of $(\mathbf{C}, U)$-frame arising from categories such that the cardinality of $\mathbf{C}_0$ is $\leq n$.

We now proceed to define the satisfaction relation for the set of formulas $\mathcal{F}_{DLT}$. Suppose given a $(\mathbf{C}, U)$-frame in Γ_n.

Definition 31. Given a set of types Typ, call *interpretation of the types* a surjective function $i' : Typ \to \mathbf{C}_0$.

[1] See for example [13] Section 2.2 p.20 : "Functional programming languages as categories".

[2] See [14] for details.

Definition 32. Given a set of action *Act*, a $(\mathbf{C}, U))$-frame and an interpretation of types, define an *interpretation of the actions* $i : Act \to Rel$ as follows:

- $i(Id_t) = U(Id_{i'(t)})$
- $i(a_{t,t'}) \in \{U(f) | f \in \mathbf{C}_1, f : i'(t) \to i'(t')\} \subseteq Rel$
- $i(\pi_{t,t''}; \pi'_{t'',t'}) = i(\pi'_{t'',t'}) \circ i(\pi_{t,t''})$[3]

Where $\circ$ is the composition of relations.

Definition 33. Given a $(\mathbf{C}, U))$-model $M = \langle W, Rel, V \rangle$, an interpretation for the actions, an interpretation for the types and an element $s \in W$ define

- $M, s \vDash_{DLT} p$ iff $s \in V(p)$
- $M, s \vDash_{DLT} t$ iff $s \in U(i'(t))$
- $M, s \vDash_{DLT} \neg p$ iff $M, s \nvDash_{DLT} p$
- $M, s \vDash_{DLT} \psi \wedge \phi$ iff $M, s \vDash_{DLT} \psi$ and $M, s \vDash_{DLT} \phi$
- $M, s \vDash_{DLT} [\pi_{t,t'}]\psi$ iff if $M, s \vDash_{DLT} t$ then $\forall s'$ s.t. $(s, s') \in i(\pi_{t,t'})$ we have $M, s' \vDash_{DLT} \psi$

We write $M, s \vDash_{DLT} \Phi$, for $\Phi \subseteq \mathcal{F}_{DLT}$, to mean that $M, s \vDash_{DLT} \psi$ for all $\psi \in \Phi$.

Notice that by its definition the semantics of $\langle \pi_{t,t'} \rangle$ is

$$M, s \vDash \langle \pi_{t,t'} \rangle \psi \text{ iff } M, s \vDash t \text{ and } \exists (s, s') \in i(\pi_{t,t'}) \text{ s.t.} M, s' \vDash \psi$$

Definition 34. Define a consequence relation $\vDash_{DLT} \subseteq \wp(\mathcal{F}_{DLT}) \times \mathcal{F}_{DLT}$ as follows: $\Phi \vDash_{DLT} \psi$ holds iff for every model in $M \in \mathcal{M}(\Gamma_n)$ and element s in that model $M, s \vDash_{DLT} \Phi$ entails $M, s \vDash_{DLT} \psi$.

Definition 35. The proof system for the logic *DLT*, the dynamic logic with types, consists of the following axioms:

1. the tautologies of classical propositional logic
2. $[\pi_{t,t'}](p \to q) \to ([\pi_{t,t'}]p \to [\pi_{t,t'}]q)$
3. $[\pi_{t,t'}]p \leftrightarrow \neg\langle \pi_{t,t'} \rangle \neg p$
4. $\langle \pi_{t,t''} \rangle \langle \pi'_{t'',t'} \rangle p \leftrightarrow \langle \pi_{t,t''}; \pi'_{t'',t'} \rangle p$
5. $\langle \pi_{t,t'} \rangle p \to t$

[3]Recall that the notation of $\circ$ is inverted with respect to the notation of ;.

6. $[\pi_{t,t'}]p \to [\pi_{t,t'}](p \wedge t')$

7. $\langle Id_t \rangle p \leftrightarrow t \wedge p$

8. $\langle Id_t \rangle p \to [Id_t]p$

9. $\bigvee_{t \in Typ} t$

Axiom 1 is self explanatory. Axioms 2 and 3 are the usual conditions on normal modal logics. Axiom 4 is the regularity condition for the composition of actions: it states that performing first action $\pi_{t,t''}$ and then action $\pi'_{t'',t'}$ is the same as performing the composite action $\pi_{t,t''};\pi'_{t'',t'}$. Axiom 5 and 6 state that if an action is performable the the types must be matching. Axiom 7 enforces the fact that the identity action is locally reflexive, that is, reflexive on the states of the right type. Axiom 8 ensures the partial functionality of the identity actions. Axiom 9 states that every state must have a type.[4]

The inference rules of this proof system are:

- Modus Ponens: from $\vdash p, p \to q$ infer $\vdash q$
- Uniform Substitution: from $\vdash \psi$ infer $\vdash \psi[p/\phi]$
- Necessitation Rule: from $\vdash p$ infer $\vdash [\pi_{t,t'}]p$

We write $\Phi \vdash_{DLT} \psi$ to mean that there is a derivation in the proof system of DLT that starts from the premises in Φ and concludes ψ.

Remark

The axioms of a category, or rather, their formulation in the syntax of DLT, can be derived in the proof system. Consider associativity:

$$\langle \pi_{t,t'} \rangle \langle \pi_{t',t''}; \pi_{t'',t'''} \rangle p \leftrightarrow \langle \pi_{t,t'}; \pi_{t',t''} \rangle \langle \pi_{t'',t'''} \rangle p$$

By Axiom 4 and the inference rules both sides are equivalent to $\langle \pi_{t,t'} \rangle \langle \pi_{t',t''} \rangle \langle \pi_{t'',t'''} \rangle p$.

The identity axiom

$$\langle \pi_{t,t'} \rangle p \leftrightarrow \langle Id_t; \pi_{t',t''} \rangle p$$

can be proved observing that the right-end side is equivalent to $\langle Id_t \rangle \langle \pi_{t',t''} \rangle p$ by Axiom 4, and to $t \wedge \langle \pi_{t',t''} \rangle p$ by Axiom 7. By Axiom 5 $t \wedge \langle \pi_{t',t''} \rangle p$ is equivalent to $\langle \pi_{t',t''} \rangle p$.

For the other identity axiom we need Axiom 4, 7 and 6.

[4] Note that the finiteness of Typ is essential if we want express this axiom in a language with finite disjunctions.

Theorem 5 (Soundness)**.** *The proof system of DLT is sound with respect to the class* Γ_n*.*

Proof. We want to show that if $\Phi \vdash_{DLT} \psi$, that is, there is a proof in the proof system from Φ to ψ, then $\Phi \vDash_{DLT} \psi$.

It is enough to show that the axioms are valid and that the inference rules preserve the satisfaction. We only show the cases of Axioms 5, 6, 7, 8 and 9; the validity of the other axioms and rules is a well know fact (see [14]).

Consider a generic $(\mathbf{C}, U)$-frame and an element s in its domain. For Axiom 5, suppose that $M, s \vDash_{DLT} \langle \pi_{t,t'} \rangle p$. Then by the definition of $\vDash_{DLT}$ we know that $M, s \vDash_{DLT} t$ and there is $(s, s') \in i(\pi_{t,t'})$ such that $M, s' \vDash_{DLT} p$. Thus t is true at s.

For Axiom 6, suppose $M, s \vDash_{DLT} \langle \pi_{t,t'} \rangle p$. Then $M, s \vDash_{DLT} t$ and there is $(s, s') \in i(\pi_{t,t'})$ such that $M, s' \vDash_{DLT} p$. By the interpretation of the actions, $i(\pi_{t,t'})$ is interpreted in *Rel* in a relation $R \subseteq U(i'(t)) \times U(i'(t'))$. Hence $s' \in U(i'(t'))$, which means $M, s' \vDash_{DLT} t' \wedge p$. So $M, s \vDash_{DLT} \langle \pi_{t,t'} \rangle (t' \wedge p)$.

Now consider Axiom 7. We have that $M, s \vDash_{DLT} \langle Id_t \rangle p$ iff $M, s \vDash_{DLT} t$ (recall that Id is of type t, t) and $M, i(Id_I)(s) \vDash_{DLT} p$. Since $i(Id_I)$ is actually the identity, we get $M, s \vDash_{DLT} p$, so $M, s \vDash_{DLT} t \wedge p$. Each step of the argument is iff, so reasoning backward we obtain the other implication.

For Axiom 8, suppose $M, s \vDash_{DLT} \langle Id_t \rangle p$. Then $M, i(Id_I)(s) \vDash_{DLT} p$ and thus $M, s \vDash_{DLT} t \wedge p$ by what we just proved. But this also means that the implication if $M, s \vDash_{DLT} t$ then $M, i(Id_I)(s) \vDash_{DLT} p$ holds. Therefore we can conclude $M, s \vDash_{DLT} [Id_t]p$.

For Axiom 9, it is sufficient to notice that the interpretation of types is surjective and every state in the Modal Logic frame is in the image of some object of the category, thus every state must have a type. □

We now prove the completeness of this proof system for the class Γ_n using the canonical model technique. It is a standard procedure in Modal Logic (see [14] section 4.2), hence we will skim over some details. We first recall a well-known proposition:

Proposition 7 ([14] p.194.)**.** *Given a proof system* P *and a class of structures* S*, if every* P*-consistent set of formulas is satisfiable on some* $M \in$ S *then* P *is complete with respect to* S*.*

Thus to prove completeness it is sufficient to find a satisfying structure for each consistent set of formulas. The idea of the canonical model is to find *one* structure satisfying *all* consistent sets of formulas. We now build up the notion of canonical model.

Definition 36. A set of formulas Φ is *maximal DLT-consistent* if it is DLT-consistent and any set of formulas properly containing it is DLT-inconsistent. If Φ is maximal DLT-consistent, we call it a MCS.

Proposition 8 (Properties of MCS)**.** *If Φ is an MCS then*

- *it is closed under Modus Ponens: if $\psi, \psi \to \phi \in \Phi$ then $\phi \in \Phi$*
- *for all formulas ϕ, $\phi \in \Phi$ or $\neg\phi \in \Phi$*
- *for all formulas ϕ, ψ, if $\phi \vee \psi \in \Phi$ then $\phi \in \Phi$ or $\psi \in \Phi$*

Proof. Straightforward check. □

Lemma 1 (Lindenbaum Lemma)**.** *If Φ is DLT-consistent then there is a MCS Φ' containing it.*

Proof. Enumerate the formulas in the language: $\psi_0, \psi_1, \ldots$. Construct a countable chain of set of formulas as follows: start from $\Phi_0 = \Phi$ and at each step $n+1$ add ψ_n if $\Phi_n \cup \{\psi_n\}$ is DLT-consistent or $\neg\psi_n$ otherwise. Put $\Phi' = \bigcup_{n\geq 0} \Phi_n$. We obtain that Φ' is an MCS and that if $\Phi' \vdash_{DLT} \psi$ then $\psi \in \Phi'$. □

Definition 37. The *canonical model* for DLT is a tuple $\langle W^{DLT}, \{R_{\pi_{t,t'}} | \pi_{t,t'} \in Act\}, V^{DLT}\rangle$ such that

- W^{DLT} is the set of all MCS
- each relation $R_{\pi_{t,t'}}$ is defined as follows: $R_{\pi_{t,t'}}ss'$ if, for all formulas $\psi \in \mathcal{F}_{DLT}$, $\psi \in s'$ entails $\langle\pi_{t,t'}\rangle\psi \in s$
- $V^{DLT}(p) = \{s \in W^{DLT} | p \in s\}$

Lemma 2. *In the canonical model, $R_{\pi_{t,t'}}ss'$ iff for all formulas ψ, $[\pi_{t,t'}]\psi \in s$ entails $\psi \in s'$.*

Proof. ($\Rightarrow$) Suppose $R_{\pi_{t,t'}}ss'$ and by contraposition suppose $\psi \notin s'$. Since s' is an MCS, $\neg\psi \in s'$. By definition of the relation, $\langle\pi_{t,t'}\rangle\neg\psi \in s$, and by consistency of s we have $\neg\langle\pi_{t,t'}\rangle\neg\psi \notin s$. Hence $[\pi_{t,t'}]\psi \notin s$.

($\Leftarrow$) Suppose for all formulas ψ, $[\pi_{t,t'}]\psi \in s$ entails $\psi \in s'$. By contradiction, suppose there is $\psi \in s'$ such that $\langle\pi_{t,t'}\rangle\psi \notin s$. By consistency of s we have $\neg\langle\pi_{t,t'}\rangle\psi \in s$, so $[\pi_{t,t'}]\neg\psi \in s$. So by assumption we get that $\neg\psi \in s'$, contradiction. □

Lemma 3 (Existence Lemma)**.** *In the canonical model, if a state s is such that $\langle\pi_{t,t'}\rangle\psi \in s$ then there is s' such that $R_{\pi_{t,t'}}ss'$ and $\psi \in s'$.*

Proof. Suppose $\langle\pi_{t,t'}\rangle\psi \in s$. Put $\Phi = \{\psi\} \cup \{\phi | [\pi_{t,t'}]\phi \in s\}$. We now prove Φ is consistent by contradiction.

If it is not, then there are $\phi_1, \ldots, \phi_n$ such that $\phi_1 \wedge \cdots \wedge \phi_n \to \neg\psi$. By generalization rule, Axiom 2 and classical propositional logic we get $[\pi_{t,t'}]\phi_1 \wedge \cdots \wedge [\pi_{t,t'}]\phi_n \to [\pi_{t,t'}]\neg\psi$. Since the antecedent is in s and s is closed under Modus Ponens we have $[\pi_{t,t'}]\neg\psi \in s$, thus $\neg\langle\pi_{t,t'}\rangle\psi$. Since $\langle\pi_{t,t'}\rangle\psi \in s$ we get a contradiction.

Since Φ is consistent, by Lindenbaum Lemma there is an MCS s' containing it. We have $\psi \in s'$ by construction. Since for all formulas ϕ, $[\pi_{t,t'}]\phi \in s$ entails $\phi \in s'$, by the previous lemma we get $R_{\pi_{t,t'}}ss'$. □

In order to have a satisfaction relation on the canonical model, however, we need an interpretation for the types and an interpretation for the actions. But we defined an interpretation of the types using the fact that the model we had was based on a $(\mathbf{C}, U)$-frame. The next proposition fixes this.

Proposition 9. *The canonical model is in $\mathcal{M}(\Gamma_n)$.*

Proof. We need to exhibit a category $\mathbf{C}$ and a functor $U : \mathbf{C} \to \mathbf{Rel}$ that give rise to the Modal Logic frame of the canonical model.

We build a category using the syntax. Define $\mathbf{C}_0 = Typ$; this takes care of the cardinality requirement on $\mathbf{C}_0$. Take an edge $f_{\pi_{t,t'}}$ between two objects t and t' for each action $\pi_{t,t'}$. We now have a directed graph, we want to make it into a category. First define $f_{\pi_{t',t''}} \circ f_{\pi_{t,t'}} = f_{\pi_{t',t''};\pi_{t,t'}}$. Then quotient this set with respect to the axioms of a category:

- $(f_{\pi_{t'',t'''}} \circ f_{\pi_{t',t''}}) \circ f_{\pi_{t,t'}} \sim f_{\pi_{t'',t'''}} \circ (f_{\pi_{t',t''}} \circ f_{\pi_{t,t'}})$
- $f_{\pi_{t,t'}} \circ f_{Id_t} \sim f_{\pi_{t,t'}} \sim f_{Id_{t'}} \circ f_{\pi_{t,t'}}$

We now take the equivalence classes to be the actual morphisms in $\mathbf{C}_1$. In this category the identity for object t is $[f_{Id_t}]$, the equivalence class of f_{Id_t}. Composition is defined as $[f_{\pi_{t',t''}}] \circ [f_{\pi_{t,t'}}] = [f_{\pi_{t',t''}} \circ f_{\pi_{t,t'}}]$. This enforces the axioms; for example

$$[f_{\pi_{t,t'}}] \circ [f_{Id_t}] = [f_{\pi_{t,t'}} \circ f_{Id_t}] = [f_{\pi_{t,t'}}]$$

Now define $U : \mathbf{C} \to \mathbf{Rel}$ as follows:

$$t \mapsto \{\Phi | \Phi MCS, t \in \Phi\}$$
$$[f_{\pi_{t,t'}}] : t \to t' \mapsto \{(\Phi, \Phi') | \forall\psi\, [\pi_{t,t'}]\psi \in \Phi \Rightarrow \psi \in \Phi'\} = R_{\pi_{t,t'}}$$

We now show that the assignment of morphisms is well defined. Suppose $f_{\pi'_{t,t'}} \sim f_{\pi_{t,t'}}$. This means that in the proof system we can prove $\langle\pi_{t,t'}\rangle\psi \leftrightarrow \langle\pi'_{t,t'}\rangle\psi$, because we can derive the (counterpart of) the axioms of a category as theorems. Now recall the definition of $R_{\pi_{t,t'}}$: $R_{\pi_{t,t'}}ss'$ if, for all formulas $\psi \in \mathcal{F}_{DLT}$, $\psi \in s'$ entails $\langle\pi_{t,t'}\rangle\psi \in s$. So if $R_{\pi_{t,t'}}ss'$ then for all formulas ψ, $\psi \in s'$ entails $\langle\pi_{t,t'}\rangle\psi \in s$, which in turn entails $\langle\pi'_{t,t'}\rangle\psi \in s$ by DLT-consistency. So $R_{\pi'_{t,t'}}ss'$. Reasoning backward we can see that $(ss') \in R_{\pi'_{t,t'}}$ entails $(ss') \in R_{\pi_{t,t'}}$, therefore the two relations are the same.

