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A Riemann-Hilbert problem with a vanishing coefficient and applications to Toeplitz operators

Abstract

We study the homogeneous Riemann-Hilbert problem with a vanishing scalar-valued continuous coefficient. We characterize non-existence of nontrivial solutions in the case where the coefficient has its values along several rays starting from the origin. As a consequence, some results on injectivity and existence of eigenvalues of Toeplitz operators in Hardy spaces are obtained.

Keywords

Riemann-Hilbert problems • Hardy spaces • Toeplitz operators • Fredholm properties • eigenvalues

MSC: 35Q15, 45E05, 30E25, 47B35

A. Perälä[1*], J. A. Virtanen[2†], L. Wolf[3‡]

1 Department of Mathematics, University of Helsinki, Helsinki 00014, Finland

2 Department of Mathematics, University of Reading, Whiteknights, P.O. Box 220, Reading RG6 6AX, U.K.

3 Department of Mathematics and Statistics, State University of New York at Albany, Albany, N.Y. 12222, U.S.A.

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1. Introduction

For $1 \le p \le \infty$, we denote the Hardy space over the circle $\mathbb{T}$ by $H^p(\mathbb{T})$; that is,

$$H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : f_k = 0 \quad \text{for } k < 0\},$$

where f_k is the kth Fourier coefficient of f. The Hardy space for the disk $H^p(\mathbb{D})$ is defined to be the class of all analytic functions in $\mathbb{D}$ for which $||f||_p < \infty$, where

$$||f||_p = \sup\{||f_r||_p : 0 \le r < 1\}$$

with $f_r(e^{i\theta}) = f(re^{i\theta})$. It is well known that $H^p(\mathbb{T})$ and $H^p(\mathbb{D})$ are isometrically isomorphic. Let P be the Riesz projection, defined by

$$P : \sum_{k=-n}^{n} f_k t^k \mapsto \sum_{k=0}^{n} f_k t^k$$

on Laurent polynomials. By the M. Riesz theorem, the projection P extends to a bounded operator of $L^p(\mathbb{T})$ onto $H^p(\mathbb{T})$ when $1 < p < \infty$. For $a \in L^\infty(\mathbb{T})$, the Toeplitz operator $T_a : H^p \to H^p$ is defined by $T_a f = P(af)$.
Coburn's theorem states that a nonzero Toeplitz operator has a trivial kernel or a dense range. It follows that for a continuous symbol $a : \mathbb{T} \to \mathbb{C}$, a point λ in $\sigma(T_a) \setminus \sigma_{\text{ess}}(T_a)$ is an eigenvalue of T_a if and only if the winding number of $\lambda - a$, $\text{wind}(\lambda - a)$, is negative. On the other hand, the question of whether λ in $\sigma_{\text{ess}}(T_a)$ $(= a(\mathbb{T}))$ is an eigenvalue

* E-mail: antti.i.perala@helsinki.fi
† E-mail: j.a.virtanen@reading.ac.uk (Corresponding author)
‡ E-mail: lwolf-christensen@albany.edu

is quite delicate and only very few results are known, most of which require strong restrictions on the behavior of the symbol a in the neighborhood of its zeros; see [14] and the references therein.
Let us denote the set of all Hölder continuous functions on $\mathbb{T}$ by C^μ. The following result easily follows from [13, Lemma 4.11] and Proposition 6.

Theorem 1.

Let $1 < p < \infty$, $\mu \in [0, 1]$, and let $a : \mathbb{T} \to \mathbb{R}$ be a continuous function such that

$$|a(t)| \le \text{const}\, \rho(t)^\mu \quad \textit{for all} \quad t \in \mathbb{T}, \tag{1.1}$$

where $\rho(t) = \text{dist}(t, \mathcal{N}_a)$, $\mathcal{N}_a = \{t \in \mathbb{T} : a(t) = 0\}$. Let $[c, d] = a(\mathbb{T})$. Then $\lambda \in (c, d)$ is not an eigenvalue of T_a if

$$p \ge \frac{2}{1+\mu}; \tag{1.2}$$

while c, d are not eigenvalues of T_a whenever

$$p \ge \frac{2}{\mu}. \tag{1.3}$$

Remark 2.

