STOCHASTIC CONTROL IN MANPOWER PLANNING

by

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A Thesis submitted for the fulfilment of the requirements of the University of London for the degree of Doctor of Philosophy.

The London School of Economics and Political Science

1984

ABSTRACT

Our concern is with control problems which arise in connection with a discrete time Markov chain model for a graded manpower system. In this model, the members of an organisation are classified into distinct classes. As time passes, they move from one class to another, or to the outside world, in a random way governed by fixed transition probabilities. The emphasis is, then, placed on examining means of reaching and then retaining the structure best adapted to the aims of the organisation, with the assumption that only the recruitment flows are subject to control.

Attainability and maintainability have received a great deal of attention in recent years. However, much of the work has been concerned with deterministic analysis, in the sense that average values are used in place of random variables. We adopt, instead, a stochastic approach to the study of these forms of control.

We present some of the problems encountered when evaluating probabilities related to the distribution of stock numbers at different steps and we give a detailed numerical comparison of different recruitment strategies.

An iterative method is developed to compute exact values of the probabilities of attaining and maintaining a structure in one step. It is designed for the special but very important case of systems in which promotion is only

possible to the next highest grade. Its efficiency makes possible the use of exact results in the comparison of the recruitment strategies, which was formerly accomplished by means of simulation techniques only.

As to the comparison itself, it emerges that the strategy which, at each step, steers the system as far as possible towards the goal is superior to all deterministic strategies. Also, this strategy is shown to come close to providing the highest level of control that is possible.

ACKNOWLEDGEMENTS

I wish to extend my deep thanks and gratitude to my supervisor, Professor D.J. Bartholomew, for his invaluable support, encouragement and guidance throughout the preparation of this work.

I am also most grateful to L'Institut National de Statistique et d' Economie Appliqueé, Rabat, Morocco for supporting me. This support has been both financial and moral and without it the completion of this work would not have been possible.

Additional financial support came from the Economic Commission for Africa and for this I am also grateful.

For their patience and application in typing this thesis, I would like to express my thanks to Mary Dobbyn and Satta Jabati.

Finally, these acknowledgements would not be complete without reference to the help and encouragement which I have generously received from my family and friends.

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CHAPTER I

INTRODUCTION

During the last two decades a good deal of attention has been devoted to problems of control in manpower planning. However, these problems are by no means so recent. Grinold and Marshall (1977), for example, argued that the construction of the pyramids of Egypt, the building of the great wall of China and the conduct of the military force operations of ancient Rome could not have been achieved without the assistance of manpower planners. But no records remain to show what their role was.

The earliest recorded work concerned with manpower systems is generally attributed to actuaries and demographers. They were the first to use mathematical techniques to forecast characteristics of distinct age groups within populations. In their original approaches, only averages and ratios of such averages were used. Human populations, their main concern, were large enough to ensure a high accuracy in the estimation of different flows and more sophisticated statistical approaches were not considered necessary. Furthermore, forecasting was the sole purpose of their techniques, rather than control. They could do little to influence the birth and death rates on which their projections depended. These made the actuarial techniques inappropriate and insufficient to handle problems concerning manpower systems in general and industrial organisation type systems in particular. Indeed, such systems are not usually as large as human populations and the ultimate function of the modern manpower planner is as much control as forecasting.

Some of the earliest work on the statistical approach to manpower planning using modelling techniques can be traced to the work of Seal (1945) and Vajda (1947) when the consideration of stratefied populations with promotion from grade to grade was first proposed. But Markov models, which are the basis of most present-day work concerning graded systems, were first used by Young and Almond (1961) and by Gani (1963). Wastage and promotion flows were considered to be governed by fixed transition probabilities and the prediction of grade sizes was therefore the sole objective of such models. In many organisations, however, the grade sizes are not free to vary and promotion and recruitment can only take place to fill vacancies as they occur. Models designed for these systems were introduced first in Bartholomew (1963) and much of their mathematical foundations were drawn from the renewal theory.

Applications and developments of the original Markov models have been pursued since then but the emphasis on predicting stocks and flows remained dominant until the end of the 1960s.

Next, attention moved to aspects of control of a manpower system. Questions of how to steer the latter toward some desired direction became the main concern during the 1970s. In the basic model, the members of a system or organisation are classified into distinct classes or groups and, as time passes, individuals move from one class to another as well as between each class and the outside world. The members of each class are homogeneous in the sense that each member of a class has the same probability of a particular move. The problem is, then, to devise means of reaching and then retaining the structure best adapted to the aims of the organisation. Control, therefore, has two aspects which are known in the literature of manpower planning as attainability and maintainability. Attainability is concerned with whether or not a goal structure can be reached and, if so, by what means. Maintainability, however, is concerned with remaining at the goal structure once it has been attained.

Theoretical formulation and early developments of this model can be found in Bartholomew (1967). Since then, the related literature has been growing rapidly and research has taken different directions. Forbes (1970) investigated the control of manpower systems under expansion or contraction. Morgan (1971) developed techniques for studying career prospects in graded systems. Charnes, Cooper and Niehaus (1972) used goal programming techniques in their model for planning the civilian manpower in the U.S. Navy

Department. Davies (1973 and 1975) placed emphasis on attainability and maintainability in many steps. He examined and gave a geometrical description of families of n-maintainable and n-attainable structures. Bartholomew (1973) advocated the use of linear and quadratic programming to tackle problems of attainability in many steps. Vajda (1975 and 1978) gave a detailed discussion of the use of linear programming methods and introduced weaker forms of maintainability in which it is sufficient to maintain the total size of some subset of grades. Grinold and Stanford (1974) developed several algorithms for calculating optimal control policies and based their approach on a combined use of dynamic programming and generalised linear programming. Further developments as well as an account of application of Markov models across a wide range of social sciences can be found in Smith (1970), Bartholomew (1976, 1979 and 1982), Grinold and Marshall (1977).

Much of the early work on control, however, was concerned with deterministic analysis in the sense that average values were used in place of random variables. Grinold and Marshall (1977) argued that the assumption in such an analysis concerning the Markovian nature of the flows is not necessary, and that the problem is essentially the same if the flow numbers are assumed to be proportional to the stocks from which they come. Their use of the term "fractional flow model" was therefore adopted in order to avoid the probabilistic connotation of the Markov chain.

The first step towards exploring the consequences of control in a stochastic environment and evaluating probabilities of groups sizes being maintainable was taken in Bartholomew (1975, 1977). Most calculations, however, were based on simulation techniques and Multinormal approximation. We propose in the following two chapters to examine new approaches for exact evaluation of the above probabilities.

Control can be exercised through three main aspects of a manpower system: probabilities of loss from the system, promotion and demotion flows, recruitment flows. But, for practical reasons that will be discussed in the remainder of this chapter and for other reasons inherent in the methods used, our attention will be focussed on control by recruitment.

A detailed comparison of different recruitment strategies based on the new evaluation methods will be the subject of chapter four.

We begin firstly with a brief review of the deterministic theory.

1.1 DEFINITIONS AND NOTATIONS

Let us consider a manpower system that consists of N individuals of whom $n_i(T)$ are in class i at time T, (i=1,2,...,k). The vector $n(T) = (n_1(T), n_2(T), \ldots, n_k(T))$ is referred to as the stock vector at T, and its elements as stock numbers.

The partition of the system into classes, grades or categories could be made in several ways and could follow different criteria, but it is essential that each individual, at each time, should belong to one and only one class. The basis of a manpower classification is generally provided by the nature of the system to be investigated and the purpose of the investigation itself.

We assume that all flows take place at integral points of time and the probability that an individual moves from i to j between one time point and the next is p_{ij} (i,j=1,2,...,k). If there are no gains or losses from or to the outside world or if the losses are immediately made good the system will be called a closed system. This assumption is generally made when concentrating on the issue of mobility of the individuals within the system. However, losses are frequently observed and recruiting new individuals is often of primordial importance and has serious and direct effects upon the evolution of a manpower system over time. Thus, since our main concern is to investigate the evolution of a manpower system over time under different recruitment strategies, it is natural to suppose that the system is open and interacts with the outside world. More precisely, we suppose that an individual in grade i may leave the system with probability

$$w_i = 1 - \sum_{j=1}^{k} p_{ij} > 0 \qquad (i = 1,2,\ldots,k),$$

and new entrants at T are allocated to various grades proportionally to a recruitment vector

$r(T) = (r_1(T), r_2(T), \ldots, r_k(T))$ with $\sum_{j=1}^{k} r_j(T) = 1$ and $r_j(T) \geq 0$ $(j=1,2,\ldots,k)$. The decision upon the total number of recruits M(T) is supposed to be made at the end of a period of time, after flows and losses have taken place.

The implicit suggestion that membership of one class is the only factor which governs the flows and wastage probabilities could be regarded as very restrictive. However, since the partition of the system into classes is left to the model-builder, this latter could include all the relevant factors which might influence different flows and losses to define an appropriate classification and overcome the restriction underlined above.

As to the way the stock numbers are related to different flows, let $n_{ij}(T)$ be the number of individuals who move from grade i to grade j during the period of time $[T-1, T[$, $T \geq 1$. We have:

$$n_j(T) = \sum_{i=1}^{k} n_{ij}(T) + M(T) \times r_j(T) \qquad j=1,2,\ldots,k \tag{1.1}$$

This equation will give, after using conditional expectations:

$$\bar{n}_j(T) = \sum_{i=1}^{k} \bar{n}_i(T-1)p_{ij} + \bar{M}(T) \times r_j(T) \quad (j=1,2,\ldots,k) \tag{1.2}$$

where the bar denotes the expected value.

Using matrix notation, the equation (1.2) becomes:

$$\bar{n}(T) = \bar{n}(T-1)\ P + \bar{M}(T) \times r(T) \tag{1.3}$$

where P denotes the k x k matrix of transition probabilities.

It can be shown that the stock vectors $\bar{n}(T)$ and $\bar{n}(T-1)$ will have the same total size if, and only if, $\bar{M}(T) = \bar{n}(T-1) * w'$, where w denotes the row vector of wastage probabilities and w' denotes its transpose. In this case, equation (1.3) becomes:

$$\bar{n}(T) = \bar{n}(T-1)P + \bar{n}(T-1) * w' * r(T) \tag{1.4}$$

If we confine ourselves to the expected values of different aggregates of a manpower system, equation (1.3) provides a concise description of the state of the system and the way in which it is changing. However, the same equation holds in the case of p_{ij} and w_i being considered as fixed proportions. Thus, any analysis which relies only on expected values of stock numbers and flows will be considered to operate in a deterministic environment and will be called a deterministic analysis.

Recruitment policies and probabilities related to different flows, as described above, might be understood to be given and not subject to any control. Consequently, it could be assumed that the only purpose of the model is to make predictions about the future of the manpower system. This is untrue; while prediction is an important feature of a manpower model, its use in advising the best actions upon different flows, to direct the evolution of a manpower system toward a desired objective, is of the utmost importance. However, this kind of control could turn out to be very difficult in practice; present environment and state of the manpower system, social and economic considerations are often the key factors which deny the management a meaningful influence on all flows or even on some of them.

The wastage flows, for example, are subject mainly to considerations, inherent in the individuals themselves and are very difficult to control. Obviously they could be influenced directly through redundancies, but this action could provoke strong reaction from within the organisation and, in any case, should not normally be planned by efficient management.

Another sensitive area of control is concerned with the promotion flows. Although the latter result from direct management decisions, an action upon them could affect seriously the career prospects of the individuals within the organisation and its implementation could be hampered by internal factors such as hostile reaction from the individuals concerned and lack of appropriate qualifications, or external factors such as legislation, competition and public image considerations. The environment of the organisation is therefore an important factor which makes any action on the internal flows very difficult and consequently infrequent.

The recruitment flows also emanate from direct management decisions but present fewer difficulties than the promotion flows. Recruitment could affect to some extent the individuals already in an organisation but on a much smaller scale than a direct action upon the promotion flows.

Through all of our work, we will confine ourselves to types of control which involve action upon only one type of flow at a time; control by recruitment arises when recruitment is the only area of direct action. If the probabilities of transition from one grade to another are the only elements for which choice is possible, the control is then by promotion. In both types of control, we will suppose that the wastage probabilities are given and that redundancies must not occur. The objectives of such control could be very varied and too numerous to be covered in general terms within the confines of one or few models. Thus, we will concern ourselves only with cases in which the purpose of the control is to reach a structure m from a structure n in one or more steps; that is with attainability. If m and n are identical, the control will be said to be concerned with maintainability.

This objective is generally pursued when the structure m presents some desirable features or when it satisfies some temporary needs. If attaining a structure m in a fixed number of steps turns out to be impossible, we will try instead to get as close as possible to m; a variety of definitions for distance between two structures will be considered.

In the remainder of this chapter, a deterministic analysis of attainability and maintainability will be made; control in a stochastic environment will be the subject of the next chapters.

1.2 MAINTAINABILITY IN A DETERMINISTIC ENVIRONMENT

In this section, we will be concerned with the evolution of a manpower system in a single period of time, say [T, T+1[, $T \geq 0$. Thus, since there is no confusion about the time considered, the suffix T will be omitted from all our notations.

A stock vector or structure n is said to be maintainable if there exist values of P, w and r such that:

$$n = nP + nw'r \qquad (1.5)$$

Because of equation (1.4), it is implied that the expected stock vector, at the end of a single period of time, could be made equal to n, if this latter was the initial stock vector at the beginning of the same period. It is also implied that the total of recruits is equal on average to the total losses.

If we could act upon all the elements P, w and r, the problem would be trivial and all structures would be maintainable. But as discussed earlier, this is almost impossible and it would be reasonable to assume that only some flows can be controlled.

1.2.1 Control by Promotion

Let us suppose that the wastage probabilities vector w and the recruitment vector r are given. Our problem is then to find a transition matrix P such that:

$$n = nP + nw'r \qquad (1.6a)$$

$$\sum_{j=1}^{k} p_{ij} = 1 - w_i \qquad i = 1,2,\ldots,k \qquad (1.6b)$$

$$p_{ij} \geq 0 \qquad i,j = 1,2,\ldots,k \qquad (1.6c)$$

Bartholomew (1973) showed that a necessary condition for the existence of a matrix P which satisfies conditions (1.6) is that:

$$n - nw'r \geq 0 \qquad (1.7)$$

let us prove that (1.7) is also a sufficient condition for the maintainability of n.

We will introduce first a new problem and show that the structure n will be maintainable if, and only if, the latter problem has a solution. Our proof will be completed by showing that (1.7) is a sufficient condition for the solvability of the new problem.