We can now see that the Modal Logic frame

- $W = \bigcup\{U(x) | x \in \mathbf{C}_0\}$
- $Rel = \{U([f_{\pi_{t,t'}}]) | \pi_{t,t'} \in Act\}$

is indeed $\langle W^{DLT}, \{R_{\pi_{t,t'}} | \pi_{t,t'} \in Act\}\rangle$. Hence the canonical model is in $\mathcal{M}(\Gamma_n)$. □

Now define the interpretation of types in the canonical model as $i'(t) = t$ and the interpretation of actions as $i(\pi_{t,t'}) = R_{\pi_{t,t'}}$.

Lemma 4 (Truth Lemma)**.** *For any state s of the canonical model, $s \vDash \psi$ iff $\psi \in s$.*

Proof. By induction on the complexity of ψ. The base case is given by the definition of V^{DLT}. The negation and conjunction follow from the properties of MCSs.

For the types we have that, by contruction of the interpretation of the types and U: $s \vDash_{DLT} t$ iff $s \in U(i'(t)) = U(t)$ iff $t \in s$.

For the modality, we first prove the left to right direction. Consider that $s \vDash_{DLT} \langle \pi_{t,t'} \rangle \psi$ iff $s \vDash_{DLT} t$ and $\exists (s, s') \in i(\pi_{t,t'}) = R_{\pi_{t,t'}}$ such that $s' \vDash_{DLT} \psi$. By induction hypothesis this happens iff $s \vDash_{DLT} t$ and $\exists (s, s') \in R_{\pi_{t,t'}}$ such that $\psi \in s'$. But by the definition of $R_{\pi_{t,t'}}$ in the canonical model, this condition entails $\langle \pi_{t,t'} \rangle \psi \in s$.

For the other direction suppose $\langle \pi_{t,t'} \rangle \psi \in s$. By DLT-conistence we have $t \in s$, and thus by what we proved $s \vDash_{DLT} t$. By the Existence lemma, there must be $(s, s') \in R_{\pi_{t,t'}}$ such that $\psi \in s'$. Applying again the induction hypothesis and the definition of the semantics we get $s \vDash \langle \pi_{t,t'} \rangle \psi$. □

We are now ready to prove the completeness theorem.

Theorem 6 (Completeness)**.** *The proof system of DLT is complete with respect to the class Γ_n, where n depends on the cardinality of Typ in the syntax.*

Proof. Take a DLT-consistent set of formulas Φ. By Lindenbaum Lemma it can be extended to an MCS Φ'. The MCS Φ' is thus a state in the canonical model. By the Truth Lemma, $\Phi' \vDash \psi$ iff $\psi \in \Phi'$. Since $\Phi \subseteq \Phi'$, we have $\Phi' \vDash \Phi$. Hence there is a model and a state satisfying Φ. Since Φ was generic, this holds for any DLT-consistent set of formulas. By Lemma 7, this entails that DLT is complete with respect to the class Γ_n. □

It is now interesting to explore possible generalization of this completeness result. For example we can ask: can we drop the cardinality restriction? More precisely, can we allow for any category and any interpretation of types (not only surjective)? If we do it then we can have a category with more objects that there are types, and thus there could be objects in the Modal Logic frame that have no type, making Axiom 9 invalid. Having infinite types also would not do, because with finite disjunction we cannot express Axiom 9 anymore.

What about dropping Axiom 9 then? This way we are not required to have a type for every object in the Modal Logic frame. This would do for soundness; indeed we can prove a soundness theorem for an wider class of Modal Logic frames.

Theorem 7. *The proof system containing Axioms 1-8 is sound with respect to the class* Γ*, where we now allow for any number of types in the syntax and any function for the interpretation of types.*

Proof. The proof is essentially the same as the proof of the Soundness Theorem. The finite number of types and the surjectivity of the interpretation of types were only needed to prove the validity of Axiom 9, so when we remove it the proof can be directly generalized. □

When we try to prove completeness, however, a problem arise. We have now additional MCSs containing $\neg t$ for all $t \in Typ$. When we want to prove that the canonical model has a $(\mathbf{C}, U)$-frame, our construction of $\mathbf{C}$ and U from the syntax will not recover those additional MCSs.

To conclude this section, we offer a remark in the opposite direction, showing a special case of the completeness theorems for a stronger proof system.

Theorem 8. *Strenghtening Axiom 5 to* $\langle \pi_{t,t'} \rangle \top \leftrightarrow t$ *(where* $\top$ *is a classical tautology) and Axiom 8 to* $\langle \pi_{t,t'} \rangle p \leftrightarrow [\pi_{t,t'}]p$ *we obtain a proof system that is sound and complete with respect to the class of* $\Gamma_n^{\mathbf{Set}} \subseteq \Gamma_n$ *containing all the frames with* $U : \mathbf{C} \to \mathbf{Set}$*.*

Proof. The new Axiom 8 enforce functionality for all the actions, and the new Axiom 5 ensure that functions are total on the set of objects of the corresponding type. Thus we have soundness. For completeness, due to these new axioms we can now prove that the relations in the canonical model are functions that are total on the set of objects of the corresponding type. So the canonical model has a Modal Logic frame that is in $\Gamma_n^{\mathbf{Set}}$. □

3.3.2 S4

In this section we prove that, given a weak language with only the diamond operator and a suitably defined consequence operation, the logic of $(\mathbf{C}, U')$-frames is exactly $S4$, the logic of reflexive and transitive Modal Logic frames.

Consider the basic syntax with only the diamond operator, that is, no types and no typed actions. In other words, the set of formulas $\mathcal{F}_\Diamond$ defined by

$$\psi := p \,|\, \neg\psi \,|\, \psi \wedge \phi \,|\, \Diamond\psi$$

Given a $(\mathbf{C}, U')$-frame, define the satisfaction relation $\vDash_\Diamond$ as usual for negation and intersection. For the diamond operator define:

$$M, s \vDash_\Diamond \Diamond\psi \text{ iff } \exists f \in Rel \text{ and } (s, s') \in f \text{ s. t. } M, s' \vDash_\Diamond \psi$$

Definition 38. A *reflexive and transitive frame* is a pair $\langle W, R\rangle$ such that W is a set and $R \subseteq W \times W$ is a reflexive and transitive relation.

A model over a reflexive and transitive frame is a reflexive and transitive frame enriched with a valuation function $V : At \to \wp(W)$.

The customary satisfaction relation $\vDash'$ for the basic modal language involves the set of formulas $\mathcal{F}_{Diamond}$ and the class $RefTr$. It is defined as:

$$M, s \vDash' \Diamond\psi \text{ iff } \exists(s, t) \in Rel \text{ s.t. } M, t \vDash \psi$$

and similarly to $\vDash_\Diamond$ in the other cases.

Definition 39. Call $\Lambda^{\mathcal{F}_\Diamond}_{RefTr}$ the logic given by the satisfaction relation $\vDash'$ over the class of reflexive transitive frames in the language of $\mathcal{F}_\Diamond$.

Because of the categorical axioms on identity and composition, *Rel* will always be "reflexive " and "transitive".

Proposition 10. *In any* $(\mathbf{C}, U)$*-frame Rel is such that*

1. *for all* $s \in W$ *there is* $f \in Rel$ *such that* $f(s) = s$
2. *for all* $s, t, k \in W$ *if there are* $f, g \in Rel$ *such that* $f(s) = t$ *and* $g(t) = k$ *then there is* $h \in Rel$ *such that* $h(s) = k$

Proof. For item 1, we must have $s \in U(I)$ for some object I of $\mathbf{C}$, hence the desired function is the identity of $U(I)$ corresponding to Id_I.

For the second condition, suppose there are $f, g \in Rel$ such that $f(s) = t$ and $g(t) = k$. Then we must have $s \in U(I)$ and $t \in U(J)$ for some objects I and J of $\mathbf{C}$ and we must also have morphisms f', g' in $\mathbf{C}$ such that $(s, t) \in U(f')$ and $(t, k) \in U(g')$. By the axioms on composition we must have in $\mathbf{C}$ a morphism $g' \circ f'$, hence we get that $(s, k) \in U(g' \circ f')$. □

It is not hard to see that a $(\mathbf{C}, U')$-frame is extremely close to a reflexive and transitive frame. Indeed, we can always recover the latter from a $(\mathbf{C}, U')$-frame taking $\bigcup Rel$.

Definition 40. Given a small category $\mathbf{C}$, a functor $U' : \mathbf{C} \to \mathbf{Sets}$ is a *singleton* functor if it maps objects to singletons:

$$I \mapsto \{*_I\}$$
$$f : I \to J \mapsto \{*_I\} \to \{*_J\}$$

We don't assume singleton functors to be injective on objects. We will use U' as a name of a generic singleton functor. Such functors map objects to singletons and morphisms to the unique functions between the two singletons. Set theoretically, each of such functions is itself a singleton $\{(*_I, *_J)\}$ consisting of one pair of objects. Notice that in general singleton functors are not faithful.

Definition 41. Call Γ_{Sing} the class of $(\mathbf{C}, U')$-frames, for $\mathbf{C}$ generic small category and U' singleton functor.

Lemma 5. *There exists a surjection between* Γ *and* $RefTr$*, the class of reflexive transitive frames.*

Proof. Given a $(\mathbf{C}, U)$-frame $\mathsf{F} = \langle W, Rel\rangle$, define $\phi : \Gamma \to RefTr$ as

$$\phi(\mathsf{F}) := \langle W', R\rangle$$

where $W' = W$ and $R = \bigcup Rel$. To show surjectivity we prove that the function ϕ restricted to the class of $(\mathbf{C}, U)$-frames where U is a singleton functor is in fact a bijection.

Given a reflexive and transitive frame, define the inverse ϕ^{-1} as

$$\phi^{-1}(\langle W', R\rangle) := \langle W, Rel\rangle$$

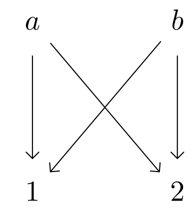
where $W = W'$ and $Rel := \{\{(s,t)\}|(s,t) \in R\}$. To see that this is a $(\mathbf{C}, U')$-frame just consider the category $\mathbf{C}$ having as objects the element of W, as morphisms the relations in Rel and define $U'(w) = \{w\}$ for $w \in W$. It is immediate to see that $\phi^{-1} \circ \phi$ is the identity on Γ_{Sing} and $\phi \circ \phi^{-1}$ is the identity on $RefTr$. □

To see that the function given by the lemma is not injective consider the example below.

Example
Consider the categories $\mathbf{C}$ and $\mathbf{D}$ defined as

- the objects of $\mathbf{C}$ are two sets $I = \{a, b\}$ and $J = \{1, 2\}$
- the objects of $\mathbf{D}$ are the same
- the morphisms of $\mathbf{C}$ are two functions $f = \{(a,1),(b,1)\}$ and $g = \{(a,2),(b,2)\}$
- the morphisms of $\mathbf{D}$ are two functions $f' = \{(a,1),(b,2)\}$ and $g' = \{(a,2),(b,1)\}$

We ignore the identity functions in this example. Take as U the inclusion functor (these categories are already categories of sets and functions). The Modal Logic frame corresponding to $\mathbf{C}$ is $\langle W = \{a, b, 1, 2\}, Rel = \{f, g\}\rangle$, and that corresponding to $\mathbf{D}$ is $\langle W = \{a, b, 1, 2\}, Rel = \{f', g'\}\rangle$. It is easy to see that the image of both under ϕ is



Theorem 9. *The logic $\Lambda_{\Gamma}^{\mathcal{F}_\Diamond}$ is equal to $\Lambda_{RefTr}^{\mathcal{F}_\Diamond}$, the logic of reflexive transitive frames in the syntax of $\mathcal{F}_\Diamond$.*

Proof. Consider a formula $\psi \in \mathcal{F}_\Diamond$. Given an $(\mathbf{C}, U')$-model M, call $\phi(M)$ the model obtained by applying ϕ to the Modal Logic frame and leaving the function V unchanged, and likewise for ϕ^{-1}. We first prove the condition

1. $M, s \vDash_\Diamond \psi$ entails $\phi(M), s \vDash' \psi$

by induction on the complexity of ψ. The base case is given, as the worlds and the valuation are the same in both models. For the step, notice that the cases of negation and conjunction follow directly from the induction hypothesis. Now consider $\Diamond\psi$.

Suppose $M, s \vDash_\Diamond \Diamond\psi$. Then there is $f \in Rel$ and $(s, s') \in f$ such that $M, s' \vDash_\Diamond \psi$. By induction hypothesis we get that $\phi(M), s' \vDash' \psi$. By the effect of ϕ on Rel, we know that there must be $(s, s') \in R$. Hence $\phi(M), s \vDash' \Diamond\psi$. This concludes the proof by induction.

Now suppose ψ is valid in $RefTr$. By contradiction, suppose ψ is not valid in Γ. Then there must be a model M based on a $(\mathbf{C}, U)$-frame and a state s of such Modal Logic frame such that $M, s \nvDash \psi$. Thus by the semantics we have $M, s \vDash \neg\psi$.

By condition 1. this entails that $\phi(M), s \vDash' \neg\psi$ for some model $\phi(M)$ based on a reflexive transitive frame. But we assumed that ψ is valid in $RefTr$, so we also have $\phi(M), s \vDash' \psi$, contradiction. Hence $\Lambda_{\Gamma}^{\mathcal{F}_\Diamond} \supseteq \Lambda_{RefTr}^{\mathcal{F}_\Diamond}$. The reverse inclusion is given by condition 1 and surjectivity of ϕ. □

This Theorem, coupled with the standard result in Modal Logic

Theorem 10 ([14])**.** *The proof system of $S4$ is complete with respect to the class of reflexive and transitive frames.*

gives us

Corollary 1. *The proof system $S4$ is complete with respect to the class of Modal Logic frames Γ.*

3.3.3 Hybrid Logic

Hybrid Logic is a strenghtening of the standard modal language with nominals for states, that is, symbols which are supposed to denote exactly one state. This additional expressive power is useful in applications where the reference to specific states is important, e.g. in applications of temporal logic, but also allows for the formulation of properties of Modal Logic frames that cannot be captured by the ordinary modal language, for example irreflexivity.

In this section we explore the connections between *LSC* logics and Hybrid Logic. Our reference on this latter topic is [9].

Given a set of nominals Nom and a set of atomic propositions At, consider the set formulas $\mathcal{F}_h$ defined as

$$\psi := t \mid p \mid \neg\psi \mid \psi \wedge \phi \mid @_t\psi \mid \Diamond\psi$$

where $p \in At$ and $t \in Nom$. This essentially the syntax of $\mathcal{F}_\Diamond$ enriched nominals and with operators $@_t$ indexed by nominals. The nominals essentially behave as types in *DLT*, but the restriction to singleton functors in the semantics will turn them into names for the objects of the Modal Logic frame.

Definition 42. Given a $(\mathbf{C}, U')$-frame in Γ_{Sing}, an *interpretation of the nominals* is a surjective function $i' : Nom \to \mathbf{C}_0$.

Observe that a $(\mathbf{C}, U')$-frame in Γ_{Sing} is a pair $\langle W, Rel\rangle$ defined as

- $W := \bigcup\{\{*_I\} | I \in \mathbf{C}_0\}$
- $Rel := \{\{(*_I, *_J)\} | f : I \to J \in \mathbf{C}_1\}$

Therefore W is a set with at most as many elements as there are objects in the category and Rel is a set of singletons containing pairs.

Definition 43. Given a $(\mathbf{C}, U')$-frame in Γ_{Sing} and an interpretation of nominals, define the consequence relation $\vDash_h$ as $\vDash_\Diamond$ on the shared language, and as in *DLT* for th nominals:

$$M, s \vDash_h t \text{ iff } t \in U'(i'(t))$$

By the fact that U' is a singleton functor we get

$$M, s \vDash_h t \text{ iff } s = *_t$$

This means that the propositions t can be satisfied only at one element of the Modal Logic frame, namely the state $*_t$ with the corresponding labels.

Hence we have *nominals* in the logic, formulas that are true at exactly one element of the Modal Logic frame. This allows us to define the semantic of the so-called *satisfaction operators*[5] $@_t$ as

[5] Despite the name, these operators are not related to the satisfaction *relation*: the former are symbols in the syntax, the latter is a metalogical notion.

$$M, s \vDash_h @_t\psi \text{ iff } \forall s' \in W\ M, s' \vDash_h t \Rightarrow M, s' \vDash_h \psi$$

We now give the usual definitions of Hybrid Logic, following [9].