The difference between (1.2) and (1.3) is explained by the simple observation that in the first case $a - \lambda$ gets both positive and negative values; while in the latter case the argument of $a - \lambda$ remains constant.

Recall that

$$\sigma(T_a : H^2 \to H^2) = \sigma_{\text{ess}}(T_a : H^2 \to H^2) = [c, d]$$

provided that a is continuous real-valued function (see [1, Sec. 2.36]). Therefore, Theorem 1 implies that T_a has no eigenvalues if a is Lipschitz continuous. If $a \in C^\mu$ with $\mu < 1$, then we can only say that there are no eigenvalues in the interior of $\sigma(T_a)$; in fact, one can construct a real-valued Hölder continuous function a so that the endpoints of $\sigma(T_a : H^2 \to H^2)$ are eigenvalues of T_a; see [12].
The next result, which follows from the main result of [15], generalizes Theorem 1 to the case when the values of a are located on two rays

$$S_k = \{z \in \mathbb{C} \setminus \{0\} : \arg z = \delta_k\}, \tag{1.4}$$

where, in general, $\delta_1 \le \delta_2 < \delta_3 < \cdots < \delta_n < 2\pi$. We also set $E_k = \{t \in \mathbb{T} : a(t) \in S_k\}$.

Theorem 3.

Let $1 < p < \infty$ and $\mu > 0$. Suppose that $a : \mathbb{T} \to S_1 \cup S_2 \cup \{0\}$ is a continuous function that satisfies (1.1). Then 0 in $\sigma_{\text{ess}}(T_a)$ is not an eigenvalue of $T_a : H^p \to H^p$ if $p > 2(\mu + (\delta_2 - \delta_1)/\pi)^{-1}$.
If, in addition, the symbol a traverses through S_j to S_k at most finitely many times, then 0 in $\sigma_{\text{ess}}(T_a)$ is not an eigenvalue of $T_a : H^p \to H^p$ if $p \ge 2(\mu + (\delta_2 - \delta_1)/\pi)^{-1}$.

Note that the setting of Theorem 1 is invariant under translation by $\lambda \in a(\mathbb{T})$, whereas the setting of Theorem 3 is not. The present approach addresses the question of whether 0 is not an eigenvalue of T_a; that is, whether T_a is injective. By inserting $S_1, \ldots, S_n$ into two sectors and applying [14, Theorem 1.3], we can show that T_a is injective if

$$p > \frac{2}{\mu + \dfrac{\max\{\delta_n - \delta_{n-1}, \ldots, \delta_2 - \delta_1\}}{\pi}}. \tag{1.5}$$

However, this bound is not optimal as we see in the following theorem. In what follows, we identify functions defined on $\mathbb{T}$ with 2π-periodic functions defined on $\mathbb{R}$. The function a is assumed to have values on rays $S_1, S_2, ..., S_n$ starting from the origin. Let $\theta(t) = \arg a(t-0)$, which is clearly a is piecewise constant function. We assume that the argument has only finitely many jumps. A point at which a goes to zero along a ray S_k and then returns to the same ray is not considered a jump. Note that since $\theta(t) = \arg a(t-0)$, the function θ remains constant when traversing over such zeros. Therefore, we allow some cases where there are infinitely many zeros, which is a natural setting in some applications. However, the Riemann-Hilbert problem cannot have a nontrivial solution if a has zeros on a set of positive measure. The change in the argument of a at t_0 is denoted by δ_{t_0}; that is,

$$\delta(t_0) = \theta(t_0 + 0) - \theta(t_0 - 0).$$

We also write

$$\delta_- = -\min\{\delta(t) : -\pi \le t \le \pi\},\ \delta_+ = \max\{\delta(t) : -\pi \le t \le \pi\} \tag{1.6}$$

and denote the largest contribution from both positive and negative jumps by δ; that is,

$$\delta = \min\{\delta_-, \delta_+\}. \tag{1.7}$$

Note that, by a simple rotation argument, we may assume that we always have $\delta \le \pi$.

