universität wien

DIPLOMARBEIT

Titel der Diplomarbeit

Consistent Cycles in Graphs

Verfasserin

Julia Wessely

angestrebter akademischer Grad

Magistra der Naturwissenschaften (Mag.rer.nat)

Wien, Juli 2012

Studienkennzahl lt. Studienblatt:	A 405
Studienrichtung lt. Studienblatt:	Mathematik
Betreuer:	Privatdoz. Mag. Dr. Bernhard Krön

Danksagung

Ich möchte meinem Betreuer Bernhard Krön für seine Unterstützung danken. Er hat mir ein spannendes Thema vorgeschlagen und sich viel Zeit genommen. Für Fragen stand er immer zur Verfügung und half mir geduldig bei den Forschungsaufgaben. Außerdem ermutigte er mich, unsere Forschungsergebnisse beim "25th LL-Seminar on Graph Theory" an der Montanuniversität Leoben in einem Vortrag zu präsentieren.

Diese Diplomarbeit wurde durch die Slowenian-Austrian Cooperation (Project SI-20/2009) unterstützt. Im Rahmen des Projekts wurde vom ÖAD (Österreichischer Austauschdienst) eine Reise nach Laibach finanziert.

Abstract

Consistent cycles in finite Graphs were introduced by J.H. Conway in 1971. He observed that if a subgroup G of the full automorphism group of a graph acts arc-transitively on it, then the number of orbits of non-trivial consistent cycles under the action of the group G on the graph is one less than the valence of the graph. A cycle is called consistent under a group of automorphisms G acting on the graph if there exists an element rotating the cycle by one step. In this diploma thesis we generalize this result of Conway to infinite graphs and groups of automorphisms acting vertex-transitively on finite and infinite graphs. In infinite graphs we also consider double-rays as cycles and define the multiplicity of orbits of consistent cycles. We state that the sum of multiplicities of orbits of consistent cycles is equal to the degree of the graph. Further we show for this conjecture that the automorphism group acting on a graph has to be closed in the full automorphism group. Therefor we consider groups of automorphisms as topological groups with the topology of point-wise convergence. For the proof of our main theorem we use Biggs' and Conway's idea of defining a symmetry tree that encodes all the information about the structure of congruence classes of consistent cycles in a given graph and for a given group of automorphisms. We show the bijective correspondence between the maximal walks in the tree and the congruence classes of consistent cycles in the graph. In a further section we discuss our results in the Diestel-Leader graph.

Zusammenfassung

Consistent cycles in endlichen Graphen wurden 1971 von J.H. Conway eingeführt. Er behauptete, dass die Anzahl der Orbits nicht trivialer *consistent cycles* in einem endlichen Graphen um eins weniger als der Knotengrad des Graphen ist, wenn eine Untergruppe der vollen Automorphismengruppe bogentransitiv auf dem Graphen agiert. Ein *cycle* wird *consistent* unter der Aktion einer Automorphismengruppe genannt, wenn es ein Gruppenelement gibt, das den *cycle* um einen Schritt rotiert. In der vorliegenden Diplomarbeit wird dieses Resultat von Conway auf unendliche Graphen und Automorphismengruppen, die knotentransitiv auf endlichen und unendlichen Graphen agieren, verallgemeinert. In unendlichen Graphen betrachten wir auch Doppelstrahlen als *cycles* und definieren die Vielfachheit für Orbits von *consistent cycles*. Wir behaupten, dass die Summe der Vielfachheiten der Orbits von *consistent cycles* gleich dem Knotengrad des Graphen ist. Weiters zeigen wir, dass für diese Behauptung die Automorphismengruppe, die auf dem Graphen agiert, in der vollen Automorphismengruppe abgeschlossen sein muss. Hierfür betrachten wir Automorphismengruppen als topologische Gruppen bezüglich der Topologie der punktweisen Konvergenz. Für den Beweis unseres Hauptsatzes verwenden wir die Idee von Biggs und Conway, einen *symmetry tree* zu definieren, der alle Infomationen über die Struktur der Kongruenzklassen von *consistent cycles* eines gegebenen Graphen und einer gegebenen Automorphismengruppe enthält. Wir zeigen die bijektive Korrespondenz zwischen den maximalen Wegen im Baum und den Kongruenzklassen von *consistent cycles* im Graphen. In einem weiteren Kapitel diskutieren wir unser Resultat im Diestel-Leader-Graphen.

Contents

Chapter 1

Introduction

1.1 Introductory examples

Consider the graph of the cube in Figure 1.1. Take a look at the cycle $[1,2,3,4]$. There is an automorphism that rotates this cycle one step, written in its cycle decomposition as permutation: $(1,2,3,4)(5,6,7,8)$.

We use the word *consistent* to describe a cycle such as the square in Figure 1.1. That is, some symmetry of the graph is consistent with a one-step rotation in the cycle. But how many different types of consistent cycles does a symmetric graph have? With *types* we mean orbits of cycles. An orbit is a congruence class under the action of an automorphism group.

Are there other orbits of consistent cycles in the cube? In fact, the cycle $[1,2,6,7,8,4]$ is consistent under the automorphism $(1,2,6,7,8,4)(3,5)$.

The Theorem of Conway states that if a group G of automorphisms acts arc-transitively on a graph of degree $d \geq 1$, then there are exactly $d-1$ orbits of non-trivial consistent cycles under the action of G. The degree of the cube in Figure 1.1 is 3. Thus there are exactly two orbits of consistent cycles, the orbit of the cycle of length 4, and the orbit of the cycle of length 6.

Consider the pictures of the dodecahedron shown in Figure 1.2 and Figure 1.3. Pay particular attention to the outside cycles: a decagon and a nonagon. The first graph has a symmetry which rotates the outer decagon one step; a $36°$ rotation. The second picture is also symmetric and has a $120°$ rotation, but there is no symmetry of the graph rotating the nonagon by a single step.

The degree of the dodecahedron is 3. Thus, by the Theorem of Conway, there are 2 orbits of non-trivial consistent cycles in this graph.

In the first picture it is seen that in the middle there is a pentagon, which is a consistent cycle. When we rotate the pentagon one step, there is a

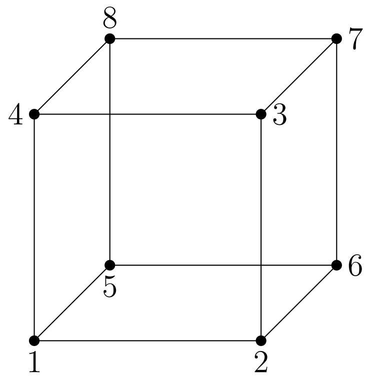

Figure 1.1: Cube

symmetry of the graph; a 72° rotation. So the decagon and the pentagon represent two orbits of consistent cycles. The nonagon in Figure 1.3 is not consistent.

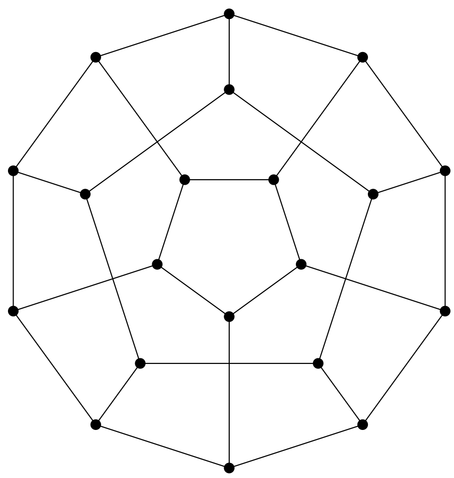

Figure 1.2: Dodecahedron

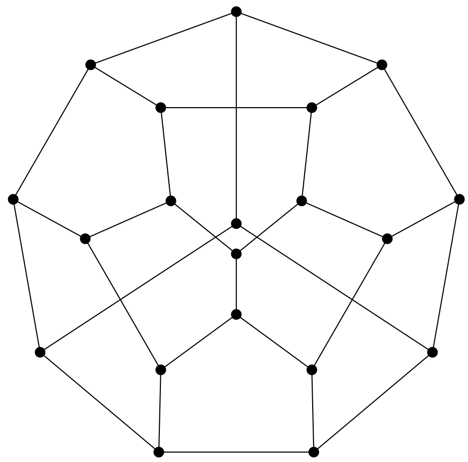

Figure 1.3: Dodecahedron

1.2 Historical background

Consistent cycles in connected, finite (undirected) graphs were introduced by J.H. Conway in his public lectures at the Second British Combinatorial Conference in 1971. He observed that, if a subgroup G of the full automorphism group of a graph acts arc-transitively on it, then the number of orbits of non-trivial consistent cycles under the action of the group G on the graph is one less than the valence of the graph.

J.H. Conway [4] presented the following result, which merges two meanings of the word "cycle". First, a cycle in a graph is a connected subgraph where each vertex has degree 2. Second, each element in a group of automorphisms acting on a graph can be written in its cycle decomposition as permutation.

Theorem 1.2.1 (Conway [4])**.** *Let G be a group of automorphisms of a d-valent graph Γ ($d \geq 2$). Assume that G acts transitively on the set of vertices of all ordered pairs of adjacent vertices. Let X be the set of all (ordered) cycles in Γ each of which is also a cycle occurring in some element of G. Then G has exactly $d-1$ orbits in its action on X.*

If $\alpha = (v_0, v_1, \ldots, v_r)$ is a cycle in a graph and also a cycle occurring in some element $g \in G$, written in its cycle decomposition as permutation, then $g(v_0) = v_1,\ g(v_1) = v_2, \ldots,\ g(v_r) = v_0$. This is what we call a consistent cycle in a graph when there exists an element $g \in G$ rotating the cycle one step.

As far as can be determined, the only written record of Conway's results may be found in Biggs' paper [1]. There, Biggs gives a sketch proof of the Theorem of Conway by defining a symmetry tree. This is a rooted tree where the leaves correspond to unique G-consistent extensions of G-consistent walks. It is a recursive method for the construction of a complete set of maximal consistent walks.

In the paper [2], the authors generalize the Theorem of Conway to arbitrary groups of automorphisms acting on finite digraphs (=directed graphs). They state that the number of G-consistent cycles starting and ending in one vertex v is equal to the number of out-neighbours of v intersecting with the G-orbit of v. They give a detailed proof by using Conway's idea of the symmetry tree to analyze the structure of the set of G-consistent walks.

The authors of [3] generalize the definition of consistent cycles to those which admit a k-step rotation and generalize the Theorem of Conway to those so-called $\frac{1}{k}$-consistent cycles. They distinguish between orbits of consistent cycles, directed cycles, and cyclets, and they give formulas for counting each. For proving these formulas they do not use Conway's symmetry tree, but they define an "overlap function".

1.3 Formulation and motivation

In this diploma thesis we generalize the Theorem of Conway to infinite graphs. We consider groups of automorphisms acting vertex-transitively on locally finite graphs. Here we assume that the group acting on a graph is closed in the full automorphism group. We show that the sum of multiplicities of orbits of consistent cycles is equal to the degree of the graph.

Section 2.1 provides some basic definitions concerning graphs, walks, cycles, automorphisms, etc.

In Section 2.2 we define consistent cyclets and orbits of consistent cycles. We state the Theorem of Conway and give further examples of orbits of consistent cycles in finite, vertex-transitive graphs.

We consider some examples of infinite, locally finite, vertex-transitive graphs in Section 2.3 and count orbits of consistent cycles there. Then we generalize the Theorem of Conway to infinite, locally finite graphs and vertex-transitive automorphism groups acting on them. In infinite graphs we extend the definition of consistent cycles to consistent double-rays and define the multiplicity of consistent cycles.

In Section 2.4 we consider groups of automorphisms as topological groups with the topology of point-wise convergence. We define a metric on the set of groups of automorphisms. Further we contend that for our assumption the group acting on the graph has to be closed in the full automorphism group. Therefor we give a counter-example of an automorphism group which is not closed acting on an infinite graph. Finally we state the main theorem, the generalization of the Theorem of Conway to locally finite graphs.

We start Section 2.5 with defining the so-called "tree of consistent cycles", which encodes all the information about the structure of consistent cycles in a corresponding graph. At first we give some lemmas of consistent walks, which are needed for the construction of the tree, followed by examples. With the help of the tree of consistent cycles we prove the bijective correspondence between the maximal walks in the tree and the orbits of consistent cycles in the graph.

Section 3.1 discusses the main-theorem in the Diestel-Leader graph. We show a result of the structure and the lengths of consistent cycles in this graph.

In Section 3.2 we generalize the definition of consistent cycles to $\frac{1}{k}$-consistent cycles and give some formulas for counting orbits of $\frac{1}{k}$-consistent cyclets, cycles, and directed cycles.

Section 3.3 considers graphs that are vertex- and edge-transitive, but not arc-transitive. Formulas for counting orbits of consistent cycles in so-called $\frac{1}{2}$-arc-transitive graphs are given.

Chapter 2

Consistent Cycles

2.1 Definitions

A *graph* Γ is a pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is an arbitrary set, called the set of *vertices*, and

$$E\Gamma \subseteq \{\{u, v\} \mid u, v \in V\Gamma,\ u \neq v\}$$

is the set of *(undirected) edges*. An *arc* is an ordered pair (u, v) of vertices where $\{u, v\}$ is an edge. We denote the set of arcs with $A\Gamma$. A *digraph* is a graph with all edges directed.

Two vertices u, v are said to be *adjacent* if there exists an edge $\{u, v\}$; in this case we write $u \sim v$. It can also be said that u is a *neighbour* of v.

A *walk* of length r (or an r-walk) from u to v in Γ is a sequence $(v_0, \ldots, v_r)$ of vertices $v_i \in V\Gamma$ such that $u = v_0$, $v = v_r$, and $v_{i-1} \sim v_i$, for $1 \leq i \leq r$. For every $v \in V\Gamma$, the sequence (v) is a walk of length 0. A walk $\alpha = (v_0, \ldots, v_r)$ of length $r \geq 2$ is *closed* if $v_r = v_0$ and *simple* if $v_i \neq v_j$ for all $i \neq j$, except possibly for $(i, j) = (0, r)$.

A graph Γ is *connected* if for every two distinct vertices u, v there exists a walk from u to v. We only consider connected graphs.

The number of neighbours of a vertex v is the *degree* of v and denoted by $\deg(v)$. If all vertices have the same degree, we call the graph *regular* or *d-valent*, where d is the degree of a vertex. In this case we call d the *degree* or the *valence* of the graph and denote it by $\deg(\Gamma)$. For a vertex v in a digraph we define the *out-neighbourhood of* v by

$$\Gamma^+(v) = \{u \in V\Gamma \mid (v, u) \in A\Gamma\}$$

and the *in-neighbourhood of* v by

$$\Gamma^-(v) = \{u \in V\Gamma \mid (u, v) \in A\Gamma\}.$$

Let $\deg^+(v) = |\Gamma^+(v)|$ and $\deg^-(v) = |\Gamma^-(v)|$ denote the *out-* and *in-degree* of v.

A *cyclet* $\alpha = (v_0, v_1, \dots, v_r)$ is a simple closed walk in a graph Γ. We define the *inverse* of α as

$$\alpha^{-1} = (v_r, \dots, v_1, v_0)$$

and the *t-shift* of α for $0 \le t \le r$ as

$$\alpha^t = (v_t, v_{1+t}, \dots, v_{r+t}),$$

considering the index modulo r. A *1-shift* is called a *shift.* A *flip* of the walk $\alpha = (v_0, v_1, \dots, v_r)$ is a transformation that maps v_k to v_{r-k} for $0 \le k \le r$.

