Application of Riccati-Bessel Functions in Light Scattering

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Abstract

Riccati-Bessel functions, written as combinations of amplitude and phase functions, have been used to re-express Mie theory. This leads to a simple physical explanation of the scattered phase angle as the sum of a phase shift arising from the optical path difference across the particle radius and an interfacial phase difference at the surface of the particle. The mathematical properties of the different phase angles are examined in detail by treating the order as a continuous variable.

1 Introduction

It has been shown that Riccati-Bessel (R-B) functions and their derivatives with respect to the wave variable $z = kr$ can be usefully represented in terms of amplitude and phase functions [1,2]. Thus for spherical R-B functions:

$$\begin{aligned} \varphi_n(z) &= M_n(z)\sin\theta_n(z) & \varphi_n'(z) &= N_n(z)\cos\phi_n(z) \\ \chi_n(z) &= -M_n(z)\cos\theta_n(z) & \chi_n'(z) &= N_n(z)\sin\phi_n(z) \\ \xi_n(z) &= -i\,M_n(z)\exp i\theta_n(z) & \xi_n'(z) &= N_n(z)\exp i\phi_n(z) \end{aligned} \tag{1}$$

in which the spatial phase angles $\theta_n(z)$ and $\phi_n(z)$ have the forms:

$$\begin{aligned} \theta_n(z) &= z - n\,\pi/2 + \gamma_n(z) \\ \phi_n(z) &= \theta_n(z) + \Delta_n(z) \end{aligned} \tag{2}$$

and the amplitudes are related by

$$M_n(z)N_n(z)\cos\Delta_n(z) = 1. \tag{3}$$

Here, $\theta_n(z)$ is fully defined by

$$\theta_n(z) = \tan^{-1}\left[-\frac{\varphi_n(z)}{\chi_n(z)}\right] \tag{4}$$

together with the boundary condition $\theta_n(0) = 0$. Also, the auxiliary phase angle $\gamma_\nu(z)$ and the phase shift $\Delta_n(z)$ (associated with differentiation w.r.t. z) are constrained by $\underset{z=0}{n\,\pi/2} \ge \gamma_n(z) > \underset{z\to\infty}{0}$ and $\underset{z=0}{\pi/2} \ge \Delta_n(z) > \underset{z\to\infty}{0}$.

From Eqs. (1), (2) and (3), three ratio functions can be derived which will be applied later when analysing light scattering. These are

$$\begin{aligned} \frac{\varphi_n(z)}{\chi_n(z)} &= -\tan\theta_n(z) \\ \frac{\varphi_n'(z)}{\varphi_n(z)} &= \frac{1}{M_n^2(z)}\left[\frac{1}{\tan\theta_n(z)} - \tan\Delta_n(z)\right] \\ \frac{\chi_n'(z)}{\chi_n(z)} &= -\frac{1}{M_n^2(z)}\left[\tan\theta_n(z) + \tan\Delta_n(z)\right]. \end{aligned} \tag{5}$$

Note that the definitions given here for spherical R-B functions differ from those given in Handbook of Mathematical Functions [2,3].

2 Re-expression of Mie theory

To apply our treatment to light scattering, we consider the case of a homogeneous sphere of radius a and refractive index m. This has external and internal size parameters $\alpha = ka$ and $\beta = mka$ respectively for light with a propagation constant of k. Hence the Mie scattering coefficients, obtained from the boundary conditions of the electromagnetic fields at the surface of the sphere, are:

$$a_n = \frac{\varphi_n'(\beta)\varphi_n(\alpha) - m\varphi_n(\beta)\varphi_n'(\alpha)}{\varphi_n'(\beta)\xi_n(\alpha) - m\varphi_n(\beta)\xi_n'(\alpha)}$$
$$b_n = \frac{m\varphi_n'(\beta)\varphi_n(\alpha) - \varphi_n(\beta)\varphi_n'(\alpha)}{m\varphi_n'(\beta)\xi_n(\alpha) - \varphi_n(\beta)\xi_n'(\alpha)} \tag{6}$$

for $n = 1, 2, 3, \ldots$. Unfortunately, Eqs. (6) contain no meaningful physics but this can be partially remedied by introducing the scattered phase angles u_n, v_n through the relations: $a_n = \frac{1}{2}(1 - \exp i2u_n)$ and $b_n = \frac{1}{2}(1 - \exp i2v_n)$ to give:

$$\tan u_n = \frac{\varphi_n'(\beta)\varphi_n(\alpha) - m\varphi_n(\beta)\varphi_n'(\alpha)}{\varphi_n'(\beta)\chi_n(\alpha) - m\varphi_n(\beta)\chi_n'(\alpha)}$$
$$\tan v_n = \frac{m\varphi_n'(\beta)\varphi_n(\alpha) - \varphi_n(\beta)\varphi_n'(\alpha)}{m\varphi_n'(\beta)\chi_n(\alpha) - \varphi_n(\beta)\chi_n'(\alpha)}. \tag{7}$$

