The London School of Economics and Political Science

Essays on Estimation of Dynamic Games

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For:

My Mother, Odete Miessi Sanches (*in memoriam*).

Declaration

I declare that my thesis consists of 19,235 words.

Statement of conjoint work

I confirm that Chapter 3 was jointly co-authored with Daniel Silva Junior and I contributed 60% of this work.

Abstract

This thesis considers estimation of discrete choice stationary dynamic games. Chapter 1 shows that when payoffs are linear in the parameters value functions are linear in the parameters and the equation system characterizing the Markovian equilibrium is linear in the parameters. This formulation allows us to estimate the model using Least Squares. We derive an optimal weight matrix for the Least Squares estimator and show that the efficient estimator is a Generalized Least Squares estimator. Chapter 2 shows that when time invariant unobservables are present the efficient estimator is a Generalized Fixed Effects estimator. Time invariant unobservables can be correlated with observed states. We do not need to impose any distributional assumption on time invariant unobservables. Our estimators have a closed form solution. In Chapter 3 we apply the framework developed in Chapters 1 and 2 to analyze the effects of the privatization of public banks on financial development. We build a dynamic entry game to analyze the Brazilian banking market. We show that profits of private banks are positively affected by the number of public branches operating in Brazilian isolated markets. The spill-over generated by public banks is quantified based on a dynamic oligopoly model. A counterfactual in which public banks are privatized is examined. It shows that the number of active branches operating in the long-run in a small market drops significantly.

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Preface

This thesis considers estimation of discrete choice stationary dynamic games. Chapter 1 shows that when payoffs are linear in the parameters Least Squares estimators are consistent estimators for payoff parameters. We derive an optimal weight matrix for the Least Squares estimator and show that the efficient estimator is a Generalized Least Squares estimator. The advantage of our formulation is that Least Square estimators have a closed form solution and do not depend on the numerical methods used in other popular estimation procedures (e.g. Hotz and Miller (1993), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others). Our estimation strategy provides globably optimal estimates that do not depend on initial guesses of parameters.

Pesendorfer and Schmidt-Dengler (2008) show that other popular estimators in the literature, including Hotz and Miller (1993), Aguirregabiria and Mira (2002) and Bajari, Benkard and Levin (2007), are Asymptotic Least Squares (ALS) estimators. We show that when payoffs are linear in the parameters ALS estimators for dynamic games are Least Squares estimators. It follows that, under the linearity of payoffs, the estimators proposed by Hotz and Miller (1993), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), are Least Squares estimators. The class of Least Squares estimators developed in this paper provides, therefore, a unified framework for a number of popular estimators for dynamic games.

Chapter 2 shows that when time invariant unobservables are present the efficient estimator is a Generalized Fixed Effects estimator. Time invariant unobservables can be correlated with observed states. We do not need to impose any distributional assumption on time invariant unobservables. Our estimators have a closed form solution.

The Generalized Fixed Effects estimator present advantages on other popular estimators for models with time invariant unobservables. Aguirregabiria and Mira (2002) also consider a model time invariant unobservables. The unobservables, however, by assumption, are uncorrelated with observed states and the estimates depend on the choice for the distribution of unobservables. GFE estimators relax these two assumptions.

GFE is less general than Arcidiacono and Miller (2011). Arcidiacono and Miller (2011) allow for time variant unobservables and propose a four step estimation procedure for payoff parameters. When time variant unobservables are not present our estimation procedure is clearly more straightforward. We do not have to estimate the distribution of unobservables and to use the four step numerical method to recover payoff parameters.

Chapter 3 applies the framework developed in Chapters 1 and 2 to analyze the effects of the privatization of public banks on financial development. We build a dynamic entry game to analyze the Brazilian banking market. We show that profits of private banks are positively affected by the number of public branches operating in Brazilian isolated markets. The spill-over generated by public banks is quantified based on a dynamic oligopoly model. A counterfactual in which public banks are privatized is examined. It shows that the number of active branches operating in the long-run in a small market drops significantly.

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Part I

Generalized Least Squares Estimators for Dynamic Games

Abstract

This paper develops Least Squares estimators for discrete choice stationary dynamic games. We show that when payoffs are linear in the parameters Least Squares estimators can consistently estimate the parameters of the model. We derive the efficient weight matrix that characterizes these estimators and the asymptotic distribution of the estimators. We show that the efficient estimator is a Generalized Least Squares estimator. The estimators have a closed form solution. We illustrate the small sample performance of our estimators using a Monte Carlo Experiment. Least Squares are much easier to implement than other popular estimators for dynamic games. This procedure simplifies the estimation of dynamic games.

1 Introduction

This paper considers the estimation of discrete choice stationary dynamic games when payoffs are linear in the parameters. Estimation of discrete choice stationary dynamic games have been studied in Hotz and Miller (1993), Hotz, Miller, Sander and Smith (1994), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others. Static models with strategic interactions (e.g. Seim (2006)), single-agent static and dynamic models (e.g. Rust (1987)) are special cases of our framework.

Pesendorfer and Schmidt-Dengler (2008) show that in dynamic games payoffs cannot be identified without restrictions. This result generalizes the identification arguments of dynamic single agent models in Magnac and Thesmar (2002). Following this evidence, most of the literature has assumed a linear, parametric structure for players' payoffs (Pesendorfer and Schmidt-Dengler (2003), Pesendorfer and Schmidt-Dengler (2008), Aguirregabiria and Mira (2007), Ryan (2012) and Collard-Wexler (2013), among many others).

We show that when payoffs are linear in the parameters value functions are linear in the parameters and the equation system characterizing the Markovian equilibrium is linear in the parameters. This formulation allows us to estimate the model using Least Squares. We derive an optimal weight matrix for the Least Squares estimator and show that the efficient estimator is a Generalized Least Squares estimator.

The advantage of our approach is that Least Squares estimators have a closed form solution and do not depend on the numerical methods used in other popular estimation procedures (e.g. Hotz and Miller (1993), Aguirregabiria and Mira (2007), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others). These methods can be computationally burdensome and can produce inconsistent estimates of the parameters of interest (Pesendorfer and Schmidt-Dengler (2010) and Srisuma (2010)). Our estimation strategy produces globaly consistent estimates of the parameters and reduces significantly the computational burden when state spaces and/or the number of parameters in the model is large.

Pesendorfer and Schmidt-Dengler (2008) show that other popular estimators in the literature, including Hotz and Miller (1993), Aguirregabiria and Mira (2002) and Bajari, Benkard and Levin (2007), are Asymptotic Least Squares (ALS) estimators. We show that when payoffs are linear in the parameters ALS estimators for dynamic games are Least Squares estimators. It follows that, under the linearity of payoffs, the estimators proposed by Hotz and Miller (1993), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and

Pesendorfer and Schmidt-Dengler (2008), are Least Squares estimators. The class of Least Squares estimators developed in this paper provides, therefore, a unified framework for a number of popular estimators for dynamic games.

We illustrate the small sample performance of our estimators using a Monte experiment. The experiment shows that Least Squares is much faster than other popular estimators.

This paper is organized as follows. Section 2 describes and solves the theoretical model. Section 3 proposes a class of Least Squares estimators for the parameters in the model. Section 4 analizes the small sample performance of our estimators based on a Monte Carlo experiment. Section 5 concludes the paper.

2 Theoretical Framework

This section describes the main elements of the model. We set up the model in a stationary discrete choice framework. Markets are treated isolately. We firstly describe the main assumptions behind the model. Subsequently we solve the model and characterize the equilibrium restrictions that are used to identify and to estimate the parameters of interest.

2.1 Assumptions

■ **Time and markets.** Time is discrete, $t = 1, 2, ..., \infty$. There is one market denoted by m.

■ **Players.** The set of players in market m is $\mathbf{N} = \{1, 2, .., N\}$. We denote each player in market m by $i \in \mathbf{N}$.

■ **Actions.** A player's action in market m, period t is denoted by $a_i^t \in \{0, 1, ..., K\}$. The $1 \times N$ vector $\mathbf{a^t} \in \mathbf{A} = \underset{i \in \mathbf{N}}{\times} a_i^t$ denotes the action profile in market m, period t. We sometimes use $\mathbf{a^t_{-i}} \in \mathbf{A_{-i}} = \underset{j \neq i, j \in \mathbf{N}}{\times} a_j^t$ to denote the actions of all players but player i. The cardinality of the action space in market m is $N_a = (K+1)^N$.

■ **State space.** The state space is discrete and finite. The state variables for player $i \in \mathbf{N}$ is composed by a vector $\mathbf{s_i^t} \in \mathbf{S_i} = \{\mathbf{1}, \mathbf{2}, ..., \mathbf{L}\}$ of exogenous variables. The state variables are publicly known to the players and to the econometrician. The vector of all players' state variables is $\mathbf{s^t} = (\mathbf{s_1^t}, \mathbf{s_2^t}, ..., \mathbf{s_N^t})$ such that $\mathbf{s^t} \in \mathbf{S} = \underset{i \in \mathbf{N}}{\times} \mathbf{S_i}$. The cardinality of the state space $\mathbf{S}$ is $N_s = L^N$.

■ **Shocks.** In each period players draw a vector of profitability shocks. We use $\xi_\mathbf{i}^\mathbf{t}$ to denote the $(K+1) \times 1$ vector $(\xi_{i0}^t, \xi_{i1}^t, ..., \xi_{iK}^t)$ of profitability shocks. The profitability shock is iid across individuals, time and actions. This is the only source of asymmetric information in the model. We denote the cumulative distribution function of $\xi_\mathbf{i}^\mathbf{t}$ by $G(\cdot)$.

■ **Payoffs.** Player i's period payoff in market m is given by $\Pi_i(\mathbf{a^t}, \mathbf{s^t}, \xi_\mathbf{i}^\mathbf{t}) = \pi_i(\mathbf{a^t}, \mathbf{s^t}) + \sum_{k=0}^{K} \xi_{ik}^t \cdot I\left(a_i^t = k\right)$, where $\pi_i(\mathbf{a^t}, \mathbf{s^t})$ denotes player's deterministic profits and $I\left(.\right)$ is an indicator function that assumes 1 if the condition (.) is satisfied and 0 otherwise.

■ **Transitions.** The vector $\mathbf{s^{t+1}}$ evolves according to the conditional cumulative density function $p(\mathbf{s^{t+1}}|\mathbf{a^t}, \mathbf{s^t})$, described by next period distribution of possible values for the vector $\mathbf{s^{t+1}}$ conditional on each $(\mathbf{a^t}, \mathbf{s^t})$. We sometimes use $\mathbf{p}$ to denote the $N_a \cdot N_s \cdot N_s \times 1$ vector of transitions, $p(\mathbf{s^{t+1}}|\mathbf{a^t}, \mathbf{s^t})$, for every possible future state $\mathbf{s^{t+1}} \in \mathbf{S}$ given all $(\mathbf{a^t} \in \mathbf{A}, \mathbf{s^t} \in \mathbf{S})$.

■ **Sequence of decisions.** The sequence of events in this game is the following:

1. States are observed by all the players.

2. Each player draws the private profitability shock $\xi_\mathbf{i}^\mathbf{t}$.

3. Actions are simultaneously chosen. Players maximize their payoffs given beliefs on competitor's actions. The total payoff of a player is given by the discounted sum of player's period payoffs. The discount rate is given by $\beta < 1$ and is the same for all players.

4. After actions are chosen the law of motion for $\mathbf{s^{t+1}}$ determines the distribution of states in the next period; the problem restarts.

Next the equilibrium for this game is characterized.

2.2 Equilibrium characterization

We restrict our attention to pure *Markovian strategies.* This means that players' actions are fully determined by the current vector of state variables. Intuitively, whenever a player observes the same vector of states it will take the same actions and the history of the game until period t does not influence player's decisions.

Player i's best response function solves the following Bellman equation:

$$\underset{a_i^t \in \{0,1,\ldots,K\}}{Max} \left\{ \begin{array}{c} \sum_{\mathbf{a_{-i}^t}} \sigma_i(\mathbf{a_{-i}^t}|\mathbf{s^t})\Pi_i(a_i^t{=}k, \mathbf{a_{-i}^t}, \mathbf{s^t}, \xi_\mathbf{i}^\mathbf{t}) + \\ \beta \mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) \mathbf{E}_\xi \mathbf{V_i}\left(\sigma_\mathbf{i}, \mathbf{p}\right) \end{array} \right\}. \tag{1}$$

Here $\Pi_i(\cdot)$ is player's period payoff; the function $\sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})$ accounts for i's beliefs on other players' actions given current states; $\sigma_\mathbf{i}$ *is a* $N_a \cdot N_s \times 1$ vector of beliefs, $\sigma_i(\mathbf{a^t}|\mathbf{s^t})$, *for* all and $\mathbf{a^t} \in \mathbf{A}$ and $\mathbf{s^t} \in \mathbf{S}$*;* $\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)$ is a $1 \times N_s$ vector containing the transitions $\sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})p(\mathbf{s^{t+1}}|a_i^t = k, \mathbf{a^t_{-i}}, \mathbf{s^t})$ and $\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right)$ is a $N_s \times 1$ vector with the expected continuation value for the player, $E_\xi V_i(\mathbf{s^{t+1}};\sigma_\mathbf{i},\mathbf{p},\pi)$, for all $\mathbf{s^{t+1}} \in \mathbf{S}$.

The value function conditional on $a_i^t = k \in \{0, 1, ..., K\}$ being played in period t is then defined as:

$$V_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) = \sum_{\mathbf{a^t_{-i}}} \sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})\pi_i(a_i^t{=}k, \mathbf{a^t_{-i}}, \mathbf{s^t}) + \beta \mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right) + \xi_{ik}^t, \tag{2}$$

and $V_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) = \tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) + \xi_{ik}^t$, where $\tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})$ comprises all the terms in (2) except the profitability shock.

We define $\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right)$ as the *ex-ante* value function, $\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right) = \mathbf{\Delta_i}\left(\tilde{\pi}_\mathbf{i} + \tilde{\mathbf{E}}_{\xi\mathbf{i}}\right)$, where $\mathbf{\Delta_i} = \left[\mathbf{I_{N_s}} - \beta\mathbf{Z_i}\right]^{-1}$; $\tilde{\pi}_\mathbf{i}$ is a $N_s \times 1$ vector stacking current payoff expected values, $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^t}|\mathbf{s^t})\pi_i(\mathbf{a^t},\mathbf{s^t})$, for every state; $\tilde{\mathbf{E}}_{\xi\mathbf{i}}$ is a $N_s \times 1$ vector stacking $\tilde{E}_\xi\left(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) = \sum_{k=0}^{K} \sigma_i(a_i^t = k|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})E\left[\xi_{ik}^t|a_i^t = k, \mathbf{s^t}\right]$ for every state; $\mathbf{I_{N_s}}$ is a $N_s \times N_s$ identity matrix; and $\mathbf{Z_i}$ is a $N_s \times N_s$ matrix stacking the $1 \times N_s$ vector $\mathbf{z}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)$ containing the transitions $\sigma_i(\mathbf{a^t}|\mathbf{s^t})p(\mathbf{s^{t+1}}|\mathbf{a^t},\mathbf{s^t})$ for every state.

The solution to problem (1) implies that player i's probability of playing action k when states are $\mathbf{s^t}$ satisfies the following equilibrium restrictions[1]:

$$\begin{aligned} P_i(a_i^t = k|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) &= Prob\left(\tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) - \tilde{V}_i^{k'}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) \geq \xi_{ik'}^t - \xi_{ik}^t, \forall k' \neq k\right) \\ &= \int 1\left\{\ \tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) - \tilde{V}_i^{k'}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})\ \geq \xi_{ik'}^t - \xi_{ik}^t, \forall k' \neq k\right\} dG\left(\varepsilon_\mathbf{i}^\mathbf{t}\right), \end{aligned} \tag{3}$$

that holds for all $k, k' \in \{0, 1, ..., K\}$, all $\mathbf{s^t} \in \mathbf{S}$ all $i \in \mathbf{N}$.

The solution to this problem is a vector of player i's optimal actions when he faces each possible configuration for the state vector $\mathbf{s^t}$ and has consistent beliefs about other players

[1] See Train (2009).

actions in the same states of the world.

By stacking up the equilibrium restrictions derived in equation (3) for every action except action $k = 0$ of every player and every possible state one can form a system of $N \cdot K \cdot N_s \times 1$ equations. This system is used to find the $N \cdot K \cdot N_s \times 1$ vector of players' beliefs.

A formal proof of the existence of this vector can be found in Pesendorfer and Schmidt-Dengler (2008). Uniqueness of this equilibrium is not, however, guaranteed. This is a common feature of games.

3 Estimation

This section develops Generalized Least Square Estimators for dynamic games. The estimation procedure is based on two standard assumptions in the literature. These assumptions are stated below.

> **Assumption E1:** ξ^t_{ik} *is drawn from a Type I Extreme Value distribution.*
>
> **Assumption E2:** $\Pi(\mathbf{a^t}, \mathbf{s^t}, \xi^t_{\mathbf{i}}) = \varphi(\mathbf{a^t}, \mathbf{s^t})\boldsymbol{\Theta}' + \sum_{k=0}^{K} \xi^t_{ik} \cdot I\left(a^t_i = k\right)$, *where* $\boldsymbol{\Theta}$ *is* $1 \times N_p$ *vector of parameters and* $\varphi(\mathbf{a^t}, \mathbf{s^t_i})$ *is a* $1 \times N_p$ *vector that depends on* $(\mathbf{a^t}, \mathbf{s^t})$.

Assumption E1 restricts the distribution of the iid profitability shock to the class of Type I Extreme Value distributions. It conveniently implies that the equilibrium restriction (3) can be written as:

$$P(a^t_i = k|\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}) = \frac{exp\left(\tilde{V}^k(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})\right)}{\sum\limits_{k'=0}^{K} exp\left(\tilde{V}^{k'}(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})\right)}.$$

This holds for any $k \in \{0, 1, ..., K\}$. Dividing both sides by $P(a^t_i = k|\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})$ and taking logs, for any $k \in \{1, ..., K\}$ the equilibrium restriction becomes:

$$q_{k0}\left(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}\right) = \tilde{V}^k(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}) - \tilde{V}^0(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}), \tag{4}$$

where $q_{k0}\left(\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}\right) = ln\left\{P(a^t_i = k|\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})\right\} - ln\left\{P(a^t_i = 0|\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})\right\}$.

This assumption also implies that $E\left[\xi^t_{ik}|a^t_i = k, \mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p}\right] = \gamma - ln\left\{P(a^t_i = k|\mathbf{s^t}; \sigma_{\mathbf{i}}, \mathbf{p})\right\}$, where γ is the Euler constant - see Hotz and Miller (1993).

Assumption 2 restricts the payoff function to be linear in the parameters. The linearity of the equilibrium restrictions is shown in the next lemma.

Lemma 1. *Under assumptions E1-E2, player i's probability of playing action* $k \in \{1, ..., K\}$ *when states are* $\mathbf{s^t}$ *satisfies the restriction* $y_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) - \mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\Theta' = 0$*, where:*

1. $y_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = q_{k0}\left(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \beta \left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right] \mathbf{\Delta_i} \mathbf{\tilde{E}}_{\xi\mathbf{i}}$;

2. $\mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \tilde{\varphi}_{k0}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) + \beta \left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right] \mathbf{\Delta_i} \tilde{\varphi}_\mathbf{i}$ *is a* $1 \times N_p$ *vector;* $\tilde{\varphi}_{k0}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \sum_{\mathbf{a^t_{-i}}} \sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t}) \left[\varphi(a_i^t = k, \mathbf{a^t_{-i}}, \mathbf{s^t}) - \varphi(a_i^t = 0, \mathbf{a^t_{-i}}, \mathbf{s^t})\right]$ *is a* $1 \times N_p$ *vector, and* $\tilde{\varphi}_\mathbf{i}$ *is a* $N_s \times N_p$ *matrix stacking* $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}}) \varphi(\mathbf{a^{t+1}}, \mathbf{s^{t+1}})$ *for all possible states.*

Proof. See appendix. □

We introduce a sequence of H auxiliary parameters containg estimates for transitions, $\mathbf{p}$, and beliefs, $\sigma_\mathbf{i}$, for all $i \in N$. Call it as $\left(\hat{\sigma}_{(\mathbf{T})}, \hat{\mathbf{p}}_{(\mathbf{T})}\right)$. We assume that:

> **Assumption E3:** *The sequence* $\left(\hat{\sigma}_{(\mathbf{T})}, \hat{\mathbf{p}}_{(\mathbf{T})}\right)$ *exists, converges in probability to* $(\sigma, \mathbf{p})$ *and is asymptotically normally distributed, that is,* $\left(\hat{\sigma}_{(\mathbf{T})}, \hat{\mathbf{p}}_{(\mathbf{T})}\right) \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} (\sigma, \mathbf{p})$ *and* $\sqrt{T}\left(\left(\hat{\sigma}_{(\mathbf{T})}, \hat{\mathbf{p}}_{(\mathbf{T})}\right) - (\sigma, \mathbf{p})\right) \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \mathbf{\Omega}\right)$*, where* $\mathbf{\Omega}$ *is a positive definite* $H \times H$ *matrix.*

This assumption follows Pesendorfer and Schmidt-Dengler (2008). Pesendorfer and Schmidt-Dengler (2008) discuss estimators for $(\sigma, \mathbf{p})$ that satisfy E3.