A *directed cycle* $\vec{C}$ is an equivalence class under the shift-relationship of a cyclet α and an *(undirected) cycle* C is an equivalence class under the inverse- and the shift-relationship of α. Note that a cyclet is different from its inverse and its t-shifts. The length of a cycle is the number of its vertices. Cycles of length 2 are called *trivial.* We consider a directed cycle of length greater 2 as a connected sub-digraph of Γ where every vertex has in- and out-degree equal to 1 and we consider an undirected cycle of length greater 2 as a connected sub-graph of Γ in which each vertex has degree 2. A pair of inverse directed cycles determines an (undirected) cycle. Note that a trivial cycle is both directed and undirected. If $\vec{C}$ is a directed cycle with vertices $\{v_0, \dots, v_{r-1}\}$ and arcs (v_i, v_{i+1}), $i \in \mathbb{Z}_r$, then a cyclet $(v_0, \dots, v_{r-1})$ is called a *representative* of $\vec{C}$. We say that α is an underlying cyclet of C. Every cyclet is a representative of a unique directed cycle, and every directed cycle of length r has r representatives, each being a shift of any of them. Similarly, every cyclet $\alpha = (v_0, \dots, v_{r-1})$ represents a unique cycle in a natural way, and every non-trivial cycle of length r has $2r$ representatives, each being a shift of a chosen representative $\alpha = (v_0, \dots, v_{r-1})$ or a shift of the inverse α^{-1}. We will denote a cycle of length r by

$$C = [v_0, \dots, v_{r-1}],\ v_i \in V\Gamma,\ v_0 \sim v_{r-1}.$$

An *automorphism* (or *symmetry*) of Γ is a function $f : V\Gamma \to V\Gamma$ such that $f(v) \sim f(u)$ if and only if $v \sim u$. This is a permutation of the vertices which preserves edges. The automorphisms of Γ form a group, denoted by $\mathrm{Aut}(\Gamma)$, under composition, and we will write the application of an automorphism g to a vertex v as v^g or $g(v)$. For a subset $A = \{v_0, \dots, v_r\}$ of $V\Gamma$ the image under an automorphism g is

$$A^g = \{v_0^g, \dots, v_r^g\}.$$

Therefore, the image of a walk $\alpha = (v_0, \ldots, v_r)$ under an automorphism g is denoted by $\alpha^g = (v_0^g, \ldots, v_r^g)$.

For a subgroup G of $\mathrm{Aut}(\Gamma)$ the *G-orbit* of a vertex $v \in V\Gamma$ is

$$v^G = \{v^g \mid g \in G\}.$$

Furthermore, for a subset A of $V\Gamma$ the G-orbit of A is the set

$$A^G = \{A^g \mid g \in G\}.$$

A subgroup G of $\mathrm{Aut}(\Gamma)$ acts on the set of (directed) cycles; two cycles will be called *G-congruent* if they belong to the same *G-orbit*.

For a subgroup $G \leq \mathrm{Aut}(\Gamma)$, we say that G acts *arc-transitively* on Γ (or that Γ is G-symmetric) if for any two arcs (u, v) and (x, y) there is a $g \in G$ such that $g(u) = x$ and $g(v) = y$. If $G = \mathrm{Aut}(\Gamma)$, then we call Γ arc-transitive. We say that a subgroup G of $\mathrm{Aut}(\Gamma)$ acts *vertex-transitively* if given any two vertices u and v there is $g \in G$ such that $g(u) = v$. The group G acts *edge-transitively* if G can map every edge to any other edge. If $G = \mathrm{Aut}(\Gamma)$, then we call Γ vertex-transitive or edge-transitive, respectively. Note, if Γ is vertex-transitive, then every vertex has the same degree.

Remark 2.1.1. If an automorphism group G acts arc-transitively on a graph, then it also acts vertex- and edge-transitively.

But what about the other direction? Is the graph necessarily G-symmetric if G acts vertex- and edge-transitively?

In finite graphs this is the case if the valence of the graph is odd (Tutte showed this by a simple counting argument). But the argument fails when the valence is even. A counter-example is the "Doyle-Holt graph" with 27 vertices and valence 4. This is the smallest graph which is a counter-example (see Section 6 in [5]). In general this does not apply to infinite graphs. The "Diestel-Leader graph" $\mathrm{DL}_{m,n}$ is vertex- and edge-transitive for all m, n, but it is not arc-transitive for $m \neq n$ (see Section 3.1).

Let G be a group of automorphisms of a graph Γ. A walk $\alpha = (v_0, \ldots, v_r)$ is called *G-consistent* if there is $g \in G$ such that

$$v_i^g = v_{i+1}$$

for all $i \in \mathbb{Z}_r$. Such a symmetry is called a *shunt (automorphism)* for α. The set of all shunts in G for α will be denoted by $\mathrm{Sh_G}(\alpha)$.

Note that consistent walks are simple. If $\alpha = (v_0, \ldots, v_r)$ is a G-consistent walk in Γ such that $v_{i_1} = v_{i_2}$ for some integers $i_1 \neq i_2$, then for some $0 \leq s \leq r$ the walk $(v_0, \ldots, v_s)$ is simple and closed, and $v_i = v_j$ whenever $i \equiv j \mod s$.

For $g, h \in G$ we let

$$g^h = h^{-1}gh$$

denote the *h-conjugate* of g. For $v \in V\Gamma$ and $G \leq \mathrm{Aut}(\Gamma)$, we let

$$G_v = \{g \in G \mid v^g = v\}$$

denote the *stabilizer* of v in G. Furthermore, for a subset $A \subseteq V\Gamma$ we define the *set-wise* stabilizer as

$$G_A = \{g \in G \mid A^g = A\}$$

and the *point wise* stabilizer as

$$G_{(A)} = \{g \in G \mid v^g = v, \ \forall v \in A\}.$$

If α is a G-consistent cyclet with a shunt automorphism $g \in G$, then its *cyclic shifts* are images of α by powers of g. Hence, if one representative of a directed cycle $\vec{C}$ is G-consistent, then all representatives of $\vec{C}$ are G-congruent and thus G-consistent.

A directed cycle is called G-consistent whenever one (and thus each) of its representatives is G-consistent. Note that a directed cycle $\vec{C}$ of length r is G-consistent if and only if the set-wise stabilizer $G_{\vec{C}}$ of its arcs acts on it as a cyclic group of order r.

Similarly, if α is a G-consistent representative of an undirected cycle C, then all the representatives of C are G-consistent, either with respect to the same shunt g as α or with respect to g^{-1}. A cycle C will be called G-consistent if one (and thus each) of its representatives is G-consistent.

Lemma 2.1.2 (Section 4 in [2])**.** *For a graph Γ and $G \leq \mathrm{Aut}(\Gamma)$ let H be a subgroup of G, and let $\alpha = (v_0, \ldots, v_r)$ be a G-consistent walk in Γ with a shunt automorphism g. If $h \in H$, then $\alpha^h = (v_0^h, \ldots, v_r^h)$ is a G-consistent walk in Γ with a shunt automorphism g^h. Hence the group H partitions the set of G-consistent walks in Γ into H-congruence classes.*

Proof. For a vertex v_i^h in α^h holds

$$(v_i^h)^{g^h} = (v_i^h)^{h^{-1}gh} = v_i^{hh^{-1}gh} = v_i^{gh} = v_{i+1}^h$$

for all $i \in \mathbb{Z}_r$. So α^h is a G-consistent walk in Γ with a shunt automorphism g^h. □

A group of automorphisms of a graph acts on the set of consistent (directed) cycles. The orbits of this action will be called *G-congruence classes* of consistent (directed) cycles.

Note that some G-consistent cycles might be trivial. Such cycles exist if there is an automorphism that transposes two adjacent vertices. For example, such an automorphism exists whenever G acts transitively on the arcs of Γ. In this case, all the trivial cycles in Γ are G-symmetric and G-congruent.

2.2 Consistent cycles in finite graphs

Theorem 2.2.1 (Conway)**.** *Let $G \leq \mathrm{Aut}(\Gamma)$ be a group acting arc-transitively on a finite graph Γ, then the number of G-orbits of non-trivial consistent cycles is one less than the degree of the graph.*

Theorem 2.2.2. *Let $G \leq \mathrm{Aut}(\Gamma)$ be a group acting vertex-transitively on a finite graph Γ, then the number of G-orbits of consistent cycles is equal to the degree of the graph.*

Note that if a graph is arc-transitive, then there exists exactly one orbit of trivial cycles, and this one is consistent.

Example 2.2.3. Consider the icosahedron together with its full automorphism group. This is a regular graph of degree 5. By Theorem 2.2.1, there are four G-orbits of non-trivial consistent cycles. The cycle $[1, 2, 3, 4, 5, 6]$ of length 6 and the cycle $[8, 10, 12]$ of length 3 in Figure 2.1 are consistent relative to the $60°$ rotation and the $120°$ rotation of the graph, respectively. The cycle $[1, 9, 2, 3, 10, 4, 12, 5, 6, 7]$ of length 10 and the cycle $[1, 2, 10, 12, 6]$ of length 5 in Figure 2.2 are consistent relative to the following shunts: the $36°$ rotation and the $72°$ rotation, respectively.

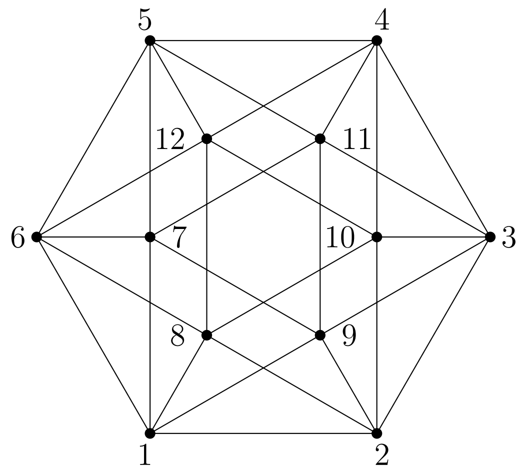

Figure 2.1: Icosahedron

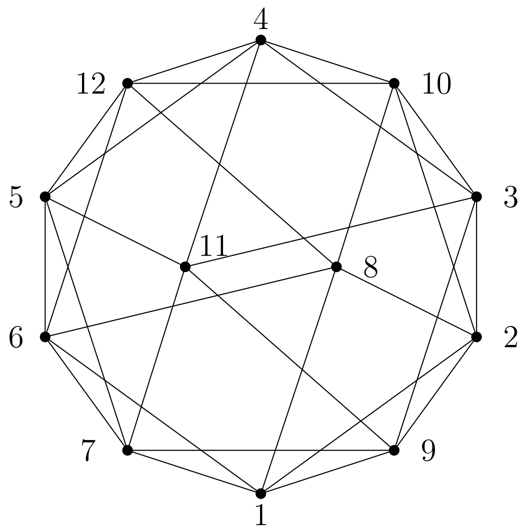

Figure 2.2: Icosahedron

Example 2.2.4. The graph $K_{4,4}$ in Figure 2.3 has degree 4. Let G be the full automorphism group. By Theorem 2.2.2, there are 4 orbits of G-consistent cycles, namely $[1, 2]$, $[1, 2, 3, 4]$, $[1, 2, 3, 4, 5, 6]$, and $[1, 2, 3, 4, 5, 6, 7, 8]$. Shunts for these cycles are each the flip around the vertical symmetry axis and the cyclic permutation of the vertices (2), $(2, 4)$, $(2, 4, 6)$, and $(2, 4, 6, 8)$, respectively.

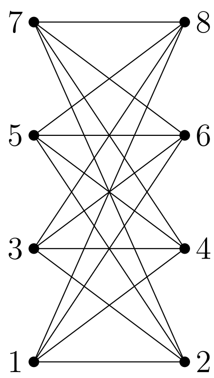

Figure 2.3: $K_{4,4}$

Example 2.2.5. Let Γ be again the $K_{4,4}$, but let G be the index-2 subgroup of $\mathrm{Aut}(\Gamma)$, isomorphic to the alternating group $\mathcal{A}_4$. This is the group of permutations of the vertices with an even number of transpositions.

The cycle $[1, 2]$ is consistent under the shunt $(1, 2)(3, 4)(5, 6)(7, 8)$, which is a concatenation of 4 transpositions of vertices.

The cycle $[1, 2, 3, 4]$ is also consistent under the action of G. A shunt may be the automorphism $(1, 2, 3, 4)(5, 6)(7, 8)(7, 5)$, which has an even number of transpositions.

The cycle $[1, 2, 3, 4, 5, 6]$ is consistent under the shunt $(1, 2, 3, 4, 5, 6)(7, 8)$. The cycles of length 8 are not consistent here because there is no shunt automorphism with an even number of transpositions of vertices.

We have another cycle of length six: $[1, 2, 3, 4, 5, 8]$ is consistent, but not in the same orbit as $[1, 2, 3, 4, 5, 6]$ because the transposition $(6, 8)$ is not in G. Thus in this example we have 4 orbits of consistent cycles.

Example 2.2.6. Consider the wheel-graph in Figure 2.4. It is vertex-transitive, but not edge-transitive under the action of the full automorphism group. The edge $\{v_1, v_2\}$ cannot be mapped to the edge $\{v_1, v_9\}$. The degree of the graph is 3. We have one orbit of non-trivial consistent cycles, the orbit of $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$, and we have two orbits of trivial consistent cycles, namely the orbits of $[v_1, v_2]$ and $[v_1, v_9]$. The cycle $[v_1, v_2, v_{10}, v_9]$ is not consistent here. Therefore we have 3 orbits of consistent cycles and thus Theorem 2.2.2 holds.

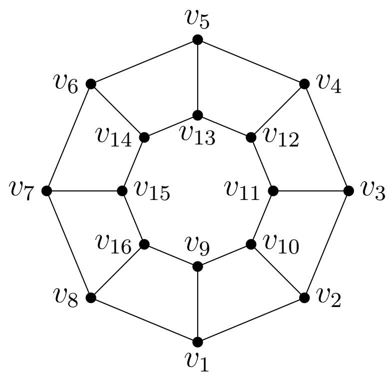

Figure 2.4: Wheel-graph

Note that Theorem 2.2.1 does not hold in Example 2.2.6 because the graph has degree 3 but there is only one orbit of non-trivial consistent cycles.

2.3 Consistent cycles in infinite graphs

We only consider *locally finite* graphs, meaning that each vertex has a finite degree.

In this section we will take a look at consistent cycles in infinite, connected graphs and discuss if Theorem 2.2.1 and Theorem 2.2.2 hold as in the finite case.

Definition 2.3.1. A *two-way infinite walk* is a sequence of vertices $(v_i)_{i \in \mathbb{Z}}$ such that $(v_i, v_{i+1}) \in E\Gamma$ for all i. Two walks $(v_i)_{i \in \mathbb{Z}}$ and $(w_i)_{i \in \mathbb{Z}}$ are identified if there is a $k \in \mathbb{Z}$ such that $v_{i+k} = w_i$ for all $i \in \mathbb{Z}$. A *double ray* $(v_i)_{i \in \mathbb{Z}}$ is a two-way infinite walk of distinct vertices. A *ray* $(v_i)_{i \in \mathbb{N}_0}$ is a one-way infinite walk of distinct vertices.

In infinite graphs we also consider double-rays as cycles in addition to finite cycles.

Now let us consider examples of vertex-transitive infinite graphs. Does the Theorem of Conway hold as it does in finite graphs?

Example 2.3.2. Let $G = \mathrm{Aut}(\Gamma)$ act on the "honeycomb lattice" in Figure 2.5. This graph has degree 3. Representatives of orbits of consistent cycles are: the hexagon $[1, 2, 3, 4, 5, 6]$ and the double-ray $[\ldots, 7, 8, 9, 5, 6, 1, 10, 11, 12, \ldots]$.

There are 2 orbits of non-trivial consistent cycles under the action of G, so Theorem 2.2.1 does in fact hold in this example.

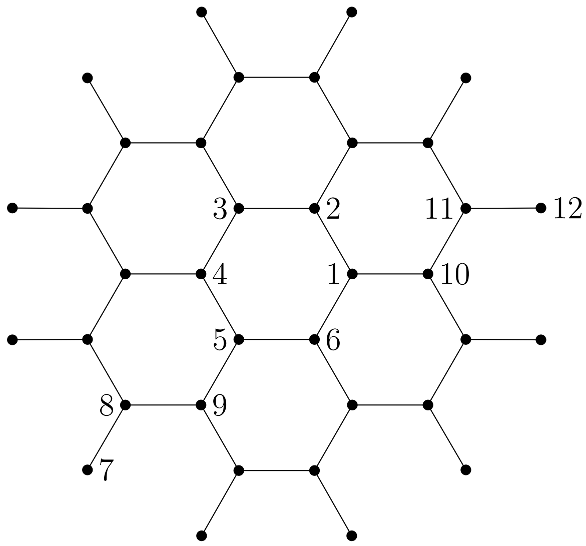

Figure 2.5: Honeycomb lattice

Example 2.3.3. Consider another example of an infinite graph, the "triangular lattice" in Figure 2.6. This graph has valence 6. Let G be the full automorphism group. There are two G-orbits of finite consistent cycles: the triangle $[13, 18, 14]$ and the hexagon $[9, 8, 14, 18, 19, 12]$. Further there are three G-orbits of infinite consistent cycles: $[\ldots, 11, 12, 13, 14, 15, \ldots]$, $[\ldots, 1, 10, 9, 13, 14, 17, 16, \ldots]$, and $[\ldots, 11, 20, 12, 19, 13, 18, 14, 17, 15, 16, \ldots]$.