Definition 44. Call Nom the set of nominals of the syntax. A *hybrid model* is a triple $\langle W, R, V\rangle$ such that

- $\langle W, R\rangle$ is a Modal Logic frame
- $V : At \cup Nom \to \wp(W)$

and the image of every nominal under V is a singleton.

A *reflexive and transitive hybrid model*, in short rfh-model, is a hybrid model where R is reflexive and transitive.

Definition 45. Given a hybrid model $M = \langle W, R, V\rangle$ the usual semantics of a nominal is defined as

$$M, s \vDash' I \text{ iff } s \in V(I)$$

As $V(I)$ is a singleton, this is tantamount to saying that I is the name of s. The semantic of $@_I$ is

$$M, s \vDash' @_I\psi \text{ iff if } t \in V(I) \text{ then } M, t \vDash' \psi$$

Definition 46. Call $\Lambda^{\mathcal{F}_h}_{rth}$ the logic of rth-models in the syntax of $\mathcal{F}_h$.

Lemma 6. *There is a function from the class of* $(\mathbf{C}, U')$*-models to to the class of rth-models.*

Proof. By Lemma 5 we know there is a bijection between the underlying Modal Logic frames, so the only thing we need to check is that the assignment of singleton to nominals is preserved. Define $\overline{\phi}$ from $(\mathbf{C}, U')$-models to rth models as follows.

Given a $(\mathbf{C}, U')$-model $M = \langle W, Rel, V\rangle$ take the Modal Logic frame $\langle W, R\rangle$ given by Lemma 5 and define V' as V on atomic proposition and as $U' \circ i'$ on nominals. □

Theorem 11. *The logic* $\Lambda^{\mathcal{F}_h}_{\Gamma_{Sing}}$ *contains the logic* $\Lambda^{\mathcal{F}_h}_{rth}$.

Proof. We first prove that, for $\psi \in \mathcal{F}_h$,

1. $M, s \vDash_h \psi$ entails $\overline{\phi}(M), s \vDash' \psi$

The proof is by induction on the complexity of ψ; we only show the cases involving the nominals and the operators $@_t$.

Suppose $M, s \vDash_h t$, with M $(\mathbf{C}, U')$-model. Then by the definition of the satisfaction relation we know that $s \in U'(i'(t))$. By the definition of $\overline{\phi}$ we know that in $\overline{\phi}(M)$ the valuation V' is such that $V'(t) = U'(i'(t))$, so we can conclude $\overline{\phi}(M), s \vDash' t$.

Now suppose $M, s \vDash_h @_t\psi$. Then we have that if $M, t \vDash_h t$ then $M, t \vDash_h \psi$. To prove $\overline{\phi}(M), s \vDash' @_t\psi$ we need to show that if $s' \in V(t)$ then $\overline{\phi}(M), s' \vDash' \psi$. Suppose $s' \in V(t)$. By definition of $\overline{\phi}$ we know $V(t) = U'(i'(t))$, so $s' \in V(t)$ and hence $M, s' \vDash_h t$. By assumption we can conclude $M, s' \vDash_h \psi$, and by induction hypothesis we can infer $\overline{\phi}(M), s' \vDash' \psi$. This concludes the induction.

Now suppose ψ is a validity on the class of rth models. By contradiction, suppose there is a $(\mathbf{C}, U')$-model M such that $M, s \vDash_h \neg\psi$. Then by 1. we know that $\overline{\phi}(M), s \vDash' \neg\psi$, contradicting our assumption. Hence $\Lambda^{\mathcal{F}_h}_{\Gamma_{Sing}} \supseteq \Lambda^{\mathcal{F}_h}_{rth}$. □

What about the other inclusion? The problem here is that in general not every rth-model is an $(\mathbf{C}, U')$-model. The point is that the valuation in an rth-model is not necessarily surjective on the set of singletons, or in other words, in an rth-model we don't necessarily have a nominal assigned to each element of W. For example, V could send all the nominals to the same singleton.

In an $(\mathbf{C}, U')$-model, on the other hand, we always have at least one nominal for each element of W. The reason is that we build the Modal Logic frame from the image of U', and at the same time U' is used as an assignment for nominals.

Example

Suppose $Nom := \{t, t'\}$. Consider the rth model $M := \langle W, R, V\rangle$ defined as

- $W := \{a, b\}$
- $V := \{(t, a), (t', a)\}$

we leave the relation and the action of V on the atomic proposition unspecified, as they are irrelevant for the sake of the example.

Now suppose we want to turn this model into an $(\mathbf{C}, U')$-model. We are given the collection of objects of $\mathbf{C}$, it is the set $Nom := \{t, t'\}$. Now, if we use the definition of V on Nom as the template for U' the resulting Modal Logic frame will only have one object and therefore we are changing the set of worlds.

Hence there is no obvious general way to construct a $(\mathbf{C}, U')$-model from a generic rth-model. However, if we impose the surjectivity of V on the set of singletons we obtain the other inclusion.

Definition 47. A *restricted reflexive transitive hybrid model*, rrth-model in short, is a rth-model such that $V \restriction Nom$ is surjective on the set of singletons.

Lemma 7. *The function $\overline{\phi}$ from $(\mathbf{C}, U')$-models to rrth-models is a bijection.*

Proof. It is easy to see that the image of $\overline{\phi}$ is inside the class of rrth models. The inverse $\overline{\phi}^{-1}$ is defined as follows.

Given an rrth-model $M = \langle W, R, V \rangle$, take the Modal Logic frame F obtained applying ϕ^{-1} to the Modal Logic frame of M. The collection of objects of the category $\mathbf{C}$ is given by Nom, the interpretation of nominals is $i'(t) = t$, the identity, and the action of U' is given by the action of V on Nom.

Notice that this does not fully specify the category $\mathbf{C}$, even with a fixed collection of objects there are different configurations of arrows that could generate the Modal Logic frame obtained via ϕ^{-1}. This however is not a concern: the fact that there exists a category generating the Modal Logic frame F ensures that it is in the class of $(\mathbf{C}, U')$-frames.

We now show that $\overline{\phi}$ is a bijection. We know that the function ϕ acting on the underlying Modal Logic frame is a bijection, so we only need to check the valuation. Suppose Nom is fixed.

Given a $(\mathbf{C}, U')$-model $M = \langle W, Rel, V \rangle$ we obtain an rrth model $\langle W, R, V' \rangle$ via $\overline{\phi}$ such that $V' \upharpoonright Nom$ is defined following the action of $U' \circ i'$. Applying the inverse we recover from V' a new U'. Taking the new interpretation of nominals to be the identity, we can see that the assignment of nominals is the same as in the original model. The valuation on the atomic propositions is left unchanged throughout the process. So even though the category and the functors are different, the model is the same after the application of $\overline{\phi}^{-1} \circ \overline{\phi}$.

The composition of $\overline{\phi} \circ \overline{\phi}^{-1}$ is also the identity, as can be easily checked. □

Definition 48. Call $\Lambda^{\mathcal{F}_h}_{rrth}$ the logic of rrth-models in the syntax of $\mathcal{F}_h$.

Theorem 12. *We have $\Lambda^{\mathcal{F}_h}_{rrth} = \Lambda^{\mathcal{F}_h}_{\Gamma_{Sing}}$.*

Proof. The proof of the inclusion $\Lambda^{\mathcal{F}_h}_{rrth} \subseteq \Lambda^{\mathcal{F}_h}_{\Gamma_{Sing}}$ is essentially the proof of Theorem 11. Since we proved that $\overline{\phi}$ is a bijection, assuming that ψ is a validity of $\Lambda^{\mathcal{F}_h}_{\Gamma_{Sing}}$ and using

1. $M, s \vDash_h \psi$ entails $\overline{\phi}(M), s \vDash' \psi$

we can conclude that it must be a validity of $\Lambda^{\mathcal{F}_h}_{rrth}$. □

3.4 Characterization of the image of U into Rel

For $(\mathbf{C}, U)$ we are given a subcategory of **Rel** and a Modal Logic frame from such subcategory, as outlined at the beginning of the chapter. Fixed $(\mathbf{C}, U)$, it is interesting to consider the following questions:

1. can we characterize the image of U independently from $\mathbf{C}$? Or in other words, can we define a subcategory of $\mathbf{Rel}$, call it $\mathbf{D}$, such that $U : \mathbf{C} \to \mathbf{D}$?

2. can we axiomatize the Modal Logic frame(s) arising from $\mathbf{D}$ with some modal language?

The first question is independent from the connection with modal logic, and thus can be asked for any category.

Considering the category **Hil** of Hilbert spaces and linear maps, an example of this question is the famous problem in Quantum Logic of the lattice-theoretic characterization of the lattice of closed linear subspaces of a Hilbert space, the so-called Hilbert lattices.

Given the category **OML** of orthomodular lattices (having as morphisms the functions preserving the orthomodular structure) we can define a functor $K : \mathbf{Hil} \to \mathbf{OML}$ the sends each Hilbert space H to the corresponding lattice L_H and each linear maps to the corresponding function between lattices (by linearity, each linear maps sends subspaces to subspaces, so it induces a function between the lattices).

We can now see that the problem of characterizing the Hilbert lattices via lattice-theoretic conditions is the same as characterizing the image of H into **OML** independently from **Hil**.

3.5 Extending LSC logics

Suppose the small category $\mathbf{C}$ has some additional categorical structure, for example, it is a monoidal category. Depending on the functor U, such structure may or may not be preserved. If it is, and is thus encoded in the corresponding $(\mathbf{C}, U)$-frame, we are now presented with the question: how do we capture this additional structure with a modal logic?

We start by making the question more precise. Suppose we can define a subcategory $\mathbf{D}$ of $\mathbf{Rel}$, such that $U : \mathbf{C} \to \mathbf{D}$. Suppose given a functor $G : \times_n \mathbf{C} \to \mathbf{C}$, where by $\times_n \mathbf{C}$ we mean the nth product category $\mathbf{C} \times \cdots \times \mathbf{C}$. Moreover, suppose there exists a functor $G' : \times_n \mathbf{D} \to \mathbf{D}$ that correspond to G, i.e., for which the following diagram commute (commutation up to natural isomorphism is sufficient)

$$\begin{CD}
\times_n \mathbf{C} @>{\times_n U}>> \times_n \mathbf{D} \\
@V{G}VV @VV{G'}V \\
\mathbf{C} @>>{U}> \mathbf{D}
\end{CD}$$

From these assumptions it follows that $\mathbf{D}$ has objects $G'(A_1, \ldots, A_n)$ and morphisms of type $G'(A_1, \ldots, A_n) \to G'(B_1, \ldots, B_n)$ with the properties inherited by G.

If we have a syntax such as that of $\mathcal{F}_{DLT}$, then we can add to the syntax a *contructor of types* G^* that takes n types and gives the type $G^*(t_1, \ldots, t_n)$. The interpretation of the new type $i'(G^*(t_1, \ldots, t_n))$ will be $G(i'(t_1), \ldots, i'(t_n))$. Similarly we can add a new operation on programs: given programs $\pi_{t_1,t'_1}, \ldots, \pi_{t_n,t'_n}$ we have the program $G^*(\pi_{t_1,t'_1}, \ldots, \pi_{t_n,t'_n})$ of type $G^*(t_1, \ldots, t_n) \to G^*(t'_1, \ldots, t'_n)$. The interpretation of the new program will be

$$i(G^*(\pi_{t_1,t'_1}, \ldots, \pi_{t_n,t'_n})) = G'(i(\pi_{t_1,t'_1}), \ldots, i(\pi_{t_n,t'_n}))$$

The commutation of the above diagram guarantees that this action has the correct types.

We can now use this syntax to express the features that the $(\mathbf{C}, U)$-frame inherits from $\mathbf{C}$.

If for example $\mathbf{C}$ is a monoidal category, and there is a functor $\bigotimes' : \mathbf{D} \times \mathbf{D} \to \mathbf{D}$ such that the following diagram commute

$$\begin{array}{ccc} \mathbf{C} \times \mathbf{C} & \xrightarrow{U \times U} & \mathbf{D} \times \mathbf{D} \\ \bigotimes \downarrow & & \downarrow \bigotimes' \\ \mathbf{C} & \xrightarrow[U]{} & \mathbf{D} \end{array}$$

then $\mathbf{D}$ is a monoidal category: all the natural isomorphisms are the coherence conditions are preserved due to the fact that functors preserve commuting diagrams and the commutation $\bigotimes' \circ U \times U = U \circ \bigotimes$. In particular, the image of the unit $U(I)$ will be the unit of $\bigotimes'$.

We can now enrich the syntax of DLT with a type constant t_u, a type costructor $\otimes$, action constants $\lambda_{t,t_u \otimes t}, \rho_{t,t \otimes t_u}, \alpha_{t,t',t''}$ (plus the action constants for their inverses) and a program operation $\otimes$. We fix the interpretation $i'(t_u) = I$.

Now we can express in the logic properties such as the naturality of $\lambda_{t,t_u \otimes t}$:

$$\langle Id_{t_u} \otimes \pi_{t,t'}; \lambda_{t,t_u \otimes t} \rangle p \leftrightarrow \langle \lambda_{t',t_u \otimes t'}; \pi_{t,t'} \rangle p$$

the fact that $\lambda^{-1}_{t_u \otimes t,t}$ is inverse of $\lambda_{t,t_u \otimes t}$:

$$\langle Id_{t_u} \rangle p \leftrightarrow \langle \lambda_{t',t_u \otimes t'}; \lambda_{t_u \otimes t,t} {-1} \rangle p$$

or bifunctoriality:

$$\langle Id_{t_1} \otimes \pi_{t_2,t'_2}; \pi_{t_1,t'_1} \otimes Id_{t'_2} \rangle p \leftrightarrow \langle \pi_{t_1,t'_1} \otimes Id_{t_2}; Id_{t'_1} \otimes \pi_{t_2,t'_2} \rangle p$$

Notice that the satisfaction relation of DLT will still work for the new types and programs.

3.6 Logics for locally small categories

Now we want to study the application of this procedure to locally small categories, in order to apply it to **FdHil**. Our procedure cannot be applied directly to a locally small category **C**, unless the functor U identifies so many objects that the image of **C** after U is a small category. Instead of dealing with this special case, we want a way to handle any locally small category. This can be done replacing the idea

one category, one frame

with the slogan

one category, *many* frames

Given a locally small category **C**, one way to approach the problem is to consider the class of Modal Logic frames generated by all the small subcategories of **C**.

In particular, every full subcategory of **C** containing only one object is small. Hence another possibility, which is actually a special case of the first, is to consider the class of Modal Logic frames generated by the class of (full subcategories corresponding to the) objects of **C**, and study the corresponding logic. In this context the typing system is of little use, because all the objects in the Modal Logic frame will be of the same type. The advantage is that if every object of the category is suppose to represent a physical system, such as in **FdHil**, then we will get a Modal Logic frame for every system. We will see in the next chapter that an application of this idea will give us the semantics of LQP.

3.6.1 Extending the logics for locally small categories

One inconvenient to this treatment of locally small categories is that, along with the typing system, we lose the possibility to express in the logic the categorical structure of the image of **C** under U. However, what was a constructor for types becomes now a constructor for Modal Logic frames, as we are given a Modal Logic frame for each object of the category. We can thus design a logic that exploits the particular way in which a Modal Logic frame is built.

Suppose that **C** is locally small, that we can characterize the image of U and that we have a functor matching the categorical structure of **C** as in the diagram

$$\begin{array}{ccc} \times_n \mathbf{C} & \xrightarrow{\times_n U} & \times_n \mathbf{D} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G'} \\ \mathbf{C} & \xrightarrow[U]{} & \mathbf{D} \end{array}$$

Notice that the image of G inside $\mathbf{C}$ is itself a category, call it $G(\mathbf{C})$, having objects like $G(A_1, \ldots, A_n)$. If we now take the class of Modal Logic frames generated by the objects of $G(\mathbf{C})$, because of the commutation of the diagram we can use the information given by the functor G' in order to know how the elements of the Modal Logic frame of $G(A_1, \ldots, A_n)$, say, are built from the elements of the Modal Logic frames of $A_1, \ldots, A_n$.

In absence of a G' with the desired property we can still consider the logic of the class of Modal Logic frames generated by the objects of $G(\mathbf{C})$. In the next chapter we will show that the logic LQP^n is obtained with a similar procedure. In this case, given a Modal Logic frame from an object $G(A_1, \ldots, A_n)$, we lack the explicit construction of the elements of the Modal Logic frame, but in the definition of the satisfaction relation in the Modal Logic frame of $G(A_1, \ldots, A_n)$ we may exploit the satisfaction in the Modal Logic frames of $A_1, \ldots, A_n$.

