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Article

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ON THE NEED FOR A NEW PLAYING DIE

ANDREW D. IRVING,* *University of Manchester*

EBRAHIM L. PATEL,** *University of Reading*

Abstract

We model the rolling of a standard die, using a Markov matrix. Though a die may be called 'fair', its initial position influences a roll's outcome. This being undesirable, a simple solution is proposed.

Keywords: markov matrix; transition matrix; playing dice; gambling

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1. Introduction

Markov chains, which are used to model stochastic processes, have been widely employed in fields as diverse as speech recognition [3] and landscape ecology [1]. A Markov chain uses an initial distribution vector in conjunction with a transition matrix to compute the probability that a system will enter a certain state, at a particular stage in the system's evolution. The initial distribution of the system's possible states is given by a row vector whose j^{th} entry denotes the probability that the system is in state j initially. Possible transitions of the system between its various states are given by a matrix whose ijth entry signifies the probability that the system will be in state j, given it had been in state i at the previous stage [2]. Here we propose a simple model for a rolling die which evolves into any one of its various states with equal probability (our basic assumption). Even so, features of the system's evolution remain of interest.

* Postal address: Alan Turing Building, School of Mathematics, University of Manchester, Oxford Road, Manchester, M13 9PL, UK. Email address a_irving@btinternet.com

** Postal address: Centre for the Mathematics of Human Behaviour, School of Mathematics and Physical Sciences, University of Reading, Whiteknights, PO Box 220, Reading, RG6 6AX, UK. Email address e.l.patel@reading.ac.uk

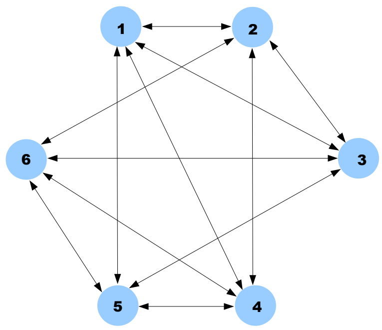

FIGURE 1: One tilt of a die can result only in the transitions shown.

1.1. Digraph

In the rolling of a playing die, such as is used in board-games and gambling, let a '*tilt*' be the smallest movement of a die such that the upper-most face changes. By definition, one tilt of a die can result in a transition between adjacent faces only. On a standard cubic die, each face has a score from 1 to 6 and four adjacent faces. Hence we know all of the possible transitions that can be achieved through a single tilt. These transitions can be described using a graph (see Figure 1), which we shall call 'G'. The nodes of G represent the possible scores on a die's upper-most face, whilst its edges represent the possible transitions between faces that can result from a single tilt. Simply put, one tilt from score r could yield score s if there is an arrow whose tail meets r and whose head meets s in Figure 1.

2. Transition matrix

Assuming that the probabilities of the 4 possible tilts of a rolling die are equal to 1/4, the Transition matrix $\mathbf{T} = [t_{i,j}]$ for the 6 possible states is given by,

$$\mathbf{T} = \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix}.$$

Each $t_{i,j}$ here denotes the probability of a transition (from score i to score j) over the course of a *single tilt* of our die. But what about the case of multiple tilts, as would occur if a die were rolled? We use a well-known property of transition matrices: consider the k^{th} power of our transition matrix, i.e. $\mathbf{T}^k = \left[t_{i,j}^{(k)}\right]$ (where $k \in \mathbb{Z}^+$), then each entry $t_{i,j}^{(k)}$ gives the probability that our die, starting on score i, will have score j after k tilts. Therefore, to model a roll involving k tilts, we raise our matrix $\mathbf{T}$ to the power k. So, for example, an entry in row i and column j of the matrix,

$$\mathbf{T}^2 = \begin{pmatrix} 1/4 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 \\ 1/8 & 1/4 & 1/8 & 1/8 & 1/4 & 1/8 \\ 1/8 & 1/8 & 1/4 & 1/4 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/4 & 1/4 & 1/8 & 1/8 \\ 1/8 & 1/4 & 1/8 & 1/8 & 1/4 & 1/8 \\ 1/4 & 1/8 & 1/8 & 1/8 & 1/8 & 1/4 \end{pmatrix}$$

would represent the probability of a transition from score i to score j over the course of a two-tilt roll of our die.