Moreover, contributions to the scattered phase can arise from only two possible sources which are:

(a) a phase shift $\theta_n(\beta) - \theta_n(\alpha)$ associated with the optical path difference of intrinsic waves across the particle radius, and

(b) an interfacial phase change u_n^i, v_n^i across the particle surface.

Thus, the total scattered phase angles are assumed to have the form

$$\begin{Bmatrix} u_n \\ v_n \end{Bmatrix} = \theta_n(\beta) - \theta_n(\alpha) + \begin{Bmatrix} u_n^i \\ v_n^i \end{Bmatrix}. \tag{8}$$

Eqs.(8) are verified by substituting for u_n and v_n in Eqs. (7) followed by the use of Eqs. (5) to eliminate $\theta_n(\alpha)$ from both sides of the equations so as to derive:

$$\tan\left[\theta_n(\beta) + u_n^i\right] = \frac{1}{\left[\dfrac{A_n^u}{\tan\theta_n(\beta)} + B_n^u\right]}$$
$$\tan\left[\theta_n(\beta) + v_n^i\right] = \frac{1}{\left[\dfrac{A_n^v}{\tan\theta_n(\beta)} + B_n^v\right]}, \tag{9}$$

where $A_n^u = \dfrac{M_n^2(\alpha)}{mM_n^2(\beta)}$, $B_n^u = \tan\Delta_n(\alpha) - A_n^u \tan\Delta_n(\beta)$ and $A_n^v = \dfrac{mM_n^2(\alpha)}{M_n^2(\beta)}$, $B_n^v = \tan\Delta_n(\alpha) - A_n^v \tan\Delta_n(\beta)$. Finally, u_n^i and v_n^i are completely separated as

$$\tan u_n^i = \frac{\left[1 - A_n^u - B_n^u \tan\theta_n(\beta)\right]\tan\theta_n(\beta)}{A_n^u + B_n^u \tan\theta_n(\beta) + \tan^2\theta_n(\beta)}$$
$$\tan v_n^i = \frac{\left[1 - A_n^v - B_n^v \tan\theta_n(\beta)\right]\tan\theta_n(\beta)}{A_n^v + B_n^v \tan\theta_n(\beta) + \tan^2\theta_n(\beta)}. \tag{10}$$

3 Results

Although the treatment above was for spherical R-B functions, it may now be generalized to all R-B functions having the form:

$$\tan u_\nu = \frac{c\,\varphi_\nu'(\beta)\varphi_\nu(\alpha) - \varphi_\nu(\beta)\varphi_\nu'(\alpha)}{c\,\varphi_\nu'(\beta)\chi_\nu(\alpha) - \varphi_\nu(\beta)\chi_\nu'(\alpha)} \tag{11}$$

in which c is a constant for all orders and $\nu = n + x$ for $0 \le x < 1;\ n = 0, 1, 2, \ldots$. Relations (8), (9) and (10) are still valid and so:

$$u_\nu = \theta_\nu(\beta) - \theta_\nu(\alpha) + u_\nu^i \tag{12}$$

$$\tan\left[\theta_\nu(\beta) + u_\nu^i\right] = \frac{1}{\left[\dfrac{A_\nu}{\tan\theta_\nu(\beta)} + B_\nu\right]} \tag{13}$$

$$\tan u_\nu^i = \frac{\left[1 - A_\nu - B_\nu \tan\theta_\nu(\beta)\right]\tan\theta_\nu(\beta)}{A_\nu + B_\nu \tan\theta_\nu(\beta) + \tan^2\theta_\nu(\beta)}, \tag{14}$$

where $A_\nu = c\dfrac{M_\nu^2(\alpha)}{M_\nu^2(\beta)}$, $B_\nu = \tan\Delta_\nu(\alpha) - A_\nu \tan\Delta_\nu(\beta)$.

Eqs. (7) have, however, been previously examined by van de Hulst [4] and this analysis too can be extended for general R-B functions so as to yield the following results.