To keep the exposition neater we abuse notation and drop the T subscript from the hat variables. From now on keep in mind that all the hat variables are indexed on T. We define $\hat{y}_{ikt} = y_k(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$ and $\mathbf{\hat{D}_{ikt}} = \mathbf{D_k}(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$.

By summing and subtracting the term $\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}'$ from the equilibrium restriction derived in Lemma 1 we write $\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}' - \hat{u}_{ikt} = 0$ or, equivalently:

$$\hat{y}_{ikt} = \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}' + \hat{u}_{ikt}, \tag{5}$$

where $u_{ikt} = \left(\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}'\right) - \left(y_{ikt} - \mathbf{D_{ikt}}\mathbf{\Theta}'\right)$, $y_{ikt} = y_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$ and $\mathbf{D_{ikt}} = \mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$.

In matrix form, stacking (5) for all players, states and actions except action $k = 0$ we write:

$$\mathbf{\hat{y}} = \mathbf{\hat{D}}\mathbf{\Theta}' + \mathbf{\hat{u}}, \tag{6}$$

where the variable $\hat{\mathbf{y}}$ is a $N \cdot K \cdot N_s \times 1$ vector stacking $\hat{y}_{ikt}$ for all individuals, actions and states, $\hat{\mathbf{D}}$ is a $N \cdot K \cdot N_s \times N_p$ matrix stacking $\hat{\mathbf{D}}_{\mathbf{ikt}}$ for all individuals, actions and states and $\hat{\mathbf{u}}$ is a $N \cdot K \cdot N_s \times 1$ vector stacking $\hat{u}_{ikt}$ for all individuals, actions and states. The asymptotic properties of $\hat{\mathbf{u}}$ are derived below.

Lemma 2. $\sqrt{T}\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{d}{\to}} \mathbf{N}(\mathbf{0}, \mathbf{\Lambda})$, *where:*

1. $\mathbf{\Lambda} = \nabla(\sigma, \mathbf{p})\, \mathbf{\Omega} \nabla'(\sigma, \mathbf{p})$*; and,*
2. $\nabla(\sigma, \mathbf{p})$ *is a* $N \cdot K \cdot N_s \times H$ *matrix of the derivatives of* $\hat{y}_{ikt} - \hat{\mathbf{D}}_{\mathbf{ikt}}\mathbf{\Theta}'$ for all $i \in \mathbf{N}$, $k \in \{1, ..., K\}$ *and* $\mathbf{s^t} \in \mathbf{S}$ with respect to the auxiliary parameters, $(\hat{\sigma}, \hat{\mathbf{p}})$, evaluated at $(\sigma, \mathbf{p})$.

Proof. See appendix. □

Equation (5) is a linear estimating equation with a well defined variance-covariance structure for the error term. Now, an additional assumption is introduced:

Assumption E4: *rank* $\left\{\hat{\mathbf{D}}'\mathbf{\Lambda}^{-1}\hat{\mathbf{D}}\right\} = N_p$.

Assumption E4 guarantees the identification of $\mathbf{\Theta}$. Under E1-E4, it readily follows that the Generalized Least Squares Estimator for $\mathbf{\Theta}$ is consistent and asymptotically normal. This result is formally stated in the next proposition.

Proposition 1. *Under assumptions E1-E4 the Generalized Least Squares Estimator,*

$$\hat{\mathbf{\Theta}}'_{\mathbf{GLS}} = \left(\hat{\mathbf{D}}'\mathbf{\Lambda}^{-1}\hat{\mathbf{D}}\right)^{-1}\left(\hat{\mathbf{D}}'\mathbf{\Lambda}^{-1}\hat{\mathbf{y}}\right),$$

is a consistent and asymptotically normal estimator for $\mathbf{\Theta}$, $\sqrt{T}\left(\hat{\mathbf{\Theta}}_{\mathbf{GLS}} - \mathbf{\Theta}\right) \underset{T\uparrow\infty}{\overset{d}{\to}} N(\mathbf{0}, \mathbf{\Xi_\Theta})$, *where* $\mathbf{\Xi_\Theta} = (\mathbf{D}'\mathbf{\Lambda}^{-1}\mathbf{D})^{-1}$ *is the asymptotic variance of* $\hat{\mathbf{\Theta}}_{\mathbf{GLS}}$. *Futhermore* $\hat{\mathbf{\Theta}}_{\mathbf{GLS}}$ *is unique and exists.*

Proof. See appendix. □

Alternatively, OLS directly on (5) is a consistent but inefficient estimator for $\mathbf{\Theta}$. In the appendix we derive the large sample properties of the OLS estimator. Because the OLS estimator does not require the computation of the weight matrix $\mathbf{\Lambda}$, it is considerably simpler than the Generalized Least Squares Estimator. When the number of states is large, this approach can be computationally attractive.

In the appendix we show that under E1 and E2 GLS estimators are equivalent to the Asymptotic Least Squares Estimators (ALS) for dynamic games developed in Pesendorfer and Schmidt-Dengler (2008). Pesendorfer and Schmidt-Dengler (2008) show that other popular estimators in the literature, including Hotz and Miller (1993), Aguirregabiria and Mira (2007) and Bajari, Benkard and Levin (2007), are ALS estimators. It follows that, under the linearity of payoffs, the estimators proposed by Hotz and Miller (1993), Aguirregabiria and Mira (2007), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), are Least Squares estimators. The class of Least Squares Estimators developed in this paper provides, therefore, a unified framework for a number of popular estimators for dynamic games.

4 Monte Carlo Experiment

We illustrate the small sample performance of our OLS estimator using the Monte Carlo experiment proposed in Pesendorfer and Schmidt-Dengler (2008). We compare the performance of the OLS estimator with the ALSE-I estimator proposed in Pesendorfer and Schmidt-Dengler (2008).

Consider the two-firm, two actions dynamic entry game analyzed in Pesendorfer and Schmidt-Dengler (2008). Time is discrete and the model has infinite horizon. Each firm i has two possible choices: Be active or not active, $a_i^t \in \mathbf{A_i} = \{0, 1\}$, where 0 corresponds to "not active" and 1 to "active". Firm i's state space in period t has four elements, denoting the actions made by both firms in period $t-1$, $\mathbf{s^t} = \mathbf{a^{t-1}} \in \{(0,0)\,;(0,1)\,;(1,0)\,;(1,1)\}$. State variables are common knowledge. The vector of states evolves over time according to the transition $\mathbf{s^{t+1}} = \mathbf{a^t}$. Firm i's period payoffs are described as follows:

$$\Pi(\mathbf{a^t}, \mathbf{s^t}, \xi_\mathbf{i}^\mathbf{t}) =$$
$$I(a_i^t = 1) \cdot \left[\pi_{0i} + \pi_{1i} a_{-i}^t + \xi_i^t\right] + I(a_i^t = 1) \cdot I(a_i^{t-1} = 0) \cdot F_i + I(a_i^t = 0) \cdot I(a_i^{t-1} = 1) \cdot W_i,$$

where the vector of parameters, $(\pi_{0i}, \pi_{1i}, F_i, W_i)$, describes respectively firm i's monopoly profits, duopoly profits, entry costs and the scrap value that the firm obtains when it leaves the market; $I(\cdot)$ is an indicator function and ξ_i^t denotes firm i's iid profitability with distribution $N(0,1)$. We denote its cdf by $\Phi(\cdot)$. The profitability shock is firm i's private information. The distribution of ξ_i^t is common knowledge.

Choices are made simultaneously to maximize the discounted sum of firm's payoffs. Without loss of generality we focus on firm i's decisions. The solution to firm i's problem when state is $\mathbf{s^t}$ implies that the probability of choosing $a_i^t = 0$ satisfies the following equilibrium restriction:

$$P_i(a_i^t = 0|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \Phi \left(\tilde{V}_i^0(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) - \tilde{V}_i^1(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) \right),$$

where $\tilde{V}_i^k(\mathbf{s_i^t}; \sigma_\mathbf{i}, \mathbf{p_i})$ denotes the value function, net of the profitability shock, conditional on action $k \in \{0, 1\}$.

We assume that the values of the parameters are $(\pi_{0i}, \pi_{1i}, F_i, W_i) = (1.2, -1.2, -0.2, 0.1)$. As in Pesendorfer and Schmidt-Dengler (2008) we assumed that W_i is known and we do not estimate it.

Using the parameter values and the best response system we solved the model for the vector of beliefs. We focused only on the symmetric equilibrium[2]. We used the vector of beliefs to simulate time series of actions. The initial state is assumed to be $(0, 0)$ and the first 250 observations are excluded from the sample. We draw time series with 100, 1000, 10000 and 100000 observations. For each configuration we repeat the exercise 1000 times. For the ALSE estimates we used 0.5 for all the three parameters as the initial guess for the non linear minimization. To use GLS and ALSE-E we firstly estimate (π_0, π_1, F) using OLS and ALSE-I respectively. Then the OLS and ALSE-I estimates are used to construct the GLS and ALSE-E optimal weight matrices. We calculate the average of the estimates across each simulation and the mean square error (MSE) of the estimates. We report the sum of the MSE for the three estimated parameters. We also report the CPU times in seconds to perform 1000 estimations.

[2]The results are quite similar for the other 2 equilibria found in Pesendorfer and Schmidt-Dengler (2008).

Table 1: Monte Carlo Results

Sample Size (T)	Estimator	F	π_0	π_1	MSE	Time (s)
100	ALSE-I	-0.246	1.113	-1.048	0.995	834
	OLS	-0.319	0.977	-0.886	0.840	2
	ALSE-E	-0.429	1.183	-1.015	6.734	1650
	GLS	-0.433	1.003	-0.888	0.658	20
1000	ALSE-I	-0.201	1.196	-1.187	0.102	828
	OLS	-0.213	1.172	-1.158	0.097	2
	ALSE-E	-0.213	1.190	-1.180	0.105	1582
	GLS	-0.230	1.178	-1.161	0.092	20
10000	ALSE-I	-0.203	1.198	-1.196	0.010	846
	OLS	-0.204	1.195	-1.193	0.009	2
	ALSE-E	-0.203	1.197	-1.196	0.009	1513
	GLS	-0.205	1.196	-1.194	0.009	20
100000	ALSE-I	-0.200	1.201	-1.200	0.001	815
	OLS	-0.200	1.200	-1.199	0.001	2
	ALSE-E	-0.201	1.200	-1.198	0.001	1470
	GLS	-0.201	1.200	-1.198	0.001	20

The results show that OLS/GLS performs better than ALSE-I/ALSE-E in all samples. In the smallest sample, OLS/GLS have much lower MSEs. The higher MSE for ALSE-E when $T = 100$ is due to outliers. These outliers increase the variance of the ALSE-E estimates. OLS/GLS delivers reasonable estimates in terms of sign and magnitude in all samples. The time to estimate the model using OLS is negligible. The time to estimate the model using ALSE-I is around 800 seconds. The difference in estimation times is explained by the time for the convergence of the non linear search algorithm used to compute ALSE-I.

The time to estimate the model using GLS/ALSE-E is larger than the time to estimate the model using OLS/ALSE-I. To use GLS/ALSE-E we need to compute the optimal weight matrices.

5 Conclusion

We show that when payoffs are linear in the parameters the equation system characterizing the Markovian equilibrium of a dynamic game is linear in the parameters. This formulation allows us to estimate the model using Least Squares. We derive an optimal weight matrix for the Least Squares estimator and show that the efficient estimator is a Generalized Least Squares estimator.

Linearity of payoffs have been used by most of the papers in the literature (Pesendorfer and Schmidt-Dengler (2003), Pesendorfer and Schmidt-Dengler (2008), Aguirregabiria and Mira (2007), Ryan (2012) and Collard-Wexler (2013), among many others).

An advantage of our approach is that Least Squares estimators have a closed form solution and do not depend on the numerical methods used in other popular estimation procedures (e.g. Hotz and Miller (1993), Aguirregabiria and Mira (2007), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others).

A Monte Carlo experiment illustrates the advantages of our estimators.

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Appendix

This appendix contains proofs.

■ **Lemma 1.**

Proof. Assumption E2 and equations (2) and (4) imply that:

$$q_{k0}\left(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) = \tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})\mathbf{\Theta}' + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)\right]\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right), \tag{7}$$

where, $\tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) = \sum_{\mathbf{a^t_{-i}}}\sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})\left[\varphi(a_i^t=k,\mathbf{a^t_{-i}},\mathbf{s^t}) - \varphi(a_i^t=0,\mathbf{a^t_{-i}},\mathbf{s^t})\right]$ is a $1\times N_p$ vector.

Substituting the *ex ante* value function into (7) we get:

$$q_{k0}\left(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)=$$
$$\tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p})\mathbf{\Theta}^{'}+\beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)-\mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)\right]\mathbf{\Delta_i}\left(\tilde{\pi}+\tilde{\mathbf{E}}_{\xi\mathbf{i}}\right)$$

Use E2 again to write $\tilde{\pi}_{\mathbf{i}}=\tilde{\varphi}_{\mathbf{i}}\mathbf{\Theta}^{'}$, where $\tilde{\varphi}_i$ is a $N_s\times N_p$ matrix stacking the vector $\sum_{\mathbf{a^{t+1}}}\sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}})\varphi(\mathbf{a^{t+1}},\mathbf{s^{t+1}})$ for all possible vector of states. Using these definitions and rearranging, the equilibrium restrictions (3) can be written as $y_k(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p})-\mathbf{D_k}(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p})\Theta^{'}=0$, where:

$$y_k(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p})=$$
$$q_{k0}\left(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)-\beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)-\mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)\right]\mathbf{\Delta_i}\tilde{\mathbf{E}}_{\xi\mathbf{i}}$$
$$\mathbf{D_k}(\mathbf{s_i^t};\sigma_{\mathbf{i}},\mathbf{p_i})=$$
$$\tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p})+\beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)-\mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)\right]\mathbf{\Delta_i}\tilde{\varphi}_{\mathbf{i}}.$$

□

■ **Lemma 2.**

Proof. Expanding $\hat{\mathbf{u}}=\left(\hat{\mathbf{y}}-\hat{\mathbf{D}}\mathbf{\Theta}^{'}\right)-\left(\mathbf{y}-\mathbf{D}\mathbf{\Theta}^{'}\right)$ around $(\sigma,\mathbf{p})$:

$$\hat{\mathbf{u}}=\nabla\left(\sigma,p\right)\left[\left(\hat{\sigma},\hat{\mathbf{p}}\right)-\left(\sigma,\mathbf{p}\right)\right]+o\left(\|\left(\hat{\sigma},\hat{\mathbf{p}}\right)-\left(\sigma,\mathbf{p}\right)\|\right),$$

where $\nabla\left(\sigma,p\right)$ is a $N\cdot K\cdot N_s\times H$ matrix of the derivatives of $\hat{y}_{ikt}-\hat{\mathbf{D}}_{\mathbf{ikt}}\mathbf{\Theta}^{'}$ for all $i\in\mathbf{N}$, $k\in\{1,...,K\}$ *and* $\mathbf{s^t}\in\mathbf{S}$ with respect to the auxiliary parameters, $(\hat{\sigma},\hat{\mathbf{p}})$, evaluated at $(\sigma,\mathbf{p})$. Therefore:

$$\begin{aligned}\hat{\mathbf{u}}&=\nabla\left(\sigma,p\right)\left[\left(\hat{\sigma},\hat{\mathbf{p}}\right)-\left(\sigma,\mathbf{p}\right)\right]+o\left(O_p\left(\frac{1}{\sqrt{T}}\right)\right)\\&=\nabla\left(\sigma,p\right)\left[\left(\hat{\sigma},\hat{\mathbf{p}}\right)-\left(\sigma,\mathbf{p}\right)\right]+o_p\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Multiplying both sides by $\sqrt{T}$:

$$\sqrt{T}\hat{\mathbf{u}}=\nabla\left(\sigma,p\right)\sqrt{T}\left[\left(\hat{\sigma},\hat{\mathbf{p}}\right)-\left(\sigma,\mathbf{p}\right)\right]+o_p\left(1\right).$$

Assumption E3 directly implies that:

$$\nabla(\sigma, p)\sqrt{T}\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \nabla(\sigma, \mathbf{p})\,\boldsymbol{\Omega}\nabla'(\sigma, \mathbf{p})\right).$$

Therefore $\sqrt{T}\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \nabla(\sigma, \mathbf{p})\,\boldsymbol{\Omega}\nabla'(\sigma, \mathbf{p})\right)$. □

■ **Proposition 1.**

Proof. Firstly we show that $\hat{\boldsymbol{\Theta}}_{\mathbf{GLS}}$ is consistent. Substitute $\hat{\mathbf{y}} = \hat{\mathbf{D}}\boldsymbol{\Theta}' + \hat{\mathbf{u}}$ into the formula for $\hat{\Theta}_{GLS}$:

$$\begin{aligned}\hat{\boldsymbol{\Theta}}'_{\mathbf{GLS}} &= \left(\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-\mathbf{1}}\left(\hat{\mathbf{D}}\boldsymbol{\Theta}' + \hat{\mathbf{u}}\right)\\ &= \boldsymbol{\Theta}' + \left(\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-\mathbf{1}}\hat{\mathbf{u}}.\end{aligned}$$

Notice now that continuity of $\hat{\mathbf{y}}$ and $\hat{\mathbf{D}}$ in $(\hat{\sigma}, \hat{\mathbf{p}})$ and E3 implies that $\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{0}_{\mathbf{N\cdot K\cdot N_s}}$, where $\mathbf{0}_{\mathbf{N\cdot K\cdot N_s}}$ is a $N \cdot K \cdot N_s \times 1$ vector of zeros and $\hat{\mathbf{D}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}$. Therefore: $\hat{\boldsymbol{\Theta}}'_{\mathbf{GLS}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \boldsymbol{\Theta}'$.

To determine the asymptotic distribution of $\hat{\boldsymbol{\Theta}}_{\mathbf{GLS}}$ write the expression above as:

$$\sqrt{T}\left(\hat{\boldsymbol{\Theta}}'_{\mathbf{GLS}} - \boldsymbol{\Theta}'\right) = \left(\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-\mathbf{1}}\sqrt{T}\hat{\mathbf{u}}.$$

Using Lemma 2 and the fact that $\hat{\mathbf{D}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}$ we have:

$$\left(\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\boldsymbol{\Lambda}^{-\mathbf{1}}\sqrt{T}\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \left(\mathbf{D}'\boldsymbol{\Lambda}^{-\mathbf{1}}\mathbf{D}\right)^{-\mathbf{1}}\right).$$

E4 guarantees the existence and unicity of $\hat{\boldsymbol{\Theta}}'_{\mathbf{GLS}}$. □

■ **Asymptotic properties of the OLS estimator.**

Proof. Under assumptions E1 and E2 and assuming that $rank\left\{\hat{\mathbf{D}}'\hat{\mathbf{D}}\right\} = N_p$, the OLS estimator for $\boldsymbol{\Theta}$ is given by $\hat{\boldsymbol{\Theta}}'_{\mathbf{OLS}} = \left(\hat{\mathbf{D}}'\hat{\mathbf{D}}\right)^{-\mathbf{1}}\left(\hat{\mathbf{D}}'\hat{\mathbf{y}}\right)$.