The problem we intend to consider derives from conditions (1.6) by multiplying each member of equation (1.6b) by n_i and introducing new variables $x_{ij}=n_i\, p_{ij}$ $(i,j=1,2,...,k)$. The concern is then to find values for all variables x_{ij} such that the following constraints are all satisfied:

$$\sum_{i=1}^{k} x_{ij} = n_j - (nw')r_j \qquad j = 1,2,...,k \qquad (1.8a)$$

$$\sum_{j=1}^{k} x_{ij} = n_i - n_i w_i \qquad i = 1,2,...,k \qquad (1.8b)$$

$$x_{ij} \geq 0 \qquad i,j = 1,2,...,k \qquad (1.8c)$$

It is not difficult to check that if there exist values $\hat{p}_{ij}$ which satisfy conditions (1.6), there will be values $\hat{x}_{ij}$ which satisfy conditions (1.8) and vice-versa; this establishes the equivalence between maintainability of n and solvability of the new problem. To prove that condition (1.7) is sufficient for the solvability of the latter problem, let us suppose that condition (1.7) holds; that is $n_j - (nw')r_j \geq 0$, $j = 1,2,...,k$. Thus, since we have also:

$$n_i - n_i w_i = n_i\ (1 - w_i) \geq 0 \qquad i=1,2,...,k$$

and $$\sum_{j=1}^{k} (n_j - nw'r_j) = \sum_{i=1}^{k} (n_i - n_i w_i),$$

the problem defined by constraints (1.8) could be seen as a balanced transportation problem and therefore admits at least one solution.

The presentation of the maintainability issue in terms of a transportation problem introduces great ease and high flexibility into the analysis; optimisation of some linear function related to the variables p_{ij} can be easily incorporated and a postoptimal analysis can be performed. It does also allow us to benefit from the simplicity of finding a solution for conditions (1.6). It does not, however, help in characterising the maintainable region better than condition (1.7).

In the above formulation, it was assumed that there are no conditions imposed upon p_{ij} other than conditions (1.6). But, situations in which many p_{ij}'s are required to be zero arise in practice. In such cases, we will concern ourselves first by finding values of x_{ij} which comply with constraints (1.8) and minimise the linear function:

$$L = \sum_{i=1}^{k} \sum_{j=1}^{k} c_{ij} x_{ij}$$

where $c_{ij} = +\infty$ if p_{ij} is required to be zero
$= $ a constant otherwise.

Then we will conclude that the maintainability problem, with the additional constraints on p_{ij}, admits a solution if and only if the minimal value of L is finite. Moreover, if the additional constraints on p_{ij} require all, except (2k-1) p_{ij} to be zero, the latter problem will have either no solution or a unique solution. The latter case will occur if the minimal value of L is finite, and the 2k-1 basic variables of the corresponding solution will be those with finite coefficients c_{ij}.

The important class of hierarchical manpower systems, where the promotion is possible only into the next higher grade, is an example in which (2k-1) p_{ij} only could take non-zero values.

1.2.2 Control by Recruitment

In this type of control we assume that the wastage probabilities vector w and the transition matrix P are both given and cannot be subject to any modification. The problem is to find a vector r such that:

$$r_i \geq 0 \quad (i=1,2, ..,k) \quad \text{and} \quad \sum_{i=1}^{k} r_i = 1,$$

and which satisfies equation (1.5).

Bartholomew (1973) showed that the vector r defined by:

$$r = n\ (I-P)/nw' \tag{1.9}$$

is the only solution to our problem if and only if $n \geq nP$. He showed also that the set of all maintainable structures with total size N, or maintainable region, is the convex hull of the points $N \times e_i \times (I-P)^{-1}/d_i$ $(i = 1,2,...,k)$, where d_i is the sum of the elements of the i^{th} row of the matrix $(I-P)^{-1}$ and e_i is the row vector with all entries equal to zero except the i^{th} element which is equal to 1; $e_i\ (I-P)^{-1}$ is thus the i^{th} row of the matrix $(I-P)^{-1}$.

More precisely, it was shown that any maintainable structure n with total size N can be expressed as:

$$n = \sum_{i=1}^{k} \frac{r_i d_i}{\sum_{j=1}^{k} r_j d_j} \; \frac{N}{d_i} \; e_i\ (I-P)^{-1} \qquad (1.10)$$

where r is the corresponding solution to equation (1.5).

Equations (1.9) and (1.10) establish a one to one correspondence between a maintainable structure n and a recruitment policy r. In particular, it can be easily checked that policies which allow recruitment only into a single grade correspond to the vertices of the maintainable region. Consequently, the recruitment policy which allocates new recruits to different grades in equal proportions will correspond to a structure well inside the maintainable region.

1.2.3 Maintainability under Growth or Contraction

The assumption that structures at successive points of time should have the same total size could be seen to be restrictive. Generally the size of a manpower system is subject to variations either imposed by external factors or freely decided upon in accordance with future planning requirements.

Let us consider the latter case and assume that the system is expanding or contracting at a constant rate, that is $N(T+1) = (1+\alpha)\ N(T)$, where $N(T)$ is the total size of the system at time T and α a constant bigger than -1. The system will be said to be under expansion if α is positive, under contraction if α is negative or of fixed total size if α is zero.

If α is non-zero, it will be impossible to maintain a structure n, but one could instead try to maintain the relative sizes of the grades. This is what Forbes (1970) termed quasi-stationarity. More precisely, our concern will be to find a vector r or a transition matrix P, according to the type of control, such that:

$$(1+\alpha)\ n = nP + (nw' + \alpha N)r \qquad (1.11)$$

where N is the total size of the structure n.

We note that the total number of recruits takes into account the total losses and the required change in the total size.

It can be shown that if a structure n is quasi-stationary under an expansion rate α, it will be also quasi-stationary under a bigger α'; α and α' are not necessarily positive. It was also shown in Bartholomew (1973), that the set of quasi-stationary structures, in the case of control by recruitment, is the convex set with k vertices proportional respectively to:

$$e_i \; (I\;(1+\alpha) - P)^{-1} \qquad\qquad (i=1,2,\ldots,k).$$

1.2.4 Partial Maintainability

The notion of maintainability as discussed above could be generalised in what Vajda (1978) called partial-maintainability; a structure is said to be partially maintainable if the total stock numbers of a group of grades is kept constant. This notion introduces a more flexible control in the context of manpower planning, but unlike maintainability, it is not repetitive; a structure which is partially maintainable after one step is not necessarily partially maintainable after two or more steps.

1.2.5 Maintainability and Limit Behaviour of $\bar{n}(t)$

The purpose of this paragraph is to prove, for fixed size systems and under fixed control policies, that the maintainable region is the set of all possible limits to $\bar{n}(t)$ when t tends to the infinity. This result will be proved to hold for either type of control, by promotion or by recruitment, and independently of the initial structure n(0).

In addition to the simplifications of the notation it introduces, the assumption that the same control policy will be used at each point of time allows the proof to be made at the same time for both types of control; according to the nature of the latter, one could consider the recruitment vector r to be given and the transition matrix P subject to control or vice-versa.

The proof that any limit to $\bar{n}(t)$ is maintainable follows directly from the definition of a limit and from equation (1.4).

To prove that any maintainable structure m is a limit to $\bar{n}(t)$ we consider again equation (1.4) and note that at each point of time t we have:

$$\begin{aligned} \bar{n}(t) &= \bar{n}(t-1)P + \bar{n}(t-1)w'r \\ &= \bar{n}\ (t-1)\ (P + w'r) \\ &= n(0)\ (P + w'r)^t \\ &= n(0)\ Q^t \end{aligned}$$

where Q = P + w'r.

It can be shown easily that the matrix Q and hence all matrices Q^t ($t \geq 1$) are stochastic.

Moreover, with the sole assumption that all w_i $(i = 1,2,\ldots,k)$ are strictly positive, we can use results from the theory of Markov chains to show that when t tends to the infinity, Q^t will converge to a stochastic matrix Π whose rows are all identical and equal to a vector $\pi=(\pi_1,\pi_2,\ldots,\pi_k)$ and that π is the unique solution to the equation $\pi=\pi Q$.

Iosifiscu (1980) showed that such kind of convergence holds for a stochastic matrix A whenever there exists a natural number t_o such that the ergodicity coefficient of A^{t_o} is strictly positive; the ergodicity coefficient $\alpha(B)$ of a stochastic matrix B is defined as:

$$\alpha(B) = \min_{i,j} \sum_{h} \min (B_{ih},B_{jh}).$$

In our case, the ergodicity coefficient of the matrix Q is itself strictly positive and therefore we are justified in using the results stated above.

Indeed, since $\sum_{i=1}^{k} r_i=1$ and $r_i\geq 0$ $(i=1,2,\ldots,k)$, there should be at least one j_o for which $r_{j_o}>0$ and consequently, given that $w_i > 0$ $(i=1,2,\ldots,k)$, all $Q_{ij_o} = p_{ij_o} + w_i r_{j_o}$ $(i=1,2,\ldots,k)$ will be strictly positive. Thus:

$$\sum_{h=1}^{k} \min (Q_{ih},Q_{jh}) \geq \min (Q_{ij_o},Q_{jj_o}) > 0 \quad (i,j=1,2,\ldots,k)$$

and hence:

$$\alpha(Q) = \min_{i,j} \left(\sum_{h=1}^{k} \min (Q_{ih}, Q_{jh}) \right) > 0.$$

The limit behaviour of Q^t as stated above means that for large values of t, $Q_{ij}^{(t)}$ will be independent of the starting grade i. It implies also, that the stock vector $\bar{n}(t)$ will converge always to a unique structure n* regardless of the initial stock vector n(0); more precisely:

$$n^* = \lim_{t\to\infty} \bar{n}(t) = n(0)\prod = \left(\sum_{i=1}^{k} n_i(0)\pi_j,\ j=1,2,\ldots,k\right)$$
$$= N(\pi_1, \pi_2, \ldots, \pi_k).$$

It follows also that n* is the only solution to the equation n* = n*Q = n* (P+w'r). Thus, according to the type of control, we can make the limit n* to be equal to m by considering either the transition matrix P or the recruitment vector r which verifies the following equation:

$$m = mP + mw'r \tag{1.12}$$

A solution to this equation exists since m is maintainable.

1.3 ATTAINABILITY IN A DETERMINISTIC ENVIRONMENT

The latter argument was about attainability in the long run and about situations in which the goal vector is maintainable. The control policies were in addition considered to be fixed and repeated at different points of time. These assumptions are unduly restrictive and one ought to investigate the attainability in much more general terms, that is how to bring a manpower system from an initial structure n to terminal structure m in a fixed number of steps, using either control by promotion or by recruitment. Obviously, except in the case of very special situations, this cannot be achieved in exact terms but only on average. Other constraints about the intermediate trajectory of the system could be also added. They are generally assumed to be linear and often of the type:

$$\bar{n}(t)g' = \alpha(t) \qquad (t = 1,2,\ldots,T)$$

where the vector g and all numbers $\alpha(t)$ are supposed to be given. Many interpretations of this equation could be made; Bartholomew (1973) imposed constraints on the total size of the system at intermediary steps and therefore considered the vector g to be equal to vector e whose all its components are equal to one. Grinold and Stanford (1974) considered other situations as well as the case where g_i $(i = 1,2,\ldots,k)$ represents the financial costs incurred by having an individual in grade i and regarded $\bar{n}(t)g'$ to be the total manpower cost at time t and $\alpha(t)$ as the corresponding budget.

The numbers $\alpha(t)$ could be supposed to be given and independent or related to $\alpha(0)$ in one way or another; the most used relation is the following:

$$\alpha(t) = \theta^t \alpha(0) = \theta^t \bar{n}(0)g' = \theta^t ng'$$

or $$\alpha(t) = \theta\, \alpha(t-1)$$

If g, for example, is the k-dimensioned vector with all its components equal to one, $\alpha(t)=\bar{N}(t)$ and hence the above relation would mean that the system is growing at rate $(\theta-1)$. We have expansion, no growth or contraction as $\theta>1$, $\theta=1$ or $\theta<1$.

A structure m is said to be attainable in T steps or T-attainable from another structure n if there exist T transition matrices P(t) and T vectors u(t) such that:

$$\bar{n}(t) = \bar{n}(t-1)\,P(t) + u(t) \qquad t=1,2,\ldots,T \qquad (1.13a)$$

$$\bar{n}(t)g' = \theta\bar{n}(t-1)g' \qquad t=1,2,\ldots,T-1 \qquad (1.13b)$$

$$\bar{n}(T) = m \qquad (1.13c)$$

$$\sum_{j=1}^{k} p_{ij}(t) = 1-w_i \qquad t=1,2,\ldots,T \qquad (1.13d)$$

$$\bar{n}(0) = n \qquad (1.13e)$$

$$u(t) \geq 0,\ \bar{n}(t) \geq 0 \qquad t=1,2,\ldots,T \qquad (1.13f)$$

where the wastage probabilities vector w, the vector g and the constant θ are supposed to be given.

The transition matrices P(t) (t = 1,2,...,T) will be supposed to be given or subject to control depending on the nature of the latter. As to the vectors u(t) (t=1,2,...,T), we should notice first that their components $u_i(t)$ are the expected recruit numbers to grade i at time t and therefore:

$$u(t) = \bar{M}(t)\,r(t) \qquad (t=1,2,\ldots,T)$$

In addition, the expected total of recruits $\bar{M}(t)$ at time t could be shown to be determined by the need to replace losses and to provide for the increase or decrease in overall size, that is:

$$\bar{M}(t) = \bar{n}(t-1)w' + \bar{N}(t)-\bar{N}(t-1) \qquad t=1,2,\ldots,T$$

Thus, even if the control has to be exclusively by promotion, we cannot assume that the vectors u(t) are fixed; they could still be influenced by the type of promotion policy pursued through the numbers $\bar{M}(t)$; the recruitment vectors r(t) only can be assumed to be fixed. On the other hand, when the transition matrices are given and the control is by recruitment, the vectors u(t) could be influenced in many ways; if there are no constraints fixing the intermediate total sizes of the manpower system, u(t) could be influenced by acting either on the vectors r(t) only, on the numbers M(T) or on both of them. In this case, we will assume that control is about r(t) and M(t) (t = 1,2,...,T) at the same time. This, however, is not possible if the total system size trajectory is imposed; the numbers M(t) cannot be subject to control and the latter can be exerted on the vectors r(t) only.