(a) Nodes of the first kind

These occur when $\theta_\nu(\beta) = p\pi$ since then $\varphi_\nu(\beta) = 0$ and

$$\tan\begin{Bmatrix} u_\nu \\ v_\nu \end{Bmatrix} = \frac{\varphi_\nu(\alpha)}{\chi_\nu(\alpha)} = -\tan\theta_\nu(\alpha) \tag{15}$$

to give $u_\nu^i, v_\nu^i = 0$ and $u_\nu, v_\nu = \theta_\nu(\beta) - \theta_\nu(\alpha)$. It may also be shown that $v_{\nu-1}$ and $v_{\nu+1}$ have the same phase as v_ν, hence all four modes satisfy the conditions

$$v_{\nu-1}, v_\nu, v_{\nu+1}, u_\nu = \theta_\nu(\beta) - \theta_\nu(\alpha) \tag{16}$$

and

$$v_{\nu-1}^i, v_\nu^i, v_{\nu+1}^i, u_\nu^i = 0. \tag{17}$$

(b) Nodes of the second kind

These are present when $\phi_\nu(\beta) = (p + 1/2)\pi$ corresponding to $\varphi_\nu'(\beta) = 0$ and

$$\tan\begin{Bmatrix} u_\nu \\ v_\nu \end{Bmatrix} = \frac{\varphi_\nu'(\alpha)}{\chi_\nu'(\alpha)} = \frac{1}{\tan\phi_\nu(\alpha)}. \tag{18}$$

Thus,

$$v_\nu, u_\nu = \phi_\nu(\beta) - \phi_\nu(\alpha) = \theta_\nu(\beta) - \theta_\nu(\alpha) + \Delta_\nu(\beta) - \Delta_\nu(\alpha) \tag{19}$$

and

$$u_\nu^i, v_\nu^i = \Delta_\nu(\beta) - \Delta_\nu(\alpha). \tag{20}$$

Such generalized expressions are valuable when investigating scattering since, by treating the order as a continuous variable, quasi-continuous graphs can be plotted of the various phase angles rather than the sparse sets of discrete points associated with spherical results alone. It should however be remembered that in the notation of general R-B functions, spherical R-B functions correspond to the orders $\nu = n + 1/2$.

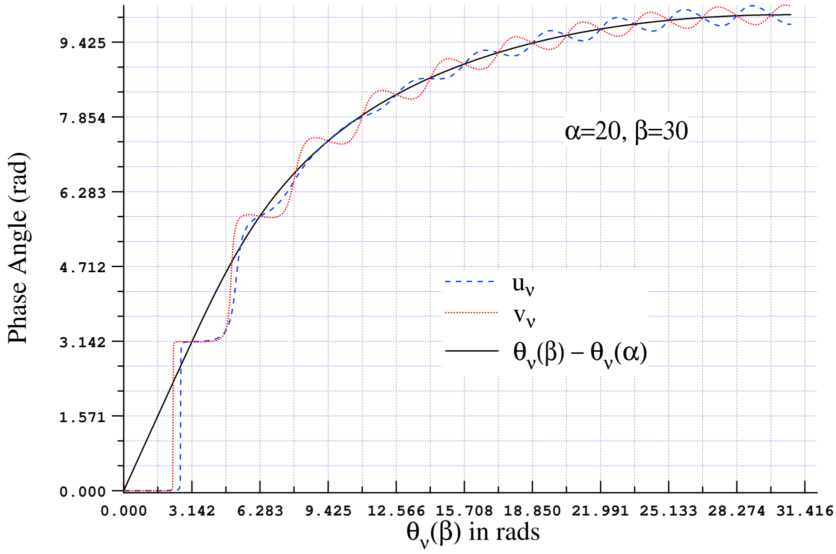

Figure 1: Display of van de Hulst's nodes of the first kind at $\theta_\nu(\beta) = p\pi$ and the second kind at $\theta_\nu(\beta) = (p + 1/2)\pi$.

Plots of the various contributions to the scattered phase angles are presented in figures 1-3 for a hypothetical R-B particle having external and internal size parameters of 20 and 30 respectively. In figure 1, nodes of the first kind are indicated by the crossing of all three traces at $\theta_\nu(\beta) = p\pi$ while for nodes of the second kind only the u_ν and v_ν curves cross at $\theta_\nu(\beta) = (p + 1/2)\pi$. The nodes are similarly recognized in figure 2 where they are displayed as a function of order and the positions of the $v_{\nu-1}$ and $v_{\nu+1}$ modes obtained. Finally figure 3 presents the interfacial phase difference as a function of order. Other features of interest are:
(a) modes make a maximum contribution to scattering whenever $u_\nu, v_\nu = (p + 1/2)\pi$ and the "half-power points" are obtained from $u_\nu, v_\nu = (p + 1/4)\pi$ and $(p + 3/4)\pi$, (b) the two step edges of height π between

orders 25 and 26 are characteristic of resonance scattering and (c) an effective cut-off order exists for the model sphere at $\nu \sim 26$.