Chapter 4

LSC logics for FdHil

In the next sections we will see how the ideas of the previous chapter can be applied to the category **FdHil**. The category **FdHil** is locally small but not small. Therefore given a functor $U : \mathbf{FdHil} \to \mathbf{Rel}$ we cannot hope to have a Modal Logic frame corresponding to the whole image of U, as in general the carrier is not a set. We can however examine the class of Modal Logic frames generated by the objects of **FdHil** via U.

In particular, we will study two different candidates for U, the corresponding Modal Logic frames and some relevant logics. The first candidate, called S, will give us the bridge between the category **FdHil** and the semantics of LQP. We expand on this connection, relating the Modal Logic frames given by S with the semantics of LQP^n and the semantics of **OQL**, the standard quantum logic. We then move to another functor, called F. The Modal Logic frames generated by F are richer: they contain both pure and strictly mixed states and also encode the information regarding the probabilities assigned by states to subspaces.

We design a logic to capture these aspects. To suggest that such logic constitutes a strengthening of LQP (of LQP^n, in multi-partite case), we prove that the validities of LQP (respectively, LQP^n) are preserved via translation.

4.1 Logics for H and S

A first possibility for the functor $U : \mathbf{FdHil} \to \mathbf{Rel}$ is the functor S defined as

$$H \mapsto \Sigma_H$$
$$L : H \to V \mapsto S(L) : \Sigma_H \to \Sigma_V$$

where Σ_H is the set of one-dimensional closed linear subspaces of H and the functions $S(L)$ are the partial functions defined as

$$S(L)(\overline{v}) = \overline{L(v)}$$

where the partiality is given by the fact that if $L(v)$ is the zero vector then $\overline{L(v)}$ is not defined. Due to this definition, the empty relation is the image of the projection to the zero subspace. The name S is chosen because the states of the Modal Logic frame are subspaces.

We now prove functoriality. By definition we have $S(L) : \Sigma_H \to \Sigma_V$, so source and target are preserved. For the identity, consider that $S(Id_H)(\overline{v}) = \overline{Id_H(v)} = \overline{v} = Id_{\Sigma_H}(\overline{v})$. For composition, consider $L : H \to V$ and $L' : V \to W$. We can check that, for $v \in H$, $S(L') \circ S(L)(\overline{v}) = S(L')(\overline{Lv}) = \overline{L' \circ Lv} = S(L' \circ L)(\overline{v})$.

We can observe that S is not injective on objects. The Hilbert space H and its dual H^* gets identified when we only consider the set of one-dimensional closed linear subspaces.

Moreover, S is not faithful. As can be readily seen, the linear maps $L(v) = 3v$ and $L'(v) = 5v$ are identified, due to the loss of scalars.

4.1.1 Mismatch of the dagger compact closed structures

We know that **Rel** is a degenerate dagger compact closed category. It is now interesting to ask: is the functor S matching the dagger compact closed structure of **FdHil** with that of **Rel**? The answer is: no.

We start considering the symmetric monoidal part. Since $\mathbb{C}$ is a one dimensional Hilbert space, there is only one one-dimensional subspace, $\mathbb{C}$ itself. Therefore $\Sigma_{\mathbb{C}}$ is a singleton containing only $\mathbb{C}$. The only two morphisms $\Sigma_{\mathbb{C}} \to \Sigma_{\mathbb{C}}$ are thus the identity and the empty relation. Hence the unit of the tensor in **FdHil** becomes the unit of the tensor in **Rel**. The idea of a preparation of a state as a map $L : \mathbb{C} \to H$ also carries over to this new setting.

The tensor in **Rel**, however, does *not* match the Hilbert space tensor, that is, the following diagram does not commute[1]

$$\begin{array}{ccc} \mathbf{FdHil} \times \mathbf{FdHil} & \xrightarrow{S \times S} & \mathbf{Rel} \times \mathbf{Rel} \\ \bigotimes \Big\downarrow & & \Big\downarrow \times \\ \mathbf{FdHil} & \xrightarrow[S]{} & \mathbf{Rel} \end{array}$$

The non-commutation is due to the fact that $\Sigma_{H \bigotimes V} \neq \Sigma_H \times \Sigma_V$ and there is no bijection between the two. Since the symmetric monoidal structure is not matching, the compact closed one cannot match.

As for the dagger structure, we have again a negative result. The diagram

[1] Where the $\times$ labeling the arrow indicates the cartesian product and the $\times$ between categories denotes the categorical product.

$$\begin{array}{ccc} \mathbf{FdHil}^{op} & \xrightarrow{S} & \mathbf{Rel}^{op} \\ \downarrow{\scriptstyle \dagger} & & \downarrow{\scriptstyle \cup} \\ \mathbf{FdHil} & \xrightarrow[S]{} & \mathbf{Rel} \end{array}$$

does not commute. Essentially, this is because the converse of a partial function is not necessarily a function. If you consider a projection P, for example, it is not hard to see that the partial function $S(P) = S(P^{\dagger})$ (recall that $P = P^{\dagger}$) cannot be the same as its converse.

As for the biproduct, again we have that the diagram

$$\begin{array}{ccc} \mathbf{FdHil} \times \mathbf{FdHil} & \xrightarrow{S \times S} & \mathbf{Rel} \times \mathbf{Rel} \\ \downarrow{\scriptstyle \bigoplus} & & \downarrow{\scriptstyle \uplus} \\ \mathbf{FdHil} & \xrightarrow[S]{} & \mathbf{Rel} \end{array}$$

does not commute. It is enough to look at the objects: given H and V, the set $\Sigma_H \uplus \Sigma_V$ is strictly "contained"in $\Sigma_{H \bigoplus V}$ (in the form of pairs $(, w0)$ and $(0, v)$). But $\Sigma_{H \bigoplus V}$ also contains all the pairs (w, v) such that $\langle w|w\rangle + \langle v|v\rangle = 1$, hence there can be no bijection between the two sets.

4.1.2 Semantics

We define a $(\mathbf{H}, S)$-frame, for $\mathbf{H}$ the full subcategory of **FdHil** containing only the Hilbert space H.

Definition 49. An $(\mathbf{H}, S)$-frame is a pair $\langle W, Rel\rangle$ defined as

- $W := \Sigma_H$
- $Rel := \{S(L) | L \in \mathbf{H}_1\}$

Hence in this case the carrier of the Modal Logic frame is the set of all one-dimensional subspaces of H. Alternatively, since such subspaces are in bijection with the unitary vectors that represent the states of a quantum systems, W is the set of all states of H.

The set Rel is the collection of partial maps generated by all linear maps from H to H.

Definition 50. An $(\mathbf{H}, S)$-model is a triple $\langle W, Rel, V\rangle$ such that

- $\langle W, Rel \rangle$ is an $(\mathbf{H}, S)$-frame

- V is a function $V : At \to \wp(W)$

Call Γ_S the class of $(\mathbf{H}, S)$-frames for $\mathbf{H}$ full subcategory of **FdHil** with one object.

4.1.3 Comparison with *LQP*

Notice that the concrete quantum dynamic frame given by a Hilbert space H is a substructure of the corresponding $(\mathbf{H}, S)$-frame: the latter has all the partial functions corresponding to linear maps of type $H \to H$, the former only those corresponding to unitary maps and projectors.

Unitary maps and projectors can be characterized categorically in **FdHil**, as we have seen. So if we interpret the programs in the syntax of *LQP* in the "right "way, that is, we send tests to the partial functions corresponding to projections and basic actions to the partial functions corresponding to unitary transformations, we get the same validities of $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_{CQDF}}$. This happens simply because all the additional relations that are in the $(\mathbf{H}, S)$-frame but not in the concrete quantum dynamic frame are not expressible in the language.

Definition 51. Call $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_S}$ the logic of $(\mathbf{H}, S)$-frames in the language of *LQP*.

Proposition 11. *We have* $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_S} = \Lambda^{\mathcal{F}_{LQP}}_{\Gamma_{CQDF}}$

Proof. Clearly each $(\mathbf{H}, S)$-model can be turned into a model over a concrete quantum dynamic frame by forgetting the partial functions corresponding to all the unnecessary linear maps. This assignment is clearly surjective and it preserve truth, since it leaves the relevant part of the semantics unchanged and the syntax is the same. Thus any validity in $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma'_S}$ must be a validity of $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_{CQDF}}$. The converse is proved by contradiction: if ψ is a validity of $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_{CQDF}}$ but not of $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_S}$ then the falsifying model can be made into a concrete quantum dynamic frame, contradicting our assumption. □

Corollary 2. *The proof system of LQP is sound with respect to the class of Modal Logic frames* Γ_S.

Therefore, given the syntax and the satisfaction relation of *LQP*, the *LSC* logic for the class of all such Modal Logic frames contains the theorems of *LQP*.

4.1.4 Comparison with LQP^n

We will now repeat the procedure employed in the previous section to obtain the semantics of LQP^n. Recall that, from a categorical perspective, a Hilbert space H is 2-dimensional if there is a unitary transformation of type $\mathbb{C} \bigoplus \mathbb{C} \to H$.

Definition 52. Call Γ_S^n the class of $(\mathbf{H}, S)$-frames generated by n-th tensor products of 2-dimensional Hilbert spaces. Call $\Lambda_{\Gamma_S^n}^{\mathcal{F}_{LQPn}}$ the corresponding logic in the language of LQP^n.

Proposition 12. *We have* $\Lambda_{\Gamma_S^n}^{\mathcal{F}_{LQPn}} = \Lambda_{\Gamma_{CQDF^n}}^{\mathcal{F}_{LQPn}}$

Proof. The proof is a straightforward generalization of the single-system case. □

Corollary 3. *The proof system of LQP^n is sound with respect to the class of Modal Logic frames Γ_S^n.*

Hence we have that the LSC logic for the class of all such Modal Logic frames in that syntax contains the theorems of LQP^n.

4.1.5 Comparison with OQL and B°

In this section we show that $(\mathbf{H}, S)$-frames can be turned into the semantics of the standard quantum logic **OQL**, and that the formulas of **OQL** can be translated into the language $\mathcal{F}_\diamond$. These connections allow us to translate all the validities of **OQL** into the validities of the $(\mathbf{H}, S)$-frames.

This section is inspired by two facts: first, it is known that we can give a modal interpretation of **OQL**; second, that the Modal Logic frames in Γ_S can be manifactured into Modal Logic frames for this modal interpretation. For the sale of brevity we skip the passage through the modal interpretation of **OQL** and show directly that $(\mathbf{H}, S)$-frames can be transformed into algebraic realizations for **OQL**.

A presentation of **OQL** and a detailed account of its modal interpretation can be found in [15].

Consider the set of formulas $\mathcal{F}_\diamond$. Define the consequence relation $\vDash_P$ as usual for negation and conjunction. For the diamond operator, set

$$M, s \vDash_P \diamond\psi \text{ iff } \exists S(P) \in \mathit{Rel} \text{ s. t. } M, S(P)(s) \vDash_P \psi$$

for $P : H \to H$ projection.

As customary, we can define an operator $\square := \neg\diamond\neg$, whose semantic definition is

$M, s \vDash_P \diamond\psi$ iff $\forall S(P) \in \mathit{Rel}$ that are defined on s we have $M, S(P)(s) \vDash_P \psi$

We will now introduce the logic **OQL**. Our main reference for this section is [15].

Definition 53. The set of formulas $\mathcal{F}_{\mathbf{OQL}}$ is defined recursively as

$$\psi ::= p \,|\, \neg\psi \,|\, \psi \wedge \phi$$

Definition 54. An *orthomodular lattice* is a structure $\mathcal{B} = \langle L, \sqsubseteq, ', \mathbf{0}, \mathbf{1} \rangle$ where

1. $\langle L, \sqsubseteq, \mathbf{0}, \mathbf{1} \rangle$ is a bounded lattice with $\mathbf{0}$ minimum and $\mathbf{1}$ maximum.
2. the unary operation $'$ is an an orthocomplement:
   - $a'' = a$
   - $a \sqsubseteq b$ entails $b' \sqsubseteq a'$
   - $a \sqcap a' = \mathbf{0}$
3. the law of orthomodularity holds: if $a \sqsubseteq c$ then $a \sqcap (a' \sqcup c) = c$

Call OML the class of orthomodular lattices.

Definition 55. An *algebraic realization* is a pair $\langle \mathcal{B}, v \rangle$ consisting of an ortholattice and a function $v : \mathcal{F}_{\mathbf{OQL}} \to \mathcal{B}$ such that $v(\neg\psi) = v(\psi)'$ and $v(\psi \wedge \phi) = v(\psi) \sqcap v(\phi)$.

Definition 56. A formula $\psi \in \mathcal{F}_{\mathbf{OQL}}$ is satisfied in a realization $\langle \mathcal{B}, v \rangle$, written $\langle \mathcal{B}, v \rangle \vDash_{\mathbf{OQL}} \psi$, iff $v(\psi) = \mathbf{1}$.

A formula $\psi \in \mathcal{F}_{\mathbf{OQL}}$ is *valid* if it is satisfied in any realization. Call $\Lambda^{\mathcal{F}_{\mathbf{OQL}}}_{OML}$ the set of validities over the class of orthomodular lattices.

Definition 57. Define a translation $\tau : \mathcal{F}_{\mathbf{OQL}} \to \mathcal{F}_{\Diamond}$ as follows:

- $\tau(p) = \Box\Diamond p$
- $\tau(\neg\psi) = \neg\Diamond\tau(\psi)$
- $\tau(\psi \wedge \phi) = \tau(\psi) \wedge \tau(\phi)$

Lemma 8. *There is a function λ from Γ_S into OML.*

Proof. Given a $(\mathbf{H}, S)$-frame, as first step forget all the partial relations that are not images of projectors on H. So in the Modal Logic frame now we only have partial functions of kind $S(P)$, for P projector.

As a second step, consider as the elements of the lattice L the codomains of the partial functions in the Modal Logic frame. The partial function corresponding to the projection to the zero subspace is the empty function, so its codomain is the emptyset. So $\emptyset \in L$. The partial function corresponding to the projection into the whole space is the identity function, so its codomain is Σ_H.

Define a partial order $\sqsubseteq$ on L by putting $X \sqsubseteq Y$ if $X \subseteq Y$, for $X, Y \in L$. Clearly it is reflexive, transitive and antisymmetric. The empty set and the whole set are the top and bottom of this order.

Define the infimum of the lattice as intersection. For the supremum, given two objects $cod(S(P_a)), cod(S(P_b))$ coming from two projection P_a, P_b, take the object $cod(S(P))$ where P is the projection into the span of the subspaces a and b.

Similarly, define the orthocomplement operation $'$ by means of the projection to the orthogonal subspace. It is easy to see that this operation satisfies the conditions of orthocomplement and orthomodularity. □

Theorem 13. *We have $\Lambda^{\mathcal{F}_\diamond}_{\Gamma_S} \supseteq \tau(\Lambda^{\mathcal{F}_{\mathbf{OQL}}}_{OML})$, that is to say, all the translations of the validities in $\Lambda^{\mathcal{F}_{\mathbf{OQL}}}_{OML}$ are validities in $\Lambda^{\mathcal{F}_\diamond}_{\Gamma_S}$.*

Proof. Given a $(\mathbf{H}, S)$-model M, extend $V : At \to \wp(\Sigma_H)$ to $V : \mathcal{F}_\diamond \to \wp(\Sigma_H)$ (we use the same letter for the extended function) by imposing $V(\neg\psi) = \Sigma_H \backslash V(\psi)$, $V(\psi \wedge \phi) = V(\psi) \cap V(\phi)$ and $V(\diamond\psi) = \{s \in \Sigma_H | \exists P\, P(s) \in V(\psi)\}$. We can now restate $M, s \vDash_P \psi$ as $s \in V(\psi)$.

Now construct an algebraic realization by taking the orthomodular lattice correponding to the Modal Logic frame of M and putting $v(\psi) = V(\tau(\psi))$ for $\psi \in \mathcal{F}_{\mathbf{OQL}} \subset \mathcal{F}_\diamond$.

We now show that v satisfies the desired properties. Given p, $V(\Box\diamond p)$ is the set of states $\{s | \forall P\, \exists P'\, P'(P(s)) \in V(p)\}$. In other words, is the set of states such that, however they are projected, can be projected back into $V(p)$. It is not hard to see that these are exactly the states contained in the subspace generated by the states in $V(p)$. Thus there is a projection that has as a codomain exactly $V(\Box\diamond p)$. Therefore v is indeed sending formulas into elements of the lattice of $\lambda(M)$.

Consider $v(\neg\psi) = V(\neg\diamond\tau(\psi))$. The latter is the set of states that have no projection into $V(\tau(\psi)) = v(\psi)$. But this is exactly the set of states orthogonal to $V(\tau(\psi)) = v(\psi)$, hence $v(\neg\psi) = v(\psi)'$. For conjunction we have $v(\psi \wedge \phi) = V(\tau(\psi \wedge \phi)) = V(\tau(\psi)) \cap V(\tau(\phi)) = v(\psi) \cap v(\phi)$.