Remark 2.1. Here we have allowed a die to 'change direction' over the course of a roll. For example, $\mathbf{T}^2$ shows that the probability of starting on a score of '1', then moving to another score before returning to '1' is 1/4. Such a transition could not take place if we did not allow a die to 'go back on itself'. At first, this seems to require a strange sort of die - one that can reverse its own momentum at will. But with Casino games such as Craps, dice are thrown into a wall which causes a change in the direction of their momentum. Our model considers a die rolled within a structure similar to that used in Casino Craps but scaled-down, say, in a flat-bottomed bowl.

2.1. Early observations

Perhaps the first thing we notice is that, just as with $\mathbf{T}^1$, values on the main diagonal of $\mathbf{T}^2$ match those on the counter-diagonal. We also observe here that, just as with $\mathbf{T}^1$, values which do not lie on either of these diagonals match one another, but take a distinct value to the 12 which lie on the main and counter diagonals.

Theorem 2.1. *For any integral power $n \geq 2$,*

$$\begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^n = \begin{pmatrix} \alpha & \beta & \beta & \beta & \beta & \alpha \\ \beta & \alpha & \beta & \beta & \alpha & \beta \\ \beta & \beta & \alpha & \alpha & \beta & \beta \\ \beta & \beta & \alpha & \alpha & \beta & \beta \\ \beta & \alpha & \beta & \beta & \alpha & \beta \\ \alpha & \beta & \beta & \beta & \beta & \alpha \end{pmatrix}$$

where $x \in \mathbb{R}_{\geq 0}$, $y \in \mathbb{R}^+ (y \neq x)$ and $\alpha, \beta \in \mathbb{R}^+ (\alpha \neq \beta)$. We call this statement $P(n)$.

Proof. A proof by induction comprises two steps: (1) a basis step and (2) an inductive step.

(1) The basis step (base case) of a proof by induction requires a proof that $P(n)$ is true for minimal n (i.e. for $n = 2$). For $P(2)$, we have the statement,

$$\begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^2 = \begin{pmatrix} 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy & 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 \\ 2y^2+4xy & 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 & 2y^2+4xy \\ 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 & 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy \\ 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 & 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy \\ 2y^2+4xy & 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 & 2y^2+4xy \\ 2x^2+4y^2 & 2y^2+4xy & 2y^2+4xy & 2y^2+4xy & 2y^2+4xy & 2x^2+4y^2 \end{pmatrix}$$

where both $2x^2 + 4y^2$ and $2y^2 + 4xy$ are clearly positive, real numbers given our definitions of x and y. But can we show that $2x^2 + 4y^2 \neq 2y^2 + 4xy$? If we suppose that $2x^2 + 4y^2 = 2y^2 + 4xy$, we find that,

$$2x^2+4y^2 = 2y^2+4xy \Longrightarrow 2x^2+2y^2-4xy = 0 \Longrightarrow 2(x^2+y^2-2xy) = 0 \Longrightarrow 2(x-y)^2 = 0 \Longrightarrow (x-y)^2 = 0$$

which is impossible for our x and y. Therefore the supposition $2x^2 + 4y^2 = 2y^2 + 4xy$ leads to a contradiction - thus $2x^2 + 4y^2 \neq 2y^2 + 4xy$. Hence we have proven the basis step - $P(2)$ is true.

(2) The inductive step of a proof by induction requires us to prove that $P(n)$ is true for

$n = q + 1$ if it is assumed true for $n = q$. For $P(q + 1)$, we have the statement,

$$\begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^{q+1} = \begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^{q} \begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^{1}$$

but our assumption that $P(q)$ is true allows us to simplify the right hand side above to give,

$$\begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^{q+1} = \begin{pmatrix} \alpha & \beta & \beta & \beta & \beta & \alpha \\ \beta & \alpha & \beta & \beta & \alpha & \beta \\ \beta & \beta & \alpha & \alpha & \beta & \beta \\ \beta & \beta & \alpha & \alpha & \beta & \beta \\ \beta & \alpha & \beta & \beta & \alpha & \beta \\ \alpha & \beta & \beta & \beta & \beta & \alpha \end{pmatrix} \begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}$$

noting that the right hand matrices commute (and so the order of their multiplication does not affect the outcome) - these matrices can be multiplied out to give the simplified equation,

$$\begin{pmatrix} x & y & y & y & y & x \\ y & x & y & y & x & y \\ y & y & x & x & y & y \\ y & y & x & x & y & y \\ y & x & y & y & x & y \\ x & y & y & y & y & x \end{pmatrix}^{q+1} = \begin{pmatrix} R & S & S & S & S & R \\ S & R & S & S & R & S \\ S & S & R & R & S & S \\ S & S & R & R & S & S \\ S & R & S & S & R & S \\ R & S & S & S & S & R \end{pmatrix}$$

where both $R = 2\alpha x + 4\beta y$ and $S = 2\alpha y + 2\beta x + 2\beta y$ are clearly positive, real numbers given our definitions of x, y, α and β. But can we show that $R \neq S$? If we suppose that $R = S$, we find that,

$$2\alpha x + 4\beta y = 2\alpha y + 2\beta x + 2\beta y \implies 2\alpha x + 2\beta y = 2\alpha y + 2\beta x \implies 2\alpha(x - y) = 2\beta(x - y) \implies \alpha = \beta$$