Theorem 4.
Suppose $1 < p < \infty$ and $\mu > 0$. Let

$$a : \mathbb{T} \to S_1 \cup \ldots \cup S_n \cup \{0\}$$

satisfy (1.1), and suppose $\delta_\pm \le \pi$. Then the Toeplitz operator $T_a : H^p \to H^p$ is injective if

$$p > \frac{2}{\mu + \frac{\delta}{\pi}} \tag{1.8}$$

provided that the symbol a traverses through any S_i to another S_j at most finitely many times.

Obviously if a traverses only along neighboring rays, then (1.8) is no different from (1.5). Also, if there are only two rays S_1 and S_2, then $\delta = \delta_2 - \delta_1$, and the condition $\delta \le \pi$ is superfluous.
The adjoint of $T_a : H^p \to H^p$ is the operator $T_{\bar{a}} : H^q \to H^q$ $(1/p + 1/q = 1)$. Since the setting of our theorem is invariant under complex conjugation, it is not difficult to construct operators T_a such that both T_a and T_a^* are injective. However, since a vanishes, such operators cannot be Fredholm. This reflects the inconvenient fact that T_a with vanishing symbol is often not normally solvable.
The proof of Theorem 4 is given in the following section. Our approach is based on that of [15]. The reason we have a strict inequality in (1.8) is related to the properties of some norm inequalities for harmonic conjugation of characteristic functions; see [2, Chap. III, Sec. 2]. We conjecture that the strict inequality in (1.8) may only be needed if the argument of the symbol has infinitely many jumps. However, our aim here is to show how some of the main results of the two-ray case (see [15]) are altered when additional rays are inserted between the two, depending on the order in which the symbol a traverses through the rays. This provides more insight into how the geometry of the sets E_k and S_k affects the nonexistence of eigenvalues in the essential spectra of Toeplitz operators. It is also of interest to know what happens when the symbol a is matrix-valued.

2. The Riemann-Hilbert problem

Let $a \in L^\infty$ and $1 < p < \infty$. The Riemann-Hilbert problem (RHP) in Hardy spaces is the problem of finding $\varphi, \psi \in H^p(\mathbb{D})$ for which

$$\varphi^* = a\overline{\psi^*} \quad a.e.\ on\ \mathbb{T}, \tag{2.1}$$

where φ^* denotes the nontangential boundary values of φ (see [2]).

The following well-known result essentially shows that the study of Toeplitz operators is closely related to the RHP in Hardy spaces. We give the proof for completeness because it is not readily available in the literature. Let us first recall a couple of useful results. For $f \in L^1$, define

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - z}\,d\tau, \quad z \in \mathbb{C}\setminus\mathbb{T}. \tag{2.2}$$

Note that such F is analytic on $\mathbb{C}\setminus\mathbb{T}$ and $F(\infty) = 0$. For $t \in \mathbb{T}$, we denote by $F^+(t)$ the boundary values of F as $z \to t$ nontangentially in $\mathbb{D}$ and by $F^-(t)$ as $z \to t$ outside of $\mathbb{T}$. According to the Plemelj formulas, if $f \in L^1$, then F is analytic in $\mathbb{C}\setminus\mathbb{T}$ and

$$F^+ = Pf \quad \text{and} \quad F^- = -Qf, \tag{2.3}$$

where $Q = I - P$ is the complementary projection (see [5, Chapter 2, Section 4]).

Proposition 5.

If H is analytic in $\mathbb{C}\setminus\mathbb{T}$, $H(\infty) = 0$ and $H^+ - H^- \in L^1$, then H is of the form (2.2) with $f = H^+ - H^-$.

Proof. Put $H_0(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - z}d\tau$. By the Plemelj formulas, $G := H - H_0$ is analytic in $\mathbb{C}\setminus\mathbb{T}$ and $G^+ - G^- = 0$. Thus, G has an analytic continuation to the whole plane and it remains to apply Liouville's theorem and use the fact that both H and H_0 vanish at infinity. □

Note that the condition in Theorem 4 is invariant under complex conjugation of the coefficient a. Therefore the RHP (2.1) is equivalent to the following

$$\overline{\psi^*} = a\varphi^*. \tag{2.4}$$

Proposition 6.