We can now observe that

1. if $M, s \nvDash_P \tau(\psi)$ for some s then $\lambda(M) \nvDash_{\mathbf{OQL}} \psi$

To see this observe that if $s \notin V(\tau(\psi)) = v(\psi)$ then $v(\psi) \neq \Sigma_H = \mathbf{1}$.

Now suppose that ψ is a validity of $\Lambda^{\mathcal{F}_{\mathbf{OQL}}}_{OML}$. If its translation is not a validity of $\Lambda^{\mathcal{F}_\diamond}_{\Gamma_S}$ then there must be a model M and a state s such that $M, s \nvDash_P \tau(\psi)$. But by what we just showed this implies that $\lambda(M) \nvDash_{\mathbf{OQL}} \psi$, which contradicts $\phi \in \Lambda^{\mathcal{F}_{\mathbf{OQL}}}_{OML}$. Therefore $\tau(\psi) \in \Lambda^{\mathcal{F}_\diamond}_{\Gamma_S}$. □

In particular we highlight the following theorem

Theorem 14 (5.2 in [15]). *For any set of formulas $T \subseteq \mathcal{F}_{\mathbf{OQL}}$ and $\alpha \in \mathcal{F}_{\mathbf{OQL}}$*

$$T \vDash_{\mathbf{OQL}} \alpha \text{ iff } \tau(T) \vDash_{\mathbf{B^o}} \tau(\alpha)$$

where τ is the translation we defined and $\mathbf{B^o}$ is an strengthening of $\mathbf{B}$, the logic of the class of reflexive and symmetric frames, with a modal version of orthomodularity: $(\alpha \wedge \neg\beta \to \Diamond(\alpha \wedge \Box\neg(\alpha \wedge \beta)))$.

Corollary 4. *The logic $\Lambda^{\mathcal{F}_\Diamond}_{\Gamma_S}$ contains all the theorems generated by the proof system of $\mathbf{B^o}$.*

4.2 Logics for H and F

The second possibility that we consider is the functor $F : \mathbf{FdHil} \to \mathbf{Rel}$ defined as

$$H \mapsto A_H$$
$$L : H \to V \mapsto F(L) : A_H \to A_V$$

The set A_H is the set of functions $s_\rho : L_H \to [0,1]$, where L_H is the lattice of closed linear subspaces of H, defined as

$$s_\rho(a) = tr(P_a\rho)$$

where P_a is the projector associated to the subspace a and ρ is a density operator on H.

A linear map $L : H \to V$ is sent to the partial function $F(L) : A_H \to A_V$ where

$$F(L)(s_\rho) = s_{\frac{L\rho L^\dagger}{tr(L\rho L^\dagger)}}$$

$F(L)(s_\rho)$ is not defined if $tr(L\rho L^\dagger) = 0$. Recall that the density operators are exactly the positive linear maps of trace 1. The operator $L\rho L^\dagger$ is still positive, and the denominator $tr(L\rho L^\dagger)$ ensures that it is an operator of trace 1. Therefore $\frac{L\rho L^\dagger}{tr(L\rho L^\dagger)}$ is again a density operator, so the function $F(L)$ is well defined. On unitary maps this definition specializes to

$$F(U)(s_\rho) = s_{U\rho U^{-1}}$$

because $tr(U\rho U^{-1}) = 1$, due to the fact that $U\rho U^{-1}$ is already a density operator.

We choose the name F because the states of the Modal Logic frames are now functions. We now prove functoriality. The preservation of source and target follows from the definition. As for the identity, notice that

$F(Id_H)(s_\rho) = s_{Id_H \rho Id_H} = s_\rho = Id_{A_H}(s_\rho)$. For composition consider that

$$\begin{aligned} F(L \circ L')(s_\rho) &= s_{\frac{L \circ L' \rho (L \circ L')^\dagger}{tr(L \circ L' \rho (L \circ L')^\dagger)}} \\ &= s_{\frac{L \circ L' \rho L'^\dagger \circ L^\dagger}{tr(L \circ L' \rho L'^\dagger \circ L^\dagger)}} \\ &= F(L)(s_{\frac{L' \rho L'^\dagger}{tr(L' \rho L'^\dagger)}}) \\ &= F(L) \circ F(L')(s_\rho) \end{aligned}$$

We now come to the discussion of the properties of F. Again we have that $F(H) = F(H^*)$. To see this, consider that the lattices L_H and L_{H^*} are the same. And even if the inner product in H^* is the complex conjugate of the inner product in H, since each function in A_H has codomain $[0,1]$, the same density operator s_ρ will give the same values in H and H^*. Therefore F is not injective on objects and we cannot hope to have a non-degenerate compact close structure matching the quantum one, as H and H^* get identified.

To see that it is not faithful consider the linear maps $L(v) = 3v$ and $L'(v) = 5v$. We can see that $F(L)(s_\rho) = s_{\frac{3\rho 3}{tr(3\rho 3)}} = s_{\frac{9\rho}{9tr(\rho)}} = s_\rho = s_{\frac{25\rho}{25tr(\rho)}} = F(L')(s_\rho)$.

For the preservation of the categorical structure, we have the same negative result that we had for S. We note that the one dimensional Hilbert space $\mathbb{C}$ is sent to the singleton containing the unique function $l : \{\{0\}, \mathbb{C}\} \to [0,1]$ such that $l(\{0\}) = 0$ and $l(\mathbb{C}) = 1$.

4.2.1 Semantics

Definition 58. A $(\mathbf{H}, F)$-frame is a pair $\langle W, Rel \rangle$ defined as

- The set W is defined as
$$W := A_H$$
- The set Rel of relations on W is defined as
$$Rel := \{F(L) | L : H \to H\}$$

Hence in this case the carrier of the Modal Logic frame is the set of functions $s_\rho : L_H \to [0,1]$ defined above. Such functions are associated to density operators on H, which represent *both pure and strictly mixed states* of the quantum system H.

The set Rel is the collection of maps generated by all the linear maps of type $H \to H$.

Definition 59. Call Γ_F the class of $(\mathbf{H}, F)$-frames for $\mathbf{H}$ full subcategory of **FdHil** with one object.

Definition 60. For $a \in L_H$, define $\mathcal{L}_1(a) := \{s_\rho \in W | s_\rho(a) = 1\}$.

Proposition 13. *For any $a, b \in L_H$, $\mathcal{L}_1(a) = \mathcal{L}_1(b)$ entails $a = b$.*

Proof. If $\mathcal{L}_1(a) = \mathcal{L}_1(b)$ then subspaces a and b contain exactly the same unitary vectors, thus $a = b$. □

Definition 61. Call *pure state* the elements of A_H that belongs to $\mathcal{L}_1(a)$ for some atom a of L_H.[2] For each atom $a \in L_H$ there is only one element of A_H in $\mathcal{L}_1(a)$, call it s^a, namely the density operator corresponding to the unitary vector lying on the one-dimensional subspace a. Call $Pure(A_H)$ the set of all pure states of A_H.

So for example if $a = \overline{|\alpha\rangle}$ and $|\alpha\rangle$ is a unitary vector then $s^a = s_{|\alpha\rangle\langle\alpha|}$.

Definition 62. An $(\mathbf{H}, F)$-model is a triple $\langle W, Rel, V\rangle$ such that

- $\langle W, Rel\rangle$ is an $(\mathbf{H}, F)$-frame
- V is a function $V : At \to \wp(Pure(A_H)) \subseteq \wp(W)$

4.2.2 A logic for Γ_F

It is now natural to ask: can we devise a logic to express the features of these Modal Logic frames? Consider the following syntax.

Given a set of atomic propositions At, build $\mathcal{F}_{\mathbf{OQL}}$ as before by recursion[3]

$$\alpha ::= p \mid \sim \alpha \mid \alpha \curlywedge \beta$$

Then put $At' := \mathcal{F}_{\mathbf{OQL}} \times [0, 1]$. We will indicate the pairs in At' as α^r. Now build the syntax as for LQP, by mutual recursion, but using At' as set of atomic propositions

$$\psi ::= \alpha^r \mid Pure \mid \neg\psi \mid \psi \wedge \phi \mid [\pi]\psi$$

The set of programs Act is defined as

$$\pi ::= U \mid \pi^\dagger \mid \pi \cup \pi' \mid \pi; \pi' \mid \psi?$$

where U belongs to a set of atomic actions $AtAct$. Call this set of formulas $\mathcal{F}_{Prob}$.

Given a $(\mathbf{H}, F)$-model we define a satisfaction relation for formulas in $\mathcal{F}_{Prob}$ as follows. In order to do so we first define a function $v : \mathcal{F}_{\mathbf{OQL}} \to L_H$ by putting

- $v(p) = \sqcup\{a | s^a \in V(p)\}$

[2] This corresponds to the usual criterion to distinguish pure states: $tr(\rho^2) = 1$.

[3] We use different symbols for the connectives not to get confused with the next step.

- $v(\sim \alpha) = v(\alpha)'$
- $v(\alpha \curlywedge \beta) = v(\alpha) \cap v(\beta)$

where $\sqcup$ is the supremum of the lattice L_H. As for LQP, we define the function i associating a relation $R \subseteq A_H \times A_H$ to every program:

- $i(U) \in \{F(U)|U : H \to H\} \subseteq Rel$
- $i(\pi^\dagger) = i(\pi)^\dagger$
- $i(\pi \cup \pi') = i(\pi) \cup i(\pi')$
- $i(\pi; \pi') = i(\pi) \circ i(\pi')$
- $i(\psi?) = F(P_{a'})$, where $a' = \sqcup\{a\text{Ł}_H|s^a \models_{Prob} \psi\}$

by mutual recursion the satisfaction relation:

- $M, s_\rho \models_{Prob} \alpha^r$ iff $s_\rho(v(\alpha)) = r$
- $M, s_\rho \models_{Prob} Pure$ iff $s_\rho \in Pure(A_H)$
- $M, s_\rho \models_{Prob} \neg\psi$ iff $M, s_\rho \not\models_{Prob} \psi$
- $M, s_\rho \models_{Prob} \psi \wedge \phi$ iff $M, s_\rho \models_{Prob} \psi$ and $M, s_\rho \models_{Prob} \phi$
- $M, s_\rho \models_{Prob} [\pi]\psi$ iff for all $(s_\rho, s_{\rho'}) \in i(\pi)$ we have $M, s_{\rho'} \models_{Prob} \psi$

Definition 63. Call $\Lambda_{\Gamma_F}^{\mathcal{F}_{Prob}}$ the logic of $(\mathbf{H}, F)$-models in the language of $\mathcal{F}_{Prob}$.

We now show that the Modal Logic frames given by F can be turned into the corresponding Modal Logic frames given by S. First we observe that

Proposition 14. *There is a natural transformation $\delta : S \to F$ defined componentwise as*

$$\delta_H = \{(a, s^a)|a \in \Sigma_H, s^a \in A_H\}$$

recall that, being a morphism in **Rel**, *δ_H is a relation of type $\Sigma_H \to A_H$. Thus the relation δ_H associates every one-dimensional linear subspace to the (function associated to) corresponding density operator.*

Proof. To prove naturality we need to show the commutativity of the following diagram for a generic linear map $L : H \to V$

$$\begin{array}{ccc} \Sigma_H & \xrightarrow{S(L)} & \Sigma_V \\ {\scriptstyle \delta_H}\downarrow & & \downarrow{\scriptstyle \delta_V} \\ A_H & \xrightarrow{F(L)} & A_V \end{array}$$

So we have to show that the composite relations $\delta_V \circ S(L)$ and $F(L) \circ \delta_H$ are equal.

Suppose the pair (a, s_ρ) is in $\delta_V \circ S(L)$. By the definition of composition on relations, this means that there is $a' \in \Sigma_V$ such that $(a, a') \in S(L)$ and $(a', s_\rho) \in \delta_V$. Call $|\alpha\rangle$ and $|\beta\rangle$ the unitary vectors belonging to a and a' respectively. By $(a, a') \in S(L)$ we can infer that $L(|\alpha\rangle) = c\,|\beta\rangle$ for some complex number c. By $(a', s_\rho) \in \delta_V$ we can infer that $s_\rho = s^{a'} = s_{|\beta\rangle\langle\beta|}$.

Now take $s^a = s_{|\alpha\rangle\langle\alpha|}$. We know that

$$F(L)(s_{|\alpha\rangle\langle\alpha|}) = s_{\frac{L|\alpha\rangle\langle\alpha|L^\dagger}{tr(L|\alpha\rangle\langle\alpha|L^\dagger)}} = s_{\frac{c|\beta\rangle\langle\beta|c^*}{cc^*}} = s_{|\beta\rangle\langle\beta|}$$

Therefore $(a, s^a) \in \delta_H$ and $(s^a, s^{a'}) \in F(L)$, so $(a, s_\rho) \in F(L) \circ \delta_H$.

For the other inclusion, suppose $(a, s_\rho) \in F(L) \circ \delta_H$. Then there is s^a such that $(a, s^a) \in \delta_H$ and $(s^a, s_\rho) \in F(L)$. Call $|\alpha\rangle$ the unitary vector belonging to a, so $s^a = s_{|\alpha\rangle\langle\alpha|}$. Say $L(|\alpha\rangle) = c\,|\beta\rangle$, for some complex number c and unitary vector $|\beta\rangle$. By $(s^a, s_\rho) \in F(L)$ we know that $L(|\alpha\rangle)$ is not the zero vector, and we get

$$F(L)(s_{|\alpha\rangle\langle\alpha|}) = s_{\frac{L|\alpha\rangle\langle\alpha|L^\dagger}{tr(L|\alpha\rangle\langle\alpha|L^\dagger)}} = s_{|\beta\rangle\langle\beta|}$$

Therefore $s_\rho = s_{|\beta\rangle\langle\beta|}$. Now taking $a' = \overline{|\beta\rangle}$ we can conclude that $(a', s_\rho) \in \delta_V$ and, since

$$S(L)(a) = S(L)(\overline{|\alpha\rangle}) = \overline{L(|\alpha\rangle)} = \overline{c\,|\beta\rangle} = a'$$

also $(a, a') \in S(L)$. Therefore $(a, s_\rho) \in \delta_V \circ S(L)$. □

Lemma 9. *There is a function η from the class $\mathcal{M}(\Gamma_F)$ to the class $\mathcal{M}(\Gamma_S)$.*

Proof. Given a $(\mathbf{H}, F)$-model, take the set of all atoms of L_H as the set W.

From any partial function $F(L) : A_H \to A_H$ take the partial function $f : \Sigma_H \to \Sigma_H$ given by $f_{F(L)}(a) = a'$, where a' is the atom corresponding to $F(L)(s^a)$. More formally, define

$$f_{F(L)} = \delta_H^{\cup} \circ F(L) \circ \delta_H$$

where δ_H is the componenet of the natural transformation from S to F and $\delta_H^{\cup}$ is its converse.

We now prove that $f_{F(L)} = S(L)$. From Proposition 14 we have $\circ F(L) \circ \delta_H = \delta_H \circ S(L)$. From this we can infer $\delta_H^{\cup} \circ F(L) \circ \delta_H = \delta_H^{\cup} \circ \delta_H \circ S(L)$. It is easy to see that $\delta_H^{\cup} \circ \delta_H$ is the identity relation on Σ_H, thus from the identity axiom we obtain $\delta_H^{\cup} \circ F(L) \circ \delta_H = S(L)$, as desired.

Therefore with this procedure we obtain the $(\mathbf{H}, S)$-frame of H. Define the valuation $V' : At \to \wp(\Sigma)$ as $V'(p) = \{a \in \Sigma_H | M, s^a \vDash_{Prob} p^1\}$. □

Definition 64. Define a translation $\tau_1 : \mathcal{F}_{LQP} \to \mathcal{F}_{Prob}$ as follows:

- $\tau_1(p) = p^1$
- $\tau_1(\neg\psi) = Pure \to \neg\tau_1(\psi)$
- $\tau_1(\psi \wedge \phi) = Pure \to (\tau_1(\psi) \wedge \tau_1(\phi))$
- $\tau_1([\pi]\psi) = Pure \to [\tau_1(\pi)]\tau_1(\psi)$

where abusing the notation we write $\tau_1(\pi)$ to mean $\tau_1(\pi) = \pi$ if π is not a quantum test and $\tau_1(\psi?) = \tau_1(\psi)?$ if $\pi = \psi?$.

Proposition 15. *If $M, s^a \vDash \tau_1(\psi)$ for all pure states s^a then $M, s \vDash \tau_1(\psi)$ for all s.*

Proof. The proof is by induction on the complexity of ψ. For $\psi = p$, suppose $M, s^a \vDash_{Prob} p^1$ for all pure states s^a. This means that p is interpreted on the whole space H, so all mixed states must assign probability 1 to p.