(noting that we can divide by $2(x - y)$ because it is non-zero by our definitions of x and y). But $\alpha = \beta$ is impossible since, by our earlier definition, $\alpha \neq \beta$. Therefore the supposition $R = S$ leads to a contradiction - thus $R \neq S$. Hence it has been shown that $P(q+1)$ holds if $P(q)$ is true. Thus, both the basis and inductive steps are proven - $P(n)$ is true for all $n \geq 2$ by induction.

Corollary 1. *Any two positive integral powers of our transition matrix* $\mathbf{T}$ *commute.*

Proof. Consider any two positive integral powers, say $\mathbf{T}^{n_1}$ and $\mathbf{T}^{n_2}$. Using Theorem 2.1, we can infer the structure of any positive integral power of $\mathbf{T}$. Hence, we can say that,

$$\mathbf{T}^{n_1}.\mathbf{T}^{n_2} = \begin{pmatrix} g_1 & h_1 & h_1 & h_1 & h_1 & g_1 \\ h_1 & g_1 & h_1 & h_1 & g_1 & h_1 \\ h_1 & h_1 & g_1 & g_1 & h_1 & h_1 \\ h_1 & h_1 & g_1 & g_1 & h_1 & h_1 \\ h_1 & g_1 & h_1 & h_1 & g_1 & h_1 \\ g_1 & h_1 & h_1 & h_1 & h_1 & g_1 \end{pmatrix} \begin{pmatrix} g_2 & h_2 & h_2 & h_2 & h_2 & g_2 \\ h_2 & g_2 & h_2 & h_2 & g_2 & h_2 \\ h_2 & h_2 & g_2 & g_2 & h_2 & h_2 \\ h_2 & h_2 & g_2 & g_2 & h_2 & h_2 \\ h_2 & g_2 & h_2 & h_2 & g_2 & h_2 \\ g_2 & h_2 & h_2 & h_2 & h_2 & g_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_1 \\ \lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_1 \end{pmatrix}$$

for some $g_1, g_2 \in \mathbb{R}_{\geq 0}$ and $h_1, h_2 \in \mathbb{R}^+$. Here $\lambda_1 = 2g_1g_2 + 4h_1h_2$ and $\lambda_2 = 2g_1h_2 + 2h_1g_2 + 2h_1h_2$. What about $\mathbf{T}^{n_2}.\mathbf{T}^{n_1}$?

$$\mathbf{T}^{n_2}.\mathbf{T}^{n_1} = \begin{pmatrix} g_2 & h_2 & h_2 & h_2 & h_2 & g_2 \\ h_2 & g_2 & h_2 & h_2 & g_2 & h_2 \\ h_2 & h_2 & g_2 & g_2 & h_2 & h_2 \\ h_2 & h_2 & g_2 & g_2 & h_2 & h_2 \\ h_2 & g_2 & h_2 & h_2 & g_2 & h_2 \\ g_2 & h_2 & h_2 & h_2 & h_2 & g_2 \end{pmatrix} \begin{pmatrix} g_1 & h_1 & h_1 & h_1 & h_1 & g_1 \\ h_1 & g_1 & h_1 & h_1 & g_1 & h_1 \\ h_1 & h_1 & g_1 & g_1 & h_1 & h_1 \\ h_1 & h_1 & g_1 & g_1 & h_1 & h_1 \\ h_1 & g_1 & h_1 & h_1 & g_1 & h_1 \\ g_1 & h_1 & h_1 & h_1 & h_1 & g_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_1 \\ \lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_2 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\ \lambda_2 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_1 \end{pmatrix}$$

Therefore, it has been shown that any two positive integral powers of our transition matrix $\mathbf{T}$ commute.