The number of periods T need not necessarily be considered fixed. In some situations, the overall exercise of the control is precisely concerned with finding the adequate value of T according to some optimality criteria that have to be defined. Bartholomew (1973) considered the case where the objective is to find the minimal number of steps T* that are necessary to attain the structure m from a given structure n. He referred to this problem as free time control. For its solution, he advised a method based on successive trials by increasing each time the value of T* until an admissible value is reached; the first trial should be for T*=1. At the final trial some optimality criteria could be added in order to differentiate between admissible control strategies. This method gives rise to two remarks. First, the full range of prescribed trials have to be done and in the same indicated order. The fact that an T-attainable structure from n is not necessarily T'-attainable from the same structure if T and T' are different, prevents us from reducing the number of trials. This leads to the second remark concerning the association between a fixed and a free time control problem; as implied, the latter could be looked upon as a collection of successive problems of the former type. This relationship between the two types of problems allows us henceforward to concentrate only on the case where T is assumed to be given.

1.3.1 Control by Recruitment

Here, the problem is to find a set of vectors $u(1), u(2), \ldots, u(T)$ and $\bar{n}(1), \bar{n}(2), \ldots, \bar{n}(T-1)$ such that:

$$\bar{n}(t) = \bar{n}(t-1)P + u(t) \qquad t=1,2,\ldots,T-1 \qquad (1.14a)$$

$$\bar{n}(t)g' = \theta\bar{n}(t-1)g' \qquad t=1,2,\ldots,T-1 \qquad (1.14b)$$

$$m = \bar{n}(T-1)P + u(T) \qquad (1.14c)$$

$$u(t) \geq 0 \qquad t=1,2,\ldots,T \qquad (1.14d)$$

$$\bar{n}(t) \geq 0 \qquad t=1,2,\ldots,T-1 \qquad (1.14e)$$

$$\bar{n}(0) = n \qquad (1.14f)$$

All transition matrices are supposed to be given and not functions of time. The stationarity assumption about the transition probabilities is not necessary but is made in order to simplify the no_ation; the models that will be discussed and the methods for their solution will be shown not to depend on that assumption.

The foregoing problem can obviously be solved by linear programming techniques. But as noted by Bartholomew (1973) the enumeration of the number of variables and constraints will show that, among all possible solutions, those provided by the simplex method would be too extreme to be acceptable in practice. He showed that, if all components of the initial stock vector are non-zero, a basic solution will have at most T+k-1 non-zero elements to be distributed among the k*T components of the vectors $u(t)$ $(t=1,2,\ldots,T)$.

This will leave almost every vector with only one non-zero component. Such a recruitment strategy will be difficult to implement in practice especially if the grade to which the recruits will be allocated keeps changing. This extreme behaviour of a basic solution could be lessened by the introduction of new constraints on the variables $u_i(t)$ to make sure that the solution will be within acceptable limits. These latter, however, have to be set with extreme care in order to avoid producing infeasibility.

Another way to select a solution which does not require sharp and repeated changes in the recruitment policies from one step to another, is to consider as the objective the minimisation of a function of the type:

$$D = \sum_{t=2}^{T} \sum_{i=1}^{k} \left| u_i(t) - u_i(t-1) \right|$$

linear programming will still be used since such a function could be written as:

$$D = \sum_{t=2}^{T} \sum_{i=1}^{k} (v_i(t) + w_i(t))$$

with

$$u_i(t) - u_i(t-1) = v_i(t) - w_i(t)$$

and $v_i(t), w_i(t) \geq 0$ for $i=1,2,\ldots,k$ and $t=2,3,\ldots,T$.

A solution to the problem defined by conditions (1.14) can also be worked out by a combined use of Dantzig-Wolfe decomposition algorithm and dynamic programming; Grinold and Stanford (1974) suggested a way of decomposition which leads to subproblems without terminal conditions on the structure $\bar{n}(t)$ and showed that these subproblems can be solved effectively by the use of dynamic programming. They showed that it is possible to define more general terminal conditions than (1.14c) or to optimise in addition a linear function related to economic or social considerations during the whole period of T steps. Incidentally, this was the main concern of that paper with different assumptions concerning the finiteness of T and the nature of the transition probabilities.

As to the foregoing problem, there was only a mention that its solution can be worked out in the same way as explained above; in their account they assumed always that an initial solution is already known. We give below a brief description of how the method above can be adapted to our case.

Let $A_T(n)$ denote the set of all T-attainable structures from n. The attainability problem is then concerned with finding a structure y such that:

$$(1) \qquad \begin{cases} y = m \\ \text{and} \\ y \in A_T(n) \end{cases}$$

To find a solution to problem (1), if it exists, we propose to solve another problem constructed from (1), that is:

$$(2) \qquad \begin{cases} \text{Min } L = (a+b)e' \\ \text{Subject to:} \\ y + a - b = m \\ y \in A_T(n) \\ a,\ b \in B \end{cases}$$

where e is a k-dimensioned vector with all its components equal to one and B a set of vectors defined by:

$$B = \{x/0 \le x_i \le m_i,\ i=1,2,\ldots,k\}.$$

If the minimal value of L is strictly positive, we will conclude that the problem (1) has no solution, otherwise a solution to (1) exists and the corresponding $\{x(t),\ u(t)\}_{t=1}^{T}$ will be also provided. The upper bounds on the components of vectors a and b are not necessary but they can be shown not to affect the feasibility of problem (1); they were added in order to avoid complications that would arise in the decomposition algorithm. Let us define the master problem as:

$$
(3)\quad
\begin{cases}
\text{Minimise } L = \sum_{j=1}^{h'} \lambda'_j \,(a^j + b^j)e' \\
\text{Subject to:} \\
\sum_{j=1}^{h} \lambda_j\, y^j + \sum_{j=1}^{h'} \lambda'_j\,(a^j - b^j) = m & \text{(E1)} \\
\sum_{j=1}^{h} \lambda_j = 1 & \text{(E2)} \\
\sum_{j=1}^{h'} \lambda'_j = 1 & \text{(E3)} \\
\lambda_j \geq 0, \quad j=1,2,\ldots,h \\
\lambda'_j \geq 0, \quad j=1,2,\ldots,h'
\end{cases}
$$

where y^1, y^2, ..., y^h are points in the attainable region $A_T(n)$ and a^j, b^j ($j=1,2,\ldots,h'$) are points in the set B, all supposed to be known. h and h' are integers greater than or equal to one. Initially they are set to one but they could change in subsequent iterations. The initial starting vectors y^1, a^1 and b^1 could be obtained by selecting first any point y^1 from $A_T(n)$ and define after a_i^1 and b_i^1 ($i = 1,2,\ldots,k$) by:

$$a_i^1 = m_i - y_i^1 \text{ and } b_i^1 = 0 \qquad \text{if } m_i \geq y_i^1$$

or $$a_i^1 = 0 \qquad \text{and } b_i^1 = y_i^1 - m_i \qquad \text{if } m_i < y_i^1$$

let $(\hat{\lambda}, \hat{\lambda}')$ denote an optimal solution of problem (3) and $\hat{u}$, $\hat{v}$ and $\hat{v}'$ denote the associated dual variables corresponding respectively to equation (E1), (E2) and (E3); $\hat{u}$ is a k-dimensioned column vector, $\hat{v}$ and $\hat{v}'$ are scalars. We need then to solve two subproblems:

$$(4)\quad \begin{cases} \text{Max } D_4 = y\hat{u} + \hat{v} \\ \text{Subject to: } y \in A_T(n) \end{cases}$$

$$(5)\quad \begin{cases} \text{Max } D_5 = (a-b)\hat{u} - (a+b)e' + \hat{v}' \\ \text{Subject to: } a, b \in B \end{cases}$$

Problem (4) can be solved directly by means of dynamic programming as shown in Grinold and Stanford (1974). The solution of problem (5) is straightforward and can be shown to be as follows:

$$\begin{aligned} a_i &= m_i,\ b_i = 0 && \text{if} && \hat{u}_i > 1 \\ a_i &= 0,\ b_i = 0 && \text{if} && -1 \leq \hat{u}_i \leq 1 \\ a_i &= 0,\ b_i = m_i && \text{if} && \hat{u}_i < -1 \quad (i=1,2,\ldots,k). \end{aligned}$$

Let D_4^* denote the maximal value of D_4 and y^* denote an optimal solution of problem (4). Let D_5^* denote the maximal value of D_5 and (a^*, b^*) an optimal solution of problem (5). If both D_4^* and D_5^* are equal to zero, then:

$$(\hat{y} = \sum_{j=1}^{h} \hat{\lambda}_j\, y^j,\ \hat{a} = \sum_{j=1}^{h'} \hat{\lambda}'_j\, a^j,\ \hat{b} = \sum_{j=1}^{h'} \hat{\lambda}'_j b^j)$$

is an optimal solution of problem (3). If on the other hand D_4^* is positive and bigger than D_5^*, we will put $y^{h+1} = y^*$, increase the value of h by one and try to solve the new problem (3). If D_5^* is positive and bigger than or equal to D_4^*, we will put $a^{h'+1}=a*$ and $b^{h'+1}=b*$, increase h' by one and try to solve the new problem (3). This process can be shown to take a finite number of iterations and then will allow us to know if the optimal value of L is positive or zero. In the latter case, a solution to problem (1) exists and from this one ought to pursue the optimisation of a linear function in order to choose eventually an alternative and better recruitment strategy.

This method has the merit of reducing the number of constraints in problem (1) from T*(k+1) equations to (k+2) equations in problem (3) and has a special appeal because of the great ease which it offers in solving the generated sub-problems. The master problem (3) itself does not need to be solved each time from scratch; the revised simplex method provides an efficient way of updating successive solutions of this problem. Nevertheless, the amount of work associated with the process of decomposition itself makes any net advantage on the linear programming doubtful especially if the number of constraints in problem (1) is not very large.

Whatever method of solution is adopted, the attainability problem could have a unique solution or could even be infeasible precisely because of reasons inherent in the model assumptions and constraints such as equation (1.14c). In such a case, the issue of selecting a less extreme recruitment strategy is irrelevant and one ought to settle for less restrictive conditions and, rather try instead to direct the manpower system towards a structure m. Exact attainability would cease to be the fundamental objective.

Bartholomew (1973) considered the attainability problem as defined by conditions (1.14) but dropped altogether the requirement that $\bar{n}(T)$ should be equal to m. To control the evolution of the system, he regarded as objective the minimisation of a distance $D(\bar{n}(T),m)$ between the terminal structure $\bar{n}(T)$ and the goal structure m. This new problem is always feasible and allows for more flexibility. It can be used to solve the original attainability problem with the additional advantage of providing, in the case of infeasibility of the latter problem, a solution with the nearest distance between $\bar{n}(T)$ and m; it suffices, for example, to put:

$$D(\bar{n}(T),m) = \sum_{i=1}^{k} (\bar{n}_i(T) - m_i)^2.$$

A distance between two structures does not have to be of the latter type and the method of solution will depend heavily on its nature. More generally, it can be defined as:

$$D(\bar{n}(T),m) = \sum_{i=1}^{k} w_i \mid \bar{n}_i(T) - m_i \mid^a \qquad a > 0$$

where w_i $(i = 1,2,\ldots,k)$ is a set of non-negative weights chosen to reflect the importance of holding deviations for grade i as small as possible. A large value of a will tend to prevent a relatively large deviation at a single or group of grades but would make the solving process very difficult if not impracticable. It is then no surprise to find, in the context of the foregoing analysis, that much of he published work is concerned only with cases where a is not bigger than 2.

Bartholomew (1973) considered both cases, a=1 and a=2. He advocated the use of linear programming in the case of the former and quadratic programming in the case of the latter. Moreover, in the special case where T is assumed to be equal to one and where the total size of the organisation is considered to be fixed, he advised a very simple way in obtaining the solution for both cases a=1 and a=2. He argued hat models with T being assumed equal to one are of the utmost importance in practice in spite of the apparently excessive restriction; putting T=1 means that, at each step, we will consider the finishing structure as a new start and try to move the system as far as possible towards the goal m. He stressed that such a strategy is fully justified if we remember the stochastic behaviour of different flows. He pointed out that even in a deterministic environment, this strategy would achieve a good control of the system; numerical comparisons between strategies that concentrate only on the step ahead and others which minimise the distance between $\bar{n}(T)$ and m, for a $T > 1$, were made to enforce his point.

Mehlmann (1980) suggested different means of solution based on dynamic programming techniques. His model considered a slightly different type of quadratic function in the definition of $D(\bar{n}(T),m)$. It allows also the use of any type of control: by promotion, by recruitment or by both of them. Unfortunately, the way the control variables are defined might lead to an optimal solution with either the recruitment vectors r(t) or the transition matrices P(t) being infeasible. In addition, the method itself requires a considerable amount of calculations including inversion of many matrices of kxk dimension, and makes additional assumptions about the regularity of the latter matrices.

But, as mentioned earlier, a more efficient use of dynamic programming can be found in Grinold and Stanford (1974) if the issue is not about reducing a distance between $\bar{n}(T)$ and m but rather about minimising a linear cost function related to the whole period of T steps.

1.3.2 Control by Promotion

In comparison with the recruitment control, it appears that much less work has been devoted to the promotion control. An explanation could find its roots in an early discussion about the nature of control by promotion, its undesirable features and the practical difficulties that would face its implementation. Other reasons related to the modelling process itself could also be invoked; modelling could be very difficult given the considerable number of variables under control or might turn out to be a duplication of models already discussed in the case of control by recruitment.

The latter case is well illustrated by Bartholomew's models; it was shown when writing conditions (1.14) as:

$$\bar{n}_j(t)=\sum_{i=1}^{k}\bar{n}_{ij}(t-1)+\bar{r}_j\ \{\sum_{i=1}^{k}\bar{n}_i(t-1)w_i+\sum_{i=1}^{k}\bar{n}_i(t)-\sum_{i=1}^{k}\bar{n}_i(t-1)\}$$

$$(j = 1,2,\ldots,k;\ t = 1,2,\ldots,T-1) \qquad (1.15a)$$

$$\sum_{i=1}^{k}\bar{n}_i(t)g_i = \theta \sum_{i=1}^{k}\bar{n}_i(t-1)g_i \quad (t = 1,2,\ldots,T-1) \qquad (1.15b)$$

$$m = \sum_{i=1}^{k}\bar{n}_{ij}(T-1) + r_j\ \{\sum_{i=1}^{k}\bar{n}_i(T-1)w_i + \sum_{i=1}^{k}m_i - \sum_{i=1}^{k}\bar{n}_i(T-1)\}$$

$$(j = 1,2,\ldots,k) \qquad (1.15c)$$

$$\bar{n}_{ij}(t) \geq 0 \quad (i,j=1,2,\ldots,k;\ t=0,1,\ldots,T-1) \qquad (1.15d)$$

$$\bar{n}(t) \geq 0 \qquad (t = 1,2,\ldots,T-1) \qquad (1.15e)$$

$$\bar{n}(0) = n \qquad (1.15f)$$

that the attainability problem can still be solved by means of mathematical programming. It was stressed that all eventual solutions resulting from this method would have extreme characteristics which would severely limit its practical value. Therefore, the previous discussion concerning model assumptions, and especially terminal conditions, remains of fundamental relevance.