There are five orbits of non-trivial consistent cycles, and thus Theorem 2.2.1 holds in this example.

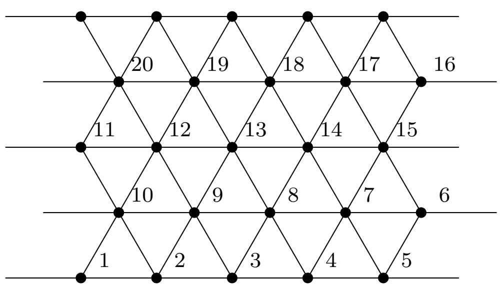

Figure 2.6: Triangular lattice

Example 2.3.4. The graph in Figure 2.7 is vertex-transitive, but not edge-transitive under the action of the full automorphism group. There are two orbits of edges, those which are within a triangle and those which are between the triangles. The valence of the graph is 3. In fact we have 3 orbits of consistent cycles. One is the orbit of triangles and the other two, the two orbits of edges, are trivial. There is no infinite consistent cycle.

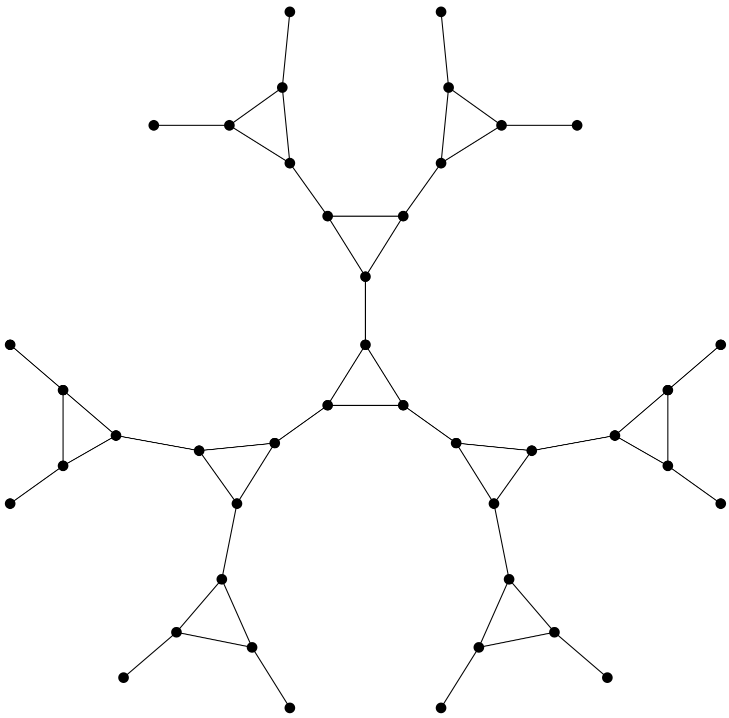

Figure 2.7: Infinite Triangles

Example 2.3.5. Consider the 3-regular tree in Figure 2.8 together with the action of $\mathrm{Aut}(\Gamma)$. The degree of the graph is 3, but it has just one orbit of non-trivial consistent cycles, the double-ray $[\ldots, 1, 2, 3, 4, 5, 6, 7, \ldots]$. In fact this cycle has multiplicity 2, which is the maximal number of consistent cycles which have the same negative tail but disjoint positive tails.

If we fix a negative tail $(\ldots, 1, 2, 3, 4, 5)$, then the cycles

$$[\ldots, 1, 2, 3, 4, 5, 6, 7, \ldots] \text{ and } [\ldots, 1, 2, 3, 4, 5, 8, 9, \ldots]$$

have the same negative tail and disjoint positive tails, $(6, 7, \ldots)$ and $(8, 9, \ldots)$, respectively. The multiplicity of the cycle is one less than the degree of the graph.

Example 2.3.6. Let the full automorphism group act on the graph "Infinite $\mathrm{K}_{2,2}$" of valence 4 in Figure 2.9. There are two finite consistent cycles, $[v_0, v_1, w_0, w_1]$ and the trivial one, as well as one infinite consistent cycle

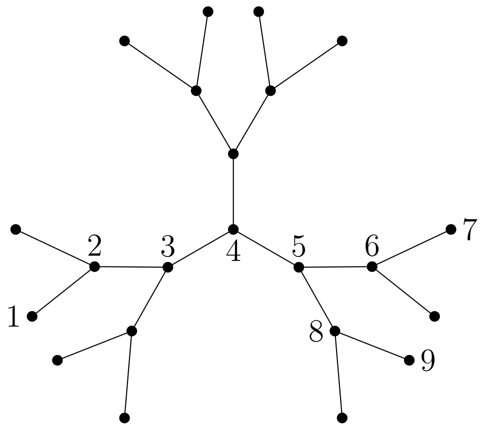

Figure 2.8: 3-regular tree

$(w_i)_{i\in\mathbb{Z}}$ with multiplicity 2. If we fix the negative tail $(w_i)_{i\leq 0}$, then there are two cycles, $(w_i)_{i\in\mathbb{Z}}$ and $(\ldots, w_{-1}, w_0, v_1, v_2, \ldots)$, which have the same negative tail, but disjoint positive tails. In this example, the degree of the graph equals the sum of the multiplicities of orbits of consistent cycles. Note that the finite cycles have multiplicity 1.

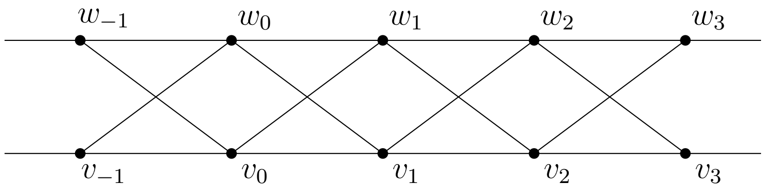

Figure 2.9: Infinite $K_{2,2}$

Let Γ be an infinite graph with a countable set of vertices and let $G \leq \mathrm{Aut}(\Gamma)$ be a group of automorphisms acting on Γ. (Note that if Γ is locally finite, then the set of vertices is countable.)

Definition 2.3.7. A *consistent cycle* is a double-ray

$$C = (v_i)_{i\in\mathbb{Z}}$$

for which there is a $g \in G$ such that $v_i^g = v_{i+1}$ for all $i \in \mathbb{Z}$. This $g \in G$ is called a *shunt* for C.

In this section we could consider a finite consistent cycle as an infinite walk $C = (v_i)_{i\in\mathbb{Z}}$, where there is a $k \in \mathbb{N}, k \geq 2$ such that $v_{i+k} = v_i$ for all i.

Lemma 2.3.8. *A consistent cycle is either finite or a double ray.*

Proof. Let $C = (v_i)_{i\in\mathbb{Z}}$ be a consistent cycle. If $v_{i_1} = v_{i_2}$ for two vertices in C, then for some shunt g is $v_{i_1+1} = v_{i_1}^g = v_{i_2}^g = v_{i_2+1}$ and also

$$v_{i_1+n} = v_{i_1}^{g^n} = v_{i_2}^{g^n} = v_{i_2+n} \text{ for all } n \in \mathbb{N}.$$

It follows that $v_i = v_j$ whenever $i \equiv j \mod k$, where $k = i_2 - i_1$. Thus C is finite. If all vertices in C are distinct, then C is a double-ray. □

Definition 2.3.9. We call $C^- = (v_i)_{i\leq 0}$ the *negative tail of* C and $C^+ = (v_i)_{i\geq 1}$ the *positive tail of* C.

Definition 2.3.10. The *multiplicity* $m(C)$ of the consistent cycle C is defined as

$$m(C) = |v_1^{G_{(C^-)}}|.$$

Thus, this is the number of possible images of v_1 under elements of G which fix the negative tail of C point-wise.

Let g be a shunt for C and $C^h = (w_i)_{i\in\mathbb{Z}}$ for some $h \in G$. Then

$$w_i^{g^h} = w_i^{h^{-1}gh} = v_i^{gh} = v_{i+1}^h = w_{i+1}.$$

Thus C^h is a consistent cycle with shunt g^h. Let $\mathcal{C}$ denote the set of all orbits of consistent cycles under the action of G.

Lemma 2.3.11. *Let $O = C^G$ be a G-orbit of some consistent cycles, then $m(O) = m(C)$ is independent of the choice of $C \in O$.*

Proof. Let $D = C^h$ be a consistent cycle, then

$$m(D) = m(C^h) = |(w_1)^{G_{(C^{h-})}}| = |(v_1^h)^{G_{(C^{-h})}}| = |v_1^{G_{(C^-)}}| = m(C).$$

Thus we can define $m(O) = m(C)$ independent of the choice of $C \in O$. □

2.4 Groups of automorphisms as topological groups

Consider the following example of an infinite graph:

Example 2.4.1. Let Γ be the "Infinite $K_{2,2}$" shown in Figure 2.9 and let $A = \{v_i \mid i \in \mathbb{Z}\}$ and $B = \{w_i \mid i \in \mathbb{Z}\}$. Instead of the full automorphism group let G be the group of automorphisms that maps all but finitely many elements from A to A, acting on Γ. This is the group consisting of: the automorphisms that map the double-ray $(v_i)_{i\in\mathbb{Z}}$ to its t-shifts for $t \in \mathbb{Z}$, the flip of the graph around the symmetry axes through (v_i, w_i) $i \in \mathbb{Z}$, and the transposition of v_i and w_i for finitely many $i \in \mathbb{Z}$.

The group G acts arc-transitively on Γ. In this example there are two finite G-consistent cycles. The cycle $[v_0, v_1]$ is consistent under the shunt of $(v_i)_{i\in\mathbb{Z}}$ and the flip around (v_1, w_1), and the cycle $[v_0, v_1, w_0, w_1]$ is consistent under the shunt of $(v_i)_{i\in\mathbb{Z}}$, the flip around (v_1, w_1), and the transposition of v_0 and w_0.

Further we have three infinite cycles. The cycles $C_1 = (v_i)_{i\in\mathbb{Z}}$ and $C_2 = (w_i)_{i\in\mathbb{Z}}$ are consistent, and $C_3 = (\ldots, v_{-1}, v_0, w_1, w_2, \ldots)$ is consistent under the shunt of $(v_i)_{i\in\mathbb{Z}}$ and the transposition of v_0 and w_0.

These three infinite cycles are representatives of three different orbits because there is no element in G that maps infinitely many vertices from A to B.

In this example we have five G-orbits of consistent cycles. This number is greater than the degree of the graph.

Remark 2.4.2. The Theorem of Conway does not hold in this example because G is not closed in $\mathrm{Aut}(\Gamma)$.

[6] The automorphism group of an infinite, locally finite graph is a topological group with the topology of point-wise convergence.

We consider automorphisms as functions from $V\Gamma$ to $V\Gamma$. In this section we will denote the image of a vertex v under a group-element g by $g(v)$.

Definition 2.4.3. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions with $f_n \in G$ for all $n \in \mathbb{N}$ and $g \in G$, where $G \leq \mathrm{Aut}(\Gamma)$. The sequence $(f_n)_{n\in\mathbb{N}}$ *converges point-wise* to g if for all $v \in V\Gamma$ there exists $N \in \mathbb{N}$ such that

$$f_n(v) = g(v) \text{ for all } n \geq N.$$

Definition 2.4.4. A $(*)$-*sequence* in G is a sequence $(f_n)_{n\in\mathbb{N}}$ such that for all $v \in V\Gamma$ there is an $N \in \mathbb{N}$ such that

$$f_n(v) = f_m(v) \text{ for all } n, m \geq N.$$

Definition 2.4.5. The group G is called *closed* in $\mathrm{Aut}(\Gamma)$ if every $(*)$-sequence in G converges.

Remark 2.4.6. We could also define a metric on the group G acting on the graph Γ. Let $V\Gamma = \{v_0, v_1, v_2, \ldots\}$. For $g, h \in G$ let

$$d(g, h) = \sum_{i=0}^{\infty} \frac{\delta(g(v_i), h(v_i))}{2^i},$$

where δ is the discrete metric

$$\delta(v,w) = \begin{cases} 1, & \text{for } v \neq w \\ 0, & \text{for } v = w. \end{cases}$$

Let us verify that d is a metric:

(M1) For all $g, h \in G$ is $d(g,h) \geq 0$. If $d(g,h) = 0$, then $\delta(g(v_i), h(v_i)) = 0$ for all i and so $g(v_i) = h(v_i)$ for all i; and it follows that $g = h$.

(M2) The function $d(g,h)$ is symmetric.

(M3) We have to show the triangle inequality:

$$d(f,g) + d(g,h) \geq d(f,h), \ \forall f,g,h \in G.$$

$$d(f,g) + d(g,h) = \sum_{i=0}^{\infty} \frac{\delta(f(v_i), g(v_i))}{2^i} + \sum_{i=0}^{\infty} \frac{\delta(g(v_i), h(v_i))}{2^i}$$

$$\sum_{i=0}^{\infty} \frac{\delta(f(v_i), g(v_i)) + \delta(g(v_i), h(v_i))}{2^i} \geq \sum_{i=0}^{\infty} \frac{\delta(f(v_i), h(v_i))}{2^i} = d(f,h)$$

For all $v \in V\Gamma$ holds

$$\delta(f(v), g(v)) + \delta(g(v), h(v)) \geq \delta(f(v), h(v))$$

because δ is the discrete metric.

Hence d is a metric and (G,d) becomes a metric space.

Definition 2.4.7. A *Cauchy sequence* in (G,d) is a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in G$ for all $n \in \mathbb{N}$ such that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that

$$d(f_n, f_m) < \epsilon \text{ for all } n, m \geq N.$$

Lemma 2.4.8. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of a group $G \leq \mathrm{Aut}(\Gamma)$ acting on a graph Γ. Then $(f_n)_{n\in\mathbb{N}}$ is a $(*)$-sequence in G if and only if it is a Cauchy sequence in (G,d).*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a $(*)$-sequence in G. For all $v \in V\Gamma$ there exists an $N \in \mathbb{N}$ such that $f_n(v) = f_m(v)$ for all $n, m \geq N$.

Let $\epsilon = \frac{1}{2^k} > 0$. For each $i \in \{0, 1, \ldots, k\}$ there exists an $N_i \in \mathbb{N}$ such that $f_n(v_i) = f_m(v_i)$ for all $n, m \geq N_i$. Let $N = \max\{N_i\}_{i\geq 0}$, then $f_n(v_i) = f_m(v_i)$ for all $0 \leq i \leq k$.

Then we have

$$d(f_n, f_m) = \sum_{i=0}^{\infty} \frac{\delta(f_n(v_i), f_m(v_i))}{2^i} = \sum_{i=k+1}^{\infty} \frac{\delta(f_n(v_i), f_m(v_i))}{2^i} =$$

$$\frac{1}{2^k} \sum_{i=1}^{\infty} \frac{\delta(f_n(v_i), f_m(v_i))}{2^i} < \frac{1}{2^k} = \epsilon$$

for all $n, m \geq N$. Thus $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (G, d).

On the other hand, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (G, d). For $\epsilon = \frac{1}{2^k} > 0$ exists an $N \in \mathbb{N}$ so that

$$d(f_n, f_m) = \sum_{i=0}^{\infty} \frac{\delta(f_n(v_i), f_m(v_i))}{2^i} < \frac{1}{2^k}, \ \forall m, n > N.$$

It follows that

$$\sum_{i=0}^{k} \frac{\delta(f_n(v_i), f_m(v_i))}{2^i} = 0 \text{ and } \delta(f_n(v_i), f_m(v_i)) = 0$$

$$\text{and thus } f_n(v_i) = f_m(v_i) \ \ \forall\ 0 \leq i \leq k.$$

Since $k \in \mathbb{N}$ is arbitrary, it follows that for all vertices v_i, $i \geq 0$ there exists an $N \in \mathbb{N}$ such that $f_n(v_i) = f_m(v_i)$ for all $n, m \geq N$, and so $(f_n)_{n\in\mathbb{N}}$ is a $(*)$-sequence in G. □

Remark 2.4.9. We want to show that in Example 2.4.1 the group G is not closed in $\text{Aut}(\Gamma)$. Consider the following Cauchy sequence

$$(f_n)_{n\in\mathbb{N}} \ , \ f_n(v_i) = \begin{cases} w_i, & \text{for } i \in \{0, \ldots, n\} \\ v_i, & \text{for } i \notin \{0, \ldots, n\}. \end{cases}$$

For a vertex $v_s \in V\Gamma$ holds $f_n(v_s) = f_m(v_s)$ for all $n, m \geq s$. Hence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, which does not converge because there is no automorphism in G which maps infinitely many elements from A to B. Thus G is not closed in $\text{Aut}(\Gamma)$.