For all the other cases, it is sufficient to observe that the constant $Pure$ is true at all pure state and false everywhere else. So if, for example, $Pure \to \neg\tau_1(\phi)$ is true at all pure states then it must be true at all states, due to the truth condition of classical implication. □

Theorem 15. *We have $\Lambda^{\mathcal{F}_{Prob}}_{\Gamma_F} \supseteq \tau_1(\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_S})$.*

Proof. As a first step we observe that due to the translation the extended definition of V' in the model $\eta(M)$ is such that

1. $V'(\psi) = \{a \in \Sigma_H | M, s^a \vDash_{Prob} \tau_1(\psi)\}$

We prove this by induction on ψ. We have the base case by definition of V' as $V'(p) = \{a \in \Sigma_H | M, s^a \vDash_{Prob} p^1\}$. For $\psi = \neg\phi$ we have $V'(\neg\phi) = \Sigma_H \backslash V'(\phi) = \{a \in \Sigma_H | M, s^a \nvDash_{Prob} \tau_1(\phi)\} = \{a | M, s^a \vDash_{Prob} \neg\tau_1(\phi)\} = \{a | M, s^a \vDash_{Prob} Pure \to \neg\tau_1(\phi)\}$.

For $\psi = \phi_1 \wedge \phi_2$ we have

$$\begin{aligned} V'(\phi_1 \wedge \phi_2) &= V'(\phi_1) \cap V'(\phi_2) \\ &= \{a | M, s^a \vDash_{Prob} \tau_1(\phi_1)\} \cap \{a | M, s^a \vDash_{Prob} \tau_1(\phi_2)\} \\ &= \{a | M, s^a \vDash_{Prob} \tau_1(\phi_1) \wedge \tau_1(\phi_2)\} \\ &= \{a | M, s^a \vDash_{Prob} Pure \to (\tau_1(\phi_1) \wedge \tau_1(\phi_2))\} \\ &= \{a | M, s^a \vDash_{Prob} \tau_1(\phi_1 \wedge \phi_2)\} \end{aligned}$$

Finally, consider the case $\psi = [\pi]\phi$. We know that the satisfaction of this kind of formulas depends on the interpretation of the actions in the model, in this case $\eta(M)$. We claim that the interpretation of the actions in $\eta(M)$, call it i' is such that

2. $i'(\pi) = f_{i(\tau_1(\pi))}$

where i is the interpretation of the actions in M. We prove this by induction on the complexity of π. Note that we are still in the inductive step of the induction proof of 1, so this constitutes a nested induction. Thus we have available the induction hypothesis of 1, call it IH_1, and the inductionhypothesis of 2, call it IH_2.

The base case $\pi = U$ is given by the definition of $f_{F(L)}$ and the fact that $\tau_1(U) = U$. We have $i'(\pi) = S(U) = f_{F(U)} = f_{i(U)} = f_{i(\tau_1(U))}$. The cases of union, composition and dagger follow from IH_2 and the definition of $f_{F(L)}$. Now consider the case of test, $\pi = \phi'?$. By definition, $i'(\phi'?)$ is the projector into the subspace $\sqcup\{a \in \Sigma_H | a \vDash_{LQP} \phi'\}$. By definition, $i(\tau_1(\phi'?)) = i(\tau_1(\phi')?)$ is the projector into the subspace $\sqcup\{a \in \Sigma_H | s^a \vDash_{Prob} \tau_1(\phi')\}$. But since the complexity of ϕ' is inferior to the complexity of $\psi = [\phi'?]\phi$, we can apply IH_1 and get that $\{a \in \Sigma_H | a \vDash_{LQP} \phi'\} = V'(\phi') = \{a \in \Sigma_H | s^a \vDash_{Prob} \tau_1(\phi')\}$. Since the two sets coincide, their span will be the same, and thus the projector will be the same, say P, hence $i'(\phi'?) = S(P) = f_{F(P)} = f_{i(\tau_1(\phi')?)}$. This concludes the nested induction.

Now that we have established $i'(\pi) = f_{i(\tau_1(\pi))}$, we can compute, using the induction hypothesis, the definition of the partial functions in $\eta(M)$ and the semantics,

$$\begin{aligned}
V'([\pi]\phi) &= \{a \in \Sigma_H | \forall (a, a') \in i'(\pi)\, a' \in V'(\phi)\} \\
&= \{a | \forall (a, a') \in f_{i(\tau_1(\pi))}\, a' \in V'(\phi)\} \\
&= \{a | \forall (a, a') \in f_{i(\tau_1(\pi))}\, M, s^{a'} \vDash_{Prob} \tau_1(\phi)\} \\
&= \{a | \forall (s^a, s^{a'}) \in i(\tau_1(\pi))\, M, s^{a'} \vDash_{Prob} \tau_1(\phi)\} \\
&= \{a | M, (s^a) \vDash_{Prob} [\tau_1(\pi)]\tau_1(\phi)\} \\
&= \{a | M, (s^a) \vDash_{Prob} Pure \to [\tau_1(\pi)]\tau_1(\phi)\} \\
&= \{a | M, (s^a) \vDash_{Prob} \tau_1([\pi]\phi)\}
\end{aligned}$$

This concludes the proof of 1.

As a second step we prove that

$$\eta(M), t \vDash_{LQP} \psi \text{ for all } t \text{ entails } M, s \vDash_{Prob} \tau_1(\psi) \text{ for all } s$$

This follow from what we just proved: if $V'(\psi) = \Sigma_H$ then $M, s^a \vDash_{Prob} \tau_1(\psi)$ for all pure states s^a, and this in turn entails, because of Proposition 15, that $M, s \vDash_{Prob} \tau_1(\psi)$ for all s.

Now suppose ψ is a validity of $\Lambda^{\mathcal{F}_{LQP}}_{\Gamma'_S}$ but its translation is not a validity in $\Lambda^{\mathcal{F}_{Prob}}_{\Gamma_F}$. Then there is a model M and a state s such that $M, s \nvDash_{Prob} \tau_1(\psi)$. But by the converse of the statement that we just proved this must entail that there is a state t in $\lambda(M)$ such that $\eta(M), t \nvDash_{LQP} \psi$. However this contradicts our first assumption, hence $\Lambda^{\mathcal{F}_{Prob}}_{\Gamma_F} \supseteq \tau_1(\Lambda^{\mathcal{F}_{LQP}}_{\Gamma'_S})$. $\square$

4.2.3 A logic for Γ^n_F

Now select the Modal Logic frames generated by the tensors of n copies of 2-dimensional Hilbert spaces and extend the language of $\mathcal{F}_{Prob}$ with the constants in $\mathcal{F}_{LQP^n}$. Following the procedure of the previous section we can encode the validities of $\Lambda^{\mathcal{F}_{LQPn}}_{\Gamma^n_S}$ into the logic of such Modal Logic frames.

Definition 65. Call Γ^n_F the class of $(\mathbf{H}, F)$-frames generated by nth tensors of 2-dimensional Hilbert spaces.

Given $H = H_1 \bigotimes \cdots \bigotimes H_n$, as in Chapter 2 take $N = \{1, 2, \ldots, n\}$.

We start defining the set of formulas $\mathcal{F}_{Prob^n}$. Given a set of atomic propositions At, we build as before

$$\alpha ::= p \,|\, \sim \alpha \,|\, \alpha \curlywedge \beta$$

Now put $At' := \mathcal{F}_{\mathbf{OQL}} \times [0, 1]$. Then we define by recursion

$$\psi ::= \top_I \,|\, \alpha^r \,|\, Pure \,|\, 1 \,|\, + \,|\, \neg\psi \,|\, \psi \wedge \phi \,|\, [\pi]\psi$$

Essentially the new symbols are the constants $\top_I$, 1 and $+$. The set of programs Act is defined as

$$\pi ::= triv_I \,|\, U \,|\, \pi^\dagger \,|\, \pi \cup \pi' \,|\, \pi; \pi' \,|\, \psi?$$

Call this set of formulas $\mathcal{F}_{Prob^n}$.

Definition 66. Define $Loc_I = \{F(L) | L = L'_I \bigotimes Id_{N\setminus I}\}$. It is the set of the images of all I-local linear maps.

The interpretation of the new action symbols is the following

- $i(triv_I) = \bigcup\{R \subseteq A_H \times A_H | R \in Loc_I\}$

Definition 67. A density operator ρ is *I-separated* if $\rho = \rho_I \bigotimes \rho_{N\setminus I}$, where ρ_I is a pure state of A_{H_I} and $\rho_{N\setminus I}$ is a pure state of $A_{H_{N\setminus I}}$. Call Sep_I the set of I-separable states of A_{H_N}.

Proposition 16. *Every separated state is pure: for all I, $Sep_I \subseteq Pure(A_H)$.*

Proof. If $\rho = \rho_I \bigotimes \rho_{N\setminus I}$ then it is the tensor of two pure states. By the fact that ρ_I and $\rho_{N\setminus I}$ are pure there must be atoms a_I and $a_{N\setminus I}$ in the respective lattices such that $\rho_I(a_I) = 1$ and $\rho_{N\setminus I}(a_{N\setminus I}) = 1$. This means that there is an atom $a_I \bigotimes a_{N\setminus I}$ in L_H and $s_\rho(a_I \bigotimes a_{N\setminus I}) = 1$. Thus $s_\rho \in Pure(A_H)$. □

The satisfaction relation $\vDash_{Prob^n}$ is defined as $\vDash_{Prob}$ on the common language and on the new symbols is:

- $M, s_\rho \vDash_{Prob^n} 1$ iff $\rho = |1_1 \ldots 1_n\rangle \langle 1_1 \ldots 1_n|$

- $M, s_\rho \vDash_{Prob^n} +$ iff $\rho = |+_1 \cdots +_n\rangle \langle +_1 \cdots +_n|$
- $M, s_\rho \vDash_{Prob^n} \top_I$ iff $\rho \in Sep_I$

Proposition 17. *$M, s_\rho \vDash_{Prob^n} \top_N$ iff $M, s_\rho \vDash_{Prob^n} Pure$*

Proof. The left to right direction is given by Proposition 16. Now suppose the state $s_\rho \in A_H$ is pure. Since there is no subspace indexed by the empty set, a state is N-separated if can be written as a pure state of $A_{H_N} = A_H$. □

Hence we could eliminate from the language the constant $Pure$, as it is now redundant. We keep it because we want to re-use the translation that we defined for the single-system fragment of the language.

Definition 68. Call $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}}$ the logic of the class Γ_F^n in the language of $\mathcal{F}_{Prob^n}$.

Lemma 10. *There is a function from the class of models $\mathcal{M}(\Gamma_F^n)$ to the class of models $\mathcal{M}(\Gamma_S^n)$.*

Proof. The function is η given from Lemma 1. Notice that the definition of the actions in $\eta(M)$, when used for the extended language, works as desired: the function $f_{F(L)}$ is I-local iff $F(L)$ is, so $triv_I$ is interpreted in the right way in $\eta(M)$. □

Definition 69. Define a translation $\tau_2 : \mathcal{F}_{LQP^n} \to \mathcal{F}_{Prob^n}$ as τ_1 for the language of LQP. For the new symbols, set:

- $\tau_2(1) = 1$
- $\tau_2(+) = +$
- $\tau_2(\top_I) = \top_N \to \top_I$

and continuing the abuse of notation of the previous section we also put $\tau_2(triv_I) = triv_I$.

Proposition 18. *If $M, s^a \vDash_{Prob^n} \tau_2(\psi)$ for all pure states s^a then $M, s \vDash_{Prob^n} \tau_2(\psi)$ for all s.*

Proof. Again we prove it by induction on ψ. We refer to Proposition 15 for the cases in the language of LQP. Consider now the cases of $\psi = 1$, $\psi = +$ or $\psi = \top_I$.

Clearly for $\psi = 1$ and $\psi = +$ we have pure states in which the formula cannot be true, those that do not correspond to $|1_1 \dots 1_n\rangle \langle 1_1 \dots 1_n|$ or $|+_1 \cdots +_n\rangle \langle +_1 \cdots +_n|$. So the antecedent of the proposition cannot be true for these two cases.

As for $\psi = \top_I$, we know $\top_N$ is true for all pure states. For $I \subset N$, however, there are always pure states that are not separated, namely the entangled states. Therefore the antecedent of the proposition cannot hold, because there are pure states that falsify $\top_I$, hence the proposition is vacuously true. Lastly, suppose $\psi = \top_N$. Then $\top_N \to \top_N$ is a tautology, by the semantic of classical implication, and thus it is true at all states in A_H. □

Theorem 16. *We have* $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}} \supseteq \tau_2(\Lambda_{\Gamma_S^n}^{\mathcal{F}_{LQP^n}})$.

Proof. The proof is similar to that of Theorem 15. We check that

$$V'(\psi) = \{a \in \Sigma_H | M, s^a \vDash \tau_2(\psi)\}$$

by induction on ψ (notice however that now V' correspond to the satisfaction relation $\vDash_{LQP^n}$). We only consider the new cases. If $\psi = 1$ then $V'(1) = \{a \in \Sigma_H | M, s^a \vDash_{Prob^n} 1\}$, the one-dimensional linear subspace of the vector $|1_1 \ldots 1_n\rangle$. Likewise, if $\psi = +$ then $V'(+) = \{a \in \Sigma_H | M, s^a \vDash_{Prob^n} +\}$.

Now suppose $\psi = \top_I$. Then $V'(\top_I)$ is the set of atoms of L_H that can be written as tensor product of an atom of L_{H_I} and an atom of $L_{H_{N \setminus I}}$. To each of these pairs of atoms is associated an atom in L_H, the tensor of them, and a pair of pure states, one for each subsystem. The tensor of the pair of pure states is a I-separated state of A_H. So $V'(\top_I) = \{a \in \Sigma_H | M, s^a \vDash_{Prob^n} \top_I\}$. Since any separable state is pure, we have $V'(\top_I) = \{a \in \Sigma_H | M, s^a \vDash_{Prob^n} \top_N \to \top_I\}$. This concludes the induction.

Thus we have that

$$\eta(M), t \vDash_{LQP^n} \psi \text{ for all } t \text{ entails } M, s \vDash_{Prob^n} \tau_2(\psi) \text{ for all } s$$

Again this follow from the fact that $V'(\psi) = \Sigma_H$ entails that $M, s^a \vDash_{Prob^n} \tau_2(\psi)$ for all pure states s^a, and this in turn entails, because of Proposition 18, that $M, s \vDash_{Prob^n} \tau_2(\psi)$ for all s.

With the usual argument by contradiction we can thus conclude that if a formula is a validity in $\Lambda_{\Gamma_S^n}^{\mathcal{F}_{LQP^n}}$ then its translation is a validity in $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}}$. □

Next we show that the validities of the class of Modal Logic frames Γ_F in the language $\mathcal{F}_{Prob}$ still hold when we extend the language and concentrate on the Modal Logic frames in Γ_F^n.

Theorem 17. $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}} \supseteq \Lambda_{\Gamma_F}^{\mathcal{F}_{Prob}}$

Proof. We know that Γ_F^n is a subclass of Γ_F, that $\mathcal{F}_{Prob} \subset \mathcal{F}_{Prob^n}$ and the satisfaction relation on the shared language is the same. Thus each validity in $\Lambda_{\Gamma_F}^{\mathcal{F}_{Prob}}$, when regarded as a formula in $\mathcal{F}_{Prob^n}$, will be true in all Modal Logic frames in Γ_F^n. □

4.3 The big picture

Calling Φ_{LQP} and Φ_{LQP^n} the sets of theorems generated by the proof systems of LQP and LQP^n, respectively, the results of this chapter can be summarized with the diagram

$$\begin{array}{ccc}
\Lambda^{\mathcal{F}_{Prob}}_{\Gamma_F} & \xrightarrow{\subseteq} & \Lambda^{\mathcal{F}_{Prob^n}}_{\Gamma^n_F} \\
\tau_1 \uparrow & & \uparrow \tau_2 \\
\Lambda^{\mathcal{F}_{LQP}}_{\Gamma_S} & \xrightarrow[\subseteq]{} & \Lambda^{\mathcal{F}_{LQP^n}}_{\Gamma^n_S} \\
\subseteq \uparrow & & \subseteq \uparrow \\
\Phi_{LQP} & \xrightarrow{\subseteq} & \Phi_{LQP^n}
\end{array}$$

The commutation of the upper square is given by the fact that τ_2 is defined as τ_1 on the formulas of $\mathcal{F}_{LQP}$. The inclusion $\Phi_{LQP} \subseteq \Phi_{LQP^n}$ is given by the fact that the proof system of LQP^n contains the proof system of LQP, so everything that can be proven in the latter can also be proven in the former.

We can thus see that the logic $\Lambda^{\mathcal{F}_{Prob^n}}_{\Gamma^n_F}$ satisifies the minimal requirement for a stronger logic able to express and reason about quantum algorithms: it preserves all results and correctness proofs of previous logics.

Chapter 5

The image of F

In the previous chapters we have seen that, among the ones we considered, the logic with most potential is that arising from the class of Modal Logic frames Γ_F^n. This motivates the study of the image of F. Along the lines of Chapter 3, we are interested in answering the following two questions:

1. can we characterize the image of F independently from **FdHil**?

2. can we define on this image a counterpart of the categorical structure of **FdHil**?

With respect to the second question, the categorical structure that we are especially interested in is the symmetric monoidal one, capturing the way in which compound systems are obtained from individual systems.