1.4 ATTAINABILITY IN A PARTIALLY STOCHASTIC ENVIRONMENT

As was pointed out earlier, the analysis in the previous sections is concerned only with the average stock numbers and can be used without any modification if all p_{ij} and w_i are to be considered as fixed proportions and not as probabilities. It does not however provide sufficient information about the actual path that the system would follow since the deterministic assumption holds rarely and variation in the flow numbers have generally to be assumed. This led Bartholomew (1975, 1977) to initiate an investigation concerning the behaviour of a manpower system in a stochastic environment. He tackled the evaluation of the probability of attaining a structure m from another structure n and compared different control strategies. These considerations will form the subject of the next chapters and a detailed discussion of the difficulties involved will be entered into. Here, we shall give a brief description of a recent model which is easier to manipulate than Bartholomew's and yet presents fewer limitations than the deterministic model by allowing for variation in some flows.

Davies (1982) considered a manpower system of fixed size N. He assumed that, of the $n_i(T)$ individuals in grade i at time T (i = 1,2,...,k; T = 0,1,2,...) a fixed proportion p_{ij} are promoted to grade j, i<j, at step T+1; no demotions are allowed. The numbers of grade i who are not promoted at step T+1 are assumed either to stay in grade i with probability p_i or to leave the system with probability $1-p_i$. All proportions p_{ij} and probabilities are considered to be given and only the recruitment vector r is assumed to be subject to control. Finally, recruitment is assumed to take place after the wastage numbers are known.

Under these conditions, a structure n(T+1) is said to be attainable in one step from another structure n(T) if, and only if,

$$n_i(T+1) \geq \sum_{j=1}^{i-1} n_j(T)p_{ji} + n_{ii}(T+1) \quad (i = 1,2,...,k)$$

where $n_{ii}(T+1)$ is the number of members of grade i at time T who will stay at the same grade at time T+1. Thus n(T+1) may or may not be attainable from n(T) depending on the values taken by the random variable $n_{ii}(T+1)$ (i=1,2,...,k). The probability P(n(T+1)/n(T)) of attaining n(T+1) in one step from n(T) is thus of importance and can be expressed as:

$$P(n(T+1)/n(T)) = \sum_{i=1}^{k} P(n_{ii}(T+1) \leq n_i(T+1) - \sum_{j=1}^{i-1} n_j(T)p_{ji}/n(T))$$

where $n_{ii}(T+1)$ are independent Binomial variables with parameters $(n_i(T)p_{ii},\ p_i)$ and $p_{ii} = 1 - \sum_{j=i+1}^{k} p_{ij}$.

Davies (1982) showed that, for an initial structure n, there is one and only one structure n^* attainable from n in one step with probability one; $n^*=nP$ and $P=\{p_{ij}\}$. However, it was pointed out that for a given goal structure n^*, there is no guarantee that all of the entries in n^*P^{-1} will be non-negative. It was shown also that the one-step probabilities decrease on lines emanating from n^*; that is, for any structure $n'(\neq n^*)$ attainable with non-zero probability in one step from n, we have:

$$o < P(n'/n) < P(\ (1-\lambda)n' + \lambda n^*/n) < 1 \qquad o < \lambda < 1$$

As to the probability $P(n(T)/n(0))$ of attaining a structure $n(T)$ from a structure $n(0)$ in T steps, its value will depend on the recruitment vectors $r(1)$, $r(2),\ldots,r(T)$ decided upon at the points of time $t=1,2,\ldots,T$. More generally, the distribution of the stock numbers at different points of time will depend largely on the adopted recruitment strategy. The purpose of this latter, could be then motivated by maximising or minimising a function, not necessarily linear, of the stock numbers according to an objective decided upon by the management.

Davies (1983) aimed at the maximisation of $P(n(T)/n(0))$ and called the maximal value of the latter as the "best T-step probability". From a previous result, the best T-step probability of attaining nP^T from n is equal to one by taking the system through the same path $(n,\ nP,\ nP^2,\ \ldots,\ nP^T)$.

But in general, no method was advised to find a path from n(0) to n(T) with the highest probability of attainability. Nevertheless, an indication about how the best T-step probabilities vary with the goal structure were obtained by numerical calculations. It was suggested that, within the regions of structures attainable with non-zero probability, the best T-step probabilities decrease on lines emanating from the structure nP^T.

Although the model discussed in this section is only partially stochastic, it gives a sounding of difficulties that would arise with stochastic flows and introduces a sample of new questions that have to be addressed. In the following chapters, we will consider the model where all the flows are stochastic, discuss different ways of evaluating probabilities of attainability and give a comparison of different recruitment strategies.

CHAPTER II

PROBABILITY OF MAINTAINING OR ATTAINING A STRUCTURE IN ONE STEP

In the context of stochastic control of a graded manpower system, the probability of maintaining or attaining a structure in one step is of fundamental importance.

Unfortunately, its evaluation has presented many difficulties and has been considered a major obstacle; Bartholomew (1977) tried to overcome the problem by using a Multinormal approximation and Davies (1982) was inclined to consider partially stochastic systems. In this chapter we will describe the computational difficulties that occur in the evaluation of the above probability. In the case of systems with upper-diagonal transition matrices, a powerful and fast approach for such evaluation will be presented. The Normal approximation and the difficulties involved in evaluating the multivariable Normal integral will be also discussed.

2.1 NOTATIONS AND ASSUMPTIONS

As in an early model, suppose the system consists of N individuals of whom $n_i(T)$ are in grade i at time T, $i = 1,2,...,k$. Time is assumed discrete and the probability that an individual moves from i to j between one time point and the next is p_{ij} $(i,j = 1,2,\dots,k)$. An individual may also leave the system from grade i with probability w_i. F(T) will denote the kxk flow matrix with elements $n_{ij}(T)$, these being the numbers moving from i to j in [T-1, T[, T≥1. The stock numbers at time T are given by the relation $n(T) = f(T) + R(T)$, where $f(T) = (\sum_{i=1}^{k} n_{ij}(T),\ j=1,2,\dots,k)$ and $R_i(T)$ is the number of new entrants to grade i at time T $(i = 1,2,\dots k)$. Decision upon recruitment is supposed to be made after flows and losses have taken place. Since control is by recruitment, a structure m will be attainable in one step from n(T) if, and only if, $f(T) \leq m$. Therefore the vector f(T) is of crucial importance and also its distribution which is needed to evaluate all probabilities of maintaining or attaining a structure.

In the following sections we will be concerned with a single period [T-1, T[, T≥1. Since there is no confusion which T is being considered, we will omit the suffix in all our notations.

The stock vector n at the beginning of the period is assumed to be known; the vector to be considered at the end of the same period will be denoted by m.

2.2 EXACT METHODS

We will present two methods to evaluate the exact value of the probability of attaining the structure m. Whilst the first method deals with the general case, the second concentrates on a particular and important class of systems in which the transition matrix is upper-diagonal.

2.2.1 General Case

The probability $P(0 \le f \le m/n)$ of attaining the structure m from a structure n in one step could be evaluated as follows:

$$P(0 \le f \le m/n) = \sum P(n_{1j} = v_j(1),\ n_{2j} = v_j(2), \ldots,\ n_{kj} = v_j(k),\quad j = 1,2,\ldots,k) \tag{2.1}$$

where the summation is over all possible vectors v(1),v(2), ...,v(k) such that:

$$\sum_{i=1}^{k} v(i) \le m \tag{2.2a}$$

$$\sum_{j=1}^{k} v_j(i) \le n_i \qquad i = 1,2,\ldots,k \tag{2.2b}$$

$$v_j(i) \ge 0 \qquad i,j = 1,2,\ldots,k \tag{2.2c}$$

The vectors $(n_{ij},\ j=1,2,\ldots,k)$ $(i=1,2,\ldots,k)$ are independent Multinomials with parameters $(n_i;p_{i1},p_{i2},\ldots,p_{ik})$ and their distribution can be evaluated directly. Thus it would be useful to express $P(0 \le f \le m/n)$ as:

$$P(0 \leq f \leq m/n) = \sum \prod_{i=1}^{k} P(n_{ij} = v_j(i),\ j = 1,2,\ldots,k) \quad (2.3)$$

However, the process of enumeration of all possible sets of k vectors v(1), v(2), ..., v(k) which comply with conditions (2.2), and the evaluation of $\prod_{i=1}^{k} P(n_{ij} = v_j(i),\ j=1,2,\ldots,k)$ for each set, would be too long; the enumeration in itself would be very complex, and some probabilities would have to be evaluated many times unless some sophisticated, and inevitably time consuming, storage procedures were introduced. Moreover, a direct evaluation of $P(n_{ij} = v_j(i),\ j=1,2,\ldots,k)$ for a particular i and vector v(i) would be very lengthy given the large number of such evaluations that would be needed. In order to avoid the problems discussed above, we propose to proceed as follows:

Step One:

For each grade i, start by considering $v(i)=(0,0,\ldots,0)$ and assume that $P(n_{ij}=0,\ j=1,2,\ldots,k)$ is equal to a constant C. Then, enumerate all possible vectors v(i) which comply with conditions (2.2b) and (2.2c). In this enumeration, a new vector v(i) would be obtained from an old vector v'(i) by adding one to an element $v'_{j_o}(i)$. This would allow the use of a recursive relation of the Multinomial distribution to evaluate the probability $P(n_{ij}=v_j(i),\ j=1,2,\ldots,k)$; more precisely we will have:

$$P(n_{ij}=v_j(i), j=1,2,..,k) = \frac{p_{ij_o}}{w_i} \frac{(n_i - \sum_{j=1}^{k} v_j'(i))}{(v_{j_o}'(i) + 1)} \times P(n_{ij}=v_j'(i), j=1,2,..,k) \quad (2.4)$$

The constant C is deduced from the fact that all probabilities should add up to one.

Step Two:

For the remaining steps and for all grades i, ignore all set of vectors v(i), enumerated in the latter step, which do not satisfy the condition (2.2a).

Step Three:

Let $i = 2$ and $s(1) = v(1)$

Step Four:

Let $P(\sum_{h=1}^{i} n_{hj} = s_j(i),\ j = 1,2,...,k) = 0$ for all integer vectors s(i), such that $0 \leq s(i) \leq m$.

Consider all combinations of vectors v(i) and s(i-1) and calculate $s(i) = s(i-1) + v(i)$. If for particular vectors $s^\circ(i-1)$ and $v^\circ(i)$, $s^\circ(i) = s^\circ(i-1) + v^\circ(i) \leq m$, add

$$P(\sum_{h=1}^{i-1} n_{hj} = s_j^\circ(i-1), j=1,2,...,k) \times P(n_{ij}=v_j^\circ(i), j=1,2,...,k)$$

to $P(\sum_{h=1}^{i} n_{hj} = s_j^\circ(i),\ j=1,2,...,k)$. This will lead to a simultaneous evaluation of all probabilities

$$P(\sum_{h=1}^{i} n_{hj} = s_j(i),\ j = 1,2,\ldots,k),\ s(i) \leq m.$$

Step Five:

If i is less than k, put i = i+1 and go to step four; otherwise, end the procedure by putting:

$$P(0 \leq f \leq m/n) = \sum_{s(k)\leq m} P(\sum_{h=1}^{k} n_{hj} = s_j(k),\ j=1,2,\ldots,k) \quad (2.5)$$

This procedure would involve too many operations for any practical use, especially if the number of grades is bigger than three or if the total size of the structures n and m are high. To have an idea about the kind of combinations that are involved, we evaluated the total number of combinations of vectors v(i) and s(i-1), (i = 2,3,...,k) that have to be considered in step four of the above procedure; the values of such numbers, obtained by enumeration, are given in table 1.

Table 1
Number of Combinations in Step Four

Number of grades	Structure to be maintained	Total Size	Number of Combinations
3	(3,3,3)	9	1 480
	(5,5,5)	15	13 272
	(8,8,8)	24	127 710
	(7,7,10)	24	145 272
	(10,10,10)	30	3 453 312
	(20,20,20)	60	16 810 332
4	(3,3,3,3)	12	14 910
	(6,6,6,6)	24	798 210
	(15,15,15,15)	60	389 491 488
5	(3,3,3,3,3)	15	124 208
	(5,5,5,5,5)	25	4 114 908
	(12,12,12,12,12)	60	4 690 850 528

However, if the transition matrix has some entries equal to zero, many $P(\sum_{h=1}^{i-1} n_{hj} = s_j(i-1),\ j = 1,2,\ldots,k)$ would also be equal to zero. Therefore the number of combinations to be considered in Step Four could be reduced, without affecting the result, by considering only s(i-1) for which the latter probability is non-zero.

In the case of k=3, a Fortran routine "General" based on the above procedure and incorporating the latest modification, has been developed to calculate the exact value of $P(0 \leq f \leq m/n)$.

This routine has been used to evaluate $P(0 \leq f \leq n/n)$ for three grades structures n in table 1. For each of these structures, the following transition matrices were considered:

$$P_1 = \begin{pmatrix} 0.7 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.7 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix} \qquad (6a)$$

$$P_3' = \begin{pmatrix} 0.8 & 0.1 & \\ & 0.8 & 0.1 \\ & & 0.8 \end{pmatrix}, \qquad P_3'' = \begin{pmatrix} 0.5 & 0.3 & \\ & 0.5 & 0.3 \\ & & 0.5 \end{pmatrix} \qquad (6b)$$

The execution time required on the CDC 7600 to carry out the evaluation of $P(0 \leq f \leq n/n)$, for each example, is given in table 2.

Table 2
Execution Times with routine "General"

Structure n	Total size	Execution time, in CP seconds			
		Matrix P_1	Matrix P_2	Matrix P_3'	Matrix P_3''
(3,3,3)	9	0.01	0.01	<0.01	<0.01
(5,5,5)	15	0.05	0.05	0.04	0.04
(8,8,8)	24	0.48	0.48	0.37	0.37
(7,7,10)	24	0.53	0.53	0.41	0.41
(10,10,10)	30	1.49	1.49	1.15	1.15
(20,20,20)	60	61.55	61.54	46.63	46.62

As would be expected, the execution times do not depend on the magnitude of the transition matrix entries but, rather, on how many and which entries are zero. However, the execution times increase sharply as the stock numbers increase. Also, given the strong relation between the execution times in table 2 and the number of combinations in table 1, we expect that the execution times will increase sharply if we increase the number of grades. Thus, it would not be suitable to use the proposed approach in the case of systems with large total size or number of grades.