So for our conjecture, we have to assume that the group acting on a graph is closed in the full automorphism group. In the following let the group be closed in the full automorphism group.

Definition 2.4.10. A topological space X is *sequentially compact* if every sequence has a convergent subsequence. A space X is *compact* if every open cover has a finite subcover. If every cover of X has a countable subcover then X is called a *Lindelöf space*.

Remark 2.4.11. A sequentially compact Lindelöf space is compact.

Lemma 2.4.12 (Lemma 1 in [6])**.** *Let G be closed in* $\mathrm{Aut}(\Gamma)$ *and x be a fixed vertex in $V\Gamma$, then the stabilizer G_x is compact.*

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G_x, and let $x_0 = x, x_1, x_2, \dots$ be an enumeration of $V\Gamma$; (this is possible because $V\Gamma$ is countable). As $g_n x = x$ for every n, and as Γ is locally finite and connected, the set $\{g_n x_k \mid n \geq 0\}$ is finite for every k. Hence there is a subsequence $(\tau_1(n))$ of $\mathbb{N}$ such that all $g_{\tau_1(n)} x_1$ coincide. Write gx_1 for this common image. Repeating this argument inductively, we get a sub-sequence $(\tau_k(n))$ of the preceding subsequence $(\tau_{k-1}(n))$ such that all $g_{\tau_k(n)}$, $n \geq 0$, send x_k to the same element of $V\Gamma$, denoted gx_k. Thus $g_{\tau_n(n)} \to g \in G$ pointwise. □

Lemma 2.4.13. *Let G be closed in* $\mathrm{Aut}(\Gamma)$ *and $A, B \subseteq V\Gamma$ with $|A| = |B|$ is finite, then $G_{A,B} = \{g \in G \mid g(A) = B\}$ is compact.*

Proof. Let $A = \{x_1, \dots, x_n\}$ and $B = \{y_1, \dots, y_n\}$. For an $i \in \{1, \dots, n\}$ let $g \in G$ such that $g(x_i) = y_i$. Then

$$G_{x_i,y_i} = \{g \in G \mid g(x_i) = y_i\} = gG_{x_i}$$

is compact because, by Lemma 2.4.12, the stabilizer is compact. It follows that the set

$$G_{(x_1,\dots,x_n),(y_1,\dots,y_n)} = \{g \in G \mid g(x_i) = y_i \text{ for all } 1 \leq i \leq n\} = \bigcap_{i=1}^{n} G_{x_i,y_i}$$

is compact. Thus the pointwise stabilizer $G_{(A)} = G_{(x_1,\dots,x_n),(x_1,\dots,x_n)}$ is also compact. There exist at most $n!$ automorphisms from A to B. Let $m = n!$ and $g_1, \dots, g_m$ be different bijections $A \to B$. Thus

$$G_{A,B} = \{g \in G \mid g(A) = B\} = \bigcup_{i=1}^{m} G_{(x_1,\dots,x_n),(g_i(x_1),\dots,g_i(x_n))}$$

is compact. □

Lemma 2.4.14. *Let $G \leq \mathrm{Aut}(\Gamma)$ be a group acting on Γ. If two infinite G-consistent cycles have the same negative tail, then they belong to the same G-orbit if G is closed in* $\mathrm{Aut}(\Gamma)$.

Proof. Let $C = (v_i)_{i\in\mathbb{Z}}$ and $D = (w_i)_{i\in\mathbb{Z}}$ be two G-consistent cycles with

$$C^- = (v_i)_{i\leq 0} = (w_i)_{i\leq 0} = D^-.$$

Let g be a shunt for C and let h be a shunt for D. Consider the following Cauchy sequence

$$(f_n)_{n\in\mathbb{N}} \text{ with } f_n = h^{-n}g^n.$$

For a vertex w_k in D^+, $k \geq 1$, and an $n \geq k$ holds

$$h^{-n}(w_k) = w_{k-n} \in D^-,$$

so $w_{k-n} = v_{k-n}$ and

$$g^n(v_{k-n}) = v_k \in C^+.$$

Therefore, for every $w_k \in D$ exists an $N \in \mathbb{N}$, $N \geq k$ with

$$f_n(w_k) = f_m(w_k) = v_k \in C \ \forall n, m \geq N.$$

For every $n \in \mathbb{N}$ the function f_n maps the walk $(w_i)_{i\leq n}$ to the walk $(v_i)_{i\leq n}$. Since G is closed in $\mathrm{Aut}(\Gamma)$, the Cauchy sequence converges to an element in G which maps D to C. Thus C and D are in the same G-orbit. □

Remark 2.4.15. We could also define $(v_i)_{i\leq n}$ for every $n \in \mathbb{N}$ as a negative tail of C and $(v_i)_{i>n}$ as the corresponding positive tail.

Lemma 2.4.16. *Let $G \leq \mathrm{Aut}(\Gamma)$ be a group that is closed in* $\mathrm{Aut}(\Gamma)$ *and acts vertex-transitively on Γ. If C is a consistent cycle with multiplicity n, then there are n consistent cycles which have the same negative tail and disjoint positive tails.*

Proof. Let $C = (v_i)_{i\in\mathbb{Z}}$ be a consistent cycle with negative tail $C^- = (v_i)_{i\leq 0}$ and a shunt $g \in G$ and $m(C) = n$. We construct a consistent cycle with negative tail C^- in the following way:

The number of out-neighbours of v_0 which are in the orbit of v_1 under the stabilizer $G_{(C^-)}$ is equal to n. Let $w_1 \in v_1^{G_{(C^-)}}$ and $w_1 = v_1^h$ for $h \in G_{(C^-)}$. There exists a consistent cycle $C^h = D = (\ldots, v_0, w_1, w_2, \ldots)$ with the shunt g^h. By Lemma 2.3.11 holds

$$n = m(C) = m(D^{g^h}) = |w_1^{G_{(D^{g^h}-)}}|,$$

where $G_{(D^{g^h-})}$ is a negative tail of $D^{g^h} = (\ldots, v_0, w_1)$. So the number of out-neighbours of w_1 which are in the orbit of w_2 under the stabilizer $G_{(D^{g^h-})}$ is also equal to n.

Choose a vertex u_2 from this orbit and let $E = (\ldots, v_0, w_1, u_2, u_3, \ldots)$ be a consistent cycle, and so on.

In each step, we have n possible vertices to choose from. Hence, there exist n consistent cycles which have the same negative tail and disjoint positive tails. □

If the group G acting on a graph Γ is closed in $\mathrm{Aut}(\Gamma)$ and C is a consistent cycle, then the multiplicity of C is the maximal number of consistent cycles which have the same negative tail as C and disjoint positive tails.

Remark 2.4.17. This is not the case if the group G is not closed in $\mathrm{Aut}(\Gamma)$. Consider Example 6.1. The multiplicity of the cycle C_1 is per definition

$$m(C_1) = |v_1^{G_{(C^-)}}| = |\{v_1, w_1\}| = 2,$$

but the number of cycles which are congruent to C under the action of the stabilizer $G_{(C^-)}$ and have disjoint positive tails is just 1. This means that C is not G-congruent to any other cycle which has the same negative tail as C and disjoint positive tail to C.

Remark 2.4.18. Note that in finite graphs every consistent cycle is finite. Finite consistent cycles have multiplicity 1. Let $C = (v_i)_{i \in \mathbb{Z}}$ with $v_i = v_{i+r}$ for some $r \in \mathbb{N}$ be a consistent cycle. Let $C^- = (\ldots, v_1, \ldots, v_r)$ and $C^+ = (v_1, \ldots, v_r, \ldots)$. Since C^- stabilizes v_1, the orbit $v_1^{G(C^-)} = v_1$. Therefore, $m(C) = |v_1^{G(C^-)}| = 1$.

Theorem 2.4.19. *Let* Γ *be a regular graph and let* $G \leq \mathrm{Aut}(\Gamma)$ *be a group that is closed in* $\mathrm{Aut}(\Gamma)$ *acting vertex-transitively on* Γ*, then*

$$\sum_{O \in \mathcal{C}} m(O) = \deg(\Gamma).$$

Note that since the multiplicity of every finite consistent cycle is 1, in finite regular graphs the number of G-orbits of consistent cycles equals the degree of the graph.

2.5 The tree of consistent cycles

This section will be introduced with lemmas of consistent walks, which are needed for the construction of the so-called tree of consistent cycles and the proof of our main theorem. The definitions in this section are as in Section 3 in [2]. Lemma 2.5.5 is similar to Lemma 3.3 in [2]. The method of construction of the tree and Theorem 2.5.14 are generalizations to infinite graphs of the construction method of the tree in Section 4 in [2] and Lemma 4.1 in [2]. In Lemma 2.5.15 there are cited parts of Theorem 5.1 in [2].

Lemma 2.5.1 (Lemma 3.1 in [2]). *Let Γ be a graph, let $G \leq \mathrm{Aut}(\Gamma)$, and let $\alpha = (v_0, v_1, \ldots, v_r)$ be a G-consistent walk in Γ. Then $\mathrm{Sh_G}(\alpha)$ is invariant under conjugation by the elements of $G_{(\alpha)}$.*

Proof. Let g be a shunt automorphism for α, and let h be an element of $G_{(\alpha)}$. Then for every $i \in \{0, \ldots, r-1\}$ we have

$$v_i^{g^h} = v_i^{h^{-1}gh} = v_i^{gh} = v_{i+1}^h = v_{i+1}.$$

Therefore, g^h is also a shunt automorphism contained in G, and the result follows. □

Definition 2.5.2. Let $\alpha = (w_0, w_1, \ldots, w_r)$. For $r \geq 1$ we define

$$\hat{\alpha} = (w_0, w_1, \ldots, w_{r-1}).$$

For a graph Γ, for a subgroup $G \leq \mathrm{Aut}(\Gamma)$, and for a G-consistent walk $\alpha = (w_0, w_1, \ldots, w_r), r \geq 1$, let

$$R_G(\alpha) = w_r^{G_{(\hat{\alpha})}} \tag{2.1}$$

denote the orbit of the last vertex of α under the stabilizer of $\hat{\alpha}$.

We define the set

$$X_G(\alpha) = \{u^g \mid u \in R_G(\alpha),\ g \in \mathrm{Sh_G}(\alpha)\}. \tag{2.2}$$

(Section 3 in [2]) Now suppose that $g_1, g_2 \in G$ are two distinct shunts for α, and for $i \in \{1, 2\}$ consider the sets

$$X_G^i(\alpha) = \{u^{g_i} \mid u \in R_G(\alpha)\}.$$

If $v \in X_G^2(\alpha)$, then there exists an element $h \in G_{(\hat{\alpha})}$ such that $v = w_r^{hg_2}$. But then

$$v = w_r^{hg_2} = w_r^{hg_2g_1^{-1}g_1} = u^{g_1}, \text{ where } u = w_r^{hg_2g_1^{-1}}.$$

Since $hg_2g_1^{-1} \in G_{(\hat{\alpha})}$, it follows that $u \in R_G(\alpha)$. Thus $v \in X_G^1$ and so $X_G^2 \subseteq X_G^1$. We can use the same argument to show that $X_G^1 \subseteq X_G^2$, and thus $X_G^1 = X_G^2$. So $X_G(\alpha)$ is independent of the choice of the shunt $g \in \mathrm{Sh_G}(\alpha)$, and therefore

$$X_G(\alpha) = \{w_r^{hg} \mid h \in G_{(\hat{\alpha})}, g \in \mathrm{Sh_G}(\alpha)\}.$$

The following graphic is taken from [1].

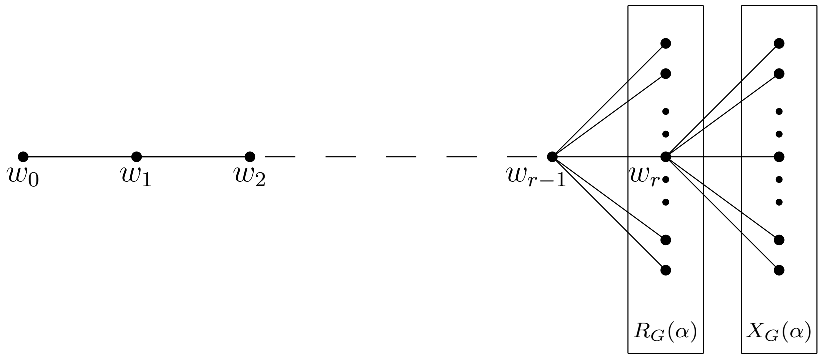

Figure 2.10

Lemma 2.5.3 (Lemma 3.2. in [2]). *Let* Γ *be a graph, let* $G \in \mathrm{Aut}(\Gamma)$, *and let* $\alpha = (w_0, \ldots, w_r)$ *be a G-consistent walk of* Γ. *Then* $X_G(\alpha)$ *is invariant under the action of the stabilizer* $G_{(\alpha)}$.

Proof. Let $v \in X_G(\alpha)$ and $f \in G_{(\alpha)}$. Then, by 2.2, there exist elements $h \in G_{(\alpha)}$ and $g \in \mathrm{Sh_G}(\alpha)$ such that $v = w_r^{hg}$, and hence

$$v^f = w_r^{hgf} = w_r^{hfg^f}.$$

But since $hf \in G_{(\hat{\alpha})}$ and $g^f \in Sh_G(\alpha)$ by Lemma 2.5.1, 2.2 implies that $v^f \in X_G(\alpha)$. □

Definition 2.5.4. Let $\alpha = (w_0, w_1, \ldots, w_r)$ be a G-consistent walk and let w be a vertex. If $\beta = (w_0, w_1, \ldots, w_r, w)$ is a G-consistent walk, then it is called a G-consistent extension of α.

Lemma 2.5.5 (Lemma 3.3 in [2]). *Let* Γ *be a graph, let* $G \leq \mathrm{Aut}(\Gamma)$, *and let* $\alpha = (w_0, \ldots, w_r)$ *be a G-consistent walk of* Γ. *Then* $w \in X_G(\alpha)$ *if and only if* $(w_0, w_1, \ldots, w_r, w)$ *is G-consistent. The number of G-consistent extensions of* α *is*

$$|X_G(\alpha)| = |R_G(\alpha)|.$$

The following proof is similar to the proof of Lemma 3.3 in [2].

Proof. If $w \in X_G(\alpha)$, then $w = w_r^{hg}$ for some $h \in G_{(\hat{\alpha})}$ and some shunt automorphism g for α. Then hg is a shunt automorphism for $(w_0, w_1, \ldots, w_r, w)$.

On the other hand, suppose that $(w_0, w_1, \ldots, w_r, w)$ is a G-consistent walk with a shunt g'. Then $w = w_r^{g'}$, and since $w_r \in R_G(\alpha)$, we have $w \in X_G(\alpha)$.

For some shunt $g \in G$ the set $X_G(\alpha) = \{u^g \mid u \in R_G(\alpha)\}$. Since for $u_1 \neq u_2 \in R_G(\alpha)$ we have $u_1^g \neq u_2^g$, and for all $u_1, u_2 \in R_G(\alpha)$ follows

$$|X_G(\alpha)| = |R_G(\alpha)|.$$

□

Definition 2.5.6. A *rooted tree* is a connected digraph T with exactly one vertex ω of in-degree 0, called the *root of T*, and with the property that for every vertex $v \in VT$ there exists exactly one walk from ω to v.

Note that a rooted tree has no cycles.

Definition 2.5.7. The *leaves* of a rooted tree are vertices with out-degree 0. An *internal* vertex of a rooted tree is a vertex that is not a leaf. A *maximal walk* in a rooted tree is a walk that starts in the root and either ends in a leaf or is infinite. The *depth* of a rooted tree is the length of the longest maximal walk.