What else do we notice about the structure of $\mathbf{T}^2$? While there is much that is similar between this structure and that of $\mathbf{T}$, we observe one clear difference. For one tilt of the die, 12 of the possible 36 transitions had no chance of happening (see $\mathbf{T}$). Yet we find that, for two tilts of the die, these same 12 transitions actually have a *greater* chance of occurring than the rest, as shown in $\mathbf{T}^2$.

Closer examination of Figure 1 shows why this is the case. For example, over two-tilts, there are twice as many ways to make the transition from a score of 1 to a score of 6 (one of the afore-mentioned 12 transitions) as there are ways to make a transition from the scores 1 to 2 (not one of the afore-mentioned 12). Hence the former transition is twice as likely to occur as the latter and this is reflected by $\mathbf{T}^2$ entries. We highlight the contrast of this example in Figure 2 (by showing the relevant subgraphs of Figure 1).

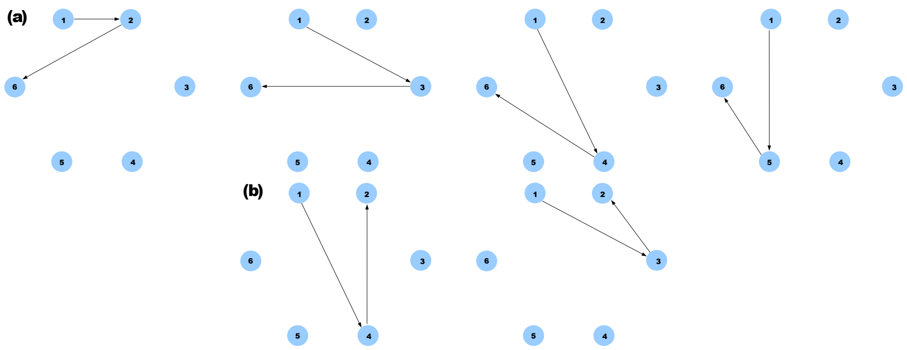

FIGURE 2: (a) All ways to go from 1 to 6 using two-tilt rolls. (b) All ways to go from 1 to 2 using two-tilt rolls.

Is this sort of phenomenon typical? That is, does the number of tilts involved in the roll of a die always have such a significant effect on the relative chances of each transition? Are we, for example, to find that an even number of tilts makes the aforementioned 12 transitions the most likely of the 36, whilst an odd number of tilts makes them the least likely? Consideration of three-tilt and four-tilt rolls,

$$\mathbf{T}^3 = \begin{pmatrix} 1/8 & 3/16 & 3/16 & 3/16 & 3/16 & 1/8 \\ 3/16 & 1/8 & 3/16 & 3/16 & 1/8 & 3/16 \\ 3/16 & 3/16 & 1/8 & 1/8 & 3/16 & 3/16 \\ 3/16 & 3/16 & 1/8 & 1/8 & 3/16 & 3/16 \\ 3/16 & 1/8 & 3/16 & 3/16 & 1/8 & 3/16 \\ 1/8 & 3/16 & 3/16 & 3/16 & 3/16 & 1/8 \end{pmatrix} \mathbf{T}^4 = \begin{pmatrix} 3/16 & 5/32 & 5/32 & 5/32 & 5/32 & 3/16 \\ 5/32 & 3/16 & 5/32 & 5/32 & 3/16 & 5/32 \\ 5/32 & 5/32 & 3/16 & 3/16 & 5/32 & 5/32 \\ 5/32 & 5/32 & 3/16 & 3/16 & 5/32 & 5/32 \\ 5/32 & 3/16 & 5/32 & 5/32 & 3/16 & 5/32 \\ 3/16 & 5/32 & 5/32 & 5/32 & 5/32 & 3/16 \end{pmatrix}$$

seems to indicate that the answer could very well be yes. In fact we can prove that this is so, noting that we have shown that it is true for $k \leq 4$.

Theorem 2.2. *For any positive integral power of our transition matrix* $\mathbf{T}$, *say* f, *the 12 entries* *which lie on the main and counter diagonal of* $\mathbf{T}^f$ *are greater than the other 24 entries of* $\mathbf{T}^f$ *when* f *is even, but smaller than the other 24 entries of* $\mathbf{T}^f$ *when* f *is odd.*