Let $a \in L^\infty$ and $1 < p < \infty$. Then the Riemann-Hilbert problem (2.4) and the problem of finding f in $\ker T_a$ are equivalent in H^p.

Proof. Note first that the study of the two operators $T_a : H^p \to H^p$ and $aP + Q : L^p \to L^p$ is equivalent in terms of their spectral properties. Indeed, $(PaP + Q)(I + QaP) = aP + Q$, where $I + QaP$ is invertible with inverse $I - QaP$, and also, since $L^p = P(L^p) \oplus Q(L^p)$, we have

$$\ker(PaP + Q) = \ker T_a, \quad \operatorname{ran}(PaP + Q) = \operatorname{ran} T_a \oplus Q(L^p).$$

Suppose that there is a function $g \in H^p$ such that $T_a g = 0$. Then, as above, $aPf + Qf = 0$ for some $f \in L^p$. Let $F(z) = (2\pi i)^{-1}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - z}d\tau$. For $|z| < 1$, define $\varphi(z) = F(z)$ and $\psi(z) = \overline{F(1/\bar{z})}$. Then F is analytic in $\mathbb{C}\setminus\mathbb{T}$ and the Plemelj formulas imply

$$a\varphi^* - \overline{\psi^*} = aF^+ - F^- = aPf + Qf = 0.$$

Conversely, suppose $a\varphi^* = \overline{\psi^*}$ for some $\varphi, \psi \in H^p(\mathbb{D})$. We define $F(z) = \varphi(z)$ for $|z| < 1$, and $F(z) = \overline{\psi(1/\bar{z})}$ for $|z| > 1$. Let $f = F^+ - F^-$. By Proposition 5, $F(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\tau)}{\tau - z}d\tau$, and so again by the Plemelj formulas, we get

$$aPf + Qf = aF^+ - F^- = a\varphi^* - \overline{\psi^*} = 0.$$

The one-to-one correspondence of the two problems follows from the fact that $F_{f_1} = F_{f_2}$ if and only if $f_1 = f_2$ almost everywhere. □

Remark 7.

The assertion of Proposition 6 remains true also if $p = 1$ provided that T_a is bounded on H^1. The proof of the case $p = 1$ is analogous to the general case.

The following outer function plays an important role in what follows. Define

$$X(z) = \exp\left(\frac{i}{4\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\theta(t)dt\right), \quad |z|<1, \tag{2.5}$$

where $\theta(t) = \arg a(t)$. Note that

$$(X^*)^{\pm 1} = \exp\left(\mp\frac{1}{2}\left(\mathcal{C}\theta(t) - i\theta(t)\right)\right)$$

(see [6, Chap. V]), where $\mathcal{C}f$ is the Hilbert transform, defined by

$$\mathcal{C}f(t) = \int_{-\pi}^{\pi} f(y)\cot\frac{t-y}{2}dy$$

for $t \in \mathbb{R}$; see [2, Chap. III], so $\left|(X^*)^{\pm 1}\right| = e^{\mp\frac{1}{2}\mathcal{C}\theta(t)}$.

Lemma 8.

Suppose a traverses through S_j to S_k at most finitely many times.
(1) If $p < \frac{2\pi}{\delta_+}$ and $q < \frac{2\pi}{\delta_-}$, then $X \in H^p$ and $X^{-1} \in H^q$.
(2) Suppose $\delta_2 > \delta_1$ and E_k are nonempty. If $p \geq \frac{2\pi}{\delta_+}$ and $q \geq \frac{2\pi}{\delta_-}$, then $X \notin H^p$ and $X^{-1} \notin H^q$.