For a vertex v, we will denote an out-neighbour of v with v^+ and an in-neighbour with v^-.

Note that in a rooted tree, a vertex that is not the root has exactly one in-neighbour.

(Section 4 in [2]) The idea behind Conway's approach is to define a rooted tree that encodes all the information about the structure of congruence classes of consistent cycles in a given graph for a given group of automorphisms. We fix a graph Γ, a group $G \leq \mathrm{Aut}(\Gamma)$, and a vertex $v \in V\Gamma$. For a rooted tree T with root ω and some $\nu \in VT$, let α_ν be the unique walk from ω to ν. Furthermore, for a function ι from VT to some set $V \subseteq V\Gamma$ and for a walk $\alpha = (\nu_0, \ldots, \nu_r)$ in T, let $\iota(\alpha) = (\iota(\nu_0), \ldots, \iota(\nu_r))$ be a walk in Γ. We define a rooted tree T with root ω, $\iota(\omega) = v$ and functions $\iota : VT \to V\Gamma$, $l : AT \to \mathbb{N}$ with the following properties:

(T1) For every $\nu \in VT$, $\iota(\alpha_\nu)$ is a G-consistent walk in Γ.

(T2) If $\iota(\alpha_\nu) = (v_0, \ldots, v_r)$ and $v_i = v_j$ for some i, j $(0 \leq i < j \leq r)$ and $\nu \in VT$, then $i = 0, j = r$, and ν is a leaf of T.

(T3) For $(\eta, \nu) \in AT$ holds

$$l(\eta, \nu) = |R_G(\iota(\alpha_\nu))|.$$

(T4) For every internal vertex $\nu \in VT$ and $\eta \in T^-(\nu)$,

$$l(\eta, \nu) = \sum_{\xi \in T^+(\nu)} l(\nu, \xi).$$

We shall construct the rooted tree T and functions ι and l recursively by defining triples (T_r, ι_r, l_r), $0 \leq r \leq s$, where s is the length of the longest consistent cycle in Γ, and T_r is a rooted tree with root ω and depth r, and $\iota_r : VT_r \to V\Gamma$, $AT_r \to \mathbb{N}$ satisfy the following:

(a) For $r \geq 0$ we have $T_r \leq T_{r+1}$, and T_{r+1} is an extension of T_r.

(b) For $r \geq 0$ the functions ι_r and l_r are restrictions of ι_{r+1} and l_{r+1} on VT_r and AT_r, respectively.

(c) Triples (T_r, ι_r, l_r) satisfy (T1)–(T4).

Because of (b) above, we shall write ι and l instead of ι_r and l_r $(0 \leq r \leq s)$. Let T_0 be the rooted tree with a single vertex ω, labeled by $v_0 = \iota(\omega)$. Observe that the conditions (T1)–(T4) are satisfied in this case: $\iota(\omega) = v_0$ is a G-consistent walk in Γ and ω is also a leaf because it has no out-neighbours. The conditions (T3) and (T4) are trivially satisfied.

For $r \geq 0$ we construct T_{r+1} in the following way: Suppose that (T_r, ι, l) is already defined such that (T1)–(T4) are satisfied. Let $\Lambda = \{\omega\}$ if $r = 0$, and let Λ be the set of all the leaves of T_r which are not labeled with v_0 if $r \geq 1$. If a vertex $\nu \in T_r$ is labeled with v_0, then ν is a leaf of T. If $\Lambda = \emptyset$, that means that all the leaves in T_r are labeled with v_0, then $s = r$, $T = T_s$ and the construction is finished.

Otherwise, define the tree T_{r+1} and extend the functions ι and l to T_{r+1} as follows. For every $\mu \in \Lambda$, consider the unique walk $\alpha_\mu = (\mu_0, \ldots, \mu_r)$ in T_r from $\mu_0 = \omega$ to $\mu_r = \mu$. Since, by the induction hypothesis, $\iota(\alpha_\mu)$ is a G-consistent walk in Γ, the stabilizer $G_{(\iota(\alpha_\mu))}$ acts on the set

$$\{\iota(\mu_r)^g \mid g \in \mathrm{Sh_G}(\iota(\alpha_\mu))\} = X_G(\iota(\alpha_\mu))$$

by Lemma 2.5.3. We can choose a set of representatives $\{w_1, \ldots, w_m\}$ of the $G_{(\iota(\alpha_\mu))}$-orbits on $X_G(\iota(\alpha_\mu))$. For every representative w_i, create a "new" vertex μ_i^+, which is an out-neighbour of μ and a "new" arc (μ, μ_i^+) of T_{r+1}. Furthermore, define $\iota(\mu_i^+) = w_i$ and let $l(\mu, \mu_i^+)$ be the size of the $G_{(\iota(\alpha_\mu))}$-orbit containing w_i.

Now, let us show that, by the induction hypothesis, the tree $T_{r+1}, r \geq 1$, satisfies (T1)–(T4).

We show that (T4) holds in T_{r+1} for $\nu = \mu$: the set $\{w_1, \ldots, w_m\} \subseteq X_G(\iota(\alpha_\mu))$ consists of the representatives of $G_{(\iota(\alpha_\mu))}$-orbits on $X_G(\iota(\alpha_\mu))$. For $1 \leq i \leq m$ let $\iota(\alpha_{w_i}) = (\iota(\mu_0), \ldots, \iota(\mu_r), w_i)$. Since

$$R_G(\iota(\alpha_{w_i})) = w_i^{G_{(\iota(\alpha_\mu))}}$$

is the orbit of w_i under the action of the stabilizer $G_{(\iota(\alpha_\mu))}$, it follows that

$$\sum_{\xi \in T^+(\mu)} l(\mu, \xi) = \sum_{1 \leq i \leq m} l(\mu, w_i) = \sum_{1 \leq i \leq m} |R_G(\iota(\alpha_{w_i})| = |X_G(\iota(\alpha_\mu))|.$$

The condition (T3) holds in T_{r+1} provided it holds for every "new" arc $(\eta, \nu) = (\mu, \mu_i^+)$. Furthermore, by the above construction, $l(\mu, \mu_i^+)$ is the size of the $G_{(\iota(\alpha_\mu))}$-orbit $w_i^{G_{(\iota(\alpha_\mu))}}$, where $w_i = \iota(\mu_i^+)$. Since

$$\iota(\alpha_\mu) = \widehat{\iota(\alpha^+_{\mu_i})},$$

by 2.1, $w_i^{G_{(\iota(\alpha_\mu))}}$ coincides with $R_G(\iota(\alpha^+_{\mu_i}))$. Hence

$$l(\mu, \mu_i^+) = |R_G(\iota(\alpha^+_{\mu_i}))|,$$

and (T3) holds in T_{r+1}.

To show that (T1) holds in T_{r+1}, let ν be a vertex of T_{r+1} and let $\alpha_\nu = (\nu_0, \ldots, \nu_t)$. If $\nu = \nu_t \in VT_r$, then (T1) holds by the induction hypothesis. If $\nu = \mu_i^+$ for some $\mu \in \Lambda$ and $i \in \{1, \ldots, m\}$, then, by the definition of T_{r+1}, we have

$$\mu_i^+ \in X_G(\iota(\alpha_\mu)).$$

Hence, by Lemma 2.5.5, $\iota(\alpha^+_{\mu_i})$ is G-consistent.

The condition (T2) holds in $T_{r+1}, r \geq 2$, because $\iota(\alpha_\mu)$ is a G-consistent walk in Γ, $v_i \neq v_j$, except for $(i, j) = (0, r)$, then $\iota(\nu) = v_r = v_0$, and it follows that ν is a leaf of T.

Remark 2.5.8. It follows that the tree T has depth at least 2 and at most s, which is the length of the longest consistent cycle in Γ. The depth of the tree of consistent cycles of a finite graph is finite. The depth of the tree of consistent cycles of an infinite graph Γ is infinite if there exist infinite consistent cycles in Γ. At each step of the construction the depth of the tree increases by 1, this implies that the construction terminates after at most s steps. Note that s can be infinite for infinite graphs, then the construction never terminates. If $T = T_r$ is the last tree constructed by this procedure, then the triple (T_r, ι_r, l) is called the *tree of consistent cycles of* Γ *with respect to* G *and* v_0. The construction of the tree terminates when all the leaves are labeled with v_0.

Remark 2.5.9. If $l(\nu_i, \nu_{i+1}) = 1$ for an arc $(\nu_i, \nu_{i+1}) \in T$, then the corresponding walk $(\iota(\nu_0), \ldots, \iota(\nu_i), \iota(\nu_{i+1}))$ has a unique G-consistent extension. It is convenient to stop the construction of the walk $(\omega, \ldots, \nu_i, \nu_{i+1})$ in the tree T if the last arc (ν_i, ν_{i+1}) is labeled by 1.

In the opposite direction, if a walk $(v_0, \ldots, v_{r-1}, v_r)$ in the graph Γ has a unique G-consistent extension, then the arc in the tree, corresponding to the arc (v_{r-1}, v_r), is labeled by 1.

In finite graphs we can, for simplicity, terminate the construction of the tree when the last arcs (ν^-, ν) are labeled with 1 for every leaf $\nu \in \Lambda$.

Lemma 2.5.10. *The tree of consistent cycles has a finite amount of maximal walks.*

Proof. Because Γ is locally finite, the vertex $v_0 \in \Gamma$ has finitely many neighbours. Therefore $l(\omega, \omega^+)$ is finite. Let (η, ν) be an arc in T. By (T4) ν can at most have $l(\eta, \nu)$ many out-neighbours. In each step of the construction, there can be added only finitely many "new" arcs to the tree. So there are only finitely many walks of maximal length. □

Remark 2.5.11. It follows that for each infinite walk $\alpha = (v_0, v_1, v_2, \ldots)$ in T there exists a minimal index N_α, where $l(v_i, v_{i+1}) = l(v_{i+1}, v_{i+2})$ for all $i \geq N_\alpha$. Let $N = \max_\alpha\{N_\alpha\}$. For the sake of simplicity, we can terminate the construction of the tree with T_N.

Example 2.5.12. We want to construct the tree of consistent cycles of the icosahedron in Figure 2.1.

We fix the arc $(1, 2)$ in the graph and label the first arc in T with $(1, 2)$. The images of the arc $(1, 2)$ in the graph under its shunts are:

$$(2, 1),\ (2, 3),\ (2, 10),\ (2, 9),\ (2, 8).$$

We label the arc $(1, 2)$ in T with 5, which is the number of these images.

The set of images of the vertex 2 under its shunts is $\{1, 3, 10, 9, 8\}$ and the orbits on this set under the stabilizer of $(1, 2)$ are $\{1\}$,$\{3, 10\}$, and $\{8, 9\}$. From each orbit we choose one representative, for example 1, 3 and 9. We create a new vertex and a new arc in T for each representative. The size of the orbit including 9 is 2, so $l(2,9) = 2$. The size of the orbit including 3 is also 2, so $l(2,3) = 2$ and $l(2,1) = 1$.

The images of the vertex 9 in the graph under the shunts of the walk $(1,2,9)$ are the vertices 1 and 3. They belong to different orbits under the action of the stabilizer of $(1,2,9)$. Hence, in the tree we have two out-neighbours of the vertex 9, labeled by 1 and 3, and two arcs $(9,1)$ and $(9,3)$ with $l(9,1) = 1$ and $l(9,3) = 1$, respectively. The images of the vertex 3 under the shunts of the walk $(1,2,3)$ are 11 and 4. They are not in the same orbit under the action of the stabilizer of $(1,2,3)$. Thus, we create two new vertices in the tree and two arcs with $l(3,4) = 1$ and $l(3,11) = 1$.

With that, we can finish the construction of the tree because for every leaf ν in T we have $l(\nu^-, \nu) = 1$, for ν^- being the in-neighbour of ν.

The walk $(1,2,9,1)$ in the tree corresponds to the orbit of consistent cycles containing $[1,2,9]$ in the graph. The arc $(1,2,9,3)$ in the tree corresponds to the orbit of consistent cycles containing $[1,2,9,3,11,4,5,12,6,8]$, which is isomorphic to the cycle $[1,9,2,3,10,4,12,5,6,7]$ (see Figure 2.1). The arc $(1,2,1)$ corresponds to the orbit of trivial cycles in the graph. The arc $(1,2,3,11)$ corresponds to the orbit containing $[1,2,3,11,7]$, which is isomorphic to $[1,2,10,12,6]$ (see Figure 2.1). The arc $(1,2,3,4)$ corresponds to the orbit of the consistent cycle $[1,2,3,4,5,6]$.

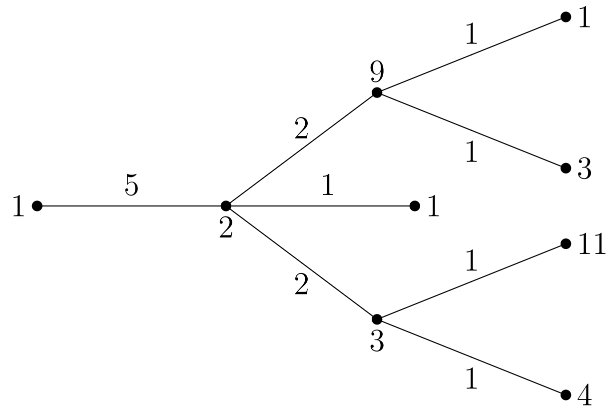

Figure 2.11: Tree of consistent cycles of the icosahedron

Example 2.5.13. We want to construct a tree of consistent cycles of an infinite graph. Consider the "Infinite $K_{2,2}$" in Figure 2.9. Let us start with the arc $(2,3)$. The size of the orbit of the vertex 3 under the stabilizer of the vertex 2 is 4. There are 4 images of the arc $(2,3)$ under its shunts: $(3,2)$,$(3,4)$,$(3,7)$, and $(3,9)$. So the first arc in the tree is labeled by 4. The arcs $(3,4)$ and $(3,9)$ are in the same orbit under the action of the stabilizer of $(2,3)$, so

$l(3,4) = 2$. In the tree the arcs $(3,2)$ and $(3,7)$ are labeled with 1, and the arc $(3,4)$ is labeled with 2.

The walk $(2,3,2)$ corresponds to the orbit of trivial cycles in the graph. The walk $(2,3,7)$ corresponds to the orbit of the consistent cycle $[2,3,7,8]$. The walk $(2,3,4)$ corresponds to the orbit of the consistent double rays which have multiplicity 2.

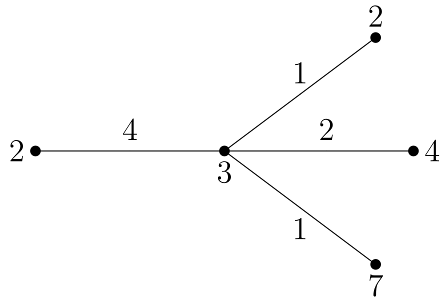

Figure 2.12: Tree of consistent cycles of the "Infinite $K_{2,2}$"

For a walk $\alpha = (v_0, v_1, \ldots)$ and an integer $i \in \{0, \ldots, n\}$ we denote $\alpha[i] = (v_0, \ldots, v_i)$.

In the following Theorem part (i) and (ii) are cited from Lemma 4.1 in [2].

Theorem 2.5.14. *Let G be closed in* $\mathrm{Aut}(\Gamma)$ *and let (T, ι, l) be the tree of consistent cycles of a graph Γ with respect to G and vertex v. Then the following hold:*

(i) For every vertex ν of T, $\iota(\nu) = v$ if and only if $\nu = \omega$ or ν is a leaf of T.

(ii) If ν is a leaf of T and $\eta \in T^{-}(\nu)$, then $l(\eta, \nu) = 1$.

(iii) There is a bijective correspondence between the maximal walks in T starting in the root and the G_v-congruence classes of G-consistent cycles in Γ starting in v.

The following proof follows the lines (partially one to one) of the proof of Lemma 4.1 in [2], with the difference that we generalize it to infinite graphs.

Proof. Part (i) follows directly from (T2) and the fact that the procedure of constructing T terminates when all the leaves are labeled by v.