Proof. For an even number of tilts, say $p \in \mathbb{Z}^+$ ($p > 4$), the possible transitions from one face to another occur with the probabilities given by the p^{th} power of our transition matrix $\mathbf{T}$, i.e. $\mathbf{T}^p$. Any such $\mathbf{T}^p$ can be expressed as the following product, $\mathbf{T}^p = \mathbf{T}^{\frac{p}{2}}.\mathbf{T}^{\frac{p}{2}}$. Therefore we know (using Theorem 2.1) that $\mathbf{T}^p$ can always take the form,

$$\begin{pmatrix} a & b & b & b & b & a \\ b & a & b & b & a & b \\ b & b & a & a & b & b \\ b & b & a & a & b & b \\ b & a & b & b & a & b \\ a & b & b & b & b & a \end{pmatrix}\begin{pmatrix} a & b & b & b & b & a \\ b & a & b & b & a & b \\ b & b & a & a & b & b \\ b & b & a & a & b & b \\ b & a & b & b & a & b \\ a & b & b & b & b & a \end{pmatrix} = \begin{pmatrix} X & Y & Y & Y & Y & X \\ Y & X & Y & Y & X & Y \\ Y & Y & X & X & Y & Y \\ Y & Y & X & X & Y & Y \\ Y & X & Y & Y & X & Y \\ X & Y & Y & Y & Y & X \end{pmatrix}$$

for some $a, b \in \mathbb{R}^+$ where $a \neq b$. On the right hand side here, $X = 2a^2 + 4b^2$ and $Y = 2b^2 + 4ab$. For $p = 2$ and $p = 4$, we have seen that $X > Y$. But suppose that, for all other p, $X \leq Y$. Then we find that,

$$X \leq Y \Longrightarrow 2a^2 + 4b^2 \leq 2b^2 + 4ab \Longrightarrow 2a^2 + 2b^2 - 4ab \leq 0 \Longrightarrow 2(a-b)^2 \leq 0 \Longrightarrow (a-b)^2 \leq 0$$

which is impossible for our a and b. Therefore the supposition $X \leq Y$ leads to a contradiction - thus X must be greater than Y for all p.

For an odd number of tilts, say $m \in \mathbb{Z}^+$ ($m > 4$), the possible transitions from one face to another occur with the probabilities given by the m^{th} power of our transition matrix $\mathbf{T}$, i.e. $\mathbf{T}^m$. Any such $\mathbf{T}^m$ can be expressed as the following product, $\mathbf{T}^m = \mathbf{T}^{\frac{m-1}{2}}.\mathbf{T}^{\frac{m-1}{2}}.\mathbf{T}^1$. Therefore we know (using Theorem 2.1) that $\mathbf{T}^m$ can always take the form,

$$\begin{pmatrix} c & d & d & d & d & c \\ d & c & d & d & c & d \\ d & d & c & c & d & d \\ d & d & c & c & d & d \\ d & c & d & d & c & d \\ c & d & d & d & d & c \end{pmatrix}\begin{pmatrix} c & d & d & d & d & c \\ d & c & d & d & c & d \\ d & d & c & c & d & d \\ d & d & c & c & d & d \\ d & c & d & d & c & d \\ c & d & d & d & d & c \end{pmatrix}\begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} = \begin{pmatrix} V & W & W & W & W & V \\ W & V & W & W & V & W \\ W & W & V & V & W & W \\ W & W & V & V & W & W \\ W & V & W & W & V & W \\ V & W & W & W & W & V \end{pmatrix}$$

for some $c, d \in \mathbb{R}^+$ where $c \neq d$. On the right hand side here, $V = 2d^2 + 4cd$ and $W = c^2 + 3d^2 + 2cd$. For $m = 3$, we have seen that $V < W$. But suppose that, for all other m, $V \geq W$. Then we find that,

$$V \geq W \implies 2d^2 + 4cd \geq c^2 + 3d^2 + 2cd \implies 0 \geq c^2 + d^2 - 2cd \implies 0 \geq (c-d)^2 \implies (c-d)^2 < 0$$

which is impossible for our c and d. Therefore the supposition $V \geq W$ leads to a contradiction - thus V must be less than W for all m.

2.2. Fairness of a roll

Here we assume our die is 'fair' - the probability of rolling a particular score is, in principle, 1 in 6 for all scores of our cubic die. But are some *rolls* 'fairer' than others? According to Theorem 2.1, the initial position of a die affects a roll's outcome (since there are distinct entries in $\mathbf{T}^k$). Thus, we propose that a roll would be more 'fair' when the distinct entries of $\mathbf{T}^k$ are closer in value. When is this the case?