To show the consistency of $\hat{\boldsymbol{\Theta}}_{\mathbf{OLS}}$ substitute $\hat{\mathbf{y}} = \hat{\mathbf{D}}\boldsymbol{\Theta}' + \hat{\mathbf{u}}$ into the formula for $\hat{\boldsymbol{\Theta}}_{\mathbf{OLS}}$:

$$\hat{\mathbf{\Theta}}'_{\mathbf{OLS}} = \left(\hat{\mathbf{D}}'\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\left(\hat{\mathbf{D}}\mathbf{\Theta}' + \hat{\mathbf{u}}\right)$$
$$= \mathbf{\Theta}' + \left(\hat{\mathbf{D}}'\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\hat{\mathbf{u}}.$$

By the same reason used to determine the consistency of $\hat{\mathbf{\Theta}}_{\mathbf{GLS}}$ we have that $\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{0}_{\mathbf{N\cdot K\cdot N_s}}$, where $\mathbf{0}_{\mathbf{N\cdot K\cdot N_s}}$ is a $N \cdot K \cdot N_s \times 1$ vector of zeros and $\hat{\mathbf{D}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}$. Therefore: $\hat{\mathbf{\Theta}}'_{\mathbf{OLS}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{\Theta}'$.

For the asymptotic distribution of $\hat{\mathbf{\Theta}}_{\mathbf{OLS}}$ write:

$$\sqrt{T}\left(\hat{\mathbf{\Theta}}'_{\mathbf{OLS}} - \mathbf{\Theta}'\right) = \left(\hat{\mathbf{D}}'\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\sqrt{T}\hat{\mathbf{u}}.$$

Using Lemma 2 and the fact that $\hat{\mathbf{D}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}$ we have:

$$\left(\hat{\mathbf{D}}'\hat{\mathbf{D}}\right)^{-1}\hat{\mathbf{D}}'\sqrt{T}\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \left(\mathbf{D}'\mathbf{D}\right)^{-\mathbf{1}}\mathbf{D}'\mathbf{\Lambda}\mathbf{D}\left(\mathbf{D}'\mathbf{D}\right)^{-\mathbf{1}}\right).$$

□

■ **Equivalence between Least Squares and Asymptotic Least Squares.**

Proof. Lemma 1 implies that player i's probability of playing action $k \in \{1, ..., K\}$ when states are $\mathbf{s^t}$ satisfies the restriction $y_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) - \mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\Theta' = 0$. Asymptotic Least Squares estimators solve (Pesendorfer and Schmidt-Dengler (2008)):

$$\underset{\mathbf{\Theta}}{min}\left[\hat{\mathbf{y}} - \hat{\mathbf{D}}\mathbf{\Theta}'\right]'\mathbf{W}\left[\hat{\mathbf{y}} - \hat{\mathbf{D}}\mathbf{\Theta}'\right],$$

where $\hat{\mathbf{y}}$ is a $N \cdot K \cdot N_s \times 1$ vector stacking $\hat{y}_{ikt} = y_k(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$ for all individuals, actions and states, $\hat{\mathbf{D}}$ is a $N \cdot K \cdot N_s \times N_p$ matrix stacking and $\hat{\mathbf{D}}_{\mathbf{ikt}} = \mathbf{D_k}(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$ for all individuals, actions and states and $\mathbf{W}$ is a $N \cdot K \cdot N_s \times N \cdot K \cdot N_s$ weight matrix.

Under some regularity conditions (Gourieroux and Monfort (1995)), the solution of the problem above implies that $\hat{\mathbf{\Theta}}'_{\mathbf{ALS}} = \left(\hat{\mathbf{D}}'\mathbf{W}^{-\mathbf{1}}\hat{\mathbf{D}}\right)^{-\mathbf{1}}\left(\hat{\mathbf{D}}'\mathbf{W}^{-\mathbf{1}}\hat{\mathbf{y}}\right)$ and that the $\mathbf{W}$ that minimizes the variance of $\hat{\mathbf{\Theta}}'_{\mathbf{ALS}}$ is equal to $\mathbf{\Lambda}$. Using $\mathbf{W} = \mathbf{\Lambda}$ the efficient ALS estimator is given by $\hat{\mathbf{\Theta}}'_{\mathbf{ALSE}} = \left(\hat{\mathbf{D}}'\mathbf{\Lambda}^{-\mathbf{1}}\hat{\mathbf{D}}\right)^{-\mathbf{1}}\left(\hat{\mathbf{D}}'\mathbf{\Lambda}^{-\mathbf{1}}\hat{\mathbf{y}}\right)$ which is equal to $\hat{\mathbf{\Theta}}'_{\mathbf{GLS}}$. □

Part II

Generalized Fixed Effects Estimators for Dynamic Games

Abstract

This paper develops Fixed Effects estimators for discrete choice stationary dynamic games with time invariant unobservables. We show that when payoffs are linear in the parameters and (time invariant) unobservables, Fixed Effects Estimators can consistently estimate the parameters of the model. We derive the efficient weight matrix that characterizes these estimators and the asymptotic distribution of the estimators. We show that under the linearity of payoffs the efficient estimator is a Generalized Fixed Effects estimator. This procedure simplifies the estimation of dynamic games.

1 Introduction

This paper considers estimation of discrete choice stationary dynamic games when payoffs are linear in the parameters and time invariant heterogeneity. Estimation of discrete choice stationary dynamic games have been studied in Hotz and Miller (1993), Hotz, Miller, Sander and Smith (1994), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others. Static models with strategic interactions (e.g. Seim (2006)), single-agent static and dynamic models (e.g. Rust (1987)) are special cases of our framework. Models with time invariant unobserved heterogeneity have been analyzed in Aguirregabiria and Mira (2002), Arcidiacono and Miller (2011) and Collard-Wexler (2013).

We show that when payoffs are linear in the parameters and (time invariant) unobservables, value functions are linear in the parameters and the equation system characterizing the Markovian equilibrium is linear in the parameters and unobservables. This formulation allows us to estimate the model using Fixed Effects estimators.

We derive an optimal weight matrix for the Fixed Effects estimator and show that the efficient estimator is a Generalized Fixed Effects (GFE) estimator. GFE estimators have a closed form solution and do not depend on the numerical methods used in other popular estimation procedures (e.g. Hotz and Miller (1993), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others). Our estimation strategy provides globably optimal estimates that do not depend on initial guesses of parameters.

The Generalized Fixed Effects estimator present advantages on other popular estimators for models with time invariant unobservables. Aguirregabiria and Mira (2002) also consider a model time invariant unobservables. The unobservables, however, by assumption, are uncorrelated with observed states and the estimates depend on the choice for the distribution of unobservables. GFE estimators relax these two assumptions.

GFE is less general than Arcidiacono and Miller (2011). Arcidiacono and Miller (2011) allow for time variant unobservables and propose a four step estimation procedure for payoff parameters. When time variant unobservables are not present our estimation procedure is clearly more straightforward. We do not have to estimate the distribution of unobservables and to use the four step numerical method to recover payoff parameters.

This paper is organized as follows. Section 2 describes and solves the theoretical model. Section 3 proposes a class of Fixed Effects estimators for the parameters in the model.

Section 4 concludes the paper.

2 Theoretical Framework

This section describes the main elements of the model. We set up the model in a stationary discrete choice framework. Markets are treated isolately. We firstly describe the main assumptions behind the model. Subsequently we solve the model and characterize the equilibrium restrictions that are used to identify and to estimate the parameters of interest.

2.1 Assumptions

■ **Time and markets.** Time is discrete, $t = 1, 2, ..., \infty$. There is one market denoted by m.

■ **Players.** The set of players in market m is $\mathbf{N} = \{1, 2, .., N\}$. We denote each player in market m by $i \in \mathbf{N}$.

■ **Actions.** A player's action in market m, period t is denoted by $a_i^t \in \{0, 1, ..., K\}$. The $1 \times N$ vector $\mathbf{a^t} \in \mathbf{A} = \underset{i \in \mathbf{N}}{\times} a_i^t$ denotes the action profile in market m, period t. We sometimes use $\mathbf{a^t_{-i}} \in \mathbf{A_{-i}} = \underset{j \neq i, j \in \mathbf{N}}{\times} a_j^t$ to denote the actions of all players but player i. The cardinality of the action space in market m is $N_a = (K+1)^N$.

■ **State space.** The state space is discrete and finite. The state variables for player $i \in \mathbf{N}$ is composed by a vector $\mathbf{s_i^t} \in \mathbf{S_i} = \{\mathbf{1}, \mathbf{2}, ..., \mathbf{L}\}$ of exogenous variables. The state variables are publicly known to the players and to the econometrician. The vector of all players' state variables is $\mathbf{s^t} = (\mathbf{s_1^t}, \mathbf{s_2^t}, ..., \mathbf{s_N^t})$ such that $\mathbf{s^t} \in \mathbf{S} = \underset{i \in \mathbf{N}}{\times} \mathbf{S_i}$. The cardinality of the state space $\mathbf{S}$ is $N_s = L^N$.

■ **Shocks.** In each period players draw a vector of profitability shocks. We use $\xi_\mathbf{i}^\mathbf{t}$ to denote the $(K+1) \times 1$ vector $(\xi_{i0}^t, \xi_{i1}^t, ..., \xi_{iK}^t)$ of profitability shocks. The profitability shock is iid across individuals, time and actions. This is the only source of asymmetric information in the model. We denote the cumulative distribution function of $\xi_\mathbf{i}^\mathbf{t}$ by $G(\cdot)$.

■ **Payoffs.** Player i's period payoff in market m is given by $\Pi_i(\mathbf{a^t}, \mathbf{s^t}, \xi_\mathbf{i}^\mathbf{t}) = \pi_i(\mathbf{a^t}, \mathbf{s^t}) + \sum_{k=0}^K \xi_{ik}^t \cdot I\left(a_i^t = k\right)$, where $\pi_i(\mathbf{a^t}, \mathbf{s^t})$ denotes player $i's$ deterministic profits and $I\left(.\right)$ is an indicator function that assumes 1 if the condition (.) is satisfied and 0 otherwise.

■ **Transitions.** The vector $\mathbf{s^{t+1}}$ evolves according to the conditional cumulative density function $p(\mathbf{s^{t+1}}|\mathbf{a^t}, \mathbf{s^t})$, described by next period distribution of possible values for the vector $\mathbf{s^{t+1}}$ conditional on each $(\mathbf{a^t}, \mathbf{s^t})$. We sometimes use $\mathbf{p}$ to denote the $N_a \cdot N_s \cdot N_s \times 1$ vector of transitions, $p(\mathbf{s^{t+1}}|\mathbf{a^t}, \mathbf{s^t})$, for every possible future state $\mathbf{s^{t+1}} \in \mathbf{S}$ given all $(\mathbf{a^t} \in \mathbf{A}, \mathbf{s^t} \in \mathbf{S})$.

■ **Sequence of decisions.** The sequence of events in this game is the following:

1. States are observed by all the players.

2. Each player draws the private profitability shock $\xi_{\mathbf{i}}^{\mathbf{t}}$.

3. Actions are simultaneously chosen. Players maximize their payoffs given beliefs on competitor's actions. The total payoff of a player is given by the discounted sum of player's period payoffs. The discount rate is given by $\beta < 1$ and is the same for all players.

4. After actions are chosen the law of motion for $\mathbf{s^{t+1}}$ determines the distribution of states in the next period; the problem restarts.

Next the equilibrium for this game is characterized.

2.2 Equilibrium characterization

We restrict our attention to pure *Markovian strategies*. This means that players' actions are fully determined by the current vector of state variables. Intuitively, whenever a player observes the same vector of states it will take the same actions and the history of the game until period t does not influence player's decisions.

Player i's best response function solves the following Bellman equation:

$$\underset{a_i^t \in \{0,1,\ldots,K\}}{Max} \left\{ \begin{array}{c} \sum_{\mathbf{a_{-i}^t}} \sigma_i(\mathbf{a_{-i}^t}|\mathbf{s^t})\Pi_i(a_i^t{=}k, \mathbf{a_{-i}^t}, \mathbf{s^t}, \xi_{\mathbf{i}}^{\mathbf{t}})+ \\ \beta \mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)\mathbf{E}_\xi \mathbf{V_i}\left(\sigma_{\mathbf{i}},\mathbf{p}\right) \end{array} \right\}. \tag{8}$$

Here $\Pi_i(\cdot)$ is player's period payoff; the function $\sigma_i(\mathbf{a_{-i}^t}|\mathbf{s^t})$ accounts for i's beliefs on other players' actions given current states; $\sigma_{\mathbf{i}}$ *is a* $N_a \cdot N_s \times 1$ vector of beliefs, $\sigma_i(\mathbf{a^t}|\mathbf{s^t})$, *for* all and $\mathbf{a^t} \in \mathbf{A}$ and $\mathbf{s^t} \in \mathbf{S}$*;* $\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_{\mathbf{i}},\mathbf{p}\right)$ is a $1 \times N_s$ vector containing the transitions $\sigma_i(\mathbf{a_{-i}^t}|\mathbf{s^t})p(\mathbf{s^{t+1}}|a_i^t = k, \mathbf{a_{-i}^t}, \mathbf{s^t})$ and $\mathbf{E}_\xi \mathbf{V_i}\left(\sigma_{\mathbf{i}},\mathbf{p}\right)$ is a $N_s \times 1$ vector with the expected continuation value for the player, $E_\xi V_i(\mathbf{s^{t+1}};\sigma_{\mathbf{i}},\mathbf{p},\pi)$, for all $\mathbf{s^{t+1}} \in \mathbf{S}$.

The value function conditional on $a_i^t = k \in \{0,1,...,K\}$ being played in period t is then defined as:

$$V_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) =$$
$$\sum_{\mathbf{a^t_{-i}}} \sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})\pi_i(a_i^t{=}k,\mathbf{a^t_{-i}},\mathbf{s^t}) + \beta \mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)\mathbf{E}_\xi \mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right) + \xi_{ik}^t, \tag{9}$$

and $V_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) = \tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) + \xi_{ik}^t$, where $\tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})$ comprises all the terms in (9) except the profitability shock.

We define $\mathbf{E}_\xi \mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right)$ as the *ex-ante* value function, $\mathbf{E}_\xi \mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right) = \mathbf{\Delta_i}\left(\tilde{\pi}_\mathbf{i} + \tilde{\mathbf{E}}_{\xi\mathbf{i}}\right)$, where $\mathbf{\Delta_i} = \left[\mathbf{I_{N_s}} - \beta \mathbf{Z_i}\right]^{-1}$; $\tilde{\pi}_\mathbf{i}$ is a $N_s \times 1$ vector stacking current payoff expected values, $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^t}|\mathbf{s^t})\pi_i(\mathbf{a^t},\mathbf{s^t})$, for every state; $\tilde{\mathbf{E}}_{\xi\mathbf{i}}$ is a $N_s \times 1$ vector stacking $\tilde{E}_\xi\left(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) = \sum_{k=0}^{K} \sigma_i(a_i^t = k|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) E\left[\xi_{ik}^t|a_i^t = k,\mathbf{s^t}\right]$ for every state; $\mathbf{I_{N_s}}$ is a $N_s \times N_s$ identity matrix; and $\mathbf{Z_i}$ is a $N_s \times N_s$ matrix stacking the $1 \times N_s$ vector $\mathbf{z}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)$ containing the transitions $\sigma_i(\mathbf{a^t}|\mathbf{s^t})p(\mathbf{s^{t+1}}|\mathbf{a^t},\mathbf{s^t})$ for every state.

The solution to problem (8) implies that player i's probability of playing action k when states are $\mathbf{s^t}$ satisfies the following equilibrium restrictions[3]:

$$\begin{aligned} P_i(a_i^t = k|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) &= Prob\left(\tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) - \tilde{V}_i^{k'}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) \geq \xi_{ik'}^t - \xi_{ik}^t, \forall k' \neq k\right) \\ &= \int 1\left\{\ \tilde{V}_i^k(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) - \tilde{V}_i^{k'}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})\ \geq \xi_{ik'}^t - \xi_{ik}^t, \forall k' \neq k\right\} dG\left(\varepsilon_\mathbf{i}^\mathbf{t}\right), \end{aligned} \tag{10}$$

that holds for all $k, k' \in \{0, 1, ..., K\}$, all $\mathbf{s^t} \in \mathbf{S}$ all $i \in \mathbf{N}$.

The solution to this problem is a vector of player i's optimal actions when he faces each possible configuration for the state vector $\mathbf{s^t}$ and has consistent beliefs about other players actions in the same states of the world.

By stacking up the equilibrium restrictions derived in equation (10) for every action except action $k = 0$ of every player and every possible state one can form a system of $N \cdot K \cdot N_s \times 1$ equations. This system is used to find the $N \cdot K \cdot N_s \times 1$ vector of players' beliefs.

A formal proof of the existence of this vector can be found in Pesendorfer and Schmidt-Dengler (2008). Uniqueness of this equilibrium is not, however, guaranteed. This is a

[3] See Train (2009).

common feature of games.

3 Estimation

This section develops Generalized Fixed Effects Estimators for dynamic games with time invariant unobserved heterogeneity. We focus on the case of time invariant unobservable components at the individual level. The estimation procedure is based on two standard assumptions in the literature. These assumptions are stated below.

> **Assumption E1:** ξ_{ik}^t *is drawn from a Type I Extreme Value distribution.*

We incorporate unobserved heterogeneity in the model through assumption E2:

> **Assumption E2:** $\Pi_i(\mathbf{a^t}, \mathbf{s^t}, \xi_\mathbf{i}^\mathbf{t}) = \varphi(\mathbf{a^t}, \mathbf{s^t})\boldsymbol{\Theta}' + \mu_{ik} + \sum_{k=0}^{K} \xi_{ik}^t \cdot I\left(a_i^t = k\right)$, *where* $\mu_{ik} = \mu_i$ *for* $k \in \{1, ..., K\}$ and $\mu_{ik} = 0$ if $k = 0$. *The variable* μ_{ik} *is assumed to be observed by the players but not by the econometrician.*

Assumption E1 restricts the distribution of the iid profitability shock to the class of Type I Extreme Value distributions. It conveniently implies that the equilibrium restriction (3) can be written as:

$$P_i(a_i^t = k|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \frac{exp\left(\tilde{V}_i^k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\right)}{\sum\limits_{k'=0}^{K} exp\left(\tilde{V}_i^{k'}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\right)}.$$

This holds for any $k \in \{0, 1, ..., K\}$. Dividing both sides by $P_i(a_i^t = k|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$ and taking logs, for any $k \in \{1, ..., K\}$ the equilibrium restriction becomes:

$$q_{ik0}\left(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) = \tilde{V}_i^k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) - \tilde{V}_i^0(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}), \tag{11}$$

where $q_{ik0}\left(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) = ln\left\{P_i(a_i^t = k|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\right\} - ln\left\{P_i(a_i^t = 0|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\right\}$.

This assumption also implies that $E\left[\xi_{ik}^t | a_i^t = k, \mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right] = \gamma - ln\left\{P_i(a_i^t = k|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\right\}$, where γ is the Euler constant - see Hotz and Miller (1993).

The linearity of the equilibrium restrictions is shown in the next lemma.