We dedicate one section to each problem, drawing the connections with the existing results in the area. To conclude we explore the link between the image of F and Correlation Models, a class of structure that have been used to model compound systems.

5.1 Characterization of the image of F

There is an important theorem characterizing the functions that we are studying.

Definition 70. Given a Hilbert space H, *measure* on its lattice L_H is a function $\mu : L_H \to [0, 1]$ such that

- $\mu(H) = 1$

- $\mu(\{0\}) = 0$

- $\mu(\bigsqcup\{a_i\}) = \sum_i \mu(a_i)$

where H is the top of the lattice, $\{0\}$ is the subspace containinig only the zero vector, $\bigsqcup$ is the supremum of the lattice and $\{a_i\}$ is a countable family of orthogonal elements of the lattice.

Theorem 18 (Gleason, [17]). *Given a Hilbert space H of dimension greater than 2, a function $\mu : L_H \to [0, 1]$ is a measure iff $\mu = s_\rho$ for some $s_\rho \in A_H$.*

This Theorem constitutes an advancement to the solution of the first problem. Let us first introduce some terminology.

Definition 71. Call *classical Hilbert lattices* the lattices L_H arising from a Hilbert spaces H over real numbers, complex numbers or quaternions.

If we were able to find an abstract characterization of classical Hilbert lattices, we could use it in combination with Gleason's Theorem to get a description of the image of F: we would take the sets of measures over such lattices. Unfortunately this has been achieved only in the case of infinite-dimensional Hilbert spaces, thanks to the following theorem.

Theorem 19 (Theorem 4.1 in [18] p.16). *Let L be an irreducible, complete, orthomodular, atomistic lattice with the covering property. If L has an orthogonal sequence of atoms $\{a_i : i = 1, 2, \dots\}$ together with another corresponding sequence of atoms $b_i \leq a_i \sqcup a_{i+1}$, with $i = 1, 2, \dots,$ such that the harmonic conjugate of b_i with respect to the pair of atoms a_i, a_{i+1} equals $b_i^\perp \sqcap (a_i \sqcup a_{i+1})$, then L is orthoisomorphic to the lattice of all closed subspaces of a real, complex or quaternionic Hilbert Space.*

We will not discuss the statement or the theorem or its proof, but we highlight two points: first, the theorem gives a characterization of classical Hilbert lattices in lattice-theoretic terms; second, the theorem is applicable only in the infinite-dimensional case due to the requirement of the existence of an infinite sequence of orthogonal atoms. This theorem is a consequence of another key result in the area:

Theorem 20 (Solér, [25]). *Let $\langle E, \langle \cdot | \cdot \rangle \rangle$ be an infinite dimensional orthomodular space over a skew field K which contains an infinite orthonormal sequence. Then K is either $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $\langle E, \langle \cdot | \cdot \rangle \rangle$ is a Hilbert space over K.*

As for the finite-dimensional case, the best we can do is a characterization of lattices arising from *generalized* Hilbert spaces, thanks to Piron's Theorem (see [22]). To the best of my knowledge, there is no result yet concerning the conditions required to characterize classical Hilbert lattices in the finite-dimensional case.

5.2 Dealing with the tensor

Due to Gleason's Theorem, the problem of the characterization of the tensor can be partially reduced to the caracterization of the tensor between Hilbert spaces. If we have an operation on lattices that given two Hilbert lattices L_H and L_V outputs the lattice $L_{H \bigotimes V}$ of the compound Hilbert space $H \bigotimes V$, taking all the measures on $L_{H \bigotimes V}$ we will obtain $A_{H \bigotimes V}$, the set of all functions induced by density operators on $H \bigotimes V$.

Unfortunately there are two problems with this strategy. The first is that Gleason's Theorem imposes a limitation on the dimension: the compound system must be of dimension greater than 2. Thus this procedure can work only if either of the two spaces has dimension greater than 2 or if both have dimension 2 (recall that the dimension of the compound system is the product of the dimensions of the components).

The second problem is that a good lattice-theoretic candidate for the tensor is not available. Indeed, negative results in this direction, see for example [23], seems to indicate that the enterprise cannot be successful.

The difficulty to find a sensible way to merge two system in a compound one has suggested that perhaps the modelling of quantum systems should start from the *assumption* of an operation to merge systems. Indeed, the study of quantum systems as symmetric monoidal categories is a representative of this perspective on the problem.

Another possible way to approach the issue is coalgebraically. More precisely, we may assume the dual of the operation construct-compound-system, namely the operation deconstruct-into-components, and try to characterize the former in terms of the latter. In the next section we explore this possibility, examining the connection of our framework with a class of structures with an inbuilt deconstruct-into-components operation.

5.3 Correlation models

In [12] Baltag and Smets defined a general class of structures meant to represent the way in which the state of a system associates results to observations or measurements. The interesting feature of these structures, called Correlation Models, is that they have an inbuilt mechanism to extract from a given state the states of the subsystems.

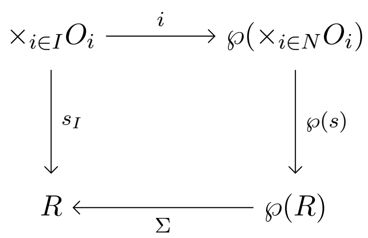
Definition 72. A *result structure* is a pair $\langle R, \sum \rangle$ where R is a set, to be undrstood as the set of possible results, and $\sum : \wp(R) \to R$ is a function, representing the compositions of results, such that

$$\sum\{\sum(A_k) | k \in K\} = \sum(\bigcup_{k \in K} A_k)$$

for $\{A_k\}_{k \in K}$ family of pairwise disjoint subsets of R.

Definition 73. Given a result structure $\langle R, \sum \rangle$ and a tuple $(O_i)_{i \in N}$ of sets of possible observations, a *correlation model* over $(R, \sum, (O_i)_{i \in N})$ is a set $A \subseteq \{s | s : \times_{i \in N} O_i \to R\}$ of functions assigning results to tuples of observations. Such functions are meant to represent states of a comopund system.

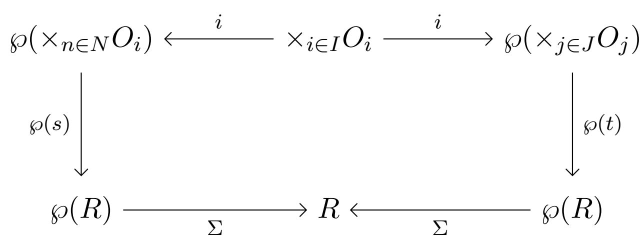
Given a correlation model $A \subseteq \{s | s : \times_{i \in N} O_i \to R\}$, we define the set of states of the subsystem indexed by $I \subseteq N$ as the set of functions s_I given by the commuting diagram



(where $\wp$ is the covariant powerset endofuntor) which is tantamount to the original definition in [12]:

$$s_I((e_i)_{i \in I}) := \sum \{s(o) | o \in \times_{i \in N} O_i, o_i = e_i \forall i \in I\}$$

this diagrammatic representation can be exploited in the definition on a relation $\overset{I}{\sim}$ by putting $s \overset{I}{\sim} t$ iff $s_I = t_I$ iff the following diagram commutes



Definition 74. We will call $\mathbf{CM_{[0,1]}}$ the category having as objects the correlation models over the result structure $[0, 1]$ (equipped with renormalized addition), that is, sets of functions $A \subseteq \{s : \times_{i \in N} O_i \to [0, 1]\}$, and as morphisms partial functions between such sets.

Proposition 19. *The category* $\mathbf{CM_{[0,1]}}$ *is a subcategory of* **Rel** *and* $F : \mathbf{FdHil} \to \mathbf{CM_{[0,1]}}$.

Proof. Being a category with sets as objects and partial functions as morphisms, $\mathbf{CM_{[0,1]}}$ is easily seen to be a subcategory of **Rel**. For the second claim, consider that each set A_H can be seen as a correlation model where the set of observation is the carrier of L_H. The condition on the subsystems is trivially satisfied, as L_H is a degenerate cartesian product with only one component. □

Indeed, the idea of the state of a system as a function assigning results to observation ties in nicely with the abstraction from Hilbert spaces given by F. It remains to be seen whether we can exploit the deconstruct-into-subsystems mechanism of Correlation Models to obtain a tensor operator.

5.4 Tensor on Correlation Models

We have the following wishlist for a tensor functor on Correlation Models:

1. it must be of type $\mathbf{D} \times \mathbf{D} \to \mathbf{D}$, where $\mathbf{D}$ is $\mathbf{CM}_{[\mathbf{0},\mathbf{1}]}$ or a subcategory
2. it must match the tensor in **FdHil**
3. given A_H and A_V, the compound correlation model must be a set of functions of type $L_H \times L_V \to [0,1]$
4. the partial trace in **FdHil** matches the subsystem operation in $\mathbf{CM}_{[\mathbf{0},\mathbf{1}]}$

The third condition is what is required to apply the deconstruct-into-subsystems mechanism of correlation models.

At first sight one might be tempted to define the set $A_H \otimes A_V$ as $\{\sum_i b_i(s^i_H, s^i_V) | b_i \in [0,1], \sum_i b_i = 1, s^i_H \in Pure(A_H), s^i_V \in Pure(A_V)\}$, where $\sum_i b_i(s^i_H, s^i_V) : L_H \times L_V \to [0,1]$ is the function defined as

$$\sum_i b_i(s^i_H, s^i_V)(a_H, a_V) = \sum_i b_i s^i_H(a_H) s^i_V(a_V)$$

If we follow this path though, in proving the matching between $A_H \otimes A_V$ and $A_{H \bigotimes V}$ we are forced to send the (probability function associated to the) density operator of an entangled states such as β_{00} to a mixture of separable states, like $\sum_i b_i(s^i_H, s^i_V)$. But this essentially would mean that entangled states are mixed states, contradicting the fact that entangled states are pure.

The reason why we cannot write the density operator of an entangled state as a mixture of density operators of separable states is that the former contains matrices that are not density operators.

Example

Consider for example the density operator of the Bell state β^{00}:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} \frac{\langle 00| + \langle 11|}{\sqrt{2}} =$$

$$= \frac{|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|}{2}$$

$$= \frac{|0\rangle \langle 0| \bigotimes |0\rangle \langle 0| + |0\rangle \langle 1| \bigotimes |0\rangle \langle 1| + |1\rangle \langle 0| \bigotimes |1\rangle \langle 0| + |1\rangle \langle 1| \bigotimes |1\rangle \langle 1|}{2}$$

The matrix $|00\rangle \langle 11|$, for example, is not of trace 1, and thus is not a density operator.

This points to a more general problem. In general the density operators in a bipartite system cannot be written in terms of the density operators of the two component systems, we need also additional terms, as highlighted in the example. Such terms have shape $|\alpha\rangle \langle\beta|$, where $|\alpha\rangle, |\beta\rangle$ are two orthogonal vectors. So it seems that to recover these terms we need to know that the functions that we see as states are density operators over some Hilbert space. If we do not, then the states of the component systems do not contain enough information to obtain the states of the bipartite system. We have however no proof for these speculations; they will be subject of future work.

Conclusions and future work

We have seen how from the category **FdHil** and the functor S we can obtain a class of Modal Logic frames for LQP and LQP^n. This constitutes the link between the two research programs that we have considered.

Our second case study, the functor F, has highlighted the possibility to obtain a richer semantics. We have designed a logic for such class of Modal Logic frames, and proven that it is an improvement with respect to LQP and LQP^n, in the sense that it has more expressive power and it contains all the (translations of) the theorems of LQP and LQP^n.

On a more general level, we have moved the first steps in the study of LSC logics. We have analyzed three examples, DLT, $S4$ and Hybrid Logic, showing how different languages, different categories and different functors can be used to characterize certain classes of $(\mathbf{C}, U)$-frames.

These results prompt two groups of questions, one related to the main topic of this thesis and one at the intersection of Category Theory and Modal Logic. We start with the former group.

1. A first problem concerns the design of a proof system of $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}}$. The application of this proof system should be the correctness proof of a quantum protocol where probability plays an essential role.

2. It would be interesting to explore the connection between $\Lambda_{\Gamma_F^n}^{\mathcal{F}_{Prob^n}}$ and other logics that have been proposed in the area, such as the calculus for dagger compact closed categories with biproducts presented in [16] and the logics proposed in [3], [2] and [1].

3. The two problems mentioned in Chapter 5 are still undecided. In particular, we would like to understand which are the conditions needed in order to have a tensor operator and, conversely, under which conditions it is impossible to have it.

4. The different functors from **FdHil** to **Rel** and the corresponding classes of Modal Logic frames represent different possible abstractions from Hilbert spaces; it would be interesting to study the connection with other abstractions from Hilbert spaces, as for example the Chu spaces described in [5] or the coalgebras in [4].

We now turn to the second, more abtract, group of questions.

1. The functors from a category **C** to **Rel** constitute a category called $\mathbf{Rel}^{\mathbf{C}}$, having as morphisms natural transformations. Therefore we can investigate the connection between the properties of this category and the properties of the corresponding logics. For example:

   - How does the existence of a natural transformation between two functors riverberate on the corresponding logics? Note that the transformation of the Modal Logic frames given by F into Modal Logic frames given by S (Lemma Chapter 4) was allowed by a natural transformation.
   - Can we characterize the logics arising from initial and terminal functors?
   - We know that most of the categorical structure of **Rel** can be lifted to $\mathbf{Rel}^{\mathbf{C}}$ via componentwise definitions. But we also know that **Rel** is a degenerate dagger compact closed category with byproducts. How does this affect the class of logics arising from the functors in $\mathbf{Rel}^{\mathbf{C}}$?
   - If we restrict our attention to the category of functors $\mathbf{Set}^{\mathbf{C}}$, i.e. the category of functors from a small cateogry into **Set**, we can see that $\mathbf{Set}^{\mathbf{C}}$ is a topos.[1] It would be interesting to investigate the interplay between Topos Theory and our procedure to obtain Modal Logic frames.

2. The main contemporary field at the intersection of Modal Logic and Category Theory is Coalgebra. We have briefly mentioned coalgebras, or rather the "coalgebraic approach", in Chapter 5, when we were discussing the characterization of the tensor. On a more general level, we are interested in understanding the link of Coalgebra with our procedure.

We will pursue these lines of research in future work.

[1] The standard reference on Topos Theory is [19].

Acknowledgements

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Appendix A

Basic Notions

In this Appendix we review the basic definitions and facts used in the thesis.

A.1 Category Theory

The reference for this section is [20].

Definition 75. A *category* $\mathbf{C}$ is a structure consisting of

- a collection of *objects*, denoted $\mathbf{C}_0$
- a collection of *morphisms*, or arrows, denoted $\mathbf{C}_1$
- two operations *dom* and *cod*, "domain" and "codomain" (sometimes called "source" and "target"), assigning an object to each morphism

We write $f : A \to B$ to mean that f is a morphism with domain A and codomain B.

- for each object A in $\mathbf{C}_0$, there is an arrow $Id_A : A \to A$, called the *identity of* A
- for each pair of morphisms f, g in $\mathbf{C}_1$ such that $cod(f) = dom(g)$ there is an arrow $g \circ f$ called the *composite*[1]

To be a category such structure is required to satisfy the following axioms:

1. Associativity: for all f, g, h in $\mathbf{C}_1$ with the correct configuration of domains and codomains,

$$(f \circ g) \circ h = f \circ (g \circ h)$$

[1] Note that the order of the morphisms in the composite is reversed with respect to the order of "application" of the morphisms.

2. Identity Axioms: for all $f : A \to B$ in $\mathbf{C}_1$,

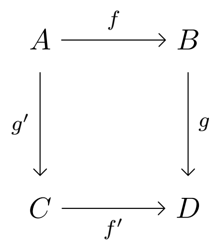
$$f \circ Id_A = f = Id_B \circ f$$

Definition 76. Call **Set** the category having sets as objects and functions as morphisms. The identity morphisms are the identity functions, the composition is the composition of functions.

Definition 77. Call **Rel** the category having sets as objects and relations as morphisms. The identity morphisms are the identity relations, the composition is the composition of relations.

Definition 78. A morphism $f : A \to B$ in a category $\mathbf{C}$ is an *isomorphism* if there is a morphism $f^{-1} : B \to A$, called the *inverse*, such that $f \circ f^{-1} = Id_B$ and $f^{-1} \circ f = Id_A$.

Definition 79. Given a category $\mathbf{C}$, A, B, C, D in $\mathbf{C}_0$ and f, g, f', g' in $\mathbf{C}_1$, a diagram such as

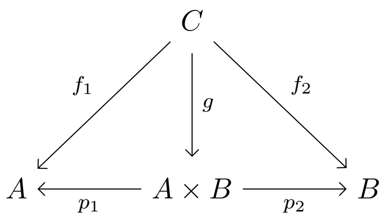


is said to *commute* if $g \circ f = f' \circ g'$.

Note that a commuting diagram can represent different mathematical statements depending on the specific category under consideration. If for example $\mathbf{C}$ is **Set**, the commutation of the diagram above means that, *for all* $x \in A$, $g \circ f(x) = f' \circ g'(x)$.