2.2.2 Case of Upper-diagonal Transition Matrices

In this section, we will suppose that:

$$p_{ij} = 0 \qquad i,j = 1,2,\ldots,k \quad \text{and} \quad j \neq i,i+1$$

This new assumption reduces considerably the previous difficulties and allows for the introduction of an iterative method to evaluate $P(0 \leq f \leq m/n)$;

let us denote by $P(0;\ m;\ h;\ j/n)$ the probability that "$0 \leq n_{11} \leq m_1,\ 0 \leq n_{12} + n_{22} \leq m_2,$

$\ldots,\ 0 \leq n_{h-1,h} + n_{h,h} \leq m_h,\ n_{h,h+1} = j/n(0)=n$",

where n_{ij} are the numbers moving from grade i to j, $m = (m_1,m_2,\ldots,m_k)$ is a fixed and known vector and $h \leq k-1$.

Thus,

$$P(0;\ m;\ h;\ j\ /n)$$

$$= \sum_{i \in I_j} P(0 \le n_{11} \le m_1,\ 0 \le n_{12} + n_{22} \le m_2,\ \ldots,\ 0 \le n_{h-1,h} + i \le m_h,\ n_{h,h} = i, n_{h,h+1} = j\ /n)$$

$$= \sum_{i \in I_j} P(0 \le n_{11} < m_1,\ 0 \le n_{12} + n_{22} \le m_2,\ \ldots,\ 0 \le n_{h-1,h} \le m_h - i\ /n)\ P(n_{h,h} = i, n_{h,h+1} = j\ /n)$$

$$= \sum_{i \in I_j} \sum_{t \in T_i} P(0 \le n_{11} \le m_1,\ 0 \le n_{12} + n_{22} \le m_2,\ \ldots,\ n_{h-1,h} = t\ /n)\ P(n_{h,h} = i, n_{h,h+1} = j\ /n)$$

$$= \sum_{i \in I_j} \sum_{t \in T_i} P(0;\ m;\ h-1;\ t/n)\ P(n_{h,h} = i,\ n_{h,h+1} = j\ /n) \qquad (2.7)$$

Where $I_j = \{\ i\ /\ 0 \le i \le \text{Min}\ (n_h - j,\ m_h)\ \}$

and $T_i = \{\ t\ /\ 0 \le t \le \text{Min}\ (m_h - i,\ n_{h-1})\ \}$.

We shall refer to the equation (2.7) as the iterative equation, since it implies an iterative way to evaluate P(0; m; k-1; j/n). This latter probability is of particular interest since it can be related to the probability of attaining m from n in one step as follows:

$$P(0 \leq f \leq m\ /n)$$

$$= P(0 \leq n_{11} \leq m_1,\ 0 \leq n_{12} + n_{22} < m_2,\ \ldots,\ 0 \leq n_{k-1,k} + n_{k,k} \leq m_k\ /n)$$

$$= \sum_{j \in J} P(0 \leq n_{11} \leq m_1,\ 0 \leq n_{12} + n_{22} \leq m_2,\ \ldots,\ 0 \leq n_{k-1,k} + n_{k,k} \leq m_k,\ n_{k-1,k} = j\ /n)$$

$$= \sum_{j \in J} P(0 \leq n_{11} \leq m_1,\ 0 \leq n_{12} + n_{22} \leq m_2,\ \ldots,\ 0 \leq n_{k,k} \leq m_k - j,\ n_{k-1,k} = j\ /n)$$

$$= \sum_{j \in J} P(0;\ m;\ k-1;\ j\ /n)\ P(0 \leq n_{k,k} \leq m_k - j\ /n) \qquad (2.8)$$

where $J = \{\ j\ /\ 0 \leq j \leq \text{Min}\ (n_{k-1},\ m_k)\ \}$

Equations (2.7) and (2.8) could be used to evaluate $P(0 \leq f \leq m\ /n)$ as follows:

Step One:

Put $h = 1$.

$P(0;\ m;\ h;\ j\ /n) = P(0 \leq n_{11} \leq m_1,\ \ n_{12} = j\ /n)$ can be deduced from the Trinomial distribution with parameters $(n_1;\ p_{11},\ p_{12})$. If $m_1 \geq n_1$, $P(0;\ m;\ 1;\ j\ /n)$ can be deduced from the Binomial distribution with parameters $(n_1;\ p_{12})$.

Step Two:

h = h+1. If h = k go to Step Four.

Step Three:

Evaluate $P(0;\ m;\ h;\ j\ /n)$ using equation (2.7) and the Trinomial distribution with parameters $(n_h;\ p_{h,h},\ p_{h,h+1})$.

Go to Step Two.

Step Four:

Evaluate $P(0 \leq f \leq m\ /n)$ using equation (2.8) and the Binomial distribution with parameters $(n_k;\ p_{k,k})$.

This algorithm could be modified to evaluate probabilities of the kind $P(a \leq f \leq b\ /n)$, where a and b are non-negative integer vectors. It would be sufficient to replace $P(0;\ m;\ h;\ j\ /n)$ by $P(a;\ b;\ h;\ j\ /n)$ and equations (2.7), (2.8) respectively by (2.9), (2.10) where:

$$\begin{aligned} & P(a;\ b;\ h;\ j\ /n) \\ &= P(a_1 \leq n_{11} \leq b_1,\ a_2 \leq n_{12} + n_{22} \leq b_2, \\ &\qquad \ldots,\ a_h \leq n_{h-1,h} + n_{h,h} \leq b_h,\ n_{h,h+1} = j\ /n) \\ &= \sum_{i \in I_j'} \sum_{t \in T_i'} P(a;\ b;\ h-1;\ t\ /n)\ P(n_{h,h}=i,\ n_{h,h+1}=j\ /n) \end{aligned} \tag{2.9}$$

and $P(a \leq f \leq b \ /n)$

$$= \sum_{j \in J'} P(a;\ b;\ k-1;\ j\ /n)\ P(a_k - j \leq n_{k,k} \leq b_k - j\ /n) \qquad (2.10)$$

$$I_j' = \{\ i\ /\ \text{Max}\ (0,\ a_h - n_{h-1}) \leq i \leq \text{Min}\ (n_h - j,\ b_h)\ \}$$

$$T_i' = \{\ t\ /\ \text{Max}\ (0,\ a_h - i) \leq t \leq \text{Min}\ (n_{h-1},\ b_h - i)\ \}$$

$$J' = \{\ j\ /\ \text{Max}\ (0,\ a_k - n_k) \leq j \leq \text{Min}\ (n_{k-1},\ b_k)\ \}$$

The probability $P(0 \leq f \leq n\ /n)$ is a special case of $P(a \leq f \leq b\ /n)$ where $a = 0$ and $b = m$. The probability of making the first move from n to m, before recruitment has taken place is $P(f = m\ /n)$, which is a special case of $P(a \leq f \leq b\ /n)$ where $a = b = m$.

A Fortran routine "MAINT", based on the latter algorithm, has been developed to evaluate $P(a \leq f \leq b\ /n)$. But before we discuss the performance of this routine, we shall present first the methods by which it evaluates Binomial and Trinomial distributions.

2.2.2.1 Evaluation of the Trinomial distribution

We started by calculating the probability density function at the mode (k_1, k_2) of the Trinomial distribution; we used a procedure in Finucan (1964) to find the mode. Then using the following recursive relations:

1° $$P(i+1, j) = \frac{n-i-j}{(i+1)} \times \frac{p_1}{p_3} P(i,j) \qquad \begin{matrix} 0 \le i \le n-1 \\ 0 \le j \le n-i-1 \end{matrix}$$

2° $$P(i-1, j) = \frac{i}{n-i-j+1} \times \frac{p_3}{p_1} P(i,j) \qquad \begin{matrix} 1 \le i \le n \\ 0 \le j \le n-i \end{matrix}$$

3° $$P(i, j+1) = \frac{n-i-j}{(j+1)} \times \frac{p_2}{p_3} P(i,j) \qquad \begin{matrix} 0 \le i \le n \\ 0 \le j \le n-i-1 \end{matrix}$$

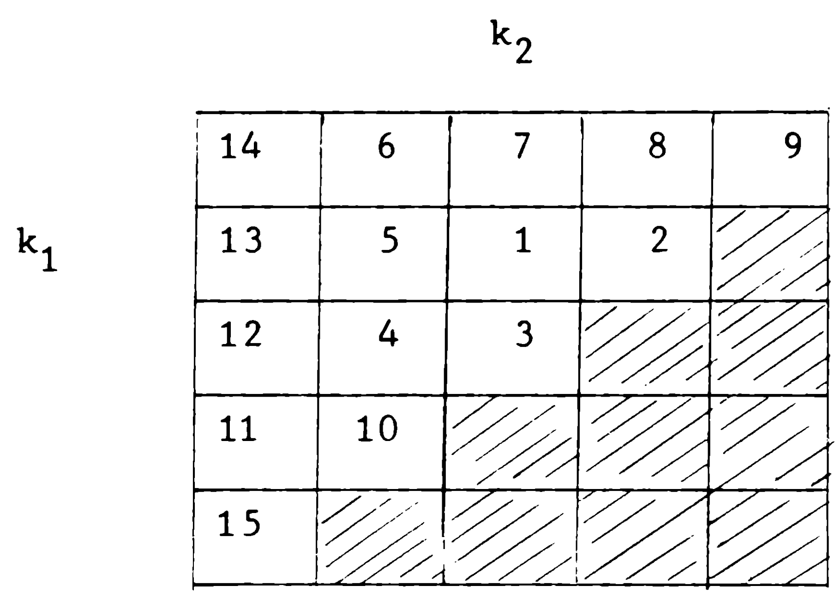
4° $$P(i, j-1) = \frac{j}{n-i-j+1} \times \frac{p_3}{p_2} P(i,j) \qquad \begin{matrix} 0 \le i \le n \\ 1 \le j \le n-i \end{matrix}$$

We calculated the probabilities P(u,v) at the remaining points, where $p_3 = 1 - p_1 - p_2$ and

$$P(u, v) = \frac{n!}{u!v!\,(n-u-v)!} p_1^u \, p_2^v \, p_3^{n-u-v}$$

However, there would be (n+1)(n+2)/2 possible (u,v) and their complete enumeration could be very lengthy and unnecessary, since most P(u,v) are nearly zero. One way to cut down significantly the amount of calculations is to choose an enumeration process which considers the more likely outcomes first and stops when the probability of occurrence of all the remaining (u,v) is less than a fixed precision ε; P(u,v) for all remaining (u,v) are set to zero.

The enumeration process of (u,v) in the routine MAINT is as follows:



where the cells are numbered according to the order in which probabilities of their occurrence were calculated. The cell number 1 corresponds to a mode of the Trinomial distribution.

2.2.2.2 Evaluation of the Binomial distribution

A similar approach to the latter method was adopted; the procedure starts by calculating P(m) for m ≈ np and then using alternately the following recursive equations:

1° $$P(j+1) = \frac{p}{(1-p)} \times \frac{(n-j)}{(j+1)} P(j) \qquad m \leq j \leq n-1$$

2° $$P(j-1) = \frac{1-p}{p} \times \frac{j}{(n-j+1)} P(j) \qquad 1 \leq j \leq m$$

to evaluate the other P(w)'s, where $P(w) = \binom{n}{w} p^w (1-p)^{n-w}$. The procedure stops when the probability of occurence of all remaining w is less than a fixed precision ε; P(w) for all remaining w are set to zero.

2.2.2.3 Precision in tne routine MAINT

The process of enumeration of (u,v) in the case of the Trinomial distribution and w in the case of Binomial distribution, especially if ε is not equal to zero, could lead to an under-evaluation of $P(a \leq f \leq b/n)$. We propose to find an upper bound α to this error;

let us denote by:

$$S(h) = \sum_{x \in X_h} P(a;\ b;\ h;\ x/n) \qquad (2.11)$$

where $h \geq 1$ and $X_h = \left\{ j/0 \leq j \leq \text{Min}\ (n_h,\ b_{h+1}) \right\}$, and by $\Delta(Z)$ the under-evaluation error in the calculation of an expression Z.

From equation (2.10) we have:

$$\Delta\ [P(a \leq f \leq b/n)]$$

$$\leq \sum_{j \in J} \Delta[P\ (a;b;k-1;j/n)]\ P(a_k - j \leq n_{k,k} \leq b_k - j/n)$$

$$+ \sum_{j \in J} P(a;b;k-1;j/n)\,\Delta[P(a_k - j \leq n_{k,k} \leq b_k - j/n)]$$

$$\leq \sum_{j \in J} \Delta[P(a;b;k-1;j/n)] + \varepsilon \sum_{j \in J} P(a;b;k-1;j/n)$$

$$\leq \quad \Delta\ [S(k-1)] + \varepsilon \qquad (2.12)$$

We will suppose that $\Delta[S(h)] \leq \alpha_h$ and attempt to find a relation between α_h and α_{h-1} for $h \geq 2$. From equations (2.9) and (2.11) we have:

$$S(h) = \sum_{j \epsilon X_h} \sum_{i \epsilon I'_j} \sum_{t \epsilon T'_i} P(a;b;h-1;t\ /n)\ P(n_{h,h}=i, n_{h,h+1}=j\ /n)$$

This would lead to:

$$\Delta\ [S(h)]$$

$$\leq \sum_{j \epsilon X_h} \sum_{i \epsilon I'_j} \Delta[\sum_{t \epsilon T'_i} P(a;b;h-1;t\ /n)]\ P(n_{h,h}=i,\ n_{h,h+1} = j\ /n)$$

$$+ \sum_{j \epsilon X_h} \sum_{i \epsilon I'_j} \sum_{t \epsilon T'_i} P(a;b;h-1;t\ /n)\ \Delta[P(n_{h,h}=i, n_{h,h+1} = j\ /n)]$$

$$\leq \sum_{j \epsilon X_h} \sum_{i \epsilon I'_j} \Delta[S(h-1)]\ P(n_{h,h} = i, n_{h,h+1} = j\ /n)$$

$$+ \sum_{j \epsilon X_h} \sum_{i \epsilon I'_j} \Delta[P(n_{h,h} = i, n_{h,h+1} = j\ /n)]$$

$$\leq \alpha_{h-1} + \epsilon$$

Thus $\alpha_h \leq \alpha_{h-1} + \epsilon,\ h \geq 2$ (2.13)

Moreover since,

$$S(1) = \sum_{j \epsilon X_1} P(a;b;1;j/n) = \sum_{j \epsilon X_1} \sum_{i \epsilon I''_j} P(n_{11}=i,\ n_{12}=j\ /n)$$

Where $I''_j = \{i/a_1 \leq i \leq \text{Min}\ (n_1 - j,\ b_1)\}$, then

$$\Delta\ [S(1)] = \sum_{j \epsilon X_1} \sum_{i \epsilon X''_j} \Delta\ [P(n_{11} = i, n_{12} = j\ /n)] \leq \epsilon$$

Thus $\alpha_1 \leq \epsilon$ (2.14)

From relations (2.12), (2.13) and 2.14) we have

$\alpha = \Delta\ [\ P(a \leq f \leq b/n)] \leq k\ \epsilon$ (2.15)

2.2.2.4 Performance of the routine MAINT

In comparison to the routine GENERAL, the routine MAINT achieves a remarkable reduction in the execution time; this latter routine has been used to evaluate $P(0 \leq f \leq n/n)$ for all structures in table 2 but only for transition matrices P_3' and P_3''. The error α in MAINT has been set to zero and the computations have been made on the CDC 7600. Under such conditions, the execution times needed for all above examples were at most equal to 0.01 CP seconds.