The first four positive integral powers of our transition matrix (i.e. $\mathbf{T}^k$ for $k \in [1, 4]$) indicate that, as the number of tilts involved in a roll increases, the chances of all possible 36 transitions become increasingly similar. For example, we observe that the difference between the chances of one-tilt transitions can be as high as 1/4 (see $\mathbf{T}$) whilst the difference between the chances of four-tilt transitions cannot exceed 1/32 (see $\mathbf{T}^4$). Generally, it would appear that this difference halves when k (the number of tilts in a roll of our die) increases by one. In fact we can prove this, noting that it has been shown for $k \leq 4$.

Theorem 2.3. *For any positive integral power of our transition matrix* $\mathbf{T}$*, say* f*, the maximal difference between any 2 of its entries is half the maximal difference between any 2 entries of* $\mathbf{T}^{f-1}$ *(when* $(f-1) \in \mathbb{Z}^+$*).*

Proof. For Theorem 2.3, there are only 2 cases: (1) where f is odd and (2) where f is even. We need to prove that Theorem 2.3 holds for both.

(1) As seen in the proof to Theorem 2.2, any $\mathbf{T}^p$ (where p is even, $p \in \mathbb{Z}^+$ and $p > 4$)

takes the form,

$$\begin{pmatrix} X & Y & Y & Y & Y & X \\ Y & X & Y & Y & X & Y \\ Y & Y & X & X & Y & Y \\ Y & Y & X & X & Y & Y \\ Y & X & Y & Y & X & Y \\ X & Y & Y & Y & Y & X \end{pmatrix}$$

where $X = 2a^2 + 4b^2$ (for some $a, b \in \mathbb{R}^+$, where $a \neq b$) is greater than $Y = 2b^2 + 4ab$. Therefore the maximal difference between any two entries in $\mathbf{T}^p$ (p even) can be expressed in the form,

$$X - Y = (2a^2 + 4b^2) - (2b^2 + 4ab) = 2a^2 + 2b^2 - 4ab = 2(a - b)^2.$$

Probabilities for the next step, i.e. for $p + 1$ tilts, are given by entries of the matrix $\mathbf{T}^{p+1}$ (where $p + 1$ is odd) where,

$$\begin{pmatrix} Y & (X+Y)/2 & (X+Y)/2 & (X+Y)/2 & (X+Y)/2 & Y \\ (X+Y)/2 & Y & (X+Y)/2 & (X+Y)/2 & Y & (X+Y)/2 \\ (X+Y)/2 & (X+Y)/2 & Y & Y & (X+Y)/2 & (X+Y)/2 \\ (X+Y)/2 & (X+Y)/2 & Y & Y & (X+Y)/2 & (X+Y)/2 \\ (X+Y)/2 & Y & (X+Y)/2 & (X+Y)/2 & Y & (X+Y)/2 \\ Y & (X+Y)/2 & (X+Y)/2 & (X+Y)/2 & (X+Y)/2 & Y \end{pmatrix}$$

since $\mathbf{T}^{p+1} = \mathbf{T}^p.\mathbf{T}^1$. From Theorem 2.2, we know that $(X + Y)/2 > Y$ here and therefore the maximal difference between any two entries in $\mathbf{T}^{p+1}$ can be expressed in the form,

$$[(X + Y)/2] - Y = (a^2 + 3b^2 + 2ab) - (2b^2 + 4ab) = a^2 + b^2 - 2ab = (a - b)^2$$

which is half the maximal difference of any two entries in $\mathbf{T}^p$. Therefore, it has been shown that the maximal difference of any two entries in $\mathbf{T}^{p+1}$ is half the maximal difference of any two entries in $\mathbf{T}^p$, where p is even.

(2) As seen in the proof to Theorem 2.2, any $\mathbf{T}^m$ (where m is odd, $m \in \mathbb{Z}^+$ and $m > 4$)

takes the form,

$$\begin{pmatrix} V & W & W & W & W & V \\ W & V & W & W & V & W \\ W & W & V & V & W & W \\ W & W & V & V & W & W \\ W & V & W & W & V & W \\ V & W & W & W & W & V \end{pmatrix}$$

where $V = 2d^2 + 4cd$ (for some $c, d \in \mathbb{R}^+$, where $c \neq d$) is less than $W = c^2 + 3d^2 + 2cd$. Therefore the maximal difference between any two entries in $\mathbf{T}^m$ (m odd) can be expressed in the form,

$$(c^2 + 3d^2 + 2cd) - (2d^2 + 4cd) = c^2 + d^2 - 2cd = (c - d)^2.$$

Probabilities for the next step, i.e. for $m+1$ tilts, are given by entries of the matrix $\mathbf{T}^{m+1}$ (where $m + 1$ is even) where,