Proof. Observe first that for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, we have

$$e^{\pi\mathcal{C}\chi_{[t_1,t_2]}(t)} = \left|\frac{\sin\frac{t-t_1}{2}}{\sin\frac{t-t_2}{2}}\right| \tag{2.6}$$

for $t \in \mathbb{R}$.
Let $x_\pm \in [-\pi, \pi]$ be such that $\delta(x_\pm) = \delta_\pm$; that is, the largest positive and negative jumps are obtained at x_+ and x_-, respectively. Then there are $\delta_m < \delta_n$ and $\epsilon > 0$ such that $\delta_n - \delta_m = \delta_+$, $\theta(t) = \delta_m$ for $x_+ - \epsilon < t < x_+$ and $\theta(t) = \delta_n$ for $x_+ < t < x_+ + \epsilon$.
If $p = \lambda\frac{2\pi}{\delta_+}$ with $\lambda < 1$, then using (2.6), we get

$$\begin{aligned}\int |X^*|^p &= \int e^{-\frac{p}{2}\mathcal{C}\theta} = \int \exp\left(-\frac{\lambda}{\delta_+}\pi\mathcal{C}(\delta_m\chi_{[x_+-\epsilon,x_+]} + \delta_n\chi_{[x_+,x_++\epsilon]} + \dots)\right)\\ &= \int\left|\frac{\sin\frac{t-(x_+-\epsilon)}{2}}{\sin\frac{t-x_+}{2}}\right|^{-\frac{\lambda\delta_m}{\delta_+}}\left|\frac{\sin\frac{t-x_+}{2}}{\sin\frac{t-(x_++\epsilon)}{2}}\right|^{-\frac{\lambda\delta_n}{\delta_+}}\cdots \leq \text{const}\int\left|\frac{1}{\sin\frac{t-x_+}{2}}\right|^{\lambda\frac{\delta_n-\delta_m}{\delta_+}} < \infty\end{aligned}$$

because $\delta_n - \delta_m = \delta_+$ and $\lambda < 1$. Thus, since X is outer and $X^* \in L^p$, $X \in H^p$. Similarly, we can show that $X^{-1} \in H^q$ if $q < 2\pi/\delta_-$.
Let $p = \frac{2\pi}{\delta_+}$ and $r_k = \delta_k/\delta_+$, and choose $\epsilon > 0$ to be sufficiently small (which we can do because θ has finitely many discontinuities). Choose $i < j$ such that $x_+ - \epsilon \in E_i$ and $x_+ + \epsilon \in E_j$. Then

$$\int_{-\pi}^{\pi}|X^*(t)|^p dt = \int_{-\pi}^{\pi} e^{-p\frac{1}{2}\mathcal{C}\delta(t)}dt = \int_{-\pi}^{\pi} e^{-r_1\pi\mathcal{C}\chi_{E_1}(t)}\cdots e^{-r_n\pi\mathcal{C}\chi_{E_n}(t)}dt \geq \text{const}\int_{x_+-\epsilon}^{x_++\epsilon}\left|\frac{1}{\sin\frac{t-x_+}{2}}\right|^{r_j-r_i}$$

which is not integrable because $r_j - r_i = 1$ $(j > i)$. Thus, $X \notin H^p$ since X is outer. Similarly, we can show $X^{-1} \notin H^q$ when $q = 2\pi/\delta_-$. □

Proof of Theorem 4. Suppose that (1.8) holds and T_a has a nontrivial kernel. Then there are nontrivial $\varphi, \psi \in H^p$ such that $\varphi^* = a\overline{\psi^*}$. It is not difficult to see that $a \neq 0$ almost everywhere; see [6, Theorem IV.C.1]. Thus, $|\mathbb{R} \setminus \cup E_k| = 0$. As in [15], we define an outer function H by setting

$$H(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|a(e^{it})|^{1/2}dt\right), \quad |z|<1. \tag{2.7}$$

Observe that $|H^*| = \sqrt{|a|}$. We also define

$$F = \frac{\varphi}{HX}, \quad G = \frac{\psi H}{X}.$$

Since

$$a = |a|e^{i\theta} = H^*\overline{H^*}X^*\overline{(X^{-1})^*}, \tag{2.8}$$

we have $F^* = \overline{G^*}$. As in the proof of Lemma 8, we can use (2.6) to show that

$$|(X^{-1})^*| = e^{\frac{1}{2\pi}\pi\mathcal{C}\theta(t)} \leq \left|\sin\frac{\rho(t)}{2}\right|^{-\frac{\delta_-}{2\pi}} \tag{2.9}$$