Lemma 1. *Under assumptions E1 and E5, player i's probability of playing action $k \in \{1, ..., K\}$ when states are $\mathbf{s^t}$ satisfies the restriction*

$$y_{ik}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) - \mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\mathbf{\Theta}' - B_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})\mu_i = 0,$$

where:

1. $y_{ik}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = q_{ik0}\left(\mathbf{s^t_i}; \sigma_\mathbf{i}, \mathbf{p_i}\right) - \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right]\mathbf{\Delta_i}\tilde{\mathbf{E}}_\xi$*;*

2. $\mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \tilde{\varphi}_{k0}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right]\mathbf{\Delta_i}\tilde{\varphi}_\mathbf{i}$ *is a* $1 \times N_p$ *vector;* $\tilde{\varphi}_{k0}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = \sum_{\mathbf{a^t_{-i}}} \sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})\left[\varphi(a^t_i = k, \mathbf{a^t_{-i}}, \mathbf{s^t}) - \varphi(a^t_i = 0, \mathbf{a^t_{-i}}, \mathbf{s^t})\right]$ *is a* $1 \times N_p$ *vector, and* $\tilde{\varphi}_\mathbf{i}$ *is a* $N_s \times N_p$ *matrix stacking* $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}})\varphi(\mathbf{a^{t+1}}, \mathbf{s^{t+1}})$ *for all possible states.*

3. $B_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) = 1 + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right]\mathbf{\Delta_i}\tilde{\varphi}_{\mathbf{i}\mu}$; $\tilde{\varphi}_{\mathbf{i}\mu}$ *is a* $N_s \times 1$ *vector stacking* $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}})I(\mathbf{a^{t+1}} : a^{t+1}_i = 1)$*, where* $I(\mathbf{a^{t+1}} : a^{t+1}_i \neq 0)$ *is an indicator function that assumes 1 if* $\mathbf{a^{t+1}}$ *is such that* $a^{t+1}_i \neq 0$ *and 0 otherwise.*

Proof. See appendix. □

We introduce a sequence of H auxiliary parameters containg estimates for transitions, $\mathbf{p}$, and beliefs $\sigma_\mathbf{i}$ for all $i \in N$. Call it as $\left(\hat{\sigma}_\mathbf{(T)}, \hat{\mathbf{p}}_\mathbf{(T)}\right)$. We assume that:

> **Assumption E3:** *The sequence* $\left(\hat{\sigma}_\mathbf{(T)}, \hat{\mathbf{p}}_\mathbf{(T)}\right)$ *exists, converges in probability to* $(\sigma, \mathbf{p})$ *and is asymptotically normally distributed, that is,* $\left(\hat{\sigma}_\mathbf{(T)}, \hat{\mathbf{p}}_\mathbf{(T)}\right) \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} (\sigma, \mathbf{p})$ *and* $\sqrt{T}\left(\left(\hat{\sigma}_\mathbf{(T)}, \hat{\mathbf{p}}_\mathbf{(T)}\right) - (\sigma, \mathbf{p})\right) \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \mathbf{\Omega}\right)$*, where* $\mathbf{\Omega}$ *is a positive definite* $H \times H$ *matrix.*

Unobservables are allowed to be correlated with observed states. The transition of $\mathbf{s^{t+1}}$ is now characterized by $p(\mathbf{s^{t+1}}|\mathbf{s^t}, \mathbf{a^t}, \mu\left(\mathbf{a^t}\right))$, where $\mu\left(\mathbf{a^t}\right)$ is a $K \times 1$ vector containing the constants μ_{ik} for every $i \in \mathbf{N}$.

Notice that to estimate the vector $\mathbf{p}$, that depends on the unobservables μ_{ik}, one can use a Dynamic Fixed Effects Logit/Probit to parametrically estimate σ and to recover μ_{ik}; then $\hat{\mu}_{ik}$ can be used as covariates in the estimation of $\mathbf{p}$. Carro (2007) provides Dynamic Fixed Effects estimators for σ and μ_{ik}. The properties of this estimator satisfy E3.

To keep the exposition neater we abuse notation and drop the T subscript from the hat variables. From now on keep in mind that all the hat variables are indexed on T. We define $\hat{y}_{ikt} = y_{ik}(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$, $\hat{\mathbf{D}}_\mathbf{ikt} = \mathbf{D_k}(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$ and $\hat{B}_{ikt} = B_k(\mathbf{s^t}; \hat{\sigma}_\mathbf{i}, \hat{\mathbf{p}})$.

By summing and subtracting the term $\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}^{'} - \hat{B}_{ikt}\mu_i$ from the equilibrium restriction derived in Lemma 1 we write $\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}^{'} - \hat{B}_{ikt}\mu_i - \hat{u}_{ikt} = 0$ or, equivalently:

$$\hat{y}_{ikt} = \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}^{'} + \hat{B}_{ikt}\mu_i + \hat{u}_{ikt}, \tag{12}$$

where $u_{ikt} = \left(\hat{y}_{ikt} - \mathbf{\hat{D}_{ikt}}\mathbf{\Theta}^{'} - \hat{B}_{ikt}\mu_i\right) - \left(y_{ikt} - \mathbf{D_{ikt}}\mathbf{\Theta}^{'} - B_{ikt}\mu_i\right)$, with $y_{ikt} = y_{ik}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$, $\mathbf{D_{ikt}} = \mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$ and $B_{ikt} = B_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p})$.

In matrix form, stacking (12) for all players, states and actions except action $k = 0$ we write:

$$\mathbf{\hat{y}} = \mathbf{\hat{D}}\mathbf{\Theta}^{'} + \left(\mathbf{I_N} \otimes \iota_{\mathbf{K \cdot N_s}}\right)\mathbf{\hat{B}}\mu + \mathbf{\hat{u}}, \tag{13}$$

where the variable $\mathbf{\hat{y}}$ is a $N \cdot K \cdot N_s \times 1$ vector stacking $\hat{y}_{ikt}$ for all individuals, actions and states, $\mathbf{\hat{D}}$ is a $N \cdot K \cdot N_s \times N_p$ matrix stacking $\mathbf{\hat{D}_{ikt}}$ for all individuals, actions and states, $\mathbf{\hat{B}}$ is a $N \cdot K \cdot N_s \times 1$ matrix stacking $\mathbf{\hat{B}_{ikt}}$ for all individuals, actions and states, $\mathbf{I_N}$ is a $N \times N$ identity matrix, $\iota_{\mathbf{K \cdot N_s}}$ is a $K \cdot N_s \times 1$ vector of ones, $\otimes$ is the Kronecker product, μ is a $N \times 1$ vector containing the constants μ_i for all individuals and $\mathbf{\hat{u}}$ is a $N \cdot K \cdot N_s \times 1$ vector stacking $\hat{u}_{ikt}$ for all individuals, actions and states.

We call $\left(\mathbf{I_N} \otimes \iota_{\mathbf{K \cdot N_s}}\right)\mathbf{\hat{B}} = \mathbf{\hat{B}}^*$, define $\mathbf{\hat{Q}} = \mathbf{I_{N \cdot K \cdot N_s}} - \mathbf{\hat{B}}^*\left(\mathbf{\hat{B}}^{*'}\mathbf{\hat{B}}^*\right)^{\mathbf{-1}}\mathbf{\hat{B}}^{*'}$, where $\mathbf{I_{N \cdot K \cdot N_s}}$ is a $N \cdot K \cdot N_s \times N \cdot K \cdot N_s$ identity matrix, and pre-multiply equation (13) by $\mathbf{\hat{Q}}$:

$$\mathbf{\hat{y}}^* = \mathbf{\hat{D}}^*\mathbf{\Theta}^{'} + \mathbf{\hat{u}}^*, \tag{14}$$

where the "star" variables are the variables in (13) pre-multiplied by $\mathbf{\hat{Q}}$. Notice that this is exactly the fixed effects transformation used in panel data models. It allow us to get rid of the unobservables μ_i.

The asymptotic properties of $\mathbf{\hat{u}}^*$ are derived in the next lemma.

Lemma 2. *Suppose that E1-E3 holds. Then* $\sqrt{T}\mathbf{\hat{u}}^* \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \mathbf{\Lambda}^*\right)$, *where:*

1. $\mathbf{\Lambda}^* = \nabla^*\left(\sigma, \mathbf{p}\right)\mathbf{\Omega}\nabla^{*'}\left(\sigma, \mathbf{p}\right)$; *and,*

2. $\nabla^*\left(\sigma, \mathbf{p}\right) = \mathbf{Q}\left[\frac{\partial\left(\mathbf{\hat{y}} - \mathbf{\hat{D}}\mathbf{\Theta}^{'} - \mathbf{\hat{B}}^*\mu\right)}{\partial(\hat{\sigma}, \mathbf{\hat{p}})}\right]_{(\hat{\sigma}, \mathbf{\hat{p}}) = (\sigma, \mathbf{p})}$ *is a* $N \cdot K \cdot N_s \times H$ *matrix and* $\mathbf{Q} = \mathbf{I_{N \cdot K \cdot N_s}} - \mathbf{B}^*\left(\mathbf{B}^{*'}\mathbf{B}^*\right)^{\mathbf{-1}}\mathbf{B}^{*'}$.

Proof. See appendix. □

Equation (14) is a linear estimating equation with a well defined variance-covariance structure for the error term. Now, an additional assumption is introduced:

Assumption E4: $rank\left\{\hat{\mathbf{D}}^{*'}\mathbf{\Lambda}^{*-1}\hat{\mathbf{D}}^{*}\right\} = N_p$.

Assumption E4 guarantees the identification of $\mathbf{\Theta}$. Under E1-E4, it readily follows that the Generalized Fixed Effects Estimator for $\mathbf{\Theta}$ is consistent and asymptotically normal. This result is formally stated in the next proposition.

Proposition 1. *Under assumptions E1-E4 the Generalized Fixed Effects Estimator,*

$$\hat{\mathbf{\Theta}}'_{\mathbf{GFE}} = \left(\hat{\mathbf{D}}^{*'}\mathbf{\Lambda}^{*-\mathbf{1}}\hat{\mathbf{D}}^{*}\right)^{-\mathbf{1}}\left(\hat{\mathbf{D}}^{*'}\mathbf{\Lambda}^{*-\mathbf{1}}\hat{\mathbf{y}}^{*}\right),$$

is a consistent and asymptotically normal estimator for $\mathbf{\Theta}$, $\sqrt{T}\left(\hat{\mathbf{\Theta}}_{\mathbf{GFE}} - \mathbf{\Theta}\right) \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} N\left(\mathbf{0}, \Xi_{\mathbf{\Theta}}\right)$, *where* $\Xi_{\mathbf{\Theta}} = \left(\mathbf{D}^{*'}\mathbf{\Lambda}^{*-1}\mathbf{D}^{*}\right)^{-1}$ *is the asymptotic variance of* $\hat{\mathbf{\Theta}}_{\mathbf{GFE}}$. *Futhermore* $\hat{\mathbf{\Theta}}_{\mathbf{GFE}}$ *is unique and exists.*

Proof. See appendix. □

The unobservables μ_i can be recovered using (13) and well known results in the panel data literature.

The Generalized Fixed Effects estimator present advantages on other popular estimators for models with time invariant unobservables. Aguirregabiria and Mira (2007) also consider a model time invariant unobservables. The unobservables, however, by assumption, are uncorrelated with observed states and the estimates depend on the choice for the distribution of unobservables. GFE estimators relax these two assumptions.

GFE is less general than Arcidiacono and Miller (2011). Arcidiacono and Miller (2011) allow for time variant unobservables and propose a four step estimation procedure for payoff parameters. When time variant unobservables are not present our estimation procedure is clearly more straightforward and the estimates do not depend on the initial choice for the distribution of unobservables.

4 Conclusion

We show that when payoffs are linear in the parameters and (time invariant) unobservables, value functions are linear in the parameters and unobservables and the equation system characterizing the Markovian equilibrium is linear in the parameters and unobservables. This formulation allows us to estimate the model using Fixed Effects estimators. We derive an optimal weight matrix for the Fixed Effects estimator and show that the efficient estimator is a Generalized Fixed Effects (GFE) estimator. GFE estimators have a closed form solution and do not depend on the numerical methods used in other popular estimation procedures (e.g. Hotz and Miller (1993), Aguirregabiria and Mira (2002), Bajari, Benkard and Levin (2007) and Pesendorfer and Schmidt-Dengler (2008), among others). Our estimation strategy provides globably optimal estimates that do not depend on initial guesses of parameters.

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Appendix

This appendix contains proofs.

■ **Lemma 1.**

Proof. Assumption E2 and equations (2) and (4) imply that:

$$q_{ik0}\left(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) = \tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})\mathbf{\Theta}' + \mu_i + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)\right]\mathbf{E}_\xi\mathbf{V_i}\left(\sigma_\mathbf{i},\mathbf{p}\right), \tag{15}$$

where, $\tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}) = \sum_{\mathbf{a^t_{-i}}}\sigma_i(\mathbf{a^t_{-i}}|\mathbf{s^t})\left[\varphi(a_i^t = k,\mathbf{a^t_{-i}},\mathbf{s^t}) - \varphi(a_i^t = 0,\mathbf{a^t_{-i}},\mathbf{s^t})\right]$ is a $1\times N_p$ vector.

Substituting the *ex ante* value function into (15) we get:

$$q_{ik0}\left(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) = \tilde{\varphi}_{k0}(\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p})\mathbf{\Theta}' + \mu_i + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t};\sigma_\mathbf{i},\mathbf{p}\right)\right]\mathbf{\Delta_i}\left(\tilde{\pi}_\mathbf{i} + \tilde{\mathbf{E}}_{\xi\mathbf{i}}\right)$$

Use E2 again to write $\tilde{\pi}_\mathbf{i} = \tilde{\varphi}_\mathbf{i}\mathbf{\Theta}' + \tilde{\varphi}_{\mathbf{i}\mu}\mu_i$, where $\tilde{\varphi}_\mathbf{i}$ is a $N_s \times N_p$ matrix stacking the vector $\sum_{\mathbf{a^{t+1}}}\sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}})\varphi(\mathbf{a^{t+1}},\mathbf{s^{t+1}})$ for all possible vector of states and $\tilde{\varphi}_{\mathbf{i}\mu}$ is a $N_s \times 1$ vector stacking $\sum_{\mathbf{a^{t+1}}} \sigma_i(\mathbf{a^{t+1}}|\mathbf{s^{t+1}})I(\mathbf{a^{t+1}} : a_i^{t+1} \neq 0)$, where $I(\mathbf{a^{t+1}} : a_i^{t+1} \neq 0)$ is an indicator function that assumes 1 if $\mathbf{a^{t+1}}$ is such that $a_i^{t+1} \neq 0$ and 0 otherwise.

Using these definitions the equilibrium restrictions (3) can be written as $y_{ik}(\mathbf{s_i^t}; \sigma_\mathbf{i}, \mathbf{p_i}) - \mathbf{D_k}(\mathbf{s_i^t}; \sigma_\mathbf{i}, \mathbf{p_i})\mathbf{\Theta}' - B_k(\mathbf{s_i^t}; \sigma_\mathbf{i}, \mathbf{p_i})\mu_i = 0$, where:

$$y_{ik}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) =$$
$$q_{ik0}\left(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right] \mathbf{\Delta_i}\tilde{\mathbf{E}}_\xi$$
$$\mathbf{D_k}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) =$$
$$\tilde{\varphi}_{k0}(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right] \mathbf{\Delta_i}\tilde{\varphi}_\mathbf{i}.$$
$$B_k(\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}) =$$
$$1 + \beta\left[\mathbf{z_k}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right) - \mathbf{z_0}\left(\mathbf{s^{t+1}}|\mathbf{s^t}; \sigma_\mathbf{i}, \mathbf{p}\right)\right] \mathbf{\Delta_i}\tilde{\varphi}_{\mathbf{i}\mu}.$$

□

■ **Lemma 2.**

Proof. Expanding $\hat{\mathbf{u}} = \left[\left(\hat{\mathbf{y}} - \hat{\mathbf{D}}\mathbf{\Theta}' - \hat{\mathbf{B}}^*\mu\right) - \left(\mathbf{y} - \mathbf{D}\mathbf{\Theta}' - \mathbf{B}^*\mu\right)\right]$ around $(\sigma, \mathbf{p})$:

$$\hat{\mathbf{u}} = \nabla(\sigma, \mathbf{p})\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] + o\left(\|(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\|\right),$$

where $\nabla(\sigma, \mathbf{p})$ is a $N \cdot K \cdot N_s \times H$ matrix of the derivatives of $\left(\hat{\mathbf{y}} - \hat{\mathbf{D}}\mathbf{\Theta}' - \hat{\mathbf{B}}^*\mu\right)$ with respect to the auxiliary parameters, $(\hat{\sigma}, \hat{\mathbf{p}})$ evaluated at $(\sigma, \mathbf{p})$.
Therefore:

$$\begin{aligned}\hat{\mathbf{u}} &= \nabla(\sigma, \mathbf{p})\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] + o\left(O_p\left(\frac{1}{\sqrt{T}}\right)\right) \\ &= \nabla(\sigma, \mathbf{p})\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] + o_p\left(\frac{1}{\sqrt{T}}\right).\end{aligned}$$

Multiplying both sides by $\sqrt{T}$:

$$\sqrt{T}\hat{\mathbf{u}} = \nabla(\sigma, \mathbf{p})\sqrt{T}\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] + o_p(1).$$

Assumption E3 directly implies that:

$$\sqrt{T}\hat{\mathbf{u}} = \nabla(\sigma, \mathbf{p})\sqrt{T}\left[(\hat{\sigma}, \hat{\mathbf{p}}) - (\sigma, \mathbf{p})\right] \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \nabla\left(\sigma, \mathbf{p}\right)\boldsymbol{\Omega}\nabla'\left(\sigma, \mathbf{p}\right)\right).$$

Therefore, $\sqrt{T}\hat{\mathbf{u}} \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \nabla\left(\sigma, \mathbf{p}\right)\boldsymbol{\Omega}\nabla'\left(\sigma, \mathbf{p}\right)\right)$. Using the fact that $\hat{\mathbf{Q}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{Q} = \mathbf{I_{N\cdot K\cdot N_s}} - \mathbf{B}^*\left(\mathbf{B}^{*\prime}\mathbf{B}^*\right)^{\mathbf{-1}}\mathbf{B}^{*\prime}$ then $\hat{\mathbf{Q}}\sqrt{\mathbf{T}}\hat{\mathbf{u}} = \sqrt{\mathbf{T}}\hat{\mathbf{u}}^* \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \mathbf{Q}\nabla\left(\sigma, \mathbf{p}\right)\boldsymbol{\Omega}\nabla'\left(\sigma, \mathbf{p}\right)\mathbf{Q}'\right)$. □

■ **Proposition 1.**

Proof. Firstly we show that $\hat{\boldsymbol{\Theta}}_{\mathbf{GFE}}$ is consistent. Substitute $\hat{\mathbf{y}}^* = \hat{\mathbf{D}}^*\boldsymbol{\Theta}' + \hat{\mathbf{u}}^*$ into the formula for $\hat{\boldsymbol{\Theta}}_{\mathbf{GFE}}$:

$$\begin{aligned}\hat{\boldsymbol{\Theta}}'_{\mathbf{GFE}} &= \left(\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}^*\right)^{-1}\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{\mathbf{-1}}\left(\hat{\mathbf{D}}^*\boldsymbol{\Theta}' + \hat{\mathbf{u}}^*\right)\\ &= \boldsymbol{\Theta}' + \left(\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}^*\right)^{-1}\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{\mathbf{-1}}\hat{\mathbf{u}}^*.\end{aligned}$$

Notice now that continuity of $\hat{\mathbf{y}}^*$ and $\hat{\mathbf{D}}^*$ in $(\hat{\sigma}, \hat{\mathbf{p}})$ and E3 implies that $\hat{\mathbf{u}}^* \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{0_{N\cdot K\cdot N_s}}$, where $\mathbf{0_{N\cdot K\cdot N_s}}$ is a $N \cdot K \cdot N_s \times 1$ vector of zeros and $\hat{\mathbf{D}}^* \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}^*$. Therefore: $\hat{\boldsymbol{\Theta}}'_{\mathbf{GFE}} \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \boldsymbol{\Theta}'$.

To determine the asymptotic distribution of $\hat{\boldsymbol{\Theta}}_{\mathbf{GFE}}$ write the expression above as:

$$\sqrt{T}\left(\hat{\boldsymbol{\Theta}}'_{\mathbf{GFE}} - \boldsymbol{\Theta}'\right) = \left(\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}^*\right)^{-1}\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{\mathbf{-1}}\sqrt{T}\hat{\mathbf{u}}^*.$$

Using Lemma 2 and the fact that $\hat{\mathbf{D}}^* \underset{T\uparrow\infty}{\overset{p}{\rightarrow}} \mathbf{D}^*$ we have:

$$\left(\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{-1}\hat{\mathbf{D}}^*\right)^{-1}\hat{\mathbf{D}}^{*\prime}\boldsymbol{\Lambda}^{\mathbf{-1}}\sqrt{T}\hat{\mathbf{u}}^* \underset{T\uparrow\infty}{\overset{d}{\rightarrow}} \mathbf{N}\left(\mathbf{0}, \left(\mathbf{D}^{*\prime}\boldsymbol{\Lambda}^{\mathbf{-1}}\mathbf{D}^*\right)^{\mathbf{-1}}\right).$$

E4 guarantees the existence and unicity of $\hat{\boldsymbol{\Theta}}'_{\mathbf{GFE}}$. □

Part III

Public Banks Improve Private Banks Performance: Evidence from a Dynamic Structural Model

Abstract

This paper shows that profits of private banks are positively affected by the number of public branches operating in Brazilian isolated markets. The spill-over generated by public banks is quantified based on a dynamic oligopoly model. A counterfactual in which public banks are privatized is examined. It shows that the number of active branches operating in the long-run in a small market drops significantly.