$$\mathbf{T}^{m+1} = \begin{pmatrix} W & (V+W)/2 & (V+W)/2 & (V+W)/2 & (V+W)/2 & W \\ (V+W)/2 & W & (V+W)/2 & (V+W)/2 & W & (V+W)/2 \\ (V+W)/2 & (V+W)/2 & W & W & (V+W)/2 & (V+W)/2 \\ (V+W)/2 & (V+W)/2 & W & W & (V+W)/2 & (V+W)/2 \\ (V+W)/2 & W & (V+W)/2 & (V+W)/2 & W & (V+W)/2 \\ W & (V+W)/2 & (V+W)/2 & (V+W)/2 & (V+W)/2 & W \end{pmatrix}$$

since $\mathbf{T}^{m+1} = \mathbf{T}^m.\mathbf{T}^1$. From Theorem 2.2, we know that $W > (V + W)/2$ here and therefore the maximal difference between any two entries in $\mathbf{T}^{m+1}$ can be expressed in the form,

$$W-(V+W)/2 = (W-V)/2 = [(c^2+3d^2+2cd)-(2d^2+4cd)]/2 = [c^2+d^2-2cd]/2 = [(c-d)^2]/2$$

which is half the maximal difference of any two entries in $\mathbf{T}^m$. Therefore, it has been shown that the maximal difference of any two entries in $\mathbf{T}^{m+1}$ is half the maximal difference of any two entries in $\mathbf{T}^m$, where m is odd.

Corollary 2. *For a positive integral power of our transition matrix* $\mathbf{T}$, *say* $f > 1$, $\mathbf{T}^f$ *has the* *form,*

$$\begin{pmatrix} \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 & (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta \\ (\gamma+\delta)/2 & \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta & (\gamma+\delta)/2 \\ (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta & \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 \\ (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta & \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 \\ (\gamma+\delta)/2 & \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta & (\gamma+\delta)/2 \\ \delta & (\gamma+\delta)/2 & (\gamma+\delta)/2 & (\gamma+\delta)/2 & (\gamma+\delta)/2 & \delta \end{pmatrix}$$

where $\gamma \in \mathbb{R}_{\geq 0}$ *and* $\delta \in \mathbb{R}^+ (\delta \neq \gamma)$ *are entries of* $\mathbf{T}^{f-1}$ *such that,*

$$\mathbf{T}^{f-1} = \begin{pmatrix} \gamma & \delta & \delta & \delta & \delta & \gamma \\ \delta & \gamma & \delta & \delta & \gamma & \delta \\ \delta & \delta & \gamma & \gamma & \delta & \delta \\ \delta & \delta & \gamma & \gamma & \delta & \delta \\ \delta & \gamma & \delta & \delta & \gamma & \delta \\ \gamma & \delta & \delta & \delta & \delta & \gamma \end{pmatrix}.$$

Proof. See the proof to Theorem 2.3, together with $\mathbf{T}^k$, $k \in [1, 4]$.

Corollary 3. *For any positive integral power of our transition matrix, say* $\mathbf{T}^k$*, the maximal difference between any 2 of its entries is* $(1/2)^{k+1}$ *(where k is the number of tilts in a roll of our die).*

Proof. Let Δ_k equal the maximal difference between any two entries in $\mathbf{T}^k$. From Theorem 2.3, we can infer the recursive formula,

$$\Delta_k = \frac{1}{2}\Delta_{k-1}.$$

We can replace Δ_{k-1} by again using Theorem 2.3,

$$\Delta_k = \frac{1}{2}\left[\frac{1}{2}\Delta_{k-2}\right] \Longrightarrow \Delta_k = \left(\frac{1}{2}\right)^2 \Delta_{k-2}.$$

but to find Δ_k we need to compute not just some, but *all*, of its previous terms. However, it is clear that, for any positive integer μ ($\mu < k$),

$$\Delta_k = \left(\frac{1}{2}\right)^{\mu} \Delta_{k-\mu}.$$

Therefore, for $\mu = k - 1$, we can find Δ_k in terms of Δ_1 since,

$$\Delta_k = \left(\frac{1}{2}\right)^{k-1} \Delta_{k-(k-1)} \Longrightarrow \Delta_k = \left(\frac{1}{2}\right)^{k-1} \Delta_1$$

where, from **T**, it is clear that $\Delta_1 = 1/4$. Thus, substitution of $\Delta_1 = 1/4$ into our equation gives the required result, i.e.

$$\Delta_k = \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{4}\right) \Longrightarrow \Delta_k = \left(\frac{1}{2}\right)^{k-1} \left(\frac{1}{2}\right)^{2} \Longrightarrow \Delta_k = \left(\frac{1}{2}\right)^{k+1}.$$

We find that, even when using a fair die, its initial position (i.e. its upper-most face at the dawn of a roll) plays a highly significant role in the outcome of a roll. As one would expect (noting Theorem 2.3 and Corollary 3), this role diminishes for more 'thorough' rolls (i.e. those involving a greater number of tilts). Thus, the more 'thoroughly' one rolls a die, the more 'fairly' one rolls it.