4 Discussion

General R-B functions have been shown to play a crucial role in understanding the physical principals underlying light scattering from a homogeneous sphere. As a consequence, the scattered phase angles u_ν, v_ν of a hypothetical R-B particle can be fully explained in terms of the optical phase associated with the optical path difference $\theta_\nu(\beta) - \theta_\nu(\alpha)$ of intrinsic waves across the particle radius and an interfacial phase difference u_ν^i, v_ν^i at the surface of the sphere. The latter functions can be calculated directly from Eqs. (10).

Furthermore, the present analysis provides a physical rationale for the sequence of steps in the form of a descending staircase reported in Ref. [1] for anomalous diffraction at a large sphere.

Other applications are light scattering from:
(i) multi-layered spheres and (after some modification)
(ii) infinite homogeneous circular cylinders, but also
(iii) acoustic scattering at a homogeneous elastic sphere and
(iv) nuclear scattering at a spherical square well.

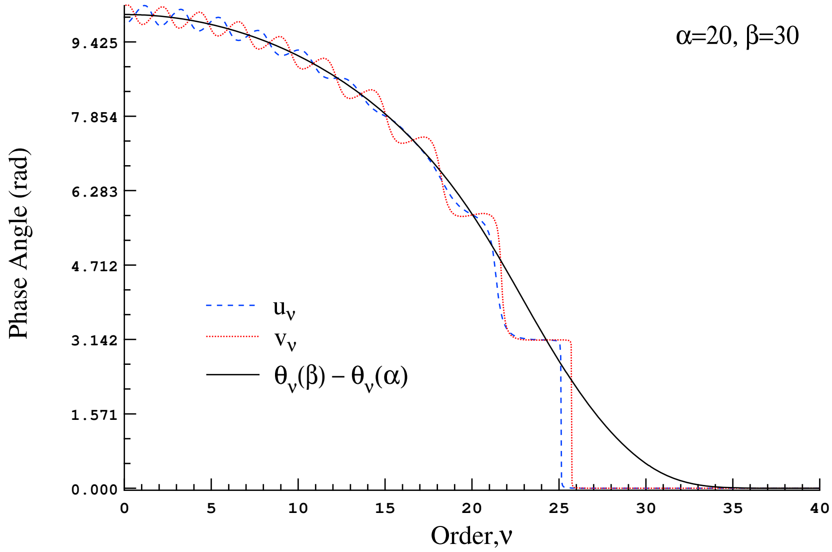

Figure 2. Comparison of the scattered phase angles u_ν, v_ν with $\theta_\nu(\beta) - \theta_\nu(\alpha)$.

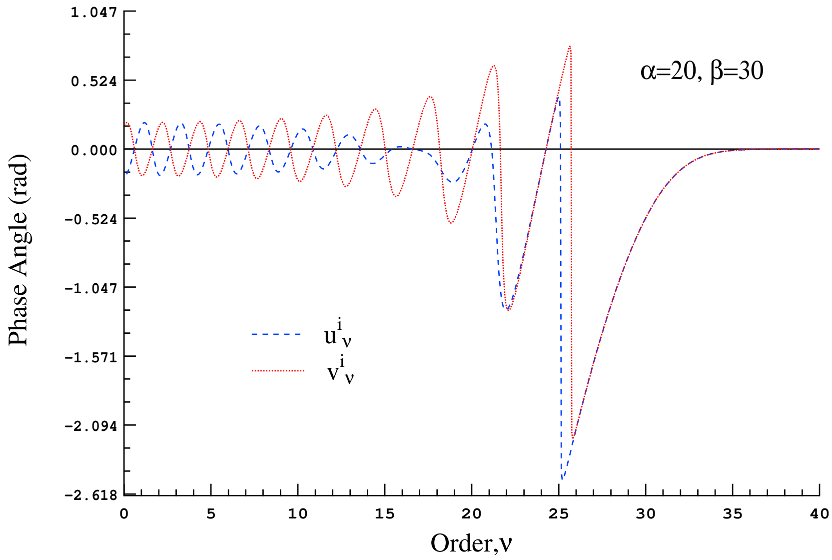

Figure 3. Interfacial phase difference at the surface of the sphere.

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