1 Introduction

The discussion about the existence of public, state owned banks has been prominent in the banking literature since the 1960's - see Barth, Caprio and Levine (2001) and La Porta, López-de-Silanes and Shleifer (2002) and Levy Yeyati, Micco and Panizza (2007).

In favour of public banks the following has been argued: (i) Public banks finance unprofitable but socially valuable investment projects; (ii) they foster financial development and (iii) they provide financial access to populations living in areas that are unattractive for private institutions. Critics of public banks argue that (i) they are used as political instruments, providing employment, credit, subsidies or other benefits in return for political assistance, and (ii) they crowd-out more efficient, more competitive private banks, slowing down the development of the financial system.

This paper examines the effects of public banks on financial development. A dynamic game between the major Brazilian public and private banks is estimated. A counterfactual experiment is used to analyze how the privatization of public banks affects the supply of banking services in small isolated markets.

Two main conclusions emerge. First, public banks generate positive profit spill-overs for private banks; second, private banks crowd-out private competitors. Our estimates show that the entry of a public bank in a given market increases the return of a private incumbent by 1.2 percent and the entry of a private bank reduces the return of a private incumbent by 0.05 to 1 percent.

The counterfactual in which public banks are sold to private players shows that the total number of active branches operating in the long-run in a typical small market drops from 3 to 0.5 on average. To guarantee that, after privatization, all small municipalities would have at least one active branch the government should give a subsidy of approximately 8% on the operational costs of private branches. We can infer that the present cost of this policy would be of approximately US$175,000[4] per market. This value is relatively small compared to the market value of Brazilian public banks. Bank of Brazil, the largest public bank in Brazil, had its market value estimated in approximately US$42 billions in 2012. This means that the resources raised with the privatization of Bank of Brazil would be sufficient to cover the subsidies for 240 thousand branches in the country or, approximately, 42 branches per Brazilian municipality.

These findings have important policy implications in developing countries. In these coun-

[4] Approximately R$350,000.

tries a large fraction of the population has no access to the banking market. Yet the access to financial services generates positive effects in terms of poverty reduction and economic growth in disadvantaged areas (Burgess and Pande (2005) and Pascali (2012)).

Our estimates do not allow us to disentangle the details of the spill-over channels. Broadly speaking, our findings are consistent with public banks (i) having monopoly over a number of important Federal funds and (ii) being driven by social, as opposed to strategic or market reasons. The first element guarantees a large volume of credit for small markets - see Feler (2012). The second induces product differentiation between public and private banks: Public and private banks target different clients - see Coelho, Melo and Rezende (2012). In this case, the amount of cheap credit and public transfers poured by the public banks in small isolated municipalities shifts the demand for banking services, making these markets more attractive for private players. This effect induces the entry of private players.

There is little prior empirical evidence of the effects of public banks on the economy. The evidence is mixed. La Porta, López-de-Silanes and Shleifer (2002) study a cross section of countries and show that the presence of public banks in the market is associated with poorly developed financial markets. They conclude that the higher the public ownership in the banking sector, the lower is the average growth of the ratio of private credit to GDP. Similar findings were obtained in Barth, Caprio and Levine (2001) who find that greater state ownership of banks with more poorly developed banks, nonbanks, and securities markets.

Levy Yeyati, Micco and Panizza (2007) extend the dataset used in La Porta, López-de-Silanes and Shleifer (2002) by including more controls and a longer period of time. They find that no robust conclusion can be drawn. The findings depend strongly on the definition of financial development, the estimator and the sample definition. They conclude that there is *"(...) no significant correlation between state-ownership of banks and credit to the private sector"* [5]. Detragiache, Tessel and Gupta (2008) confirm the findings in Levy Yeyati, Micco and Panizza (2007).

The effects of public banks on development are analyzed in Cole (2007) and Feler (2012). Cole (2007) finds that the nationalization of public banks in India increased the amount of credit but had no effect on real outcomes. Feler (2012) analyses the privatization of state banks in Brazil. His findings are close to the findings in Cole (2007): Privatization led to a significant reduction of credit supply in local markets but it did not affect local GDP.

Coelho, Mello and Rezende (2012) analyze the effects of public banks on the performance of private banks. They extend the traditional Bresnahan and Reiss (1991) framework and

[5] Levy Yeyati, Micco and Panizza (2007).

estimate an empirical model using a cross-section of entry and exit movements in Brazilian municipalities. This approach is related to ours, but in that paper the analysis is static and relies purely on cross-sectional variation. Based on the negative and significant but small coefficient related to the number of public banks in the profit function of private banks, they conclude that public banks do not affect competition in the market.

This paper builds a dynamic entry game in which the major Brazilian public and private banks are the players. The dynamic structure of the model is strongly supported by our data and can be intuitively rationalized by the existence of substantial entry costs in the market. At each period these players have information about the state variables and decide simultaneously to be active or not active in a given market by maximizing an inter-temporal profit function. Entrants pay a fixed cost. We assume that the profit function of the major private players is asymmetrically affected by public and private competitors. This allows us to understand how public banks influence the performance of private players.

We use data from 1002 isolated markets in Brazil during 1988-2010 to estimate the decision rules for public and private banks. We infer the model primitives that rationalize these decision rules in a dynamic oligopoly game. Importantly, we evaluate the market equilibrium under different counterfactual scenarios. We report consistent *ex ante* estimates of the effects of changes in the banking market structure on market outcomes. Our model is valuable to predict policy changes. By relying on micro data from a single market, we are able to reduce the market heterogeneity present in cross country regressions, which, as reported by Levy Yeyati, Micco and Panizza (2007), causes important bias in the conclusions obtained by the existing literature.

Methodologically our paper is related to the empirical industrial organization literature that studies the estimation of dynamic games - see Aguirregabiria and Nevo (2010), Bajari, Hong and Nekipelov (2010) and Pesendorfer (2010) for a rich discussion on the topic. Applications that are similar to ours are also found in Pesendorfer and Schmidt-Dengler (2003) for small businesses in Austria, Dunne, Klimek, Roberts and Xu (2009) for dentists and chiropractors in the US, Gowrisankaran, Lucarelli, Schmidt-Dengler and Town (2010) for hospitals in the US, Collard-Wexler (2013) for the concrete industry in the US, Ryan (2012) for the cement industry in the US and Kalouptsidi (2013) for the shipping industry. Other applications include Maican and Orth (2012), Minamihashi (2012), Lin (2011), Fan and Xiao (2012), Nishiwaki (2010), Arcidiacono, Bayer, Blevins, and Ellickson (2012), Jeziorski (2012), Snider (2009), Suzuki (2012), Sweeting (2011) and Beresteanu, Ellickson and Misra (2010).

Based on the results of Chapters 1 and 2 Least Squares estimators are used to recover the structural parameters. This approach allows us to avoid the use of numerical methods in the estimation of the structural parameters. Doing so we reduce significantly the computational burden.

This paper is organized as follows. The next section describes our dataset and the Brazilian banking market. Section 3 shows reduced form evidence of competition between public/private players. Sections 4 and 5 describe the theoretical model, the empirical model and our main results. Section 6 discusses the fitting of the empirical model and our counterfactual analysis. The last section concludes the paper.

2 Data and Institutional Background

The data come from the Brazilian Central Bank and from the Brazilian Ministry of Labour. The Brazilian Central Bank database follows the activities of all Brazilian banks since 1900. These data contain the opening and closing dates[6] and the name of the chain that operates each branch for all branches opened since 1900 in all Brazilian municipalities. A measure of market size is constructed by using data from the Brazilian Ministry of Labour containing the total payroll in the formal sector[7] for all Brazilian cities since 1985. The payroll data is deflated using the official inflation index, IPCA-IBGE. All the values are in R$ of 2011. The information about banking and economic activity is annual.

Following Bresnahan and Reiss (1991) our analysis examines small isolated markets. We select municipalities[8] that are at least 20 km away from the nearest municipality. State capitals and metropolitan areas are excluded. We also excluded municipalities that had more than 10 bank branches since 1900. This selection leaves us with 1002 isolated small markets, corresponding, roughly, to 20% of all Brazilian municipalites. The idea of isolated markets enable us to obtain a clear measure of the potential demand for each branch.

The market size data starts in 1985. We exclude 1986 and 1987 from our sample because a major macroeconomic shock caused by two heterodox stabilization plans[9] disorganized severely the Brazilian economy in those years. Our final sample consists of observations for 1002 isolated municipalities in the period 1988-2010. Because in most of our municipalities

[6]For the branches that were closed.

[7]Number and wage of employees in the formal sector of the economy.

[8]From now on we use municipality/market interchangeably.

[9]Cruzado Plan in 1986 and Bresser Plan in 1987.

the chains have at most one branch we focus on entry and exit patterns.[10]

■ **Sample statistics.** The next table illustrates the basic statistics of our sample.

Table 2: Basic Sample Statistics 1988-2010

	Average	Std
Active Branches (**)	1560.2	142.4
Entry (**)	50.5	49.1
Exit (**)	59.5	55.1
Sample Market Size (*) (**)	10,404.85	
Municipalities	1002	
Sample Municipalities/Total Number of Municipalities	18%	
Municipalities × Periods	22041	

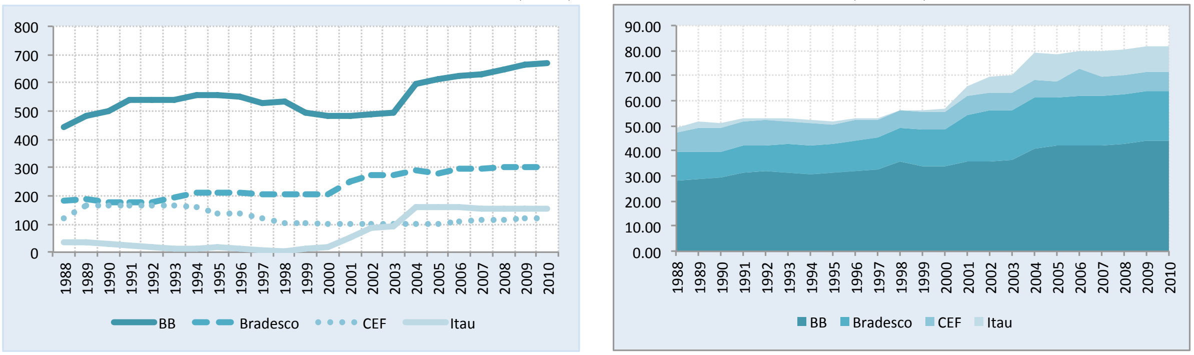
Note: * R$ millions of 2011. ** Yearly averages.

Our sample is composed by 1002 isolated markets. This corresponds to approximately 18% of the total number of municipalities in Brazil. The number of branches in this sample is 1560 per year on average. Entry is observed 50 times per year and exit 59 times. The yearly market size measured by the annual payroll of the formal workers of all the municipalities in the sample is of R$ 10.4 billions of 2011. This value is relatively small because by excluding state capitals and metropolitan regions the richiest cities in the country are left aside.

The Brazilian banking market is basically dominated by four big institutions: Two of them, Bank of Brazil, BB, and Caixa Economica Federal, CEF, are public and controlled by the federal government while the other two, Bradesco and Itau, are privately held. The next figures show the number of branches that are controled by these institutions and the market share, measured in terms of the number of active branches in the sample, of these four players.

[10]In the municipalities that had more than one branch operated by the same chain, which correspond to less than 4% of the total number of municipalities and around 0.2% of our sample, we aggregated the branch level information for each player that had more than one branch in the same market. The exclusion of these municipalities does not change our results. Therefore we kept this information in the dataset.

Figure 1: Number of Branches (left) and Market Share (right) - "Big" Four



Note: Number of active branches per year (left) and fraction (right) of these branches over the total number of active branches in our sample.

These four players hold more than 80% of the active branches in our sample. The share of Bradesco and Itau increased substantially over time. From 2000 to 2010 Bradesco's market share measured in terms of active branches increased from 13% to 20%. Itau's share increased from 1% to 10%. Part of the expansion is explainable by the acquisitions of privatized smaller public institutions. Bank of Brazil, BB, also experienced an increase in the number of branches. This expansion is mainly driven by the social policy of Lula's government (2003-2010), which tried to expand the presence of public banks in small markets.

Table 3: Average Monthly Payroll and Number of Public/Private Players

Number of Private	Number of Public	Mun Payroll	Observations
0	0	0.227	4592
0	1	0.478	6496
0	2	0.680	2511
1	0	0.515	1555
1	1	1.333	1696
1	2	1.677	1107
2	0	1.034	149
2	1	2.419	552
2	2	2.202	380

Note: Average market size is the monthly average payroll of the municipality and is measured in R$ millions of jan/2011 according to the number of players in the market. Sample period: 1988-2010. Each observation corresponds to a municipality in a given year. We showed in the table only the most frequent market structures. This correposnds to around 90% of the total number of observations.

Table 3 reports (i) the frequency distribution of each market configuration (number of observations corresponding to each market structure) and (ii) the average market size (monthly average payroll of the municipality in R$ millions of 2011) corresponding to each market structure. These numbers illustrate that:

1. Public players are located more frequently in small markets (as measured by the municipality average payroll) than private players; and,
2. Public players are frequently the only providers of financial services in these isolated markets (the frequency of public monopolies - 6496 observations - is the highest in the sample).

The Brazilian government has launched some programs that aim to "popularize" basic financial services in poorer areas, which includes the supply of basic services (current account, for example) and of credit lines to small farmers and firms. This feature may explain the fact that the frequency of markets where the public player is a monopolist (the only provider of financial services) is quite high in our sample. In addition, the empirical evidence indicates that public banks are much less productive than their private counterparts (Nakane and Weintraub (2005)). This means that the presence of public banks in smaller markets is not explained by cost advantages of public players.

■ **Institutional background.** The Brazilian banking market is large. In 2012 Itau was considered the 8th largest bank in the world in terms of market value (with a market value of US$88 billions); Bradesco was the 17th largest (market value of US$64 billions) and Bank of Brazil was the 31st largest (market value of US$42 billions)[11].

As pointed out in Coelho, Melo and Rezende (2012) there are important differences in the objectives of public and private banks. Private banks are essentially profit oriented. By legal mandate public banks focus their operations on market segments that are not profitable for private banks. This suggests the existence of product differentiation in the market. In what follows we describe the "social" role of public banks in Brazil[12].

Bank of Brazil (BB) has expanded enormously its operations in smaller and poorer areas of the country based on central government policies aiming to "popularize" banking services among poor workers and small businesses. BB plays an important role as the provider of government funds to the Brazilian agriculture[13]. Also, to expand its capillarity in isolated

[11] http://www.relbanks.com/worlds-top-banks/market-cap. Access: November 12, 2012.

[12] also present a detailed discussion on the role of Brazilian public banks.

[13] The total amount of agricultural credit provided by this player in 2010 reached more than US$ 26 billions. Moreover, BB is the main bank in the Pronaf, a program created to supply credit for small businesses (agriculture, fishing, turism, and handcraft) in rural areas at a very low interest rate. The total credit availble for the program increased from US$ 1 billion in 1999 to US$ 7 billions in 2010. All banks

areas, BB created a DSR (Regional Development Program). The DSR provides a set of tools for small entrepreuners, including a business plan, technical support and credit[14].

Caixa Economica Federal (CEF) has monopoly over a number of different government funds and services, such as the FGTS, Bolsa Família, PIS[15] and the Federal Lottery[16]. FGTS is a Brazilian fund created in 1966 to provide assistance to unemployed people[17] . These resources are allocated in two main areas: Housing and sanitation. The government gives the investments guidelines in order to finance strategic areas with lack of credit. CEF is also responsible for the distribution of the benefits from Bolsa Família[18], a program that gives to poor families a monthly income. It was created to reduce the poverty in the most backward areas of the country.

Summarizing, the descriptive analysis suggests that the major Brazilian public banks have been used by the federal government to improve the financial access of isolated markets. The financial access includes new branches in smaller and poorer markets and the injection of cheap credit in these areas. These operations are not profitable for private banks. This suggests the existence of product differentiation in the market.

3 Reduced Form Analysis

We estimate a series of reduced form logit models to explain entry/exit movements of the biggest public and private players using the sample of isolated municipalities. We focus on the behavior of the 4 largest players: Bank of Brazil (BB), Caixa Economica Federal (CEF), Bradesco and Itau.

Two pooled logit models are estimated: One for the public players, BB and CEF; the

in Brazil are allowed to take part in the program, however, BB distributes around 65% of the total Pronaf credit.

[14]In 2007 the program supported 2800 business plans and distributed US$ 1.7 billions in credit.

[15]PIS is a tax to cover unemployment benefits. Their assets were around US$14 billions in December 2010.

[16]The Federal Lottery provided a gross revenue of US$5.2 billions in 2010. It is used to fund sports.

[17]The main source of funding is the monthly compulsory deposit that every private employer must do in the name of each employee. These values constitutes a fund and the worker have access to the money deposited in his/her name only in some special conditions: Unemployment, chronic disease, and for buying a house (if the worker does not own another house). CEF is responsible for the whole operation of the fund - from the tax collection to the payments for the benefited workers. Since the FGTS universe includes all formal workes (except public servants) the total size of the fund is considerably high - around US$90 billions in December 2008.

[18]In 2006 the program served around 11 million families or approximately 44 thousand individuals. The public expenditure with the program is around 0.5% of the Brazilian GDP and is growing steadily since its creation.

other for private players, Itau and Bradesco. Our logit specification is:

$$
\begin{aligned}
P(a_{imt} = 1 | a_{imt-1}, n^{pub}_{mt-1}, n^{pri}_{mt-1}, \mathbf{x_{mt}}; \rho, \mu) = \\
\Lambda(\rho_0 + \rho_1 a_{imt-1} + \rho_2 n^{pub}_{mt-1} + \rho_3 n^{pri}_{mt-1} + \rho_\mathbf{4} \mathbf{x_{mt}} + \mu_t + \mu_m + \mu_{mt} + \mu_i)
\end{aligned} \tag{16}
$$

The dependent variable, a_{imt}, is the action of player i in municipality m, period t. It assumes 1 if player i was active in that municipality/period and zero otherwise. a_{imt-1} indicates the action of the same player in that municipality in the prior period. n^{pub}_{mt-1} is the number of **public** competitors in the previous period in market m[19]; n^{pri}_{mt-1} is the number of **private** competitors in the previous period in market m[20]; $\mathbf{x_{mt}}$ is a vector of municipality characteristics; μ_t are time effects; μ_m are market specific effects; μ_{mt} captures market/time specific effects and μ_i are player specific effects. $\Lambda(\cdot)$ is the logistic distribution. The greek letters denote parameters to be estimated. The data include all muncipalities where player i was active for at least one period[21].

The estimation of dynamic binary response models with market fixed effects produces biased coefficients - see Carro (2007), Wooldridge (2010). To avoid this bias in our analysis, instead of including market dummies, 27 state dummies (one for each Brazilian state) were included in the model. States dummies are used to capture time invariant heterogeneity across municipalities of different states. Time effects are captured by year dummies. Time varying market effects are captured by the interaction of state dummies and a trend variable.

The vector $\mathbf{x_{mt}}$ includes municipality payroll, transfers of the Federal and State governments to the municipality, municipal government expenditure and agricutural production of the municipality. Municipality payroll is a measure of market size. The inclusion of transfers and municipal expenditure controls for the fact that entry of public banks can be correlated with an increase of Federal/State investment in the municipality, which also can affect entry of private banks. Agricultural production is included because a large fraction of the income in our isolated municipalities comes from agricultural activities. This variable is a different

[19]Mathematically, $n^{pub}_{mt-1} = \sum_{j \in i_{pub}, j \neq i} a_{jmt}$, where i_{pub} is the set of public players.