3. Final thoughts

An object's motion is almost invariably affected by its structure. Here we have seen the significance of a die's shape with respect to the evolution of its movements. This significance is perhaps seldom more apparent than in the matrix associated with those transitions which can be achieved through a single tilt (see **T**). The distinct entries of **T** convey a simple fact - not all faces of a standard cubic die are adjacent. Is it this property of a standard die which produces interesting results here (e.g. the significance of whether the number of tilts involved in a roll is odd or even)? Not entirely.

The kinds of bias seen here (towards different transitions) for a standard die do not result from the number of its faces, so much as from the manner in which those faces are labelled. That is, each one of a standard die's faces has its own distinct label (a score from '1' to '6'). But what if this were not the case? What if a standard die had just two distinct labels associated with its six faces? Say we used the labels '1' and '2' only, where scores on opposite faces summed to three (giving us a die with three faces labelled '1' and their opposing faces labelled '2'). Then our transition matrix -

say $\hat{\mathbf{T}} = \left[\hat{t}_{i,j}\right]$ - for single tilts of our 'relabelled' die is given by,

$$\begin{array}{c} \text{Score} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{2} \\ \mathbf{1} \\ \mathbf{2} \end{array} \quad \begin{array}{c} \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{pmatrix} \end{array} = \hat{\mathbf{T}}.$$

Whilst entries of $\hat{\mathbf{T}}$ match those of our earlier transition matrix $\mathbf{T}$, note the labelling of $\hat{\mathbf{T}}$'s borders which corresponds to our proposed 'new scores' for a standard die's six faces. Our relabelled die can yet experience $6^2 = 36$ facial transitions (as shown in $\hat{\mathbf{T}}$ where each $\hat{t}_{i,j}$ signifies the probability that a single tilt of our relabelled die could result in a change from face i to face j), but with only two distinct scores - '1' and '2' - our die enables only $2^2 = 4$ possible transitions between scores. Indeed, the conditional probabilities of these four transitions are communicated by $\hat{t}_{i,j}$, e.g. the transition from a score of '1' to a score of '2' via one tilt occurs with probability $\hat{t}_{1,2} + \hat{t}_{1,4} + \hat{t}_{1,6}$ (and equally $\hat{t}_{3,2} + \hat{t}_{3,4} + \hat{t}_{3,6}$ or $\hat{t}_{5,2} + \hat{t}_{5,4} + \hat{t}_{5,6}$). Thus the following transition matrix can be derived from $\hat{\mathbf{T}}$,

$$\begin{array}{c} \text{Score} \\ \mathbf{1} \\ \mathbf{2} \end{array} \quad \begin{array}{c} \begin{array}{cc} \mathbf{1} & \mathbf{2} \end{array} \\ \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \end{array} = \mathbf{P}$$

where $\mathbf{P} = \left[p_{i,j}\right]$. Each $p_{i,j}$ gives the probability that our relabelled cubic die undergoes a transition from a score of i to a score of j. Each entry of $\mathbf{P}^k = \left[p_{i,j}^{(k)}\right]$ (where $k \in \mathbb{Z}^+$) gives the probability of a transition from score i to score j over k tilts of our relabelled die. And it is easy to see that all positive integral powers of $\mathbf{P}$ are equal to $\mathbf{P}$ itself, i.e. that $p_{i,j}^{(k)} = p_{i,j}$ for all i, j and k. Therefore, since all $p_{i,j}$ are equal, we can say that a transition between any two scores (be they equal or distinct) over the course of any number of tilts occurs with the same probability when using our relabelled die. So although our relabelled die experiences the same kinds of bias as a standard die with respect to its facial transitions (see $\hat{\mathbf{T}}$), it is free from the bias involved in the transitions between a standard die's scores (see $\mathbf{P}$). Unlike those of a standard die, all rolls of our relabelled die are 'fair'. Our relabelled die therefore seems preferable for general use.

References

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