To show part (ii), let ν be a leaf of T and let $\eta = \nu^-$ be the in-neighbour of ν. As before, let α_ν be the unique walk in T from the root ω to the leaf ν, and let $\alpha = \iota(\alpha_\nu)$. Since $\iota(\nu) = v$, it follows by (T3) and 2.1 that

$$l(\eta, \nu) = |R_G(\alpha)| = |v^{G_{(\hat{\alpha})}}|.$$

However, since the walk $\hat{\alpha}$ starts in v, the stabilizer $G_{(\hat{\alpha})}$ fixes v, and thus the right-hand side of the above equality is 1.

Let $\mathcal{B}$ be the set of all the maximal walks in T starting in the root, and let $\mathcal{C}$ be the set of G_v-congruence classes of G-consistent cycles in Γ starting in v. We shall prove Part (iii) by defining a pair of functions $\gamma : \mathcal{B} \to \mathcal{C}$ and $\delta : \mathcal{C} \to \mathcal{B}$, and by showing that one is the inverse of the other.

Let us define the function γ: For a walk $\beta \in \mathcal{B}$, let $\gamma(\beta)$ be the element of $\mathcal{C}$ containing $\iota(\beta)$. If β is finite, then $\beta = \alpha_\nu$ for a leaf $\nu \in T$ and thus $\iota(\nu) = v$. The walk $\iota(\alpha_\nu)$ is indeed closed and starts in v. It is G-consistent because of (T1). If $\beta = (\nu_0, \nu_1, \ldots)$ is an infinite maximal walk, then we want to show that $\iota(\beta)$ is a G-consistent ray.

For some $n \in \mathbb{N}$ the walk $\iota(\beta[n])$ is consistent because of (T1). The set of all shunts for $\iota(\beta[n])$ is

$$\mathrm{Sh_G}(\beta[n]) = G_{(\iota(\nu_0),\ldots,\iota(\nu_n)),(\iota(\nu_1),\ldots,\iota(\nu_{n+1}))} =$$

$$= \{g \in G \mid g(\iota(\nu_i)) = \iota(\nu_{i+1}) \text{ for all } 0 \leq i \leq n\}.$$

This set is compact by Lemma 2.4.12 for all n. Since

$$\mathrm{Sh_G}(\beta[i+1]) \subseteq \mathrm{Sh_G}(\beta[i])$$

for all i, the sequence $(\mathrm{Sh_G}(\beta[i]))_{i\in\mathbb{N}}$ is a descending sequence of compact sets. Hence there exists an element

$$g \in \bigcap_{n=0}^{\infty} (\mathrm{Sh_G}(\beta[i]))_{i\in\mathbb{N}}$$

because the intersection of a descending sequence of compact sets is not empty. This g is a shunt for $\iota(\beta)$, so the ray $\iota(\beta)$ is consistent. There exists a unique orbit of consistent cycles containing $\iota(\beta)$ by Lemma 2.4.14 .

Now, let us define the function $\delta : \mathcal{C} \to \mathcal{B}$. For an element $\alpha \in \mathcal{C}$ let $\alpha = (\ldots, v_{-1}, v_0, v_1, \ldots)$ be a consistent cycle in Γ containing the vertex v_0. We shall recursively define a sequence $\nu_0, \nu_1, \nu_2, \ldots$ of vertices in T, and G-consistent cycles $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \ldots$ such that for every $s \in \{0, 1, \ldots\}$ the following two conditions will be satisfied:

(P1) $\alpha^{(s)}$ is a G-consistent cycle in Γ which is G_v-congruent to α.

(P2) The sequence $(\nu_0, \ldots, \nu_s)$ is a walk in T and $\iota(\nu_i) = \alpha^{(s)}[i]$ for every $i \in \{0, \ldots, s\}$. This means that the cycles $\alpha^{(s-1)}$ and $\alpha^{(s)}$ coincide on the first $s-1$ vertices for every $s \in \{0, 1, 2, \ldots\}$.

Let $\nu_0 = \omega$ and $\alpha^{(0)} = \alpha$. Then (P1) holds for $s = 0$ because $\alpha^{(0)} = \alpha$ is G_v-congruent to α; and (P2) holds because (v_0) is a walk in T and $\iota(v_0) = \alpha^{(0)}[0] = \alpha[0] = v_0$.

Suppose now that for some $t \in \{1, 2 \ldots\}$, the vertices $\nu_0, \ldots, \nu_{t-1}$ and the walks $\alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(t-1)}$ are already defined such that (P1) and (P2) are satisfied for each $s \in \{0, \ldots, t-1\}$. Then let

$$\alpha^* = (\iota(\nu_0), \ldots, \iota(\nu_{t-1}))$$

be the walk in Γ consisting of the first t vertices of $\alpha^{(t-1)}$, and let $u = \alpha^{(t-1)}[t]$ be the $(t+1)$-th vertex of $\alpha^{(t-1)}$. Since $\alpha^{(t-1)}$ is G-consistent, so is the extension of α^* by u. Hence, by Lemma 2.5.5, we have $u \in X_G(\alpha^*)$. By the construction of T, each out-neighbour of ν_{t-1} in T uniquely corresponds to an element of the complete set of representatives of the $G_{(\alpha^*)}$-orbits on $X_G(\alpha^*)$. So there exists a unique vertex $\nu_t \in \Gamma^+(\nu_{t-1})$ such that u is in the $G_{(\alpha^*)}$-orbit of $\iota(\nu_t)$. Therefore, there exists $g \in G_{(\alpha^*)} \leq G_v$ such that $u^g = \iota(\nu_t)$. Let $\alpha^{(t)} = (\alpha^{(t-1)})^g$. The condition (P1) is satisfied for $s = t$ because $\alpha^{(t)}$ is G_v-congruent to $\alpha^{(t-1)}$, which is G_v-congruent to α by induction hypothesis, therefore $\alpha^{(t)}$ is G_v-congruent to α and thus G-consistent. Since ν_t is an out-neighbour of ν_{t-1}, the sequence $(\nu_0, \ldots, \nu_{t-1}, \nu_t)$ is a walk in T. By induction hypothesis, $(\alpha^{(t-1)}) = \iota(\nu_i)$ for $i \in \{0, \ldots, t-1\}$, and $\alpha^{(t)} = (\alpha^{(t-1)})^g$ for an element $g \in G_{(\alpha^*)}$. Therefore,

$$\iota(\nu_i) = \alpha^{(t-1)}[i] = \alpha^{(t)}[i] \text{ for } i \in \{0, \ldots, t-1\}.$$

Since $\alpha^{(t)}[t] = \iota(\nu_t)$, it follows that $\alpha^{(t)}[i] = \iota(\nu_i)$ for $i \in \{0, \ldots, t\}$. Thus (P2) is satisfied for $s = t$.

By the above recursive procedure, we have constructed a walk $\alpha_T = (\nu_0, \nu_1, \nu_2, \ldots)$. By (P1) and (P2), we obtain that $\iota(\alpha_T)$ is G-consistent in Γ and G_v-congruent to α. In particular, if α is closed and starts in v, also $\iota(\alpha_T)$ is closed and starts in v. But then $\iota(\nu_r) = v$, and by Part (i) ν_r is a leaf of T. If α is infinite, then $\alpha^{(s)}$ is infinite for all $s \geq 1$ and α_T is an infinite maximal walk in T. We may thus define the δ-image of the G_v-congruence class α^{G_v} to be the maximal walk α_T. It remains to show that this definition does not depend on the choice of the representative of α^{G_v}.

Suppose therefore that $\beta = \alpha^g$, for some $g \in G_v$. Then $\beta = (w_0, w_1, \ldots)$ for $w_0 = v$ and some vertices w_i, $i \geq 2$. Let $\beta_T = (\mu_0, \mu_1, \ldots)$ be the walk in T obtained from β in the same way as α_T is from α. Then, by definition,

$\delta(\beta) = \beta_T$. If $\mu_i = \nu_i$ for all $i \geq 0$, then δ maps the G_v-congruence classes of α and β to the same element, as required. Therefore, assume that there exists an index $r \geq 2$ such that $\mu_r \neq \nu_r$, and let $t \in \{0, \ldots, r-1\}$ be the smallest integer such that $\mu_t \neq \nu_t$. Since $\mu_0 = \omega = \nu_0$ and $\mu_1 = \nu_1$, it follows that $t \geq 2$. Also since $v_i = \iota(\nu_i) = \iota(\mu_i) = w_i$ for all $i \in \{0, \ldots, t-1\}$, it follows that the automorphism g, mapping α to β, belongs to the stabilizer $G_{(\alpha^*)}$ of the walk $\alpha^* = (v_0, \ldots, v_{t-1})$. In particular, v_t and w_t are in the same $G_{(\alpha^*)}$-orbit. By the construction of T, there exists a unique vertex ν_t in T such that v_t is in the $G_{(\alpha^*)}$-orbit of $\iota(\nu_t)$, and there exists a unique vertex μ_t in T such that w_t is in the $G_{(\alpha^*)}$-orbit of $\iota(\mu_t)$. Since v_t and w_t are in the same $G_{(\alpha^*)}$-orbit, it follows that $\nu_t = \mu_t$, which contradicts to our assumption on t. This shows that δ is a well-defined function from $\mathcal{C}$ to $\mathcal{B}$.

To finish the proof, let us show that the functions $\gamma : \mathcal{B} \to \mathcal{C}$ and $\delta : \mathcal{C} \to \mathcal{B}$ are inverse to each other. Observe first that every walk $\beta \in \mathcal{B}$ is the δ-image of the G_v-congruence class of $\iota(\beta)$, where β is a unique walk in T, beginning with the root ω. Hence δ is surjective, and γ is its inverse provided that it is its left inverse. Now, let α^{G_v} be an element of $\mathcal{C}$, and let $\alpha_T = (\nu_0, \nu_1, \ldots)$ be the corresponding walk in T. By definition of δ, we have $\delta(\alpha^{G_v}) = \alpha_T$. Since by definition $\gamma(\alpha_T)$ is the element of $\mathcal{C}$ containing $\iota(\alpha_T)$, it follows that

$$\gamma(\delta(\alpha^{G_v})) = \gamma(\alpha_T) = \iota(\alpha_T)^{G_v}.$$

□

The following Lemma and its proof are parts of the proof of Theorem 5.1 in [2].

Lemma 2.5.15. *Let Γ be a graph, let $v \in \Gamma$ be a vertex, and let G be a group acting vertex-transitively on Γ. There is a bijective correspondence between the set $\mathcal{C}_v$ of G_v-congruence classes of G-consistent walks in Γ starting in v and the set $\mathcal{C}$ of G-congruence classes of G-consistent directed cycles in Γ.*

Proof. For a walk $\alpha = (v, v_1, \ldots, v_{r-1}, v)$ in Γ let $\vec{C}(\alpha)$ denote the corresponding directed cycle in Γ. We define the function

$$\phi : \mathcal{C}_v \to \mathcal{C} \text{ by } \phi(\alpha^{G_v}) = \vec{C}(\alpha)^G.$$

This function is well defined, that is, independent of the choice of the representative of α^{G_v}. Since G is vertex-transitive, every element of $\mathcal{C}$ has a representative of the form $\vec{C}(\alpha)$, where α starts and ends in v. Hence ϕ is surjective. Let us now show that it is also injective. Suppose that $\phi(\alpha^{G_v}) = \phi(\beta^{G_v})$ for two G-consistent closed walks $\alpha = (v_0, \ldots, v_{r-1}, v_0)$ and $\beta = (w_0, \ldots, w_{r-1}, w_0)$ such that $v_0 = w_0 = v_r = w_r = v$. Then there exists

$g \in G$ such that α^g is some shift of β. In fact, by multiplying g by some powers of a shunt for β, we may assume that g maps α to β. But then g fixes v, and so $\alpha^{G_v} = \beta^{G_v}$. This shows that ϕ is a bijection. $\square$

In the following we use the restiction of the tree of consistent cycles as in Remark 2.5.11.

Lemma 2.5.16. *Let T be the tree of consistent cycles. Let C be a consistent cycle in Γ, let ν_n be the corresponding leaf in T, and let $\alpha = (\nu_0, \ldots, \nu_n)$ be the unique walk from the root ν_0 to the leaf ν_n. Then*

$$l(\nu_{n-1}, \nu_n) = m(C).$$

Proof. Let $C = (v_i)_{i\in\mathbb{Z}}$ and $\iota(\alpha) = C[n] = (v_0, \ldots, v_n)$ and $C[k] = (v_0, \ldots, v_k)$ for $k \geq n$. Since

$$l(\nu_{i-1}, \nu_i) = l(\nu_i, \nu_{i+1}) \ \forall i \geq n,$$

it follows that

$$R_G(C[n]) = R_G(C[k]) \ \forall k \geq n.$$

Let g be a shunt for C and let $(C[k]^{g^{-k}})_{k\geq n}$ be a sequence of walks. Since G is closed, the sequence $(C[k]^{g^{-k}})_{k\geq n}$ converges to $C^- = (v_i)_{i\leq 0}$. For all $k \geq n$ holds

$$R_G(C[k]) = |v_k^{G_{(C[k-1])}}| = |v_1^{G_{(C[k-1]-k+1)}}| = |v_1^{G_{(C^-)}}| = m(C).$$

Thus

$$m(C) = R_G(C[n]) = l(\nu_{n-1}, \nu_n).$$

$\square$

Theorem 2.5.17. *Let Γ be a regular graph. Let $G \leq \mathrm{Aut}(\Gamma)$ be a group that is closed in $\mathrm{Aut}(\Gamma)$ and acts vertex-transitively on Γ, and let $\mathcal{C}$ be the set of orbits of G-consistent cycles. Then*

$$\sum_{O\in\mathcal{C}} m(O) = \deg(\Gamma).$$

Proof. By $(T4)$ holds

$$\sum_{\xi\in L} l(\xi^-, \xi) = \sum_{\xi\in T^+(\omega)} l(\omega, \xi),$$

where L is the set of the leaves and ω is the root in T. By Lemma 2.5.16,

$$\sum_{\xi \in L} l(\xi^-, \xi) = \sum_{O \in \mathcal{C}} m(O),$$

where $\mathcal{C}$ is the set of all orbits of consistent cycles. Since $\sum_{\xi \in T^+(\omega)} l(\omega, \xi)$ is the sum of the sizes of orbits of the out-neighbours of a fixed vertex under its stabilizer, which is equal to the degree of the graph, holds

$$\sum_{O \in \mathcal{C}} m(O) = \deg(\Gamma).$$

This completes the proof of our main-theorem. □

Chapter 3

Applications

3.1 The Diestel-Leader graph

Consider a pair of infinite regular trees T_m and T_n for arbitrary $m, n \in \mathbb{N}$, where T_m has degree $m+1$ and T_n has degree $n+1$, arranged in horizontal levels. These levels are called horocycles $(H_i)_{i\in\mathbb{Z}}$. The horocycles are infinite, and for all $i, j \in \mathbb{N}$ there exists an automorphism of the graph $\mathrm{T}_{m,n}$ that maps H_i to H_j.

We denote the graph consisting of T_n and T_m by $\mathrm{T}_{m,n}$. Subgraphs of $\mathrm{T}_{2,2}$ and $T_{2,3}$ are illustrated in Figure 3.1 and Figure 3.3.

A vertex in the Diestel-Leader graph $\mathrm{DL}_{m,n}$ is a pair of vertices (u, v), where $u \in \mathrm{T}_n$ and $v \in \mathrm{T}_m$. Two vertices (u_1, v_1) and (u_2, v_2) in $\mathrm{DL}_{m,n}$ are adjacent if and only if $u_1 \sim u_2$ and $v_1 \sim v_2$.

For every vertex $(u, v) \in H_i$ in $\mathrm{DL}_{m,n}$, the vertex $u \in \mathrm{T}_m$ has one neighbour in $H_{i-1} \in \mathrm{T}_{m,n}$ and m neighbours in $H_{i+1} \in \mathrm{T}_{m,n}$, and $v \in T_n$ has n neighbours in $H_{i-1} \in \mathrm{T}_{m,n}$ and one in $H_{i+1} \in \mathrm{T}_{m,n}$. So for a vertex (u, v) in H_i in $\mathrm{DL}_{m,n}$, there exist n pairs of vertices in H_{i-1} and m pairs of vertices in H_{i+1} which are adjacent to (u, v). Therefore the vertex (u, v) has n neighbours in H_{i-1} and m neighbours in H_{i+1}, and thus $\mathrm{DL}_{m,n}$ has degree $m + n$.