Table 3 shows the effect of increasing the value of α on the execution times needed to evaluate $P(0 \leq f \leq n/n)$, using the routine MAINT, in the case of upper diagonal transition matrices.

- Matrix P_k' where:

$$p_{i,i}' = 0.8, \; p_{i,i+1}' = 0.1, \; w_i' = 0.1 \quad 1 \leq i \leq k-1$$

$$p_{k,k}' = 0.8, \; w_k' = 0.2 \tag{2.16a}$$

- Matrix P_k'' where:

$$p_{i,i}'' = 0.5, \; p_{i,i+1}'' = 0.3, \; w_i'' = 0.2 \quad 1 \leq i \leq k-1$$

$$p_{k,k}'' = 0.5, \; w_k'' = 0.5 \tag{2.16b}$$

P_3' and P_3'' have already been defined by relation (2.6b). The computations have been made on the CDC 7600.

Table 3
Accuracy and Execution time with routine MAINT

Number of grades	Structure n	Execution times, in CP seconds					
		α = 0		α = 10^{-4}		α = 10^{-2}	
		P'_k	P''_k	P'_k	P''_k	P'_k	P''_k
3	(100,100,100)	0.34	0.42	0.02	0.10	0.02	0.05
	(200,200,200)	2.56	3.20	0.10	0.44	0.06	0.24
	(500,500,500)	41.60	51.21	0.72	2.81	0.37	1.47
	(1000,1000,1000)	441.96	486.16	2.90	12.33	1.45	6.44
	(5000,5000,5000)	>1200	>1200	87.43	369.82	44.11	193.18
4	(100,100,100,100)	0.68	0.83	0.05	0.22	0.03	0.12
	(200,200,200,200)	5.12	6.42	0.20	0.91	0.12	0.51
	(500,500,500,500)	83.58	102.46	1.52	5.77	0.80	3.12
	(1000,1000,1000,1000)	883.60	970.57	6.00	25.47	3.05	13.47
	(5000,5000,5000,5000)	>1200	>1200	180.46	760.98	93.43	406.61
5	(100,100,100,100,100)	1.02	1.25	0.07	0.33	0.05	0.18
	(200,200,200,200,200)	7.68	9.63	0.31	1.41	0.18	0.79
	(500,500,500,500,500)	125.68	153.50	2.31	8.82	1.24	4.81
	(1000,1000,1000,1000,1000)	>1200	>1200	9.19	38.69	4.82	20.97
	(5000,5000,5000,5000,5000)	>1200	>1200	277.42	1165.39	144.61	631.70
30	(1000,1000... 1000,1000)	>1200	>1200	101.33	427.60	59.71	259.54

Table 3 shows that a substantial reduction in the execution times can be achieved by fixing α, even at a low level, so long as this is not at zero. We noticed that the execution time depends on the magnitude of the transition matrix entries. This dependency increases when α, and thus ϵ, becomes non-zero. This could be explained by the way the Trinomial distributions are evaluated; the enumeration in the case of Trinomial variable with parameters $(h;\ p_1,\ p_2)$ and for a fixed ϵ, is shorter if p_1, p_2 and $(1-p_1-p_2)$ are very different, rather than if they are similar; In the former case, fewer outcomes (u,v) would have P(u, v) non-negligible and all of these outcomes would be concentrated around the mode $(k_1,\ k_2)$. Table 3 also shows that the execution time increase linearly with the number of grades and that this latter is not a real problem in evaluating probabilities of the type $P(0 \leq f \leq n/n)$. Potential problems may arise only in the case of systems with very high stock numbers. However, table 3 shows that the execution times are reasonably low and yet the stock numbers are too high for most practical situations.

Unfortunately, the routine MAINT deals only with a particular type of transition matrices and we still have to find another method to cope with the general case; the Normal approximation seems to be the most suitable.

2.3 NORMAL APPROXIMATION

Bartholomew (1977) showed, by means of simulation and in the case of some numerical examples, that the Multinormal distribution does approximate reasonably well to the distribution of $f = (\sum_{i=1}^{k} n_{ij},\ j = 1,2\ldots k)$

It was suggested that

$$P\ (a \leq f \leq b/n) \approx P\ (a_j - 0.5 \leq Y_j \leq b_j + 0.5,\ j = 1,2,\ldots k) \tag{2.17}$$

where $Y = (Y_1, Y_2, \ldots . Y_k)$ has the multivariate Normal distribution with mean vector μ and variances-covariances matrix A with:

$$\mu_j = \sum_{h=1}^{k} n_h\ P_{hj} \qquad j = 1,\ 2,\ \ldots k$$

and

$$A_{ii} = \sum_{h=1}^{k} n_h\ P_{hi}(1-P_{hi}) \qquad i = 1,\ 2,\ \ldots k$$

$$A_{ij} = \sum_{h=1}^{k} n_h\ P_{hj}P_{hj} \qquad i,j = 1,\ 2,\ \ldots k;\ i \neq j$$

However, the evaluation of the multivariate Normal integral is not easy in itself. Milton (1972) developed a Fortran program using a multidimensional iterated Simpson's quadrature adapted to the Multinormal integral. Unfortunately, from our own experience, this program failed very often in evaluating integrals of five or more variables. In the case of fewer variables, its failure was less frequent but still occasional. Bohrer and Schervish (1981), Schervish (1984) discuss these shortcomings in some detail. In 1981 the NAG library contained new Fortran routines to evaluate multi-dimensional definite integrals for general functions. We found routine D01FCF to be suitable for our needs. This routine is based on automatic adaptive procedures which involve subdivision of the region of integration into subregions, concentrating the divisions in those parts of the region where the integrand is worst behaved, and apply numerical integration rules separately to each subregion.

The routine D01FCF was used successfully to evaluate the right hand side of relation (2.17), for values of k up to the value eight. A potential relative error β, equal to 1%, was accepted in such evaluations and all computations were made on the CDC7600. The evaluations took a fraction of a second for $k \leq 5$, a few seconds for $6 \leq k \leq 7$ and just over 60 seconds for $k = 8$. These times increased sharply when a higher accuracy was required; they reached 120 CP seconds when we considered the case of k equal to five and set β equal to 0.1%. However, since a relative error of 1%

is already a very small error, and given the purpose of an approximation method, we see no need to require higher accuracy than $\beta = 0.01$, especially if it is costly in time. As to the quality of the Multinormal approximation, table 4 confirms the positive conclusions in Bartholomew (1977); it gives, in the case of several numerical examples, the exact and approximate values of the probability of maintaining a structure. The approximate values were obtained by the use of routine D01FCF with $\beta = 0.01$. The exact values were obtained by the use of the routine MAINT in the case of upper diagonal transition matrices P'_k and P''_k and the routine General in the case of matrices P_1 and P_2. These matrices are defined by relations (2.6) and (2.16).

This table shows that whilst the quality of the approximation improves with the size of stock numbers, it does not seem to be affected by the number of grades. In most cases the relative error in the approximation is small, except in the case of matrices P'_k where it reaches 13%. This could be explained by the combination of big differences among the elements of these matrices and low stock numbers.

In the case of upper diagonal transition matrices, if n_1 is equal to zero, the variance-covariance matrices would be singular. We would then modify the equation (2.17) as follows:

Table 4

Comparison of the multivariate Normal approximation (NA) to $P(0 \leq f \leq n/n)$ with the exact value (EV)

		P_1			P_2		
k	n	EV	NA	$\frac{NA-EV}{EV} \times 100$	EV	NA	$\frac{NA-EV}{EV} \times 100$
3	(5,5,5)	0.364	0.362	-1	0.479	0.482	1
	(10,10,10)	0.389	0.385	-1	0.577	0.581	1
	(20,20,20)	0.478	0.476	-0.4	0.729	0.745	2

		P'_k			P''_k		
k	n	EV	NA	$\frac{NA-EV}{EV} \times 100$	EV	NA	$\frac{NA-EV}{EV} \times 100$
3	(5,5,5)	0.667	0.606	-9	0.686	0.681	-1
	(10,10,10)	0.678	0.649	-4	0.759	0.771	2
	(20,20,20)	0.747	0.727	-3	0.860	0.870	1
4	(5,5,5,5)	0.533	0.477	-11	0.559	0.556	-1
	(10,10,10,10)	0.548	0.522	-5	0.656	0.657	0.2
	(20,20,20,20)	0.639	0.631	-1	0.797	0.800	0.4
5	(5,5,5,5,5)	0.425	0.376	-12	0.455	0.454	-0.2
	(10,10,10,10,10)	0.443	0.420	-5	0.566	0.570	1
	(20,20,20,20,20)	0.549	0.537	-2	0.737	0.741	1
6	(5,5,5,5,5,5)	0.339	0.296	-13	0.370	0.370	0
	(10,10,10,10,10)	0.358	0.337	-6	0.489	0.496	1
	(20,20,20,20,20)	0.468	0.458	-2	0.682	0.687	1

$$P\ (a \leq f \leq b/n) \approx P\ (a_j - 0.5 \leq Y_j \leq b_j + 0.5,\ j = 2,\ldots k)$$

where $Y'=(Y_2,\ldots,Y_k)$ has a multivariate Normal distribution.

Finally, in the case of upper diagonal transition matrices, there is no gain in computing time by using the Multinormal approximation unless the stock numbers are very high (exceeding a thousand). The routine MAINT would be quicker in most cases.

2.4 MAINTAINABLE AND ATTAINABLE REGIONS IN A STOCHASTIC ENVIRONMENT

In our discussion of the deterministic theory, we referred to results which Bartholomew (1973) obtained in the case of control by recruitment concerning maintainable regions. We saw that these latter are convex and that the position of a structure within these regions gives a clear indication concerning the recruitment policy that should be adopted for maintaining that structure. Davies (1973) made a detailed investigation concerning the attainable and maintainable regions. He considered, in particular, the set of all structures that are attainable in T steps from a given structure, $T \geq 1$. He found that all regions for $T=1,2....$ are convex but pointed out that in general no inclusion relationship exists between them except in the case of the initial structure being maintainable in one or many steps. In the above analysis, all structures can be divided into two distinct groups: maintainable (or attainable) structures and the others. In a stochastic environment, such classification is not possible; the maintainability (or attainability) of a structure may or may not hold depending on the actual values taken by the random flows. Davies (1982) faced with the same problem in the case of a partially stochastic model, considered a classification which distinguishes between structures depending on whether their probabilities of attainability are zero or not. He proved that the attainable structures with non-zero probabilities form a convex set and that the attainable region as defined in the deterministic theory is included in the latter set. In the case of a completely stochastic environment, however, all structures have a non-zero probability of maintainability (or attainability) and hence the latter classification is inappropriate. Nevertheless, this difficulty can be easily removed by considering regions where all structures have a probability of maintainability (or attainability) greater than a given constant α, $0 < \alpha < 1$. We will refer to the newly defined regions as α-maintainable (or α-attainable) regions. As in Davies (1983) we will try to investigate the properties of these latter, especially their nature and their relationship with the corresponding regions in the deterministic theory. No theoretical results are available for such investigation but much insight can be gained from a recourse to graphical representation. This however will not permit us to look into systems with a number of grades greater than three. In addition, so as to reduce the computing time in the evaluation of all required probabilities, it is desirable to consider only systems with upper-diagonal transition matrices. Finally, we will focus our discussion on the case of T being fixed at one. Greater values of T, as will be argued in the next chapter, would make the evaluation of the needed probabilities very lengthy and would require the examination of an extensive range of recruitment strategies.