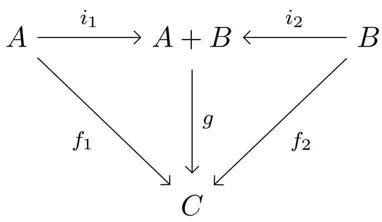
Definition 80. An object A in a category $\mathbf{C}$ is *initial* if for every object B in $\mathbf{C}_0$ there is a unique arrow $A \to B$. An object A is *terminal* (sometimes also called *final*) if for every object B there is a unique arrow $B \to A$.

Definition 81. Given objects A, B in a category $\mathbf{C}$, their *product* is an object $A \times B$ equipped with two morphisms $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ such that for all objects C in $\mathbf{C}_0$ and morphisms $f_1 : C \to A$ and $f_2 : C \to B$ there exists a unique morphism $g : C \to A \times B$ such that the triangles in the following diagram commute



A category has products if for every pair of objects their product exists in the category.

Definition 82. Given objects A, B in a category $\mathbf{C}$, their *coproduct* is an object $A{+}B$ equipped with two morphisms $i_1 : A \to A{+}B$ and $i_2 : B \to A{+}B$ such that for all objects C in $\mathbf{C}_0$ and morphisms $f_1 : A \to C$ and $f_2 : B \to C$ there exists a unique morphism $g : A + B \to C$ such that the triangles in the following diagram commute



A category has coproducts if for every pair of objects their coproduct exists in the category. Notice that the coproduct is the dual notion of the product: it is obtained by reversing all the arrows in the definition of product.

Definition 83. Given two categories $\mathbf{C}$ and $\mathbf{D}$, a *functor* G from $\mathbf{C}$ to $\mathbf{D}$, written $G : \mathbf{C} \to \mathbf{D}$, is a pair of functions

1. G_0 from $\mathbf{C}_0$ to $\mathbf{D}_0$
2. G_1 from $\mathbf{C}_1$ to $\mathbf{D}_1$

such that

- for all f in $\mathbf{C}_1$, $dom(G_1(f)) = G_0(dom(f))$ and $cod(G_1(f)) = G_0(cod(f))$; it *preserves domains and codomains*
- for all A in $\mathbf{C}_0$, $G_1(Id_A) = Id_{G_0(A)}$; it *preserves identities*
- for all f, g in $\mathbf{C}_1$ $G_1(f \circ g) = G_1(f) \circ G_1(g)$; it *preserves compositions*

It is customary to abuse the notation and drop the subscript from the functor. When specifying a functor G we write

$$\begin{aligned} A &\mapsto \dots \\ f : A \to B &\mapsto \dots \end{aligned}$$

where the first line describes the action of G_0 and the second line the action of G_1.

Proposition 20. *Functors preserve commuting diagrams: if $G : \mathbf{C} \to \mathbf{D}$ and*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f']{} & D \end{array}$$

is a commuting diagram in $\mathbf{C}$ then

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle G(g')}\downarrow & & \downarrow{\scriptstyle G(g)} \\ G(C) & \xrightarrow[G(f')]{} & G(D) \end{array}$$

is a commuting diagram in $\mathbf{D}$. As a consequence, functors preserve isomorphisms.

Definition 84. Call **Cat** the category having categories as objects and functors as morphisms.

Definition 85. Given two categories $\mathbf{C}$ and $\mathbf{D}$, the *product category* $\mathbf{C} \times \mathbf{D}$ is composed as follows

- the objects are pairs of objects (C, D), where C is in $\mathbf{C}_0$ and D in $\mathbf{D}_0$
- the morphisms are pairs of morphisms (f, g), where f is in $\mathbf{C}_1$ and g in $\mathbf{D}_1$

the identity is the pair of identities and the composite is the pairwise composite. The category $\mathbf{C} \times \mathbf{D}$ is the product of $\mathbf{C}$ and $\mathbf{D}$ in the category **Cat** in the sense of Definition 81.

Definition 86. Given categories $\mathbf{C}, \mathbf{D}$ and $\mathbf{E}$, a *bifunctor* is a functor of type $G : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$.

Definition 87. Given two functors $G, G' : \mathbf{C} \to \mathbf{D}$, a *natural transformation* δ from G to G', written $\delta : G \to G'$, is a family of morphisms in $\mathbf{D}$ indexed by the objects of $\mathbf{C}$ satisfying the following conditions.

- for A in $\mathbf{C}_0$, the morphism δ_A is of type $G(A) \to G'(A)$;
- for all $f : A \to B$ in $\mathbf{C}_1$, the following diagram commutes

$$\begin{array}{ccc} G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle \delta_A}\downarrow & & \downarrow{\scriptstyle \delta_B} \\ G'(A) & \xrightarrow[G'(f)]{} & G'(B) \end{array}$$

  A *natural isomorphism* is a natural transformation such that every morphism in the family is an isomorphism.

A.2 Quantum Mechanics

Our reference for this section is [21]. Along with the formalism we introduce the postulates of Quantum Mechanics.

Definition 88. A *field with involution* $\mathbb{K}$ is a field with a map $()^* : \mathbb{K} \to \mathbb{K}$ such that, for all $k, k' \in \mathbb{K}$, $k^{**} = k$, $(k + k')^* = (k)^* + (k')^*$ and $(kk')^* = (k')^*(k)^*$.

Definition 89. A *Hilbert spaces* is a pair $\langle E, \langle \cdot|\cdot \rangle \rangle$ such that

1. E is a vector space over some field with involution $\mathbb{K}$
2. $\langle \cdot|\cdot \rangle : E \times E \to \mathbb{K}$ is an inner product:
   - $\langle v| \sum_i c_i v_i \rangle = \sum_i c_i \langle v|v_i \rangle$, for $c_i \in \mathbb{K}$
   - $\langle v|w \rangle = \langle w|v \rangle^*$
   - $\langle v|v \rangle \geq 0$, with equality iff $|v\rangle = 0$

We will mainly use Hilbert spaces over complex numbers, where $*$ is complex conjugation.

Proposition 21. *The set of closed linear subspaces of a Hilbert space H forms a lattice under set-theoretic inclusion. Such lattice is indicated with L_H.*

Definition 90. Two vectors $|v\rangle, |w\rangle$ are called *orthogonal* if $\langle v|w\rangle = 0$.

Definition 91. The one-dimensional linear subspaces of a Hilbert space H are called *rays*; if $|v\rangle$ is a vector in H then the ray containing $|v\rangle$ is indicated with $\overline{|v\rangle}$.

Definition 92. A *unitary* vector in a Hilbert space is a vector $|v\rangle$ such that $\langle v|v\rangle = 1$. There is a bijecton between unitary vectors and rays. A *qubit* is a unitary vector of the 2-dimensional Hilbert space over complex numbers.

To distinguish them from generic vectors, we sometimes indicate unitary vectors with $|\phi\rangle, |\alpha\rangle, |\beta\rangle, \ldots$. Qubits are usually expressed in the computational basis $\{|0\rangle, |1\rangle\}$.

Postulate 1 (First Postulate)**.** *An isolated physical systems is represented by a Hilbert space. The states of the system correspond to the unitary vectors of the Hilbert space.*

Because of this postulate and the aforementioned bijection we use the terms "state", "unitary vector" and "ray" as synonyms.

Definition 93. A *linear map* $L : E \to E'$ between two vector spaces is a map such that $L(\sum_i c_i |v_i\rangle) = \sum_i c_i L(|v_i\rangle)$. Given a linear map $L : H \to V$, its *adjoint map* is the unique map $L^\dagger : V \to H$ such that $\langle v|L(w)\rangle = \langle L^\dagger(v)|w\rangle$. A linear map is *Hermitian* or *self-adjoint* if $L = L^\dagger$. A *unitary map* or transformation is a linear map $U : H \to V$ such that $U^\dagger U = Id_H$. Unitary maps are sometimes called *gates*.

We sometimes drop the parenthesis between a linear map and its argument.

Proposition 22. *A linear map U is unitary iff it preserves the inner product: $\langle v|w\rangle = \langle U(v)|U(w)\rangle$.*

Examples of matrices corresponding to unitary maps on the Hilbert space of one qubit (two in the case of $CNOT$) are:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

X, Y, Z are called *Pauli matrices*, H is called $Hadamard$ and $CNOT$ is short for controlled-not.

Postulate 2 (Second Postulate)**.** *The evolutions of an isolated physical systems are represented by unitary maps.*

Definition 94. A *projector* on a Hilbert space H is a linear map $P : H \to H$ that is both Hermitian and idempotent, that is, $P \circ P = P$.

Postulate 3 (Third Postulate)**.** *A measurement on a physical system represented by H is modelled as a collections of Hermitian maps $\{M_m : H \to H\}$ indexed by the possible outcomes m. This collection is required to satisfy the following requirements:*

- *if the state of the system is $|\phi\rangle$ then the probability of observing outcome m is*
$$p(m) = \langle \phi | M_m^\dagger M_m \phi \rangle$$
- $\sum_m M_m^\dagger M_m = Id_H$
- *the state of the system after the measurement is*
$$\frac{M_m |\phi\rangle}{\sqrt{\langle \phi | M_m^\dagger M_m \phi \rangle}}$$

In this thesis we will consider *projective* measurements, that is, measurement as Hermitians $M = \sum_m m P_m$, with P_m projectors. The probability of observing outcome m then becomes

$$p(m) = \langle \phi | P_m \phi \rangle$$

Definition 95. Given two Hilbert spaces H and V over the same field, the *tensor product* of the two spaces, indicated with $H \bigotimes V$, is the Hilbert space $\langle E, \langle \cdot | \cdot \rangle \rangle$ where

- E is the nm-dimensional vector space (where n is the dimension of H and m the dimension of V) having as basis the vectors $|v_i\rangle \bigotimes |w_j\rangle$, where $|v_i\rangle$ is a basis of H and $|w_j\rangle$ is a basis of V
- the inner product is defined as
$$\langle v_1 \bigotimes w_1 | v_2 \bigotimes w_2 \rangle_{H \bigotimes V} = \langle v_1 | v_2 \rangle_H \langle w_1 | w_2 \rangle_V$$

The vectors of a tensor product are sometimes also written as $|v_i\rangle |w_j\rangle$ and $|v_i w_j\rangle$.

Postulate 4 (Fourth Postulate)**.** *The state space of a compound physical system is represented by the tensor product of the Hilbert space corresponding to the component systems.*

Proposition 23. *There is a bijection between the linear maps of type $H \to V$ and the states of $H^* \bigotimes V$, where H^* is the conjugate space of H, i.e., the space having the same set of vectors and the same addition operation but with scalar multiplication and inner product defined as*

$$c \cdot_{H^*} v := c^* \cdot_H v \qquad \langle v_1 | v_2 \rangle_{H^*} := \langle v_2 | v_1 \rangle_H$$

where $v, v_1, v_2 \in H$ and c^* is the complex conjugate of c.

Definition 96. Consider the Hilbert space of two qubits, the 4-dimensional space over complex numbers. The vectors

$$|\beta_+^{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad |\beta_-^{00}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$|\beta_+^{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \quad |\beta_-^{01}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

form a basis for this Hilbert space, called the *Bell basis*.

Definition 97. The *trace* of a linear map L represented by a n-by-n matrix $\{c_{ij}\}$ is defined as

$$tr(L) = \sum_{i=1}^{i=n} c_{ii}$$

Proposition 24. *The trace has the following properties:*

- $tr(L + L') = tr(L) + tr(L')$
- $tr(cL) = ctr(L)$
- $tr(LL') = tr(L'L)$

Definition 98. A linear map is *positive* if, for every vector $|v\rangle$, $\langle v|L(v)\rangle$ is a real non-negative number.

Proposition 25. *For any linear map L, $L^\dagger L$ is a positive operator.*

We write $|v\rangle \langle w|$ to indicate the matrix resultiong from the matrix multiplication of the matrix $|v\rangle$ and the transpose of the matrix $|w\rangle$.

Definition 99. Suppose a physical system is in one of the states $|\phi_i\rangle$, with i index, with probabilities p_i. We suppose $\sum_i p_i = 1$, so the states comprise all the possibilities. The linear map

$$\rho = \sum_i p_i \, |\phi_i\rangle \langle \phi_i|$$

is called a *density operator* of H

Proposition 26. *A linear map is a density operator iff it is a positive map of trace 1.*

Proposition 27. *For any P projector and $\rho = \sum_i p_i |\phi_i\rangle \langle\phi_i|$ density operator,*

$$tr(P\rho) = \sum_i p_i \langle\phi_i|P\phi_i\rangle$$

A.3 Modal Logic

The reference for this section is [14].

Definition 100. A *relational structure* is a tuple whose first element is a set, usually denoted with W, and the remaining elements are relations, indicated with the symbol R equipped with subscripts.

Elements of W are usually called *states* or worlds. A relational structure with one relation is sometimes called *Kripke frame*; we will use the term "Modal Logic frame", sometimes abbreviated in "frame" as a synonym for relational structure.

An example of relational structure are the *labelled transition systems*, pairs $\langle W, \{R_a | a \in A\}\rangle$ where W is a non-empty set, A is a non-empty set of labels, and for each $a \in A$ we have $R_a \subseteq W \times W$. Transition systems can be used as a model for computation: the states can be viewed as possible states of a computer, the labels represent programs and $(s, s') \in R_a$ means that there is an execution of the program a starting from state s and terminating in state s'. Since the programs that we want to model are deterministic, the relations in the labelled transititon systems are required to be partial functions.

Modal Logic is a logic designed to handle relational structures. The modalities are the syntactic counterparts of the relation, in the sense that each modality is associated to a relation.

Definition 101. The set of formulas $\mathcal{F}$ of the *basic modal language* is defined starting from a set of atomic propositions At:

$$\psi := p \,|\, \neg\psi \,|\, \psi \wedge \phi \,|\, \Diamond\psi$$

where $p \in At$. This expression means that a formula in the language is either an atomic proposition, a negated formula, a conjunction of two formulas or a formula with the diamond operator on the front. The operator $\Box$ is defined as $\Box := \neg\Diamond\neg$.

The basic modal language is used for relational structures with one relation. If we want to deal with multiple relation we can enrich the language with more diamond operators, one for each relation.

Definition 102. Given a set of atomic propositions At, a *model* is a pair $M = \langle \mathsf{F}, V \rangle$ consisting of a Modal Logic frame and a valuation function $V : At \to \wp(W)$.

Definition 103. Given a model M over a Kripke frame, the *satisfaction relation* $\vDash \subseteq W \times \mathcal{F}$ for the basic modal language is defined recursively as

- $M, s \vDash p$ iff $s \in V(p)$
- $M, s \vDash \neg\psi$ iff $M, s \nvDash \psi$
- $M, s \vDash \psi \wedge \phi$ iff $M, s \vDash \psi$ and $M, s \vDash \phi$
- $M, s \vDash \Diamond\psi$ iff $\exists(s, s') \in R$ such that $M, s' \vDash \psi$

where R is the relation in the frame.

By its definition, the satisfaction relation of $\Box$ is

$$M, s \vDash \Box\psi \text{ iff } \forall(s, s') \in R_n \text{ we have } M, s' \vDash \psi$$

This satisfaction relation is easily extendable to cover the cases where the relational structure has many relations and the language has many diamond operators: just put

$$M, s \vDash \Diamond_n\psi \text{ iff } \exists(s, s') \in R_n \text{ such that } M, s' \vDash \psi$$

where R_n is the relation associated to the diamond $\Diamond_n$.

Definition 104. A formula is *valid in a state* of a Modal Logic frame if it is satisfied in every model over that Modal Logic frame, that is, for every valuation. A formula is *valid in a frame* if it is valid at every state of the Modal Logic frame. A formula is *valid in a class of frames* if it is valid in every Modal Logic frame of the class.

The association between diamond operators and relations can be made more formal. The labels for the diamond operators are sometimes called *programs* or *actions*, indicated with π and written $\langle\pi\rangle$ ($[\pi]$) instead of $\Diamond_\pi$ ($\Box_\pi$). Call Act the set of such labels.

Definition 105. An *interpretation of the actions* in a Modal Logic frame F is a function $i : Act \to \{R_i\}$ from the labels to the collection of the relations in the Modal Logic frame.

The satisfaction relation for the modalities can then be expressed in general as

$$M, s \vDash \langle\pi\rangle\psi \text{ iff } \exists(s, s') \in i(\pi) \text{ such that } M, s' \vDash \psi$$

Suppose the collection of relation in the Modal Logic frame is closed under some kind of operation on relations, for example the composition of relation $\circ$. We can model this closing the set *Act* with respect to an operation of the same ariety, call it ;, and imposing

$$i(\pi;\pi') = i(\pi) \circ i(\pi')$$

This pattern can be generalized to operations of any ariety. Note that this only affects the association of relations to labels i, the definition of satisfaction for the modalities remains unchanged. Along this lines, we can reproduce the algebraic structure of the collection of relation inside the language, as an algebraic structure on the set of labels for the modalities.

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