for $t \in E_1 \cup \cdots \cup E_n$. Therefore,

$$|H^*(t)||(X^{-1})^*(t)|^s \leq \text{const}\,|a(t)|^{1/2}\rho(t)^{-\frac{s\delta_-}{2\pi}} \leq \text{const}\,\rho(t)^{\frac{\mu}{2}-\frac{s\delta_-}{2\pi}} \leq \text{const}$$

where $s = \mu\pi/\delta_-$. Similarly, if $r = \mu\pi/\delta_+$, $|H^*(t)||X^*(t)|^r \leq$ const. Using (1.8) and the assumption that $\delta_- \leq \pi$, we get

$$1-\frac{1}{p'} = \frac{1}{p} < \frac{\mu}{2}+\frac{\delta_-}{2\pi} \leq 1-\frac{\delta_-}{2\pi}+\frac{\mu}{2} \implies \frac{\delta_-}{2\pi}-\frac{\mu}{2} < \frac{1}{p'},$$

and so $(1-s)p' < (2\pi)/\delta_-$. Therefore,

$$||G||_1 \leq \text{const}\left|\left||(X^{-1})^*|^{1-s}\psi^*\right|\right| \leq \text{const}\left|\left||(X^{-1})^*|^{1-s}\right|\right|_{p'}||\psi^*||_p = \text{const}\left|\left|(X^{-1})^*\right|\right|_{(1-s)p'}^{1-s}||\psi^*||_p < \infty$$

by Lemma 8. Since $F^* = \overline{G^*}$, we also have $G^* \in L^1$. We can also show that $F, G \in H^p$ for some $p < 1$. Thus, an application of Smirnov's theorem implies that $G, F \in H^1$. Consequently, G is a nonzero constant.
Now (1.8) implies $(2\pi/\delta_+ - 1 - r)p' < 2\pi/\delta_+$ and so we can choose a $q > 2\pi/\delta_+$ be such that $0 < (q-1-r)p' < 2\pi/\delta_+$. By Lemma 8,

$$||G^*(X^*)^q||_1 = \left|\left|H^*(X^*)^s(X^*)^{q-1-s}\psi^*\right|\right| \leq \text{const}\,||X^*||_{(q-1-r)p'}^{q-1-r}||\psi^*||_p < \infty,$$

but $G^*(X^*)^q = \text{const}(X^*)^q \notin L^1$ by the same lemma, which is a contradiction. □

Remark 9.
Theorem 4 shows that T_a is injective if $p > \frac{2}{\mu+\frac{\delta}{\pi}}$. We can show that the condition is sharp; that is, if

$$1 \leq p < \frac{2}{\mu+\frac{\delta}{\pi}}, \tag{2.10}$$

then we can construct symbols $a \in C^\mu$ such that the kernel of T_a is nontrivial. Indeed, let a be in C^μ such that $a(t) \in S_1 \cup \ldots S_n \cup \{0\}$ and $|a|^{-1} \in L^q$ for $q < \mu^{-1}$. Recall the outer functions X and H defined in (2.5) and (2.7). Since $|H^*| = |a|^{1/2}$ and H is outer, we have $H \in H^\infty$ and $H^{-1} \in H^{2q}$ for $q < \mu^{-1}$. Let

$$\varphi = HX, \quad \psi = H^{-1}X.$$

Then $\varphi^* = a\overline{\psi^*}$ (see (2.8)), and so $0 \in a(\mathbb{T}) = \sigma_{\text{ess}}(T_a)$ is an eigenvalue of T_a. Using Lemma 8 and Hölder's inequality, we see that $\varphi \in H^p$ and $\psi \in H^p$ provided that $1 \leq p < 2/(\mu + \delta_+/\pi)$. It is obvious that there are symbols such as those above with an additional property that $\delta_+ \leq \delta_-$. Thus, if (2.10) holds, then the kernel of T_a may be nontrivial.