The automorphisms in $\mathrm{DL}_{m,n}$ must preserve or invert the order of the sequence $(H_i)_{i\in\mathbb{Z}}$ of horocycles. All infinite consistent cycles in $\mathrm{DL}_{m,n}$ are double-rays $(v_i)_{i\in\mathbb{Z}}$, where for a $k \in \mathbb{Z}$ the vertices $v_i \in H_{i+k}$ for all $i \in \mathbb{Z}$. A shunt of an infinite consistent cycle maps H_i to H_{i+1} for all i.

A flip of an infinite consistent cycle is only a possible automorphism in $\mathrm{DL}_{m,n}$ for $m = n$, and not for $m \neq n$. This automorphism results a 180°-rotation of the graph. If $m \neq n$ two vertices can only be transposed within one horocycle.

The Diestel-Leader graph is vertex- and edge-transitive. For $m = n$ the graph $\mathrm{DL}_{m,n}$ is arc-transitive. An automorphism that transposes two adjacent vertices v in H_i and u in H_{i+1} inverts the order of horocycles.

The flip of $(H_i)_{i\in\mathbb{Z}}$ is an automorphism that swaps two adjacent vertices. For $m \neq n$ the graph $\mathrm{DL}_{m,n}$ is not arc-transitive; the 180°-rotation is no automorphism in this case.

The Diestel-Leader graph for $m \neq n$ is an example of a graph which is vertex- and edge-transitive, but not arc-transitive (see Remark 2.1.1).

In the following examples let us consider consistent cycles in the Diestel-Leader graphs $\mathrm{DL}_{2,2}$ and $\mathrm{DL}_{2,3}$.

Example 3.1.1. The graph $\mathrm{DL}_{2,2}$ has valence 4. Consider Figure 3.2. The 1-arc is a trivial consistent cycle because $\mathrm{DL}_{2,2}$ is arc-transitive. A double-ray with vertices $(w_i)_{i\in\mathbb{Z}}$, where $w_i \in H_i,\ i \in \mathbb{Z}$, is consistent and has multiplicity 2. The finite cycle $[(u,x),(w,y),(u,z),(v,y)]$ of length 4 is consistent. An automorphism containing the flip of the graph around the horizontal axis through the edges between H_1 and H_2 and the permutation of the vertices (u,x) and (u,z) is a shunt for this cycle. The set of neighbours of the vertices (v,y) and (w,y) which are in H_3 permute with the set of neighbours of (u,x) and (u,z) which are in H_0. Thus the sum of multiplicities of orbits of consistent cycles is 4.

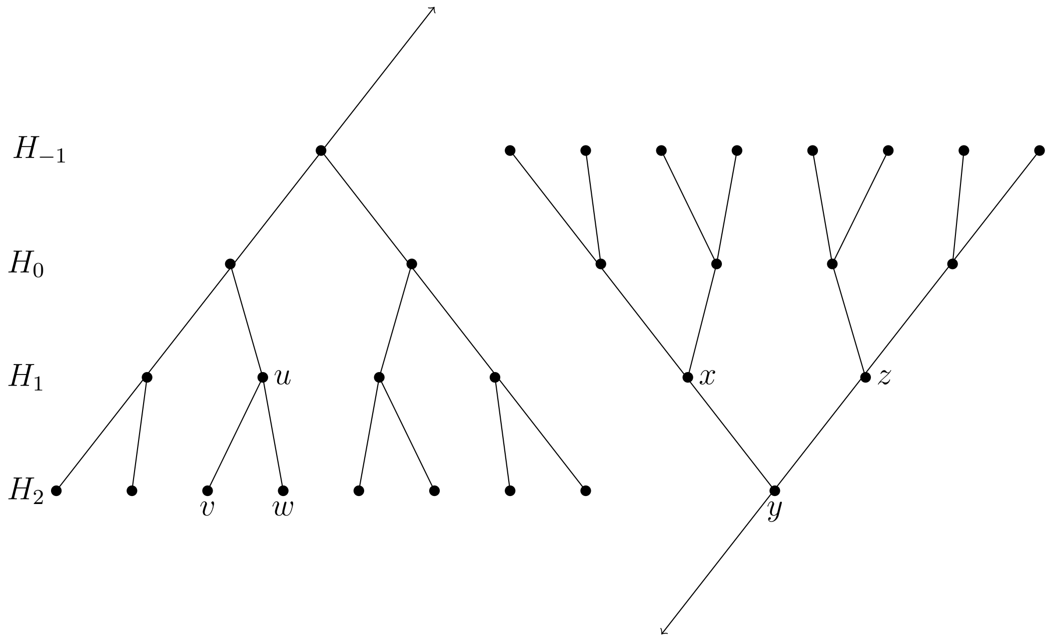

Figure 3.1: $\mathrm{T}_{2,2}$

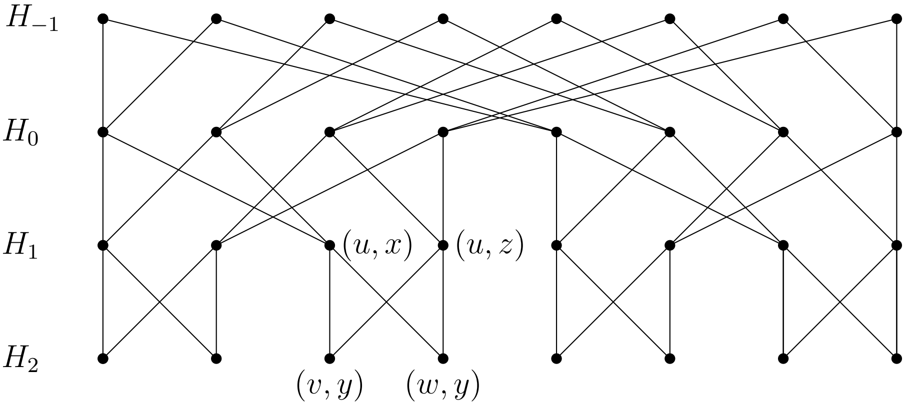

Figure 3.2: $\mathrm{DL}_{2,2}$

Example 3.1.2. The graph $\mathrm{DL}_{2,3}$ in Figure 3.4 has valence 5. In Figure 3.4 we see that the 1-arc is not consistent there because the graph has no horizontal symmetry-axis. There are no finite consistent cycles in this graph; the cycles of length 4 and 6 are not consistent here. A double ray $(w_i)_{i\in\mathbb{Z}}$ with $w_i \in H_i,\ i \in \mathbb{Z}$ is consistent and has multiplicity 2. It is not in the same orbit as its inverse, which is also consistent and has multiplicity 3. Thus, the sum of multiplicities of orbits of consistent cycles is 5.

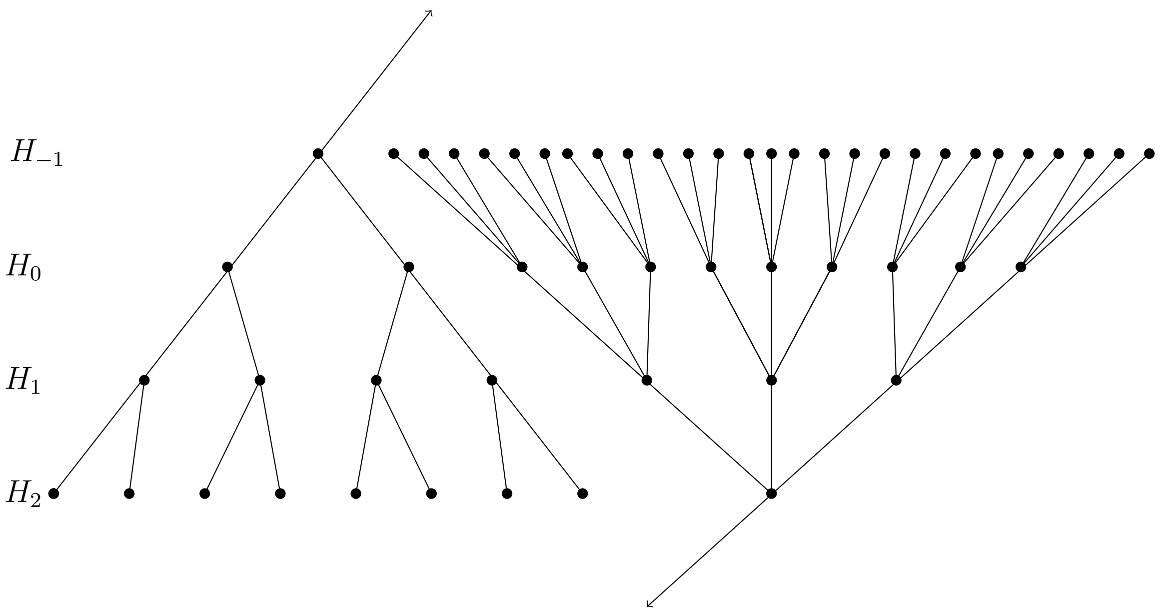

Figure 3.3: $\mathrm{T}_{2,3}$

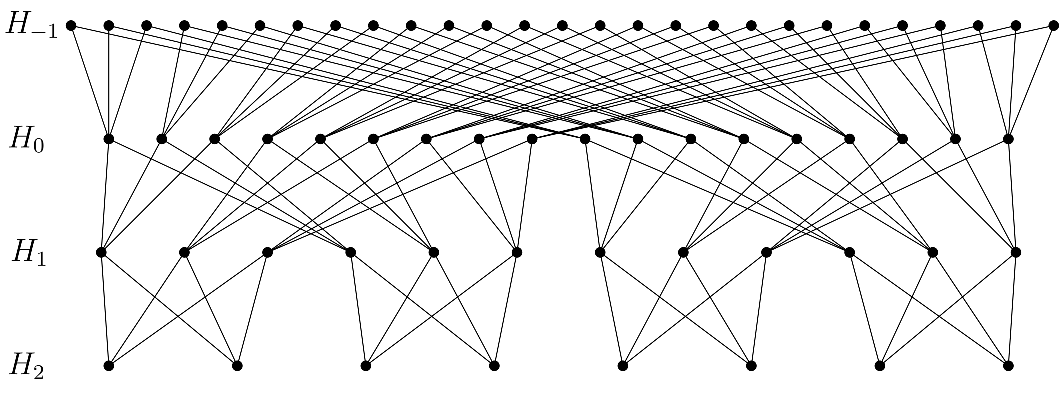

Figure 3.4: $\mathrm{DL}_{2,3}$

Definition 3.1.3. The graph $\mathrm{K}_{m,n}$ is a bipartite graph with a vertex set A of m vertices and a vertex set B of n vertices. Each vertex in A is adjacent to all vertices in B, and each vertex in B is adjacent to all vertices in A.

For $m \neq n$ the vertices in A have degree n and the vertices in B have degree m. Therefore $\mathrm{K}_{m,n}$ is not vertex-transitive for $m \neq n$ and thus it has no consistent cycles.

For $m = n$ all vertices have the same degree. The graph $\mathrm{K}_{n,n}$ has n consistent cycles, which have lengths $2, 4, 6, \ldots, 2n$ under the action of the full automorphism group. (See for example the graph $\mathrm{K}_{4,4}$ in Example 2.2.4).

Lemma 3.1.4. *In the Diestel-Leader graph* $\mathrm{DL}_{m,n}$ *for arbitrary* m, n *the sum of the multiplicities of orbits of consistent cycles under the action of the full automorphism group is equal to the degree of the graph, which is* $m + n$*.*

(i) For $m = n$ *the* $\mathrm{DL}_{m,n}$ *has* n *orbits of consistent cycles of lengths* $2, 4, 6, \ldots, 2n$ *and one orbit of infinite consistent cycles with multiplicity* n*.*

(ii) For $m \neq n$ *the* $\mathrm{DL}_{m,n}$ *has two orbits of infinite consistent cycles. One has multiplicity* m *and the other one has multiplicity* n*.*

Proof. The $\mathrm{DL}_{m,n}$, restricted to two horocycles H_i and H_{i+1}, splits into disjoint subgraphs isomorphic to $K_{m,n}$. Each of those graphs isomorphic to $\mathrm{K}_{m,n}$ has n vertices $\{v_1, \ldots, v_n\}$ in the horocycle H_i and m vertices $\{w_1, \ldots, w_m\}$ in H_{i+1}. Each of those n vertices in H_i has n neighbours in H_{i-1} and m neighbours in H_{i+1}. Analogously, each of the m vertices in H_{i+1} has m neighbours in H_{i+2} and n neighbours in H_i. Each of the vertices in H_i together with its neighbours in H_{i-1} belong to another graph isomorphic to $\mathrm{K}_{m,n}$. These graphs are disjoint for all of the n vertices in H_i. The same holds for the vertices in H_{i+1} together with their neighbours in H_{i+2}.

(i) For $m = n$ the $\mathrm{K}_{m,n}$ has n consistent cycles of lengths $2, 4, 6, \ldots, 2n$ under the action of the full automorphism group. These are also consistent cycles in $\mathrm{DL}_{n,n}$. An automorphism that transposes the set of vertices $\{v_1, \ldots, v_n\}$ in H_i and the set $\{w_1, \ldots, w_m\}$ in H_{i+1} also transposes the whole horocycles H_i and H_{i+1}. Thus, it also swaps H_{i-k} and H_{i+1+k} for all $k \in \mathbb{Z}$. Since all graphs isomorphic to $\mathrm{K}_{n,n}$ between two consecutive horocycles are disjoint, they can be permuted arbitrarily. The shunt automorphisms map H_{i-k} to H_{i+1+k} for all $k \in \mathbb{Z}$ and permute the graphs isomorphic to $\mathrm{K}_{n,n}$ between the particular horocycles. Thus $\mathrm{DL}_{n,n}$ has n finite consistent cycles. The double-ray $(w_i)_{i \in \mathbb{Z}}$ with $w_i \in H_i$ is consistent under the shunt mapping H_i to H_{i+1} for every

$i \in \mathbb{Z}$ and a convenient permutation of the vertices. The orbit of those double-rays has multiplicity n.

(ii) For $m \neq n$ there are no finite consistent cycles in $\mathrm{DL}_{m,n}$. An automorphism that transposes the vertex $v_i \in H_i$ and the vertex $v_{i+1} \in H_{i+1}$ also maps the horocycle H_i to H_{i+1} and H_{i-1} to H_{i+2}, respectively. The vertex v_i has n neighbours in H_{i-1}, and v_{i+1} has m neighbours in H_{i+2}. Thus, there is no automorphism that transposes the vertices v_i and v_{i+1}. The double-ray $(w_i)_{i \in \mathbb{Z}}$ with $w_i \in H_i$ is also consistent in $\mathrm{DL}_{m,n}$ for $m \neq n$ and has multiplicity n. It is not in the same orbit as its inverse. The orbit of the inverse double-ray has multiplicity m.

□

3.2 Orbits of $\frac{1}{k}$-consistent cycles

Definition 3.2.1. Let Γ be a finite graph and G be a group acting on Γ and $k \in \mathbb{N}$. A cyclet $\alpha = (v_i)_{i \in \mathbb{Z}}$ is called $\frac{1}{k}$-*consistent* if there exists $g \in G$ such that $v_i^g = v_{i+k}$ for all i. Such an automorphism is called a $\frac{1}{k}$-shunt for α.

Lemma 3.2.2. *If a cyclet α is $\frac{1}{k}$-consistent, then α is $\frac{1}{nk}$-consistent for all $n \in \mathbb{N}$.*

Proof. Let $g \in G$ be a $\frac{1}{k}$-shunt for α. Then g maps v_i to v_{i+k}, and g^n maps v_i to v_{i+nk} for all $n \in \mathbb{N}$. So g^n is a $\frac{1}{nk}$-shunt for α. □

Lemma 3.2.3. *If a cyclet α is $\frac{1}{k}$-consistent and $\frac{1}{l}$-consistent for $k, l \in \mathbb{N}$ and $j = \gcd(k, l)$, then α is also $\frac{1}{j}$-consistent.*

Proof. Let $g \in G$ be a $\frac{1}{k}$-shunt and let $h \in G$ be a $\frac{1}{l}$-shunt for α. Because $j = \gcd(k.l)$, it follows that $j = nk + ml$ for certain $n, m \in \mathbb{Z}$. The automorphism $g^n h^m$ maps v_i to $v_{i+nk+ml} = v_{i+j}$ for all i. So $g^n h^m$ is a $\frac{1}{j}$-shunt for α. □

Note that for a finite cyclet of length r we do not require that k is a divisor of r.

