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EXPLICIT SINGULAR VISCOSITY SOLUTIONS OF THE ARONSSON EQUATION

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ABSTRACT. We establish that when $n \geq 2$ and $H \in C^1(\mathbb{R}^n)$ is a Hamiltonian such that some level set contains a line segment, the Aronsson equation $D^2u : H_p(Du) \otimes H_p(Du) = 0$ admits explicit entire viscosity solutions. They are superpositions of a linear part plus a Lipschitz continuous singular part which in general is non-C^1 and nowhere twice differentiable. In particular, we supplement the C^1 regularity result of Wang and Yu [W-Y] by deducing that strict level convexity is necessary for C^1 regularity of solutions.

RÉSUMÉ. Nous démontrons que, pour $n \geq 2$ et un Hamiltonien $H \in C^1(\mathbb{R}^n)$ tel qu'au moins une de ses lignes de niveau contienne un segment de droite, l'équation de Aronsson $D^2u : H_p(Du) \otimes H_p(Du) = 0$ admet des solutions de viscosité explicites définies sur $\mathbb{R}^n$. Elles sont superpositions d'une partie linéaire et d'une partie continue, lipschitzienne, singulière qui, en général, n'est pas C^1 et est nulle part deux fois dérivable. Plus précisément, nous complétons le résultat de régularité établit par Wang et Yu [W-Y] en montrant que la stricte convexité des lignes de niveau est nécessaire pour que les solutions soient C^1.

1. INTRODUCTION

Let $H \in C^1(\mathbb{R}^n)$ be a Hamiltonian function and $n \geq 2$. We discuss aspects of the C^1 regularity problem of viscosity solutions to the Aronsson equation, which is defined on smooth $u \in C^2(\mathbb{R}^n)$ by

$$(1) \qquad \mathcal{A}[u] := D^2u : H_p(Du) \otimes H_p(Du) = 0.$$

Here, $\mathcal{A}[u]$ is understood as $\sum_{i,j=1}^n D_{ij}^2 u \, H_{p_i}(Du) \, H_{p_j}(Du)$ and $H_{p_i} = D_{p_i}H$. Formula (1) defines a quasilinear highly degenerate elliptic PDE. It arises in L^∞ variational problems of the supremal functional $E_\infty(u, \Omega) := \|H(Du)\|_{L^\infty(\Omega)}$, as well as in other contexts (Barron-Evans-Jensen [BEJ]). When $H(p) = \frac{1}{2}|p|^2$, (1) reduces to the ∞-Laplacian:

$$(2) \qquad \Delta_\infty u := D^2u : Du \otimes Du = 0.$$

Under reasonable convexity, coercivity and regularity assumptions on H, there exists a unique continuous solution of the Dirichlet problem with Lipschitz boundary data, interpreted in the viscosity sense of Crandall-Ishii-Lions [CIL]. Moreover, any continuous viscosity solution to (1) is actually Lipschitz continuous. The C^1 regularity problem for (1) however remains open. Wang and Yu [W-Y] established that when $n = 2$, H is in $C^2(\mathbb{R}^2)$ with $H \geq H(0) = 0$ and it is uniformly convex on the plane (i.e. there exists $a > 0$ such that $H_{pp} \geq aI$), then continuous viscosity solutions of (1) over $\Omega \subseteq \mathbb{R}^2$ are in $C^1(\Omega)$. When $n > 2$, viscosity solutions are

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linearly approximatable at all scales in the sense of De Pauw-Koeller [DePK], having approximate gradients. In the special case of Δ_∞ and when $n = 2$, solutions are $C^{1+\alpha}$ (Savin [S], Evans-Savin [E-S]). Recently, Evans and Smart established everywhere differentiability of ∞-Harmonic functions [E-Sm].

Herein we prove that when a level set $\{H = c\}$ of H contains a straight line segment, there exists an entire viscosity solution of (1) given as superposition of a linear term plus a rather arbitrary Lipschitz continuous term. The latter may *not* be C^1; moreover, it may well be nowhere twice differentiable with Hessian realized only as singular distribution and not as Radon measure, as we show by examples.

We note that our *only* assumption is H being constant along a line segment but arbitrary otherwise. This suffices for these solutions to appear. Actually, they arise as almost everywhere solutions of the Hamilton-Jacobi equation

$$(3) \qquad\qquad H(Du) \; = \; c.$$

In order to keep the proof self-contained and direct, we work with the second order PDE (1) ignoring the relation between viscosity solutions of (1) and differentiable solutions of (3). We just notice that in the classical C^2 context, the identity

$$(4) \qquad\qquad D^2u : H_p(Du) \otimes H_p(Du) \; = \; H_p(Du) \cdot D\big(H(Du)\big)$$

suffices to imply $\mathcal{A}[u] = 0$, whenever $H(Du) = c$. Let us now state our result.