[20]Mathematically, $n^{pri}_{mt-1} = \sum_{j \in i_{pri}, j \neq i} a_{jmt}$, where i_{pri} is the set of private players.

[21]Our estimation approach is based on the potential markets for each player. The potential market is defined based on the super efficient estimator described in Pesendorfer and Schmidt-Dengler (2003). We defined that market m is a potential market for player i if $\max_t \{a_{imt}, t = 1900, 1901, .., 2010\} = 1$, or, in other words, market m is a potential market for player i if she entered in that market at least for one period since 1900.

indicator of market size.

3.1 Private players

Table 4 reports the estimates of equation (16) for the private players, Bradesco and Itau. Only the marginal effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} evaluated at the sample means are reported. The model fit is good, with Pseudo-R2 of 87%-91%. Strikingly, the number of public banks increases the entry probabilities of the private players by 10%-14%. The effects are very significant and robust across specifications. The inclusion of state dummies and the interaction between state dummies and the time trend increases this effect.

Interestingly, the number of private competitors reduces the entry probabilities of private players. This effect is around -6.4% in the specification with the full set of controls. It is statistically significant at 5%.

Table 4: Marginal Effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} on the Entry Probabilities of Private Players, Bradesco and Itau

	(I)	(II)	(III)	(IV)
Nº Public	0.10544***	0.12839***	0.13829***	0.13237***
	[0.01]	[0.02]	[0.02]	[0.02]
Nº Private	-0.03532	-0.01930	-0.06206**	-0.06409**
	[0.02]	[0.03]	[0.03]	[0.03]
Player Dummy	Yes	Yes	Yes	Yes
Time Dummies	Yes	Yes	Yes	Yes
State Dummies	No	Yes	Yes	Yes
Trend*State Dummies	No	No	Yes	Yes
Transfers, Expenditure, Agric. Prod.	No	No	No	Yes
Observations	15,919	15,229	15,229	15,217
Pseudo R2	0.87	0.87	0.91	0.91

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Marginal effects calculated at the sample means. Clustered standard errors in brackets. All the models have lagged activity, number of public and private competitors and municipality payroll. Transfers correspond to the total transfers of Federal and State governments to the municipality. Expenditure corresponds to municipal government expenditure. Agricultural Production is the total agricultural production of each municipality.

As robustness check, the model was estimated in a subsample of municipalities that had (i) at least one public player and (ii) at least one and at most three public players in any time period. Tables 10 and 11 in the appendix report the results for each subsample.

The pattern of results remains unchanged when compared to the estimates in Table 4. This strategy is used to control for unobservable characteristics of markets with and without public players. In the subsample with at least one and at most three public players the market heterogeneity is reduced. Markets with roughly the same number of public players have similar observable characteristics.

3.2 Public players

Table 5 reports the estimates of equation (16) for the public players, Bank of Brasil and Caixa Economica Federal. The model fit is good, with Pseudo-R2 statistics around 85%. Strikingly, entry probabilities of public players are barely affected by the number of public and the number of private competitors in each market. Although significant in some specifications the marginal effects of the number of public and the number of private competitors are very small when contrasted with the estimates in Table 4. In all the specifications the marginal effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} are small in magnitude, being below 1% and 0.5% respectively. In specification (IV) in Table 4 these effects were respectively 13% and -6.4%. These estimates imply that the marginal effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} are around 13 times larger for private banks than for public banks. This pattern is robust to the inclusion of state dummies, the interaction of state dummies and the time trend variable, public transfers, municipal expenditure and agricultural production.

Table 5: Marginal Effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} on the Entry Probabilities of Public Players, BB and CEF

	(I)	(II)	(III)	(IV)
Nº Public	0.00498***	0.00871***	0.00934***	0.00936***
	[0.00]	[0.00]	[0.00]	[0.00]
Nº Private	0.00016	0.00105	0.00455**	0.00455**
	[0.00]	[0.00]	[0.00]	[0.00]
Player Dummy	Yes	Yes	Yes	Yes
Time Dummies	Yes	Yes	Yes	Yes
State Dummies	No	Yes	Yes	Yes
Trend*State Dummies	No	No	Yes	Yes
Transfers, Expenditure, Agric. Prod.	No	No	No	Yes
Observations	20,357	20,357	20,357	20,350
Pseudo R2	0.83	0.84	0.87	0.87

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Marginal effects calculated at the sample means. Clustered standard errors by municipality in brackets. All the models have lagged activity, number of public and private competitors and municipality payroll. Transfers correspond to the total transfers of Federal and State governments to the municipality. Expenditure corresponds to municipal government expenditure. Agricultural Production is the total agricultural production of each municipality.

4 Theoretical Model

This section sets up and solves a dynamic entry game between the major Brazilian banks. Motivated by the data, the game considers entry and exit decisions. In the data a chain has typically at most one branch in each municipality[22]. We focus on the behavior of two public banks, Bank of Brazil and Caixa Economica Federal, and two private banks, Bradesco and Itau. In 2010, these players had more than 80% of the total number of active branches in our sample.

The model captures the features documented by the reduced forms. Dynamics can be rationalized by high entry costs[23]. Importantly, the model allows for different behavior of public and private players.

We estimate the primitives that rationalize the behavior of private banks using a dynamic oligopoly game. We do not structurally model the behavior of public banks. Entry decisions of public banks are assumed to do not depend on the number of public and the number of private competitors in the market. There are two explanations behind this assumption. First, the reduced form analysis suggests that entry probabilities of public banks are barely affected by the number of public and the number of private competitors in the market. Second, the literature recognizes that public banks are not necessarily profit maximizers. The behavior of public banks can depend on political and social reasons - see Levy Yeyati et al (2007), La Porta et al (2002) and Barth et al (2001).

At each period private players have information about the state variables and decide simultaneously to be active or not active in a given market by maximizing an inter-temporal profit function. Private players know that the entry of public banks do not depend on the actions of public and private competitors. Private entrants pay a fixed cost. The profit function of the major private players is assumed to be asymmetrically affected by public and private competitors. This allows us to understand how public banks influence the performance of private players.

Closed related models were applied in Pesendorfer and Schmidt-Dengler (2003), Dunne, Klimek, Roberts and Xu (2009), Gorisankaran, Lucarelli, Schmidt-Dengler and Town (2010), Collard-Wexler (2013), Ryan (2012) and Kalouptsidi (2013), among others. Aguirregabiria

[22]As described above only in 4% of these municipalities one chain had more than 1 branch during the same period.

[23]Market analysts point out that the returns of branches in small markets is quite low. Lower returns in these markets are explained by high fixed and operational costs and by reduced revenues - see Gonçalves and Sawaya (2007), Gouvea (2007) and Andrade (2007). This explains why the number of bank branches is small in the most backward areas of the country.

and Nevo (2010), Bajari, Hong and Nekipelov (2010) and Pesendorfer (2010) present a rich discussion on the estimation of dynamic games.

4.1 Assumptions

■ **Players.** There are two private players, Bradesco and Itau. The set of private players is $i_{pri} \in \{Bradesco, Itau\}$. There are two public players, Bank of Brazil, BB, and Caixa Economica Federal, CEF. The set of public players is $i_{pub} \in \{BB, CEF\}$[24].

■ **Time and markets.** Time is discrete, $t = 1, 2, ..., \infty$ and there are $m \in \mathbf{M} = \left\{1, 2, 3, .., \overline{M}\right\}$ markets.

■ **Actions.** A player's action in market m, period t is denoted by $a^t_{im} \in \{0, 1\}$, where 0 means that player is inactive; 1 means that player is active. The $1 \times N$ vector $\mathbf{a^t_m}$ denotes the action profile in market m, period t. We sometimes use $\mathbf{a^t_{-im}}$ to denote the actions of all players but player i.

■ **State space.** The state space is discrete and finite. The state variables consist of a vector $\mathbf{x^t_m}$, of exogenous variables, and of the previous period actions. We call the vector of states $\mathbf{s^t_m} = (\mathbf{x^t_m}, \mathbf{a^{t-1}_m})$. The state variables are publicly known to the players and to the econometrician.

■ **Transitions.** The vector $\mathbf{s^t_m}$ evolves according to the transition matrix $p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, \mathbf{a^t_m})$, described by next period distribution of possible values for the vector $\mathbf{s^t_m}$ conditional on each possible current state and actions in municipality m.

■ **Unobservables.** In each period players draw a profitability shock ε^t_{im}. The shock is privately observed while the distribution is publicly known.

■ **Payoffs of private players.** ***Private*** player's period payoff is:

$$\Pi(\mathbf{a^t_m}, \mathbf{s^t_m}; \mathbf{\Theta_{im}}) = \begin{cases} \pi_{im}(\mathbf{a^t_m}, \mathbf{x^t_m}) \\ +\mathbf{1}(a^t_{im} = 1) \cdot \varepsilon^t_{im} \\ +\mathbf{1}(a^t_{im} = 1) \cdot \mathbf{1}(a^{t-1}_{im} = 0) \cdot F_i \end{cases} \tag{17}$$

Here $\pi_{im}(\mathbf{a^t_m}, \mathbf{x^t_m})$ denotes player $i's$ deterministic profits in market m, F_i are fixed costs

[24]We also estimated a version of the model including a fringe of public/private players. The inclusion of these players does not change our results but increases substantially the state space of our model. This imposes computational difficulties to solve the model and to make conterfactual analysis. By this reason we do not include these players in the model.

and ε_{im}^t is a profitability shock. $\mathbf{\Theta}$ denotes the parameters in the model including F_i.

This specification captures the main aspects of our idea. In this formulation, an incumbent that decides to stay in the market have profits given by $\pi_{im}(a_{im}^t = 1, \mathbf{a^t_{-im}}, \mathbf{x^t_m}) + \varepsilon_{im}^t$. An entrant have the same profit as the incumbent plus a negative entry cost F_i. That is, any player that was outside the market and decides to entry pays a fixed cost F[25].

The term $\pi_{im}(\mathbf{a^t_m}, \mathbf{x^t_m})$ is a linear function of exogenous states and actions[26]:

$$\pi_{im}(\mathbf{a^t_m}, \mathbf{x^t_m}) =$$
$$\left\{\pi_{0i} + \pi_{1i}^{pub}\left(\sum_{j \in i_{pub}} a_{jm}^t\right) + \pi_{1i}^{pri}\left(\sum_{j \neq i, j \in i_{pri}} a_{jm}^t\right) + \pi_{2i} x_m^t\right\} \cdot \mathbf{1}(a_{im}^t = 1) \tag{18}$$

Here $\pi_{ji}^k \in \mathbb{R}^k$ are parameters and x_m^t is a demand shifter. This specification allows for different "competition" effects of public and private players.

The profitability shock ε_{im}^t is assumed to have three components:

$$\varepsilon_{im}^t = \mu_{im} + \eta_{it} + \xi_{im}^t$$

Here μ_{im} is a term that varies only across markets and players but not over time, η_{it} is a time varying player specific term and $\xi_{im}^t \sim EV(0,1)$ is an idiosyncratic shock iid across individuals, time and markets. This is the only source of asymmetric information in the model. The first and the second elements of ε_{im}^t are known to the players and capture respectively (i) the correlation of the profitability shocks in the same market across time and (ii) correlation of the profitability shock across time in different markets. Both effects are empirically justified by the significance of state and year dummies in the reduced forms analyzed above.

The time varying shock is included to capture the fact that the decision structure of the chains can be centralized: First, the "general" conditions of the economy are observed; second the decision in which municipality(ies) to enter/exit is taken. The model captures the feature that a better (worse) macroeconomic landscape can increase (decrease) the probability of being active in all available markets.

[25] We assume that players leaving the market get a scrap value equal to zero. This hypothesis was also used in Collard-Wexler (2013).

[26] Similar structures were used in Pesendorfer and Schmidt-Dengler (2003, 2008), Ryan (2012) and Collard-Wexler (2009).

We impose a structure in the time effect. We further assumed that $\eta_{it} = \eta_i \bar{x}_t$, where $\overline{x}_t = \sum_m x_{mt}$, is the total payroll of the municipalities in our sample in a given year. The process for the shock is:

$$\varepsilon_{im}^t = \mu_{im} + \eta_i \bar{x}_t + \xi_{im}^t$$

The parameters of interest are $\mathbf{\Theta_{im}} = \left\{ F_i, \pi_{0i}, \pi_{1i}^{pub}, \pi_{1i}^{pri}, \pi_{2i}, \mu_{im}, \eta_i \right\}$.

We do not structurally model the behavior of public banks. Entry decisions of public banks are assumed to do not depend on the number of public and the number of private competitors in the market. We do not specify the payoff structure of public players.

■ **Sequence of decisions.** The sequence of events of the game is the following:

1. States are observed by all the players.

2. Each player draws a private profitability shock ε_{im}^t.

3. Actions are simultaneously chosen. Private players maximize their payoffs given beliefs on competitor's actions. The total payoff of a private player is given by the discounted sum of player's period payoffs. The discount rate is given by $\beta < 1$ and is the same for all players.

4. After actions are chosen the law of motion for $\mathbf{s_m^t}$ determines the distribution of states in the next period; the problem restarts.

Next the equilibrium for this game is characterized.

4.2 Equilibrium characterization

We restrict our attention to pure *Markovian strategies*. This means that players' actions are fully determined by the current vector of state variables. Intuitively, whenever a player observes the same vector of states it will take the same actions and the history of the game until period t does not influence player's decisions.

■ **Public players.** Public players are assumed to be exogenous. Entry probabilities of public players are known to the private players and do not depend on the actions of other public/private players.

■ **Private players.** Private player i's best response solves the following Bellman equation:

$$\underset{a_{im}^t\in\{0,1\}}{Max}\sum_{\mathbf{a^t_{-im}}}\sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m})\left\{\begin{array}{c}\Pi(\mathbf{a^t_m},\mathbf{s^t_m};\mathbf{\Theta_{im}})+\\ \beta\cdot\sum_{\mathbf{s^{t+1}_m}}p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m},\mathbf{a^t_m})\cdot V(\mathbf{s^{t+1}_m};\sigma_{im}(\cdot),\mathbf{\Theta_{im}})\end{array}\right\} \tag{19}$$

Here $\Pi(\mathbf{a_t},\mathbf{s_t};\mathbf{\Theta})$ is given by (17) and (18) and $V_{im}(\cdot)$ is the continuation value for the player given current states and actions. In practice we use the *ex-ante* value function. Its derivation is in the appendix. The function $\sigma_{im}(\cdot|\mathbf{s^t_m})$ accounts for i's beliefs on -i's actions given current states.

Firstly, players' best response functions are characterized. We define: $\Pi_{im}(\mathbf{a^t_m},\mathbf{s^t_m}) = \widetilde{\pi}_{im}(\mathbf{a^t_m},\mathbf{s^t_m}) + \mathbf{1}(a^t_{im}=1)\cdot\xi^t_{im}$, where $\widetilde{\pi}_i(\mathbf{a_t},\mathbf{s^t_m})$ comprises all the terms in (17) but the iid part of the profitability shock and, $\widetilde{\Pi}_{im}(\mathbf{a^t_m},\mathbf{s^t_m}) = \widetilde{\pi}_{im}(\mathbf{a^t_m},\mathbf{s^t_m}) + \beta\cdot\sum_{\mathbf{s^{t+1}_m}}p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m},\mathbf{a^t_m})\cdot V_{im}(\mathbf{s^{t+1}_m};\sigma_{im}(\cdot))$. Then the i's best response is implicitly defined by[27]:

$$H_{im}(a^t_i=1|\mathbf{s^t_m};\sigma_{im}(\cdot)) =$$
$$1-exp\left\{-exp\left\{\begin{array}{c}\sum_{\mathbf{a^t_{-im}}}\sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m})\cdot\widetilde{\Pi}_{im}(a^t_{im}=1,\mathbf{a^t_{-im}},\mathbf{s^t_m})-\\ \sum_{\mathbf{a^t_{-im}}}\sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m})\cdot\widetilde{\Pi}_{im}(a^t_{im}=0,\mathbf{a^t_{-im}},\mathbf{s^t_m})\end{array}\right\}\right\} \tag{20}$$

The solution to this problem is a vector of player i's optimal actions when he faces each possible configuration for the state vector $\mathbf{s^t_m}$ and has consistent beliefs about other players actions in the same states of the world.

By stacking up best responses for every player and every state one can form a system of $1\times N_s$ equations, where N_s expresses the number of different possible states in this market. This system is used to find the $1\times N_s$ vector of players' beliefs. A formal proof of the existence of this vector can be found in Pesendorfer and Schmidt-Dengler (2008). Uniqueness of this equilibrium is not, however, guaranteed. This is a common feature of entry games. The estimation procedure is designed to deal with the multiplicity of equilibria.

5 Econometric Model

This section describes identification and the estimation procedure. We use the OLS derived in Chapter to the recover payoff parameters. Our representation of the problem avoids

[27]For a proof see the appendix.

the use of numerical methods in the estimation procedure and reduces significantly the computational burden.

5.1 Identification and estimation

Following the CCP approach (Hotz and Miller (1993)) we firstly identify the vector of entry probabilities for public and private players and the transitions directly from the data. For the identification of entry probabilities we need to introduce two assumptions:

> ***Assumption (i):*** *There are no unobserved common knowledge states.*
>
> ***Assumption (ii):*** *The same equilibrium is played in all available markets.*

These identifying assumptions follow Ryan (2012). Pesendorfer (2010), Aguirregabiria and Nevo (2010) and Bajari, Hong and Nekipelov (2010) discuss the importance of the assumptions above.

Arcidiacono and Miller (2011) relaxes assumption (i). Our reduced form evidence suggests that our results are robust to the inclusion of market, time and player level unobservables. This mitigates our concern with unobservables. Regarding assumption (ii), we could deal with the multiplicity problem by estimating the model for each market we observe in our sample, as proposed by Pesendorfer and Schmidt-Dengler (2008). The main problem is that because most of these markets are quite stable, i.e. we do not observe frequent entry and exit movements, we could not accurately identify the reduced form parameters and, therefore, the structural parameters associated to most of our markets. In our application it is necessary to pool the data of different markets. The same is done in Collard-Wexler (2013) and Ryan (2012).

Under these assumptions the identification of $\mathbf{\Theta_{im}}$ follows from Pesendorfer and Schmidt-Dengler (2008).

Using Lemma 1 of Chapter 1 we represent the equilibrium system (20) as a linear function of the parameters and estimate the model using OLS. The linear representation of the equilibrium system (20) is derived in the appendix.

5.2 CCPs and state space

Following the CCP approach the empirical implementation of the model depends on (i) the estimation of beliefs and actions for each player, respectively, $H_{im}(a_i^t = 1|\mathbf{s_m^t}; \sigma_{im}(\cdot))$ and

$\sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m})$ and (ii) the estimation of a transition process for the exogenous states, $p^s_m(\cdot)$. Next our estimation procedure for these elements is discussed.

5.2.1 Reduced form estimation of beliefs

We estimated equation (16) pooling the two private players, Bradesco and Itau. The data include the markets where Bradesco was active for at least one period and the markets where Itau was active for at least one period[28].

Instead of including year dummies we included $\overline{x}_t = \sum_m x_{mt}$, the total payroll of the municipalities in our sample in a given year, to control for the correlation in the decisions of private players in the same period of time. Instead of including state dummies we constructed 4 categories of markets. This keeps the state space of the structural model reduced. The market categories are defined according to the number of potential competitors in a given market [29]. More specificaly, if N_m is the number of potential competitors in municipality m then $M_{1m} = 1$ if $N_m \leq 2$; $M_{2m} = 1$ if $2 < N_m \leq 4$; $M_{3m} = 1$ if $4 < N_m \leq 6$; and $M_{4m} = 1$ if $N_m \geq 7$. With this definition we can substitute $\mu_m = \sum_{k=1}^{4} \gamma_k M_{km}$. The same strategy was used in Collard-Wexler (2013). The vector $\mathbf{x_{mt}}$ includes only municipality payroll.

Entry decisions for public players were estimated using the same specification but excluding n^{pub}_{mt-1} and n^{pri}_{mt-1} from the set of covariates. We pooled the two public players, Bank of Brazil and Caixa Economica Federal. The data include the markets where Bank of Brazil was active for at least one period and the markets where Caixa Economica Federal was active for at least one period.