Corollary 3.2.4 (Section 2 in [3])**.** *Let α be a cyclet of length r. If $k' = \gcd(k, r)$, then α is $\frac{1}{k}$-consistent if and only if it is $\frac{1}{k'}$-consistent.*

Proof. Because α has length r, it is $\frac{1}{r}$-consistent. If α is $\frac{1}{k}$-consistent, it is also $\frac{1}{k'}$-consistent by Lemma 3.2.2.

On the other hand, if $k' = \gcd(k, r)$, then $k = k'n$ for $n \in \mathbb{N}$. If α is $\frac{1}{k'}$-consistent, it follows by Lemma 3.2.3 that α is $\frac{1}{k}$-consistent. □

Definition 3.2.5. A cyclet is called *precisely* $\frac{1}{k}$-consistent if it is $\frac{1}{k}$-consistent, but not $\frac{1}{l}$-consistent for any $l < k$.

From Lemma 3.2.2 and Lemma 3.2.3 follows that if a cyclet is $\frac{1}{k}$-consistent and precisely $\frac{1}{l}$-consistent, then l is a divisor of k.

Observe that if a cyclet is $\frac{1}{k}$-consistent, then all of its shifts and their inverses are also consistent.

Since a cycle is the set of all shifts and inverses of an underlying cyclet and a directed cycle is the set of all shifts of an underlying cyclet, we will define a (directed) cycle to be $\frac{1}{k}$-consistent if any (and thus each) underlying cyclet is $\frac{1}{k}$-consistent.

It follows that the number of orbits of consistent cycles is greater equal to the number of orbits of directed cycles, and the number of orbits of directed cycles is greater equal to the number of orbits of cycles.

Definition 3.2.6. A $\frac{1}{k}$-consistent directed cycle is said to be *G-symmetric* provided that there exists an automorphism in G sending it to the inverse of one of its shifts, and it is said to be *G-chiral* otherwise.

(Section 7 in [3]) Note that not all representatives of a G-consistent cycle need to be G-congruent. Namely, it might happen that a representative α is not G-congruent to its inverse and thus to any of the cyclic shifts of α^{-1}. A G-consistent cycle in which all of its representatives are G-congruent will be called *G-symmetric*. A G-consistent cycle which is not G-symmetric is called *G-chiral*. Note that a non-trivial cycle of length r is G-chiral if and only if the set-wise stabilizer of its edges acts on it as a cyclic group of order r, and it is G-symmetric if and only if the set-wise stabilizer acts on it as a dihedral group of order $2r$. An (undirected) cycle can be considered as a pair of directed cycles. If the directed cycles are chiral, then two orbits of directed cycles correspond to one orbit of undirected cycles. If the directed cycles are symmetric, then there is a one-to-one correspondence between orbits of cycles and orbits of directed cycles. We denote the number of orbits of chiral directed cycles with c and the number of orbits of symmetric directed cycles with s. The number of orbits of directed cycles is $c + 2s$ and the number of orbits of undirected cycles is $c + s$.

Let G be a group of automorphisms. Contrary to a consistent cyclet, for $k > 1$ the group G cannot map a precisely $\frac{1}{k}$-consistent cyclet to its t-shift for $t < k$ because then the cyclet would also be $\frac{1}{t}$-consistent. If a directed cycle α is precisely $\frac{1}{k}$-consistent for $k > 1$, the underlying cyclets are not all in the same G-orbit. They split into k orbits of precisely $\frac{1}{k}$-consistent cyclets and the shifts $\alpha, \alpha^1, \ldots, \alpha^{k-1}$ form a complete set of representatives of G-orbits of $\frac{1}{k}$-consistent cyclets. The group G acts on the set of $\frac{1}{k}$-consistent directed cycles in a natural way. One orbit of precisely $\frac{1}{k}$-consistent directed cycles corresponds to k orbits of precisely $\frac{1}{k}$-consistent cyclets. So, contrary to consistent cycles, there is a difference between counting "orbits of directed cycles" and "orbits of cyclets". The following theorems give formulas for counting "orbits of directed cycles" and "orbits of cyclets ", respectively.

Definition 3.2.7. The *girth* of a graph is the length of the smallest closed walk.

In the following theorems of orbits of $\frac{1}{k}$-consistent cyclets we assume that the graph has girth at least $2k+1$. From this follows that if $\alpha = (v_0, \ldots, v_r)$ is a walk of length r and $\beta = (w_0, \ldots, w_s)$ is a walk of length s, where $r, s \leq k$, and $v_0 = w_0,\ u_r = u_s$, then $\alpha = \beta$.

Theorem 3.2.8 (Theorem 2.1 in [3]). *Let $G \leq \mathrm{Aut}(\Gamma)$ be a group acting arc-transitively on a finite graph Γ with girth at least $2k+1$ for $k \in \mathbb{N}$ and degree d, then the number of orbits of non-trivial $\frac{1}{k}$-consistent cyclets is exactly $(d-1)^k$.*

Theorem 3.2.9 (Theorem 5.1 in [3]). *Let k be a positive integer, let Γ be a finite graph of girth at least $2k+1$, and let G be an arc-transitive subgroup of $\mathrm{Aut}(\Gamma)$. Then the number of G-orbits of directed $\frac{1}{k}$-consistent cycles in Γ is exactly*

$$\frac{1}{k}\sum_{l|k}\varphi(\frac{k}{l})(d-1)^k,$$

where φ denotes the Euler totient (and the summation is taken over all positive divisors of k).

Remark 3.2.10. In these two theorems the trivial cycles are not counted. In arc-transitive graphs a trivial cycle is clearly $\frac{1}{k}$-consistent for all k.

Example 3.2.11. Consider another view of the dodecahedron in Figure 3.5. This graph has girth 5 and degree 3. By Theorem 3.2.8, there are 4 orbits of non-trivial $\frac{1}{2}$-consistent cyclets under the action of the full automorphism group.

We can see that the $60°$-rotation is an automorphism which rotates the outer cycle C of length 12 by 2 steps. So C is $\frac{1}{2}$-consistent. Since it is not consistent, there is no automorphism that maps the cyclet

$$\alpha = \ (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

to its shift $\alpha^1 = (2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1)$. So the cycle C splits into two orbits of cyclets. One consists of all even t-shifts of α and all odd t-shifts of α^{-1}. The other one consists of all odd t-shifts of α and all even t-shifts of α^{-1}. The cycle C is chiral because G does not map α to its inverse, but rather to the shift of its inverse.

The orbits of consistent cyclets of length 5 and of length 10 are also $\frac{1}{2}$-consistent. Thus we have 4 orbits of $\frac{1}{2}$-consistent cyclets.

By the formula in Theorem 3.2.9, there are 3 orbits of consistent directed cycles under the action of the full automorphism group. In addition to the orbits of directed cycles of lengths 5 and 10, the directed cycle of length 12 represents only one orbit here.

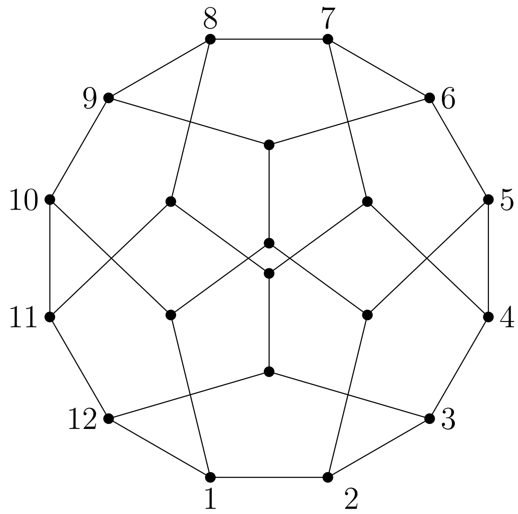

Figure 3.5: Dodecahedron

Example 3.2.12. We consider an example of an infinite graph. In Section 2.3 we study consistent cycles in the "Honeycomb lattice". Now let's have a look at $\frac{1}{2}$-consistent cyclets. This graph has girth 6 and valence 3. The two consistent cyclets (see Figure 2.5) are of course also $\frac{1}{2}$-consistent. The cycle $[\ldots, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots]$ in Figure 3.6 splits into two orbits of precisely $\frac{1}{2}$-consistent cyclets. One consists of the even shifts of the cyclet $(\ldots, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots)$, and the other one consists of the odd shifts of this cyclet. We have 4 orbits of non-trivial consistent cyclets, and Theorem 3.2.8 holds as in the finite case.

Remark 3.2.13. If the condition on the girth fails to be true, the formula in Theorem 3.2.8 does not hold in general. For example, the graph of the tetrahedron shown in Figure 3.7 has 2 consistent cyclets $[1, 2, 3]$ and $[1, 2, 4, 3]$. By Theorem 3.2.8, there would be 4 orbits of $\frac{1}{2}$-consistent cyclets, but there is no $\frac{1}{2}$-consistent cyclet which is not consistent.

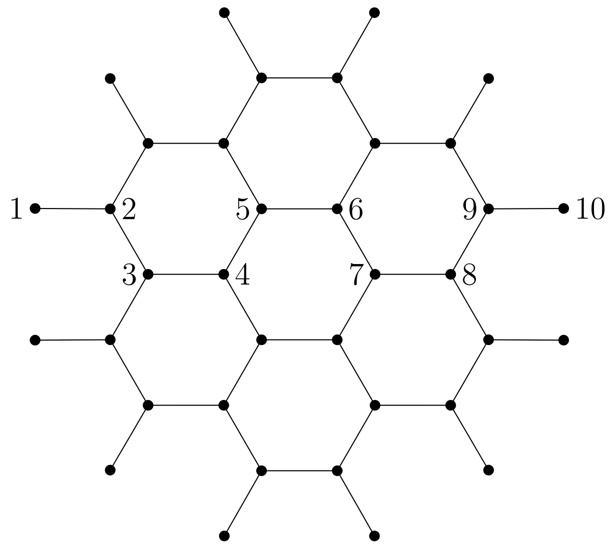

Figure 3.6: Honeycomb lattice

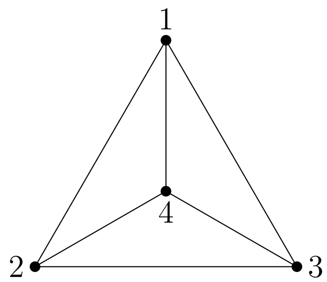

Figure 3.7: Tetrahedron

3.3 Consistent cycles in $\frac{1}{2}$-arc-transitive graphs

Definition 3.3.1. We say that a subgroup G of $\mathrm{Aut}(\Gamma)$ is $\frac{1}{2}$-arc-transitive if it acts transitively on the set of vertices and the set of edges, but intransitively on the set of arcs.

For a group G acting $\frac{1}{2}$-arc-transitively every G-consistent directed cycle is chiral. We define a $\frac{1}{k}$-consistent (undirected) cycle as pair of the two underlying $\frac{1}{k}$-consistent directed cycles.

Proposition 3.3.2 (Proposition 5.3 in [5]). *If Γ is a d-valent finite graph admitting a $\frac{1}{2}$-arc-transitive group of automorphisms G, then the number of G-orbits of consistent undirected cycles is $\frac{d}{2}$.*

Note that $\frac{1}{2}$-arc-transitive finite graphs always have even degree (see Remark 2.1.1).

Theorem 3.3.3 (Theorem 4.3 in [5]). *Let k be a positive integer, let Γ be a finite graph of girth at least $2k+1$ and valence d, and let G be a $\frac{1}{2}$-arc-transitive subgroup of* $\mathrm{Aut}(\Gamma)$. *Then there are exactly $(d-1)^k+1$ G-orbits of $\frac{1}{k}$-consistent cyclets in Γ.*

Note that if G is not arc-transitive, then the trivial cycles are not consistent. If we counted trivial cycles in Theorem 3.2.8, we would get the same result as for arc-transitive group-actions.

Theorem 3.3.4 (Theorem 5.2 in [5]). *Let k be a positive integer, let Γ be a finite graph of girth at least $2k+1$ and valence d, and let G be a $\frac{1}{2}$-arc-transitive subgroup of* $\mathrm{Aut}(\Gamma)$. *Then the number of orbits of directed $\frac{1}{k}$-consistent cycles is exactly*

$$\frac{1}{k}\sum_{l|k}\varphi(\frac{k}{l})((d-1)^k+1),$$

where φ denotes the Euler totient (and the summation is taken over all positive divisors of k).

Example 3.3.5. Consider the graph in Figure 3.8. Let the group acting on the graph be the group consisting of the automorphisms which rotate the graph and map the vertices from the inner cycle to the outer cycle. This group acts vertex-transitively and edge-transitively, but not arc-transitively because there is no element that flips an edge. The degree of the graph is 4. By Proposition 3.3.2, there are 2 orbits of consistent cycles: $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8]$ and $[v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}]$.

The cycle $[v_1, v_2, v_9, v_{10}]$, which is consistent under the action of the full automorphism group, is not consistent here.

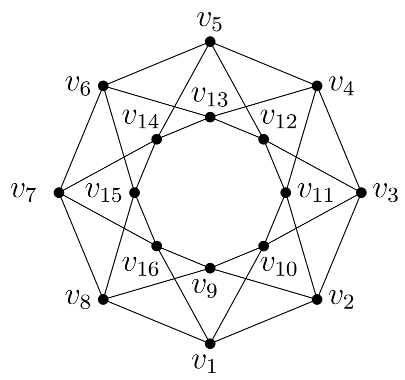

Figure 3.8

In Section 6.1 in [5] the authors study finite graphs of valence 4. By Theorem 3.3.3 and Theorem 3.3.4, there exist precisely four orbits of consistent cyclets in such graphs and the same number of orbits of consistent directed cycles under the action of an $\frac{1}{2}$-arc-transitive group. Since G is not arc-transitive, none of these orbits is symmetric. Hence there are precisely two orbits of G-consistent cycles. In the rest of the section they describe the orbits of consistent cycles in the Doyle-Holt graph. With its 27 vertices, this is the smallest $\frac{1}{2}$-arc-transitive graph (under the action of the full automorphism group). There are illustrations of the Doyle-Holt graph on pages 8 and 9 in [5].

Bibliography

[1] N. L. Biggs, Aspects of symmetry in graphs, in *Algebraic Methods in Graph Theory,* Vol. I, II (Szeged, 1978), Colloq. Math. Soc. János Bolyai, 25, North-Holland, Amsterdam-New York, 1981, 27–35.

[2] Š. Miklavič, P. Potočnik, and S. Wilson, Consistent cycles in graphs and digraphs, *Graphs Combin.* **23**, (2007), no. 2, 205–216.

[3] Š. Miklavič, P. Potočnik, and S. Wilson, Overlap in consistent cycles, *J. Graph theory* **55**, (2007), no. 1, 55–71.

[4] W. M. Kantor, Cycles in graphs and groups, *Amer. Math. Monthly* **115** (2008), no. 6, 559–562.

[5] M. Boben, S. Miklavič and P. Potočnik, Consistent cycles in $\frac{1}{2}$-arc-transitive graphs, *The electronic Journal of Combinatorics* **16**, (2009), no. 1, #R5

[6] W. Woess, Topological groups and infinite graphs, *Discrete Mathematics* **95** (1991), no. 1–3, 373–384

Curriculum Vitae

Personal information

Name	Julia Wessely
Date of Birth	14th of March, 1984
Place of Birth	Vienna, Austria
Nationality	Austrian
email	a0404215@unet.univie.ac.at

Education

Since 10/2006	Teacher education programme in Mathematics, University of Vienna
Since 10/2004	Diploma programme in Mathematics, University of Vienna
2002-2004	Kolleg für Mode und Bekleidungstechnik Wien XVI
06/2002	Matura
1994-2002	Realgymnasium mit sportlichem Schwerpunkt Wien XVII
1991-1994	Primary school Wien XVI

Employment

Since 03/2010	Tutor at the Faculty of Scientific Computing, University of Vienna
03/2009	Co-worker at the exhibition "Imaginary", Vienna
Since 2004	Private tuition at IFL Dr. Rampitsch and Lernquadrat

Other activities

09/2011	Talk about *Consistent cycles* in the "25th LL-Seminar on Graph Theory", Montanuniversität Leoben