Theorem 1. *We assume that $H \in C^1(\mathbb{R}^n)$, $n \geq 2$ and there exists a straight line segment $[a, b] \subseteq \mathbb{R}^n$ along which H is constant. Then, for any $f \in W^{1,\infty}_{loc}(\mathbb{R})$ satisfying $\|f'\|_{L^\infty(\mathbb{R})} < 1$, the formula*

$$(5) \qquad\qquad u(x) \; := \; \frac{b+a}{2} \cdot x \; + \; f\left(\frac{b-a}{2} \cdot x\right), \quad x \in \mathbb{R}^n,$$

defines an entire viscosity solution $u \in W^{1,\infty}_{loc}(\mathbb{R}^n)$ of the Aronsson equation.

We deduce that the existence of the non-C^1 solutions (5) implies the following

Corollary 2. *Strict level convexity of the Hamiltonian H is necessary to obtain C^1 regularity of viscosity solutions to the Aronsson PDE in all dimensions $n \geq 2$.*

In particular, the uniform convexity assumption of Wang and Yu [W-Y] can not be relaxed to mere convexity, unless if strict level-convexity is additionally assumed.

We observe that C^1 regularity of solutions is not an issue of regularity of H; the singular solutions (5) persist even when $H \in C^\infty(\mathbb{R}^n)$. The sensitive dependence of regularity on the convexity of H is a result of the geometric degeneracy structure of the PDE $\mathcal{A}[u] = 0$ which in view of (4) can be rewritten as the perpendicularity condition $H_p(Du) \perp D\big(H(Du)\big)$. Also, the singular solutions persist for arbitrarily small straight line segments, as long as the segments do not trivialize to a point.

2. Proofs

For the definition and the properties of viscosity solutions we refer to Crandall-Ishii-Lions [CIL]. We will first prove Theorem (1) for smooth functions f and then deduce the full result by approximation.

Lemma 3. *Let u be given by (5) with $f \in C^2(\mathbb{R})$ satisfying $\|f'\|_{L^\infty(\mathbb{R})} < 1$. Then,*
(i) $Du(\mathbb{R}^n) \subseteq (a, b)$, i.e. the range of its gradient Du is valued in the open segment $(a, b) = \{x \in \mathbb{R}^n \mid x = \lambda a + (1 - \lambda)b, \; \lambda \in (0, 1)\}$,

(ii) $H_p\big(Du(\mathbb{R}^n)\big) \subseteq (span[b-a])^\perp$, *i.e. the gradient of H restricted on $Du(\mathbb{R}^n)$ is normal to (a,b).*

Proof of Lemma 3. By differentiating (5), we have

$$
(6) \qquad Du(x) \;=\; \frac{b+a}{2} \;+\; \frac{1}{2}\, f'\left(\frac{b-a}{2}\cdot x\right)(b-a),
$$

for all $x \in \mathbb{R}^n$. By rearranging (6), we have

$$
(7) \qquad Du(x) = \left(\frac{1}{2} - \frac{1}{2}f'\left(\frac{b-a}{2}\cdot x\right)\right) a \;+\; \left[1 - \left(\frac{1}{2} - \frac{1}{2}f'\left(\frac{b-a}{2}\cdot x\right)\right)\right] b.
$$

Since $\|f'\|_{L^\infty(\mathbb{R})} < 1$, there exists a $\delta > 0$ such that

$$
(8) \qquad \delta \;\leq\; \frac{1}{2} - \frac{1}{2}f'\left(\frac{b-a}{2}\cdot x\right) \;\leq\; 1-\delta,
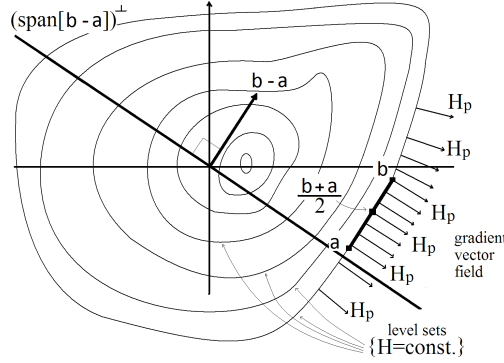$$

for all $x \in \mathbb{R}^n$. Hence, $Du(x)$ is for all $x \in \mathbb{R}^n$ a strict convex combination of a and b. Thus, (i) follows. Since H is constant on $[a,b]$, there exists $c \in \mathbb{R}$ such that, for all $t \in (0,1)$, we have

$$
(9) \qquad H\big(tb + (1-t)a\big) \;=\; c.
$$

Since $H \in C^1(\mathbb{R}^n)$, we may differentiate to find

$$
(10) \qquad \frac{d}{dt}\Big(H\big(tb + (1-t)a\big)\Big) \;=\; (b-a)\cdot H_p\big(tb + (1-t)a\big),
$$

for all for $0 < t < 1$. Hence, we obtain that $(b-a)\cdot H_p(q) = 0$ for all $q \in (a,b)$. Since by (i) we have $Du(\mathbb{R}^n) \subseteq (a,b)$, (ii) follows as well. $\qquad\square$