3. Toeplitz operators on H^1

According to Coburn's lemma, a nonzero bounded Toeplitz operator T_a on H^p has a trivial kernel or a dense range. When $p = 2$, this result was proved for Hölder continuous symbols in [7], for continuous symbols in [4, 10], and for bounded symbols in [3]. The case $1 < p < \infty$ for bounded symbols is in [11]. Because of the duality argument used in the proof of Coburn's lemma in the most general case, there seems to be no obvious way to extend the result to the case $p = 1$. However, there is an alternative approach due to Vukotić [17], which we recall next.
We write

$$\mathcal{P} \ker T_a = \{pf : p \in \mathcal{P}, f \in \ker T_a\},$$

where $\mathcal{P}$ is the set of all analytic polynomials.

Theorem 10.
Let $1 \leq p < \infty$ and suppose that T_a is a nontrivial bounded Toeplitz operator on H^p. If T_a is not one-to-one, then

$$T_a\big(\operatorname{span}\{\mathcal{P} \ker T_a\}\big) = \mathcal{P}. \tag{3.1}$$

Proof. The proof given in [17] also works when $p \neq 2$. We only comment on the case $p = 1$. Put $\chi_n(z) = z^n$. The main idea is still the observation that the rank of the commutator

$$[T_a, T_{\chi_1}] = T_a T_{\chi_1} - T_{\chi_1} T_a$$

is at most one; that is, it can be showed (see [17]) that

$$T_a(\chi_1 f) - \chi_1 T_a f = T_a(\chi_1 f)(0),$$

where

$$T_a f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{a(e^{i\theta}) f(e^{i\theta})}{1 - ze^{-i\theta}} d\theta \quad (z \in \mathbb{D})$$

is the analytic extension of $T_a f$. All the algebraic properties used in [17] remain true in the case $1 \leq p < \infty$. $\square$

Coburn's lemma for Toeplitz operators on H^1 now follows directly from the preceding theorem.

Corollary 11.
If T_a is bounded on H^1, then either its kernel is trivial or its range is dense.

It is well known that continuity of a is not sufficient for T_a to be bounded on H^1. The most natural substitute for the class of continuous functions is the algebra

$$C \cap VMO_{\log},$$

where $VMO_{\log}$ is the space of functions of logarithmic vanishing mean oscillation; see [9, 16] for the definition. Observe that

$$C^\mu \subset VMO_{\log} \subset VMO.$$

If $a \in C \cap VMO_{\log}$, then

$$\sigma_{\Phi(H^1)}(T_a) = a(\mathbb{T}) \tag{3.2}$$

and

$$\operatorname{ind} T_a = -\operatorname{ind} a \tag{3.3}$$

provided that $a(t) \neq 0$ for any $t \in \mathbb{T}$.

Proposition 12.
Let $a \in C \cap VMO_{\log}$. Then

$$\sigma_{H^1}(T_a) \setminus \sigma_{\Phi(H^1)}(T_a) = \{\lambda \in \mathbb{C} \setminus a(\mathbb{T}) : \operatorname{ind}(\lambda - a) \neq 0\}. \tag{3.4}$$

A point $\lambda \in \sigma_{H^1}(T_a) \setminus \sigma_{\Phi(H^1)}(T_a)$ is an eigenvalue of T_a if and only if $\operatorname{ind}(\lambda - a) < 0$, in which case the multiplicity of λ is the number $-\operatorname{ind}(\lambda - a)$.

Proof. Apply (3.2), (3.3), and Corollary 11. □

As in the case $1 < p < \infty$, the situation regarding the (non)existence of eigenvalues embedded in the essential spectra of Toeplitz operators is a much more difficult question. One reason that the approach used in the previous section cannot be applied here is related to the role that conjugate exponents play in the proof of Theorem 4. All known results are restricted to real-valued symbols that satisfy Hölder or a slightly weaker condition. We give one condition, which is based on the following result (see [13]).
Let $a : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function which satisfies the condition in (1.1) with $\mu = 1$. If a changes sign, the Riemann-Hilbert problem (2.1) has no solutions in H^1. Let us see what this means in terms of eigenvalues. Using a similar argument as in the proof of Proposition 6, it is easy to see that no point in the interior of the essential spectrum of T_a can be an eigenvalue of T_a. In particular, if a is Lipschitz continuous and if λ is in the essential spectrum of T_a, then λ may be an eigenvalue only if it is one of the endpoints of $a(\mathbb{T})$.

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