Two graphical representations are possible. the first is in three dimensional space where each structure is represented in X-Y plane by its barycentric co-ordinates and

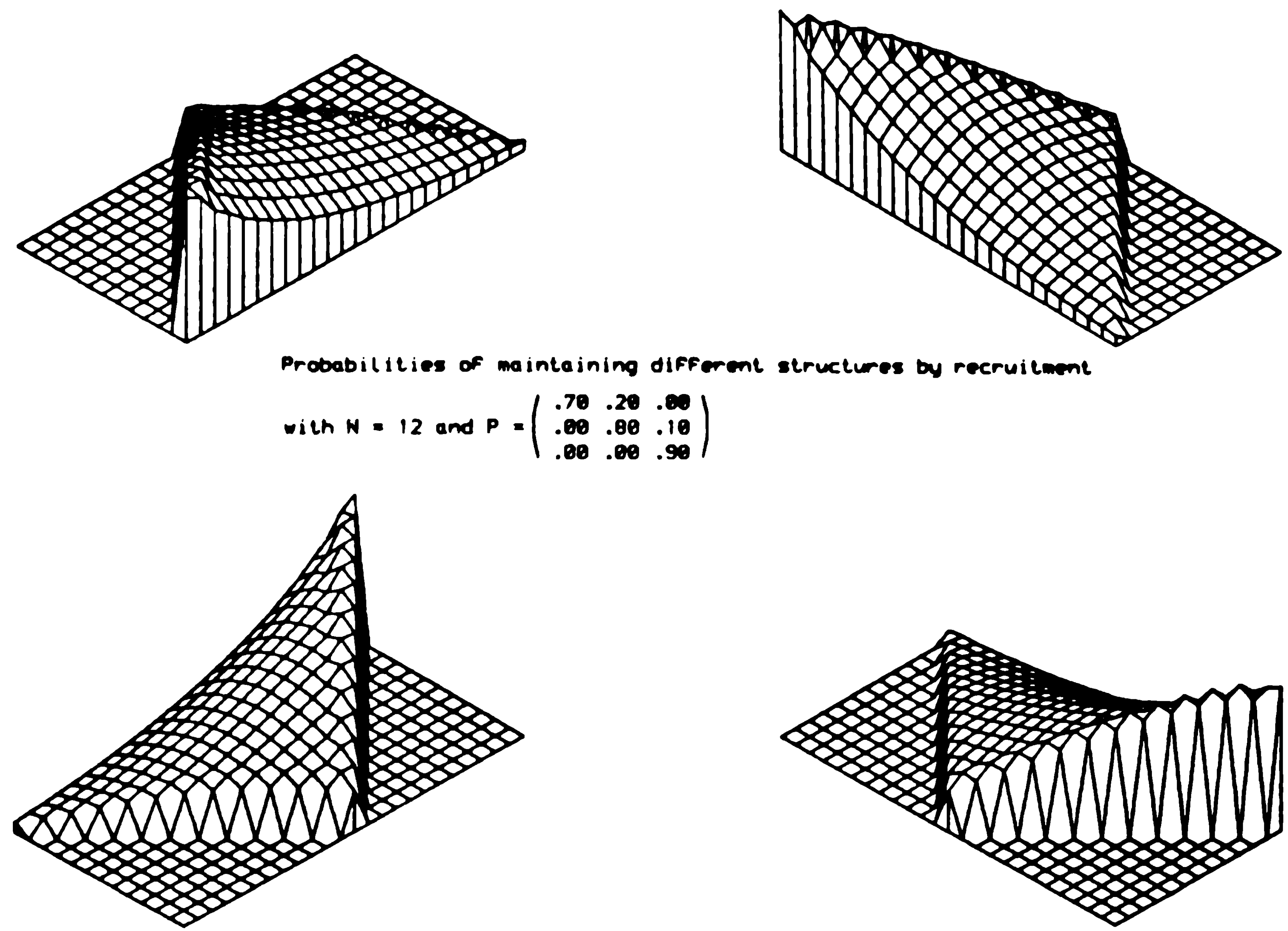

Probabilities of maintaining different structures by recruitment
with N = 12 and P = (.70 .20 .00 / .00 .80 .10 / .00 .00 .90)

Figure 1.1

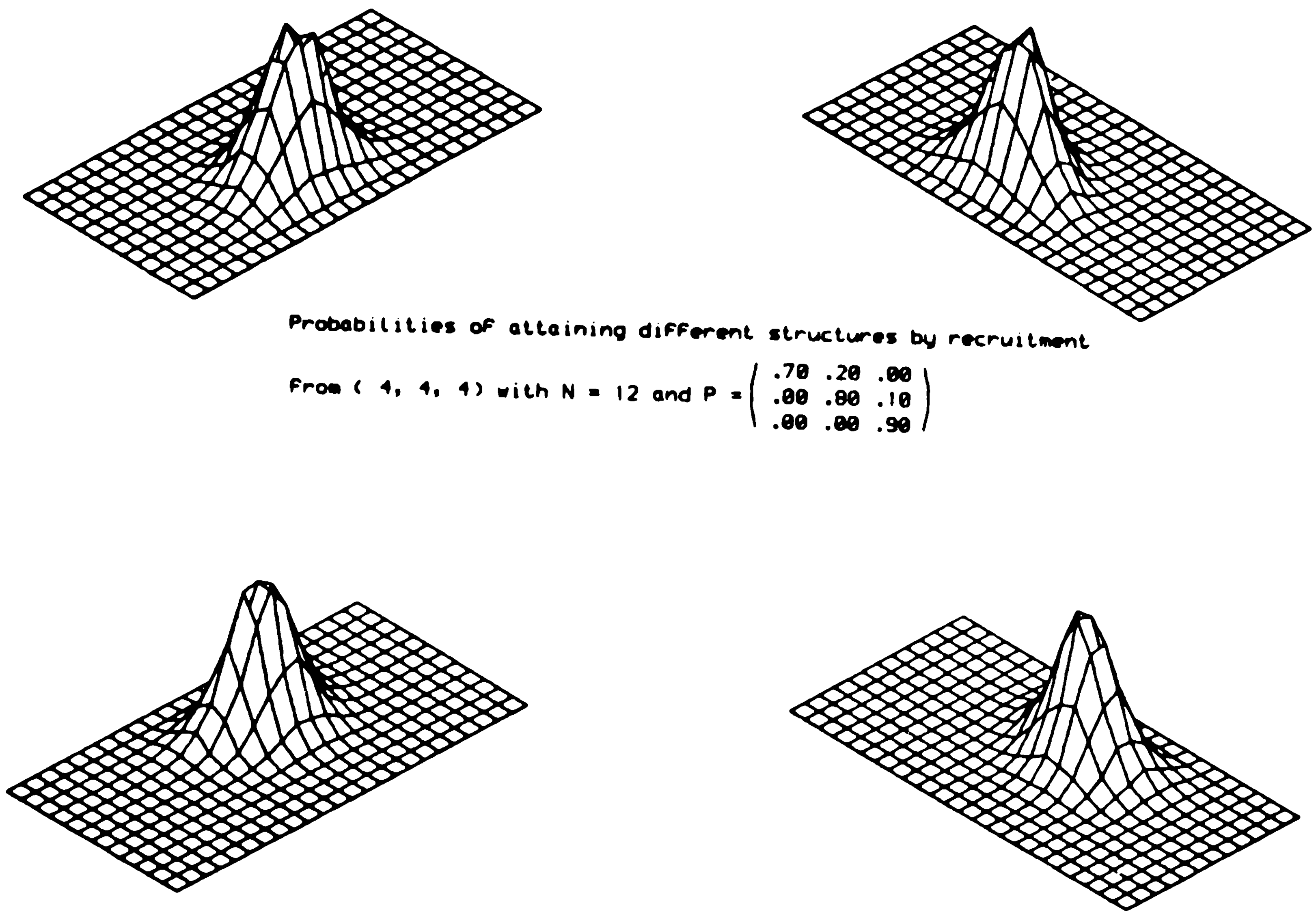
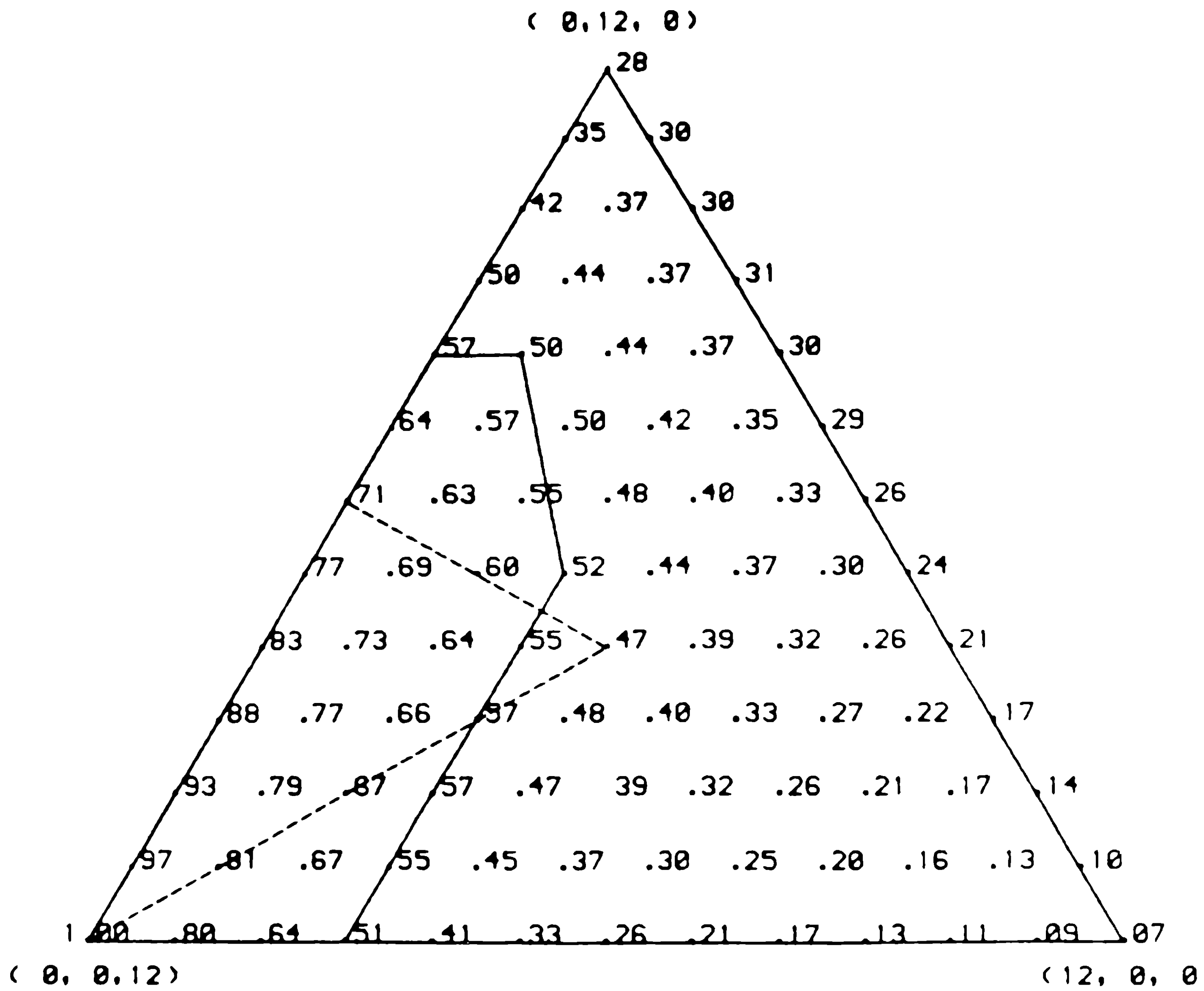

Probabilities of attaining different structures by recruitment
From (4, 4, 4) with N = 12 and P = (.70 .20 .00 / .00 .80 .10 / .00 .00 .90)

Figure 1.2

the probabilities are regarded as the heights of the surface at these points. Gino-Surf, a library of subroutines for displaying three-dimensional surfaces was used for producing an isometric projection of this surface. Figures 1.1 and 1.2 show a typical projection as produced by Gino-Surf. In each figure, the four pictures correspond to the same projection but viewed from different angles. Figure 1.1 shows that the structures with higher probabilities of maintainability are top heavy and suggests that α-maintainable regions are convex for any level α. The latter property holds also in the case of α-attainable regions as can be deduced from figure 1.2. This representation, however, does not allow for an accurate description of the probabilities nor does it deal directly with the α-maintainable (or α-attainable) regions; its main function is primarily pictorial. In the second type of plotting, each structure is also identified by its barycentric co-ordinates but, in addition, its probability of maintainability or attainability is printed at these co-ordinates.

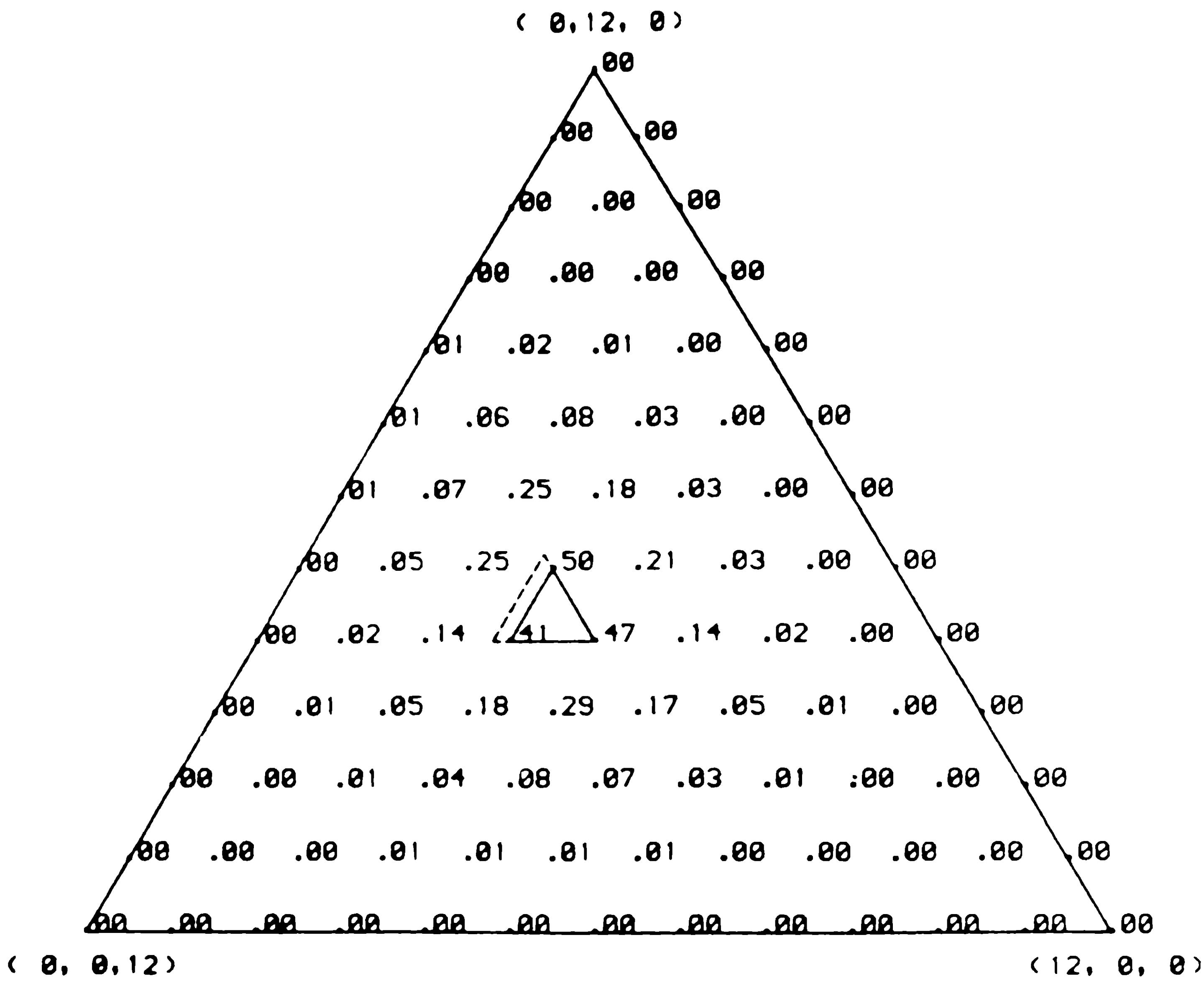
To identify the borders of an α-maintainable or α-attainable region, we used a Fortan program based on the assumption that these regions are convex. Beginning with a list of all eligible structures, the program finds their convex hull and finally draws the edges of the region by joining the determined vertices with straight lines. Should the convexity assumptions happen to be false, we would have within the boundary of the region a structure with probability less than or equal to α. But from a wide range of



.50 - Maintainable region with N = 12 and

$$P = \begin{pmatrix} .70 & .20 & .00 \\ .00 & .80 & .10 \\ .00 & .00 & .90 \end{pmatrix}$$

Figure 2.1



.30 - Attainable region from (4, 4, 4) with

$$P = \begin{pmatrix} .70 & .20 & .00 \\ .00 & .80 & .10 \\ .00 & .00 & .90 \end{pmatrix}$$

Figure 2.2

examples which were considered, no single case was found to contradict the above assumption. Figures 2.1 and 2.2 show a typical graphical representation as produced by the program discussed above. The dashed lines are the edges of the maintainable (or attainable) region as defined in the deterministic theory.