Lemma 4. *Let u be given by (5) with $f \in C^2(\mathbb{R})$ satisfying $\|f'\|_{L^\infty(\mathbb{R})} < 1$. Then, (5) defines a $C^2(\mathbb{R}^n)$ solution of the Aronsson PDE (1).*

Proof of Lemma 4. By (5) and our assumption, the Hessian $D^2u(x)$ exists for all $x \in \mathbb{R}^n$. By differentiating (6), we have

$$
(11) \qquad D^2u(x) \;=\; \frac{1}{4}\, f''\left(\frac{b-a}{2}\cdot x\right)(b-a)\otimes(b-a).
$$

We now calculate using (11) and (6):

$$
\begin{aligned}
\mathcal{A}[u](x) \ &= \ D^2 u(x) : H_p\big(Du(x)\big) \otimes H_p\big(Du(x)\big) \\
&= \frac{1}{4}\, f''\left(\frac{b-a}{2}\cdot x\right)(b-a)\otimes(b-a) : \\
&\qquad : H_p\left(\frac{b+a}{2}\ +\ \frac{1}{2}\, f'\left(\frac{b-a}{2}\cdot x\right)(b-a)\right)\otimes \\
&\qquad \otimes H_p\left(\frac{b+a}{2}\ +\ \frac{1}{2}\, f'\left(\frac{b-a}{2}\cdot x\right)(b-a)\right).
\end{aligned}
$$
(12)

By employing Lemma 3, we have

$$
\begin{aligned}
\mathcal{A}[u](x) \ &= \ \left\{(b-a)\cdot H_p\left(\frac{b+a}{2}+\frac{1}{2}\, f'\left(\frac{b-a}{2}\cdot x\right)(b-a)\right)\right\}^2 \cdot \\
&\qquad \cdot \frac{1}{4}\, f''\left(\frac{b-a}{2}\cdot x\right) \\
&= 0
\end{aligned}
$$
(13)

and the Lemma follows. $\qquad\square$

Hence, in the case of smooth u the PDE (1) is satisfied because the Hessian $D^2 u$ is normal to $H_p(Du)\otimes H_p(Du)$ in the space of symmetric matrices. Now we conclude with the general case of merely Lipschitz f.

Proof of Theorem 1. Let u be given by (5) with $f\in W^{1,\infty}_{loc}(\mathbb{R})$ and $\|f'\|_{L^\infty(\mathbb{R})} < 1$. Let η^ε, $\varepsilon > 0$, be the standard mollifier and define $f^\varepsilon := f * \eta^\varepsilon \in C^\infty(\mathbb{R})$. Let also u^ε be given by (5) with f^ε in the place of f. Then, $f^\varepsilon \longrightarrow f$ in $C^0(\mathbb{R})$ as $\varepsilon \to 0$ and hence $u^\varepsilon \longrightarrow u$ in $C^0(\mathbb{R}^n)$ as $\varepsilon \to 0$. Moreover,

$$
\|f^{\varepsilon\prime}\|_{L^\infty(\mathbb{R})} \ \leq \ \operatorname*{ess\,sup}_{x\in\mathbb{R}} \int_{\mathbb{R}} |f'(x-y)|\, |\eta^\varepsilon(y)|dy
$$
(14)

and hence $\|f^{\varepsilon\prime}\|_{L^\infty(\mathbb{R})} \leq \|f'\|_{L^\infty(\mathbb{R})} < 1$. Consequently, by Lemmas 3 and 4, all u^ε are smooth entire solutions to PDE (1): $\mathcal{A}[u^\varepsilon] = 0$. By the stability of viscosity solutions, we have $\mathcal{A}[u] = 0$ on $\mathbb{R}^n$ in the viscosity sense and Theorem 1 follows. $\quad\square$

Example 5. *The choice $f(t) := \frac{1}{2}|t|$ for $|t| \leq 1$ and $f(t+2) = f(t)$ gives a non-C^1 solution u to the PDE (1). The choice $f(t) := \frac{1}{2}\int_0^t K_{\alpha,\nu}(s)ds$ with $K_{\alpha,\nu}\in C^0(\mathbb{R})$ the singular function of* [K] *gives a nowhere twice differentiable solution u to (1) with $D^2 u$ existing only as a singular first order distribution.*

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