We estimated the logits for the samples 1988-2010 and 1996-2010. The sample 1996-2010 excludes the hyperinflation period and allows us to focus on the more recent market trends. The coefficients in the logits for public/private players are in the appendix.

5.2.2 State space and transitions for exogenous states

Two different identification strategies to estimate the structural model are exploited. The first strategy excludes time and market effects as in Ryan (2012). The advantage of this formulation is that it uses a reduced state space. The second formulation uses market dummies and the sample payroll to control for market and time effects. Both strategies

[28]This follows the definition of potential markets defined in section 3.

[29]Our definition of potential competitor is based on the super efficient estimator in the section 3 - i.e. the number of potential competitors in municipality m is equal to the maximum number of players that were active for at least one period in municipality m since 1900.

are based on the empirical CCP estimates. Only the structural parameters for the private players, Bradesco and Itau, are estimated.

■ **Strategy 1: Model without time and market effects.** The state space for any private player, $i \in i_{pri}$, is composed by the following elements:

$$\mathbf{s_i^t} \in \left\{a_i^{t-1}, \left\{\mathbf{a_j^{t-1}}\right\}_{j\neq i}, x^t, \{\mathbf{I}(i=k)\}_{k\in i_{pri}}\right\}$$

Here $\{\mathbf{I}(i=k)\}_{k\in i_{pri}}$ is a set of private players dummies and x^t is the municipality payroll. The other elements are the actions of player i in period $t-1$, a_i^{t-1}, and the actions of player i's competitors in period $t-1$, $\left\{\mathbf{a_j^{t-1}}\right\}_{j\neq i}$. The variable x^t is discretized in 10 deciles.

The law of motion for x^t is estimated by a simple auto-regressive ordered logit. This formulation for the law of motion of x^t ignores potential effects of banks, either public or private, on municipality income.

The state space of this model is composed by $2 \cdot 2^3 \cdot 10 \cdot 2 = 320$ elements.

■ **Strategy 2: Model with time and market effects.** The second model includes time and market effects. Market effects are captured by 4 market dummies. Time effects are captured by the sample payroll. Market dummies and the sample payroll variables were defined above - see section 5.2.1. The state space in this model is:

$$\mathbf{s_{im}^t} \in \left\{a_{im}^{t-1}, \left\{\mathbf{a_{jm}^{t-1}}\right\}_{j\neq i}, x_m^t, \bar{x}^t, \{\mathbf{I}(i=k)\}_{k\in i_{pri}}, \{\mathbf{I}(m=k)\}_{k=1}^4\right\}$$

Here $\{\mathbf{I}(m=k)\}_{k=1}^4$ is a set of market dummies for the 4 market types; $\bar{x}^t = \sum_{m=1}^4 w_m x_m^t$ is the sample payroll, where, w_m is the number of markets of type m and x_m^t is the average payroll of type m markets in period t; a_{im}^{t-1} is player i's action in a market of type m in $t-1$, $\left\{\mathbf{a_{jm}^{t-1}}\right\}_{j\neq i}$ are the actions of player i's competitors in the same market in period $t-1$ and $\{\mathbf{I}(i=k)\}_{k\in i_{pri}}$ is a set of dummies for each private player.

The law of motion for x_m^t is calculated using an auto-regressive ordered logit structure. The variable x_m^t is discretized in four percentiles for each market type. A model for each market type was estimated. Finally $\bar{x}^t$ was calculated using $\bar{x}^t = \sum_{m=1}^4 w_m x_m^t$ under the assumption that w_m is fixed over time.

The estimation of the model with market dummies is time consuming because the inclusion of the sample payroll, which depends on the realization of the payroll variable in every market, increases exponentially the dimension of the state space. The state space has $2 \cdot 2^3 \cdot 4^4 \cdot 2 \cdot 4 = 32768$ elements.

5.3 Results

We assumed that β, the discount factor, is equal to 0.90. To focus on the more recent trends of the market we used the CCPs and transitions estimated with the 1996-2010 sample. The CCPs of the non strategic public players correspond to models I and II in the second block of Table 12 - those estimated using the sample 1996-2010. For the private players the CCPs are given by models I and II in the second block of Table 13. We used the OLS estimator to estimate the model.

Parameters are estimated in units of the scale factor in the EV distribution and do not have a level interpretation. Only relative magnitudes matter. Standard errors of the parameters were calculated by block bootstraping CCPs and transitions 100 times. The structural model was estimated 100 times, one for each block bootstrap draw of beliefs and transitions. The standard error across this set of parameters was calculated. A similar procedure was applied in Ryan (2012) and Collard-Wexler (2013).

Table 6 reports the structural parameters. The first column corresponds to the model without market unobservables. The second column shows the model with market dummies and the sample payroll, estimated according to strategy 2.

Table 6: Structural Parameters for Private Players

	(I)	(II)
	Profit Components	
N Public	0.0605	0.0726
	[0.01]	[0.00]
N Private	-0.0487	-0.0256
	[0.01]	[0.00]
Market Payroll*	-0.0001	0.0019
	[0.00]	[0.00]
Constant	-0.3720	-0.5821
	[0.01]	[0.05]
	Shock Components	
Sample Payroll*		0.0004
		[0.00]
Market 1		0.2377
		[0.04]
Market 2		0.2104
		[0.03]
Market 3		0.1176
		[0.02]
	Entry/Player Costs	
Entry Costs	-4.9272	-5.7442
	[0.09]	[0.02]
Dummy Bradesco	-0.0270	-0.0245
	[0.01]	[0.02]
Observations	320	32768

Note: (*) Sample payroll measured in R$ billions of 2011; market payroll measured in R$ millions of 2011. Standard-errors in brackets. Standard errors obtained from 100 block bootstraps of beliefs and transitions. Parameters are measured in units of standard deviations of the iid profitability shock.

Qualitatively, both specifications produce similar results. The main difference is that in model I, the market payroll coefficient is negative but small and not significant. All models predict that the entry of a new private competitor reduces the profits of the private incumbent. The entry of a new public player increases the profits of a private incumbent. The constant term, which measures operational costs, is negative and relatively larger in the second model. Entry costs are also negative and relatively larger in the second model. The contribution of the components of the shock in the second model is relatively important. The coefficient attached to the sample payroll is positive. This means that increases in the sample income shifts to the right the distribution of the shock and increases entry rates.

Market effects are positive.

To facilitate the interpretation of these results, the next table reports the estimates as percentage of entry costs. This means that coefficients are divided by the absolute value of the entry costs.

Table 7: Structural Parameters as Percentage of the Entry Costs

	(I)	(II)
	Profit Components	
N Public	1.228%	1.264%
N Private	-0.988%	-0.446%
Market Payroll*	-0.002%	0.033%
Constant	-7.550%	-10.134%
	Shock Components	
Sample Payroll*		0.007%
Market 1		4.138%
Market 2		3.663%
Market 3		2.047%
	Entry/Player Costs	
Entry Costs	100.000%	100.000%
Dummy Bradesco	-0.549%	-0.427%

Note: (*) Sample payroll measured in R$ billions of 2011; market payroll measured in R$ millions of 2011.

Again, the predictions of both models are quite close. In particular, entry of a new public player increases profits of the private incumbent in around 1.3% of the entry costs. Entry of a new private player reduces profits of the private incumbent in around 0.45-0.9% of the entry costs.

The next table provides profit estimates for the private banks using the structural parameters. These parameters allow us to estimate a measure of return over entry costs. We also simulate the number of years necessary to recover entry costs.

Table 8: Average Period Profits and Return to Fixed Costs in Private Monopoly Markets

	(I)	(II)
Period Profits in Std Deviations	0.1936	0.2493
Period Profits as % of Entry Costs	3.930%	4.339%
Years to Recover Entry Costs	26.0	23.0

Note: Average profits of a private monopoly in a small market (market in the lower market payroll decile). Period profits as % of entry costs corresponds to the period profit in std deviations divided by the entry cost. To calculate the number of years to recover the entry costs we assumed a discount rate of 0.9, that the market payroll is increasing steadily at 3% per period and a monopoly structure every period.

The average period payoff of the private banks in monopoly markets is computed in the the first line of the table. The results show that the second model predicts larger profits. The second line shows that model I predicts returns to the entry investment of 4% in monopoly markets. Model II predicts that returns over entry costs are slightly larger, around 4.4%[30].

The third line shows the number of years necessary to recover entry costs. We assumed $\beta = 0.9$ and that municipality and sample payroll are growing steadily at 3% per year. We accumulated the discounted payoffs and computed the number of years that are necessary to recover the estimated entry costs. Model I predicts that in monopoly markets it takes on average 26 years to the private player recover the entry investment. Model II predicts that a private player needs on average 23 years to cover the entry cost[31].

5.4 Discussion

Two remarkable facts arise from our analysis:

1. Public players complement private players;

2. Private players crowd-out other private players.

[30] To calculate profits we fixed the sample payroll at its 2010 average value in smaller markets, that is in markets of type 1. For model II we assumed that the market dummy for markets of type 1 is equal to one. Thus the results are calculated for markets type 1. The sample payroll is used only to compute profits in model II and is equal to the sample payroll of 2010.

[31] This means that entry barriers are quite high. A recent expansion plan of Bank of Brazil illustrates this point. BB set down R$1 billion to construct 600 new branches in the Brazilian territory. This implies that on average each new branch costs R$1.66 million.

Notice also that the potential demand of a small market is quite small: The average yearly payroll of a market in our sample was R$9 million in 2010 and only a small fraction of the population demands banking services. In 2011 the Institute of Applied Economic Research (IPEA, 2011), an institute of the Brazilian federal government, estimated that around 40% of the Brazilian population has no access to any kind of banking services. This percentual can be even large in the markets represented in our sample.

The first result shows that profit of public banks are positively affected by the number of public branches in the same market. Our estimates do not allow us to disentangle the details of the spill-over channels. Broadly speaking, our findings are consistent with public banks (i) having monopoly over a number of important Federal funds and (ii) being driven by social, as opposed to strategic or market reasons. The first element guarantees a large volume of credit for small markets - see Feler (2012). The second induces product differentiation between public and private banks: Public and private banks target different clients - see Coelho, Melo and Rezende (2012). In this case, the amount of cheap credit and public transfers poured by the public banks in small isolated municipalities shifts the demand for banking services, making these markets more attractive for private players. This effect induces the entry of private players.

The crowd-out effect is consistent with Coelho et al (2012). They found that the competition between the Brazilian private banks is relatively tough. Our results reinforce the view that private banks are competitive.

6 Model Fit and Counterfactuals

This section uses the structural model to construct a policy experiment. We are interested in the following question: What happens with the supply of private financial services in small isolated markets when public banks are privatized?

First we solve the model using the estimated parameters. The solution to the model is a vector of N_s entry probabilities that solves the system of implicit best responses given by equation (20).

For models with a large state space this exercise is not computationally feasible. The state space of model II has dimension $N_s = 32768$. The solution for this model showed to be beyond our computational possibilities. The time to solve the model increases exponentially with the state space. From now on, we use only model I, that has a reduced state space ($N_s = 320$), to compute the counterfactual experiments.

6.1 Model fit

We solved the system (20) for private banks entry probabilities. This system is non linear. This means that its solution is not necessarily unique.

To check how the multiplicity affects our conclusions, we proceed in the following way:

First, we solve model I for the entry probabilities using the logit probabilities as the initial guess; second, we perturbed the logit probabilities; third we computed again the solution for the model using the "perturbed" vector of logit probabilities as the initial guess; fourth, we compared the "perturbed" solution with the original solution[32]. Doing so we find that the solutions were identical for any initial guess.

We compare the solution obtained from the structural model with the logit probabilities for all available states. The next table illustrates some statistics of our predictions.

Table 9: Fitted vs Sample (Logit) Probabilities

	(I)
Correlation Fitted and Logit Probabilities	99.91%
Average Sum of Squared Errors	0.05%
Average Sum of Errors	0.82%

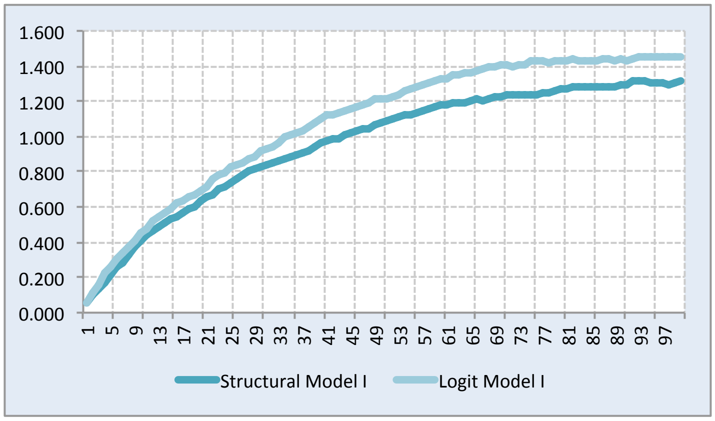
Note: Correlation between the probabilities obtained from the solution of model I and the logit model (model I, Table 13 for 1996-2010) for each state (320 states). Average sum of squared errors gives the sum of the squared difference between the logit and the model probabilities for each state averaged across states. The average sum of errors gives the sum of the differences between the logit and the model probabilities averaged across the 320 states.

In the first line the correlation between the logit probabilities and the solution of the structural model for all states is calculated. The second shows the average squared difference between the logit and the structural probabilities. The third line shows the average sum of these differences. The fitting of the model is very good. The correlation between the logit and structural probabilities is high. The average error of the structural probabilities is below 1%.

We performed an additional exercise. We took the smallest market in terms of sample payroll and assumed that in the first period all the four banks are out of the market. We used firstly the probabilities predicted by the logit models and simulated 1000 paths 100 periods ahead of private banks actions and then we constructed an average path taking the mean across the 1000 paths. We did the same using the probabilities predicted by the structural models. The next figure compares the paths implied by the logit and by the structural model.

[32]Firstly we multiplied the original guesses (calculated from the CCPs showed above) by several factors between 0 and 1. We also started the model with a "fixed" guess, where the probabilities for all the states and for all the players are equal to 0.25, 0.5 and 0.75. We used the same procedure to compute the counterfactuals.

Figure 2: Number of Private Banks 100 Periods Ahead - Model I



Note: Number of private banks in a small market starting from a state where all the competitors are out of the market. Paths 100 periods ahead simulated 1000 times using the structural and the logit probabilities for model I. The figure shows number of private branches averaged over 1000 simulations.

The figure shows that the path obtained from the structural model is close to the path obtained from the reduced form logit models in the first 30 periods. Subsequently the path of the structural model is below the path produced by the logit model. The structural model predicts that after 100 years this small market, without any public/private branch in operation in the first period, will have on average 1.32 private branches. The logit predicts that the same market will have 1.45 branches.

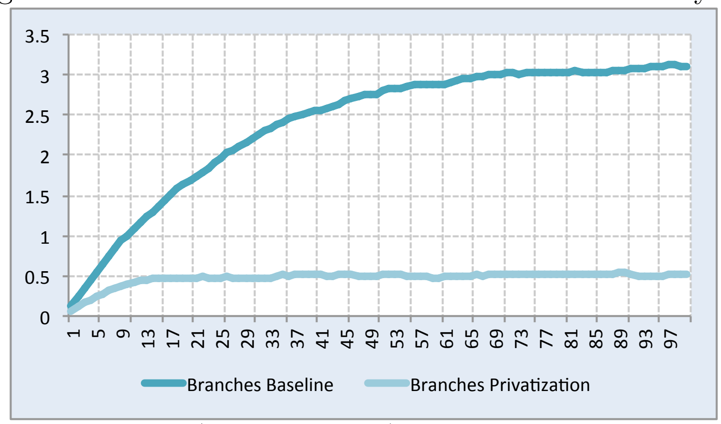
Next we use this model to construct counterfactuals.

6.2 Counterfactual: Privatization of public banks

This section analyzes the effects of the privatization of both public banks on the total supply of financial services in small isolated markets. We assumed that each public bank is bought by different players: BB is bought by one player and CEF by the other. We assumed that the coefficient attached to the number of public competitors in the structural model is equal to the coefficient attached to the number of private competitors. The entry probabilities of public players, instead of being generated by an exogenous process, are calculated according to the system of best responses showed in equation (20).

We calculated the equilibrium probabilities for 4 players. Now this depends on the solution of a system of 640 equations and 640 unknown variables. To check how multiplicity affects our conclusions we used the procedure described in Section 6.1. In all experiments the resulting equilibrium did not change. These probabilities are used to simulate 1000 paths 100 periods ahead. The next figure shows the path for the total number of branches, public plus private, after and before the privatization. We computed this path for a small market where the initial state is characterized by zero active players.

Figure 3: Counterfactual: Privatization of Public Players



Note: Number of branches (public plus private) in a small market starting from a state where all the competitors are out of the market (baseline and privatization counterfactual). Paths 100 periods ahead simulated 1000 times using the structural probabilities for model I. The figure shows number of private branches averaged over 1000 simulations. Branches privatization shows the total number of branches if public branches are privatized. Branches baseline is the total number of branches (public plus private) using the structural model I for private players and the non strategic behavior assumption for public players (calculated based on the logits in Table 12, model I, sample 1996-2010).

The exercise shows that in the long-run the total number of active branches in small municipalities drops from 3 to 0.5 on average. This means that with the privatization around 50% of the Brazilian small municipalities would not be attended by any bank branch. To assure that all these small municipalities would have at least one bank branch in the counterfactual world where public banks are bought by strategic players the government should give a subsidy of 8% over the operational costs of all active branches in the market. Using the fact that the structural model predicts that operational costs are around 7.55% of entry costs and an estimate of R$1 million for the entry costs we calculated that the present cost of this policy is around R$349,463.51 per municipality[33]. This value is relatively small compared to the market value of Brazilian public banks. BB's market value in 2012 was approximately US$88 billions. This means that the resources raised with the privatization of BB would be sufficient to cover the subsidies for 240 thousand branches in the country or, approximately, 42 branches per municipality.

[33]Present values for a time horizon of 100 years and using a discount factor of 0.9 per year.

7 Conclusions

This paper explores microdata of 1002 isolated markets in Brazil during 1988-2010 to estimate a dynamic entry game for public and private banks. We compute the market equilibrium in different counterfactual scenarios. Our setup allow us to make consistent *ex ante* analyses on the effects of changes in the banking market structure on variables of interest.

We obtained two main conclusions. First, public banks generate positive profit spill-overs for private banks; second, private banks crowd-out private competitors. Our estimates show that the entry of a public bank in a given market increases the return of a private incumbent by 1.2 percent and the entry of a private bank reduces the return of a private incumbent by 0.05 to 1 percent.

The counterfactual in which public banks are sold to private players shows that the total number of active branches operating in the long-run in a typical small market drops from 3 to 0.5 on average. To guarantee that, after the privatization, all small municipalities would have at least one active branch the government should give a subsidy of approximately US$175,000 for each small market. This value is relatively small compared to the market value of Brazilian public banks. BB's market value in 2012 was approximately US$88 billions. This means that the resources raised with the privatization of BB would be sufficient to cover the subsidies for 240 thousand branches in the country or, approximately, 42 branches per municipality.

Our estimation procedure improves Pesendorfer and Schmidt-Dengler (2008). We show that the OLS estimator gives us consistent estimates of the structural parameters. This approach allows us to avoid the use of numerical methods in the estimation of the structural parameters. Doing so we reduce significantly the computational burden.

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Appendix 1: Proofs

■ **Model solution.** We omit the term $\mathbf{\Theta}$ from the equations to simplify the notation and write $V(\mathbf{s_m^{t+1}};\sigma_{im}(\cdot),\mathbf{\Theta_{im}}) = V_{im}(\mathbf{s_m^{t+1}};\sigma_{im}(\cdot))$ and $\Pi(\mathbf{a_m^t},\mathbf{s_m^t};\mathbf{\Theta_{im}}) = \Pi_{im}(\mathbf{a_m^t},\mathbf{s_m^t})$. We futher simplify the notation as follows.

1. Isolate the iid shock from (1) writing $\Pi_{im}(\mathbf{a_m^t},\mathbf{s_m^t}) = \widetilde{\pi}_{im}(\mathbf{a_m^t},\mathbf{s_m^t}) + \mathbf{1}(a_{im}^t = 1)\cdot\xi_{im}^t$, where $\widetilde{\pi}_i(\mathbf{a_t},\mathbf{s_m^t})$ comprises all the terms in (17) but the iid part of the profitability shock.

2. With this we can write the deterministic part of the objective function simply as:

$$\widetilde{\Pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) = \widetilde{\pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) + \beta \cdot \sum_{\mathbf{s^{t+1}_m}} p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, \mathbf{a^t_m}) \cdot V_{im}(\mathbf{s^{t+1}_m}; \sigma_{im}(\cdot)) \tag{21}$$

3. Using these facts rewrite i's problem in the following way:

$$\underset{a^t_i \in \{0,1\}}{Max} \sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-i}}|\mathbf{s_t}) \left\{ \widetilde{\Pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) + \mathbf{1}(a^t_i = 1) \cdot \xi^t_{im} \right\} \tag{22}$$

Now, we analyze i's decision process. Given beliefs and states the expected payoff derived from $a^t_{im} = 1$ and $a^t_{im} = 0$ is respectively:

$$E\Pi_{im}(a^t_{im} = 1|\mathbf{s^t_m}; \sigma_{im}(\cdot)) =$$
$$\xi^t_{im} + \sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 1, \mathbf{a^t_{-im}}, \mathbf{s^t_m})$$

and,

$$E\Pi_{im}(a^t_{im} = 0|\mathbf{s^t_m}; \sigma_{im}(\cdot)) =$$
$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m})$$

Therefore, given states and beliefs, $a^t_{im} = 1$ iff:

$$\xi^t_{im} + \sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 1, \mathbf{a^t_{-im}}, \mathbf{s^t_m})$$
$$\geq$$
$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m})$$

Now using $\xi^t_{im} \backsim EV(0,1)$ we can find a closed form solution for i's best response function given states and beliefs:

$$H_{im}(a_i^t = 1|\mathbf{s_m^t}; \sigma_{im}(\cdot)) =$$

$$1 - exp\left\{-exp\left\{\begin{array}{c} \sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 1, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) - \\ \sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) \end{array}\right\}\right\}$$

■ **Value function.** Following Pesendorfer and Schmidt-Dengler (2008) we define the value function as the continuation value for profits before shocks are observed and actions are taken. Mathematically:

$$\begin{aligned} V_{im}(\mathbf{s^t_m}; \sigma_i(\cdot)) = \sigma_{im}(a^t_{im} = 1|\mathbf{s^t_m}) \cdot E\left[\xi^t_{im}|a^t_{im} = 1, \mathbf{s^t_m}\right] + \\ \sum_{a^t_{im}, \mathbf{a^t_{-im}}} \sigma_{im}(a^t_{im}, \mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) \end{aligned} \tag{23}$$

In this formulation $E\left[\xi^t_{im}|a^t_{im} = 1, \mathbf{s^t_m}\right]$ is the (conditional) expectation operator with respect to the iid EV payoff shock and $\sigma_{im}(a^t_{im}, \mathbf{a^t_{-im}}|\mathbf{s^m_t})$ is the joint probability distribution of players' actions given states.

Now if we stack up (23) for every one of the N_s states of the we find a system of N_s equations in N_s unknown variables, in this case the value function for each state. The solution to this (linear) system provides us with a unique continuation value for each state as a function of parameters, beliefs and the distribution of future states. The solution for any state $\mathbf{s^t_m}$ can be easily plugged in equation (20), leaving us with a (non-linear) system of N_s equations with N_s unknown beliefs.

Before proceeding we are going to insert (21) into (23):

$$\begin{aligned} V_{im}(\mathbf{s^t_m}; \sigma_i(\cdot)) = \sigma_{im}(a^t_{im} = 1|\mathbf{s^t_m}) \cdot E\left[\xi^t_{im}|a^t_{im} = 1, \mathbf{s^t_m}\right] + \\ \sum_{\mathbf{a^t_{im}}} \sigma_{im}(\mathbf{a^t_{im}}|\mathbf{s^t_m}) \cdot \left\{\widetilde{\pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) + \beta \cdot \sum_{\mathbf{s^{t+1}_m}} p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, \mathbf{a^t_m}) \cdot V_{im}(\mathbf{s^{t+1}_m}; \sigma_{im}(\cdot))\right\} \end{aligned}$$

Call $\sum_{\mathbf{a^t_{im}}} \sigma_{im}(\mathbf{a^t_{im}}|\mathbf{s^t_m}) \cdot \widetilde{\pi}_{im}(\mathbf{a^t_m}, \mathbf{s^t_m}) = \mathbf{\Sigma_{im}}(\mathbf{a^t_m}|\mathbf{s^t_m}) \cdot \mathbf{\Gamma_{im}}(\mathbf{a^t_m}, \mathbf{s^t_m})$, where $\mathbf{\Sigma_{im}}(\mathbf{a^t_m}|\mathbf{s^t_m})$ is a 1×2^N vector of probabilities for all possible tuple of players actions given states, $\mathbf{\Gamma_{im}}(\mathbf{a^t_m}, \mathbf{s^t_m})$ is the $2^N \times 1$ vector of (non stochastic) profits in that given state for all possible combination of players actions and $\sum_{\mathbf{a^t_{im}}} \sum_{\mathbf{s^{t+1}_m}} \sigma_{im}(\mathbf{a^t_{im}}|\mathbf{s^t_m}) \cdot p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, \mathbf{a^t_m}) = \mathbf{Z_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$, where $\mathbf{Z_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$ is a $1 \times N_s$ vector containing the (weighted) probabilities of each state when the states are $\mathbf{s^t_m}$ and $\Psi_\mathbf{i}(\mathbf{s^t_m}) \equiv \sigma_{im}(a^t_i = 1|\mathbf{s^t_m}) \cdot E\left[\xi^t_{im}|a^t_{im} = 1, \mathbf{s^t_m}\right]$. Using this we can rewrite the equation above as:

$$V_{im}(\mathbf{s^t_m}; \sigma_{im}(\cdot)) =$$
$$\Psi_\mathbf{i}(\mathbf{s^t_m}) + \mathbf{\Sigma_{im}}(\mathbf{a^t_m}|\mathbf{s^t_m}) \cdot \mathbf{\Gamma_{im}}(\mathbf{a^t_m}, \mathbf{s^t_m}) + \beta \cdot \mathbf{Z_m}(\mathbf{s^{t+1}_m}|\mathbf{a^t_m}, \mathbf{s^t_m}) \cdot \mathbf{V_{im}}(\mathbf{s^{t+1}_m}; \sigma_\mathbf{im}(\cdot))$$

In this equation $\mathbf{V_{im}}(\sigma_{im}(\cdot))$ is a $N_s \times 1$ vector of value functions in which each line represents a value function for a particular state. Now, stacking up these equations for any possible state in the economy we can write:

$$\begin{aligned} \mathbf{V_{im}}(\sigma_{im}(\cdot)) &= \mathbf{\Psi_{im}} + \mathbf{\Sigma\Gamma_{im}} + \beta \cdot \mathbf{Z_m} \cdot \mathbf{V_{im}}(\sigma_\mathbf{im}(\cdot)) \\ &= \left[\mathbf{I} - \beta \cdot \mathbf{Z_m}\right]^{-1} \cdot \left[\mathbf{\Psi_m} + \mathbf{\Sigma\Gamma_{im}}\right] \end{aligned} \tag{24}$$

The existence of the inverse matrix in the first bit of (24) follows from the dominant diagonal property - see Pesendorfer and Schmidt-Dengler (2008). Here $\mathbf{\Psi_{im}}$ is a $N_s \times 1$ vector stacking all possible values of $\Psi_\mathbf{i}(\mathbf{s^t_m}) \equiv \sigma_{im}(a^t_i = 1|\mathbf{s^t_m}) \cdot E\left[\xi^t_{im}|a^t_{im} = 1, \mathbf{s^t_m}\right]$, $\mathbf{\Sigma\Gamma_{im}}$ is a $N_s \times 1$ vector of expected (non stochastic) profits for each state (where the expectation is taken across each possible vector of actions conditional on states) and $\mathbf{Z_m}$ is a $N_s \times N_s$ transition matrix for all states.

■ **Linearity of the value function.** Start from equation (24). The only term that depends on parameters is $\mathbf{\Sigma}\Gamma_{im}$. Each line in the $N_s \times 1$ vector $\mathbf{\Sigma\Gamma_{im}}$ (represented as $\mathbf{\Sigma_{im}}(\mathbf{a^t_m}|\mathbf{s^t_m}) \cdot \mathbf{\Gamma_{im}}(\mathbf{a^t_m}, \mathbf{s^t_m})$) corresponds to the expected (non stochastic) profits for each state (where the expectation is taken across each possible vector of actions conditional on states). Having in mind the parametrization in (17) and (18) it is straightforward to see that for any state $\mathbf{s^t_m}$, $\mathbf{\Sigma_{im}}(\mathbf{a^t_m}|\mathbf{s^t_m}) \cdot \mathbf{\Gamma_{im}}(\mathbf{a^t_m}, \mathbf{s^t_m}) = K(\mathbf{s^t_m})'\Theta_{im}$, where $K(\mathbf{s^t_m})'$ is a $1 \times N_p$ vector of variables that depends on the beliefs for all players and states, where N_p is the number of parameters in the model. Now, to simplify the notation, write $\left[\mathbf{I} - \beta \cdot \mathbf{Z_m}\right]^{-1} = \Delta$, a $N_s \times N_s$ matrix that depends on the transition matrix $\mathbf{Z_m}$. Going back to (24) and using these facts it is easy to see that $\mathbf{V_{im}}(\sigma_{im}(\cdot)) = \Delta \cdot \mathbf{\Psi_{im}} + \Delta \cdot K \cdot \Theta_{im}$, where K is a $N_s \times N_p$ matrix of variables constructed by stacking up $K(\mathbf{s^t_m})'$ for all states.

■ **Linearity of the best response function.** Start from equation (20). Use the definition of $\widetilde{\Pi}_{im}(\mathbf{a_m^t}, \mathbf{s_m^t})$ to write:

$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) =$$
$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) +$$
$$\beta \cdot \sum_{\mathbf{a^t_{-im}}} \sum_{\mathbf{s^{t+1}_m}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, a^t_{im} = 0, \mathbf{a^t_{-im}}) \cdot V_{im}(\mathbf{s^{t+1}_m}; \sigma_{im}(\cdot))$$

Now we can use the equation above and rewrite the transition matrix (containing beliefs and transition between states) as $\sum_{\mathbf{a^t_{-im}}} \sum_{\mathbf{s^{t+1}_m}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot p^s_m(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}, a^t_{im} = 0, \mathbf{a^t_{-im}}) = \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$, where $\mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$ is $1 \times N_s$ vector containing the (weighted) probabilities of each state when the states are $\mathbf{s^t_m}$ and player i is choosing $a^t_{im} = 0$. Using the same argument that we used to state the linearity of the value function, it is easy to see that $\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) = K(\mathbf{s^t_m}, a^t_{im} = 0)' \Theta_{im}$, where $K(\mathbf{s^t_m}, a^t_{im} = 0)'$ is a $1 \times N_p$ vector of variables when states are $\mathbf{s^t_m}$ and player i is choosing $a^t_{im} = 0$. Therefore we can write:

$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) =$$
$$K(\mathbf{s^t_m}, a^t_{im} = 0)' \Theta_{im} + \beta \cdot \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) \cdot V_{im}(\mathbf{s^{t+1}_m}; \sigma_{im}(\cdot))$$

By substituting the value function using $\mathbf{V_{im}}(\sigma_{im}(\cdot)) = \Delta \cdot \mathbf{\Psi_{im}} + \Delta \cdot K \cdot \Theta_{im}$, it is easy to see that:

$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im} = 0, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) =$$
$$K(\mathbf{s^t_m}, a^t_{im} = 0)' \Theta_{im} + \beta \cdot \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) \cdot [\Delta \cdot \mathbf{\Psi_{im}} + \Delta \cdot K \cdot \Theta_{im}]$$

Analogously, we can write $\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$ as the equivalent of $\mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})$ when $a^t_{im} = 1$ and $K(\mathbf{s^t_m}, a^t_{im} = 1)$ as the equivalent $K(\mathbf{s^t_m}, a^t_{im} = 0)$ when the action is $a^t_{im} = 1$. With this we can calculate $\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im}=1, \mathbf{a^t_{-im}}, \mathbf{s^t_m})$ as:

$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \widetilde{\Pi}_{im}(a^t_{im}=1, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) =$$
$$K(\mathbf{s^t_m}, a^t_{im}=1)^{'}\Theta_{im} + \beta \cdot \mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) \cdot [\Delta \cdot \mathbf{\Psi_{im}} + \Delta \cdot K \cdot \Theta_{im}]$$

This means that the terms inside the inner curly brackets of (20) reduces to:

$$\sum_{\mathbf{a^t_{-im}}} \sigma_{im}(\mathbf{a^t_{-im}}|\mathbf{s^t_m}) \cdot \left[\widetilde{\Pi}_{im}(a^t_{im}=1, \mathbf{a^t_{-im}}, \mathbf{s^t_m}) - \widetilde{\Pi}_{im}(a^t_{im}=0, \mathbf{a^t_{-im}}, \mathbf{s^t_m})\right] =$$
$$\left\{ \begin{array}{c} [K(\mathbf{s^t_m}, a^t_{im}=1) - K(\mathbf{s^t_m}, a^t_{im}=0)]^{'} + \\ \beta \cdot [\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})] \cdot \Delta \cdot K \end{array} \right\} \cdot \Theta_{im} +$$
$$\beta \cdot \left[\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})\right] \cdot \Delta \cdot \mathbf{\Psi_{im}}$$

Insert this formula into (20) and take logs and then write the LHS of the linearized expression as $y_i(\mathbf{s^t_m}) = ln\left\{ln\left[\frac{1}{1-H(a^t_i=1|\mathbf{s_t};\sigma_i(\cdot),\mathbf{\Theta})}\right]\right\}$. Therefore:

$$y_i(\mathbf{s^t_m}) = \beta \cdot \left[\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})\right] \cdot \Delta \cdot \mathbf{\Psi_{im}} +$$
$$\left\{ \begin{array}{c} [K(\mathbf{s^t_m}, a^t_{im}=1) - K(\mathbf{s^t_m}, a^t_{im}=0)]^{'} + \\ \beta \cdot [\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})] \cdot \Delta \cdot K \end{array} \right\} \cdot \Theta_{im}$$

Given that elements in $\mathbf{\Psi_{im}}$ are known we can write $y^*_i(\mathbf{s^t_m}) = y_i(\mathbf{s^t_m}) - \beta \cdot [\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})] \cdot \Delta \cdot \mathbf{\Psi_{im}}$. Also, to simplifiy the notation, use:

$$\left\{ \begin{array}{c} [K(\mathbf{s^t_m}, a^t_{im}=1) - K(\mathbf{s^t_m}, a^t_{im}=0)]^{'} + \\ \beta \cdot [\mathbf{Z^1_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m}) - \mathbf{Z^0_m}(\mathbf{s^{t+1}_m}|\mathbf{s^t_m})] \cdot \Delta \cdot K \end{array} \right\} = M_i(\mathbf{s^t_m})$$

Therefore, $y^*_i(\mathbf{s^t_m}) = M_i(\mathbf{s^t_m}) \cdot \Theta_{im}$.

Appendix 2: Reduced Forms

Table 10: Marginal Effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} on the Entry Probabilities of Private Players (Bradesco and Itau) - Subsample $n^{pub} \geq 1$

	(I)	(II)	(III)	(IV)
Nº Public	0.11467***	0.12303***	0.15040***	0.14810***
	[0.02]	[0.02]	[0.03]	[0.03]
Nº Private	-0.04343*	-0.02822	-0.07914**	-0.07935**
	[0.02]	[0.03]	[0.03]	[0.03]
Player Dummy	Yes	Yes	Yes	Yes
Time Dummies	Yes	Yes	Yes	Yes
State Dummies	No	Yes	Yes	Yes
Trend*State Dummies	No	No	Yes	Yes
Transfers, Expenditure, Agric. Prod.	No	No	No	Yes
Observations	9,348	9,164	9,164	9,162
Pseudo R2	0.87	0.88	0.92	0.92

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Marginal effects calculated at the sample means. Clustered standard errors at the municipality level in brackets. All the models have lagged activity, number of public and private competitors and municipality payroll. Subsample $n^{pub} \geq 1$ includes all municipalities that had at least one public player in every period.

Table 11: Marginal Effects of n^{pub}_{mt-1} and n^{pri}_{mt-1} on the Entry Probabilities of Private Players (Bradesco and Itau) - Subsample $1 \leq n^{pub} \leq 3$

	(I)	(II)	(III)	(IV)
Nº Public	0.19269***	0.18806***	0.20575***	0.21391***
	[0.03]	[0.04]	[0.04]	[0.04]
Nº Private	-0.05677*	-0.06105*	-0.12490***	-0.11986***
	[0.03]	[0.04]	[0.05]	[0.05]
Player Dummy	Yes	Yes	Yes	Yes
Time Dummies	Yes	Yes	Yes	Yes
State Dummies	No	Yes	Yes	Yes
Trend*State Dummies	No	No	Yes	Yes
Transfers, Expenditure, Agric. Prod.	No	No	No	Yes
Observations	7,301	7,117	7,117	7,115
Pseudo R2	0.87	0.88	0.92	0.92

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Marginal effects calculated at the sample means. Clustered standard errors at the municipality level in brackets. All the models have lagged activity, number of public and private competitors and municipality payroll. Subsample $1 \leq n^{pub} \leq 3$ includes all municipalities that had at least one and at most three public players in every period.

Appendix 3: CCPs

Table 12: CCP Logit for Public Players (BB and CEF)

	(I)	(II)	(I)	(II)
	Sample:1988-2010		Sample:1996-2010	
Lagged Activity	6.75***	6.73***	6.88***	7.22***
	[0.09]	[0.10]	[0.12]	[0.14]
Market Payroll	0.03***	0.02***	0.04***	0.02***
	[0.00]	[0.00]	[0.00]	[0.00]
Sample Payroll		0.07***		0.17***
		[0.01]		[0.01]
Market 1		-1.15***		-1.17***
		[0.26]		[0.32]
Market 2		-1.09***		-1.45***
		[0.23]		[0.29]
Market 3		-0.64***		-0.88***
		[0.22]		[0.26]
Dummy BB	1.35***	1.68***	1.82***	2.22***
	[0.11]	[0.13]	[0.14]	[0.17]
Constant	-3.70***	-3.56***	-4.55***	-5.58***
	[0.12]	[0.25]	[0.16]	[0.33]
Observations	20,357	20,357	13,680	13,680
Pseudo R2	0.793	0.796	0.815	0.827

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Clustered standard errors in brackets. Model I does not include sample payroll and market dummies. Model II includes these variables.

Table 13: CCP Logit for Private Players (Bradesco and Itau)

	(I)	(II)	(I)	(II)
	Sample:1988-2010		Sample:1996-2010	
Lagged Activity	7.44***	7.32***	8.05***	8.17***
	[0.12]	[0.13]	[0.18]	[0.20]
N Public	0.25***	0.25***	0.46***	0.63***
	[0.05]	[0.07]	[0.06]	[0.08]
N Private	-0.42***	-0.40***	-0.57***	-0.35**
	[0.10]	[0.11]	[0.12]	[0.15]
Market Payroll	0.02***	0.01***	0.01***	0.01***
	[0.00]	[0.00]	[0.00]	[0.00]
Sample Payroll		0.08***		0.03**
		[0.01]		[0.01]
Market 1		-0.42		0.98**
		[0.32]		[0.38]
Market 2		-0.12		0.85***
		[0.26]		[0.32]
Market 3		0.13		0.51**
		[0.21]		[0.26]
Dummy Bradesco	0.05	0.04	-0.51***	-0.52***
	[0.11]	[0.11]	[0.13]	[0.13]
Constant	-3.84***	-4.43***	-3.41***	-4.84***
	[0.11]	[0.39]	[0.13]	[0.46]
Observations	15,919	15,919	10,595	10,595
Pseudo R2	0.828	0.831	0.830	0.831

Note: (***) Significant at 1%; (**) significant at 5%; (*) significant at 10%. Clustered standard errors in brackets. Model I does not include sample payroll and market dummies. Model II includes these variables.