It was observed, in the case of maintability, that the maintainable regions are generally included in α-maintainable regions of α close to 0.5. In the case of attainability, however, no similar conclusion could be drawn given the strong dependency of the probabilities on the initial stock vectors. Nevertheless, in both cases it is clear that the probabilities within the maintainable and attainable regions are higher than those outside these regions.

CHAPTER III

PROBABILITY OF MAINTAINING OR ATTAINING A STRUCTURE IN MANY STEPS

In this chapter, we will be concerned with the evaluation of $P(0 \leq f(h) \leq m/n)$, the probability of attaining a structure m from a structure n in h steps, $h \geq 2$, where n is the stock vector at time T=0 and $f(h)=(\sum_{i=1}^{k} n_{ij}(h), j=1,2,...,k)$, $n_{ij}(h)$ being the flow numbers from i to j in [h-1, h[.

In the case of h being equal to one, we knew the distribution of the vectors $(n_{ij}(h), \quad j = 1,2,...,k)$, $(i = 1,2,... k)$. However, we were unable to describe analytically the distribution of their sum. For $h \geq 2$, we do not know even the distribution of the vectors themselves; this distribution depends on the stock vector at T = h-1, which in its turn depends on all flows in the previous periods. They all depend on recruitment vectors decided upon at the end of preceding periods.

As in the case of h = 1, we will examine in turn two methods in evaluating $P(0 \leq f(h) \leq m)$:

- Exact approach
- Multinormal approximation

But firstly, we will review different types of recruitment policies and strategies.

3.1 RECRUITMENT STRATEGIES

A recruitment policy is a set of rules to obtain the recruitment vector at a given time T. A recruitment strategy over a period is a sequence of such policies. The recruitment vector at each time is assumed to be non-negative, otherwise we will be dealing with a highly undesirable situation in which redundancies have to take place.

Instead of introducing hypothetical recruitment strategies, we will concentrate on a few types which are the most relevant to our work and those which will be used most extensively in the control context.

3.1.1 Deterministic and Fixed Strategies

Firstly, generalising from the notion of a T*-maintainable path in Davies (1975), let us suppose that there is a sequence of non-negative vectors m(T) and R(T) such that:

$$m(T) = m(T-1)\,P + R(T), \quad (1 \leq T \leq T^*) \qquad (3.1)$$

where $m(0)=n$ and $m(T^*)$ is a fixed goal vector that we wish to attain in a fixed number of steps T^*. (T^* is not necessarily equal to h). Then $\{ n, m(1), m(2), \ldots, m(T^*) \}$ will be said to form a T^*-attainable path from n to $m(T^*)$. The recruitment strategy defined by (3.1) will be called a deterministic strategy. This definition supposes that $m(T^*)$ could be attained from n in T^* steps and a T^*-attainable path from n to $m(T^*)$ is known.

Under a deterministic strategy the vectors $m(T)$, $(T = 1,2,\ldots,T^*)$ will be successively attained but only on average. In particular, if n belongs to the maintainable region and $m(T^*) = n$, $\{ n, n, \ldots, n \}$ will be a T^*-maintainable path for n and the corresponding recruitment strategy will be defined by:

$$R(T) = n(I-P), \quad (T = 1,2,\ldots,T^*) \qquad (3.2)$$

The recruitment strategy defined by (3.2), and denoted by F_1 in Bartholomew (1977), will keep the total size of the system constant only on average. However, if the losses are known before the recruitment vectors have to be chosen, we can improve F_1 by making recruitment equal losses and allocating the new recruits in the same proportions as in (3.2). This strategy, denoted by F_2 in

Bartholomew (1977), will ensure at each time that the total size of the system is kept constant and the structure n is maintained on average.

Both strategies F_1 and F_2 allocate recruits in fixed proportions, and therefore are described as "fixed".

3.1.2 Adaptive Strategies

An adaptive strategy should try at each time T, to readjust the evolution of the system and use R(T) which minimises the distance between n(T) and a fixed vector m(T); the vectors m(T) could belong to a T*-attainable path from n to m(T*), or they could be all equal to a fixed vector m.

If we consider a squared distance function, we would choose R(T) which minimises:

$$D(m(T),\ n(T)) = \sum_{j=1}^{k} (\ m_j(T) - f_j(T) - R_j(T)\ )^2,$$

which is given by:

$$R_j(T) = \text{Max}\ (\ 0,\ m_j(T) - f_j(T)\ ),\ (j=1,2,\ldots,k).$$

This strategy assumes that f(T) is known at time T, when a decision upon R(T) has to be made. Bartholomew (1977) introduced this strategy in the case of m(T)=n and denoted it by S_2^o. But since S_2^o tends to increase the total size of the system, a policy S_2^1 which aims to minimise the same distance D(n(T), m(T)) as in S_2^o, but with an additional constraint:

$$\sum_{j=1}^{k} R_j(T) = \sum_{j=1}^{k} n_j - \sum_{j=1}^{k} f_j(T)$$

was introduced. It ensures that the total size of the system remains fixed.

At time T, the recruitment vector R(T) is then a solution to an integer quadratic programming problem. An optimal solution should try to make the differences $(n_j - f_j(T) - R_j(T))$ $(j = 1,2,\ldots,k)$, as close to each other as possible. The following algorithm achieves this aim:

1° Put $a_j = n_j - f_j(T)$ and $S = \sum_{j=1}^{k} a_j$

2° Set $R_j(T) = 0$ $\quad j = 1,2,\ldots,k$

3° If S is equal to zero, stop.

4° Find the maximum a_j, then put:

$$R_j(T) = R_j(T) + 1$$
$$a_j = a_j - 1$$
$$S = S - 1$$

Go to step 3.

This algorithm can be improved in the case of a large S; its proof and the improvement are fully discussed in Appendix A.

3.1.3 Rounding R(T) to an Integer Vector

Some definitions of R(T) in the above examples, may lead to a vector which is not integer. One way of overcoming this difficulty is to round up or down the elements of R(T) with probabilities chosen to ensure that the expectation of the new random vector R*(T) is equal to R(T). These transformations could be done in many ways and the results obtained could be dependent on them.

In the case of k=3, we were able to find a general way to obtain an R* vector. Two cases are considered:

(i) Case 1: $S = \sum_{j=1}^{3} R_j(T)$ is integer;

we define:

$$[R(T)] = ([R_j(T)] , j = 1,2,3)$$

and $\alpha = R(T) - [R_j(T)]$

Since $\alpha_j < 1$ (j=1,2,3), we have $\sum_{j=1}^{3} \alpha_j < 3$, and since S is integer, $\sum_{j=1}^{3} \alpha_j = S - \sum_{j=1}^{3} [R_j(T)]$ is integer. Therefore $\sum_{j=1}^{3} \alpha_j$ can take only three possible values 0, 1 or 2.

- If $\sum_{j=1}^{3} \alpha_j = 0$:

we put R*(T) = R(T) with probability equal to one.

- If $\sum_{j=1}^{3} \alpha_j = 1$:

we put:

$$R^*(T) = [R(T)] + \begin{cases} (1,0,0) \text{ with probability } \alpha_1 \\ (0,1,0) \text{ with probability } \alpha_2 \\ (0,0,1) \text{ with probability } \alpha_3 \end{cases}$$

We can check that $E(R^*(T)) = R(T)$ and $\sum_{j=1}^{3} R^*_j(T) = S$

- If $\sum_{j=1}^{3} \alpha_j = 2$:

we put:

$$R^*(T) = [R(T)] + \begin{cases} (0,1,1) \text{ with probability } (1-\alpha_1) \\ (1,0,1) \text{ with probability } (1-\alpha_2) \\ (1,1,0) \text{ with probability } (1-\alpha_3) \end{cases}$$

We can check that $E(R^*(T)) = R(T)$ and $\sum_{j=1}^{3} R^*_j(T) = S$.

<u>(ii) Case 2</u>: S is not integer;

we define: $S' = [S] + 1$

$S'' = [S]$

and put $p = S - [S]$

Thus $S = pS' + (1-p) S''$

we define: $R'(T) = \frac{S'}{S} R(T)$

$$R''(T) = \frac{S''}{S} R(T)$$

R'(T) and R"(T) could be dealt with as in the previous case. Thus:

$$R^*(T) = \begin{cases} R'^*(T) \text{ with probability } p \\ R''^*(T) \text{ with probability } (1-p) \end{cases}$$

we can check that $E(R^*(T)) = R(T)$ and $\sum_{j=1}^{3} R^*_j(T) = S$.

These transformations have the advantage of being general, simple and the most natural way of rounding a real vector R(T) to an integer vector R*(T) such that:

$$E(R^*(T)) = R(T)$$

$$\text{and} \quad \sum_{j=1}^{3} R^*_j(T) = \sum_{j=1}^{3} R_j(T)$$

Unfortunately, for k bigger than three, no similar methods could be found; to illustrate the difficulties involved, let us consider the following example when k=4. Let us define $\alpha = R(T) - [R(T)]$ and suppose that $\sum_{j=1}^{4} \alpha_j = 2$. Thus the vector R*(T) should have the following distribution:

$$R^*(T) = [R(T)] + \begin{cases} (1,1,0,0) \text{ with probability } \beta_1 \\ (1,0,1,0) \text{ with probability } \beta_2 \\ (1,0,0,1) \text{ with probability } \beta_3 \\ (0,1,1,0) \text{ with probability } \beta_4 \\ (0,1,0,1) \text{ with probability } \beta_5 \\ (0,0,1,1) \text{ with probability } \beta_6 \end{cases}$$

Such that:

$$\beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_3 \geq 0,\ \beta_4 \geq 0,\ \beta_5 \geq 0,\ \beta_6 \geq 0$$

$$\begin{aligned} \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 &= 1 \\ \beta_1 + \beta_2 + \beta_3 &= \alpha_1 \\ \beta_1 + \beta_4 + \beta_5 &= \alpha_2 \\ \beta_2 + \beta_4 + \beta_6 &= \alpha_3 \\ \beta_3 + \beta_5 + \beta_6 &= \alpha_4 \end{aligned}$$

Obviously, a solution for such a problem could be found by means of linear programming techniques, but this process of rounding R(T) in itself will make the calculations very lengthy.

3.2 EXACT METHOD

3.2.1 Basic Formulae and Limitations

Unlike the case of $h = 1$, we were unable to find a direct and exact method to evaluate the probability $P(0 \leq f(h) \leq m/n)$, of attaining a structure m from an initial structure n in h steps, $h \geq 2$. But indirectly, this probability could be evaluated by the use of the following recursive relation:

$$P(0 \leq f(h) \leq m/n) = \sum_{v \in A} P(f(1)=v/n)P(0 \leq f(h) \leq m/n(1)=v+R)$$
$$= \sum_{v \in A} P(f(1)=v/n)P(0 \leq f(h-1) \leq m/n') \quad (3.4)$$

where A is the set of all structures that could be reached at time T=1, before recruitment has taken place, R is the recruitment vector at the end of the first step of the considered period, and n'=v+R. In this relation, we have supposed that once the outcome of f(1) is known, we will be able to provide a unique recruitment vector R according to a given recruitment strategy. However, in some situations, as shown in section (3.1.3), R could be a random vector. In this case, relation (3.4) could be modified as follows:

$$P(0 \leq f(h) \leq m /n)$$
$$= \sum_{v \in A} P(f(1)=v/n) \sum_{r \in E} P(R=r/f(1)=v)P(0 \leq f(h-1) \leq m/n') \quad (3.5)$$

where E is the set of all possible outcomes of R given that f(1)=v.

To see how this recursive approach could be used, let us start with a simple case and suppose that under the recruitment strategy, the total size N of the system will be kept constant and only relation (3.4) will be needed. In this case, the calculations will be made backwards; we will start by evaluating $P(0 \leq f(1) \leq m/n')$ for all possible integer vectors n' whose total size is equal to N; $P(0 \leq f(1) \leq m/n')$ for a particular n', will be used whenever n(h-1) is equal to n'. Since there would be many paths $\{n, n(1), \ldots, n(h-2), n'\}$ from n to n' in (h-1) steps, it is better to evaluate $P(0 \leq f(1) \leq m/n')$ once and to store it for subsequent use. For the same reason, using the relation (3.4), we will evaluate successively for $t=2,\ldots,h-1$, the probabilities $P(0 \leq f(t) \leq m/n')$ for all possible n'. The probabilities $P(0 \leq f(h-1) \leq m/n')$ will be then used to evaluate $P(0 \leq f(h) \leq m/n)$. We should point out that in using the relation (3.4) to evaluate $P(0 \leq f(t) \leq m/n')$, the vector R stands for R(h-t+1). The evaluation of $P(f(1)=v/n')$ and $P(0 \leq f(1) \leq m/n')$ could be done by one of the two exact methods discussed in the previous chapter.

A serious difficulty in this approach, is the high number of evaluations of probabilities of type $P(f(1)=v/n')$; if $h \geq 3$, these probabilities should be evaluated, at least once, for all possible vectors n', whose total size is equal to N, and for all possible outcomes v of f(1) from each of these vectors n'. More precisely, if we denote by M_1 the number of vectors n' and by M_2 the number of vectors v mentioned above, we will have:

$$M_1 = \binom{N + k - 1}{k - 1} \quad \text{and} \quad M_2 = \binom{N + k}{k} \; ;$$

Thus, there will be a need to evaluate $M_3=M_1 x M_2$ probabilities of the kind P(f(1)=v/n').

Table 5 gives the values of M_1, M_2 and M_3 in the case of k=3 and some values of N.

Table 5
Number of evaluations of P(f(1)=v/n')

N	M_1	M_2	M_3	M_4
10	66	286	18 876	8 294
15	136	816	110 976	47 328
18	190	1 330	252 700	106 666
20	231	1 771	409 101	171 787
24	325	2 925	950 625	396 045
25	351	3 276	1 149 876	478 296
50	1 326	23 426	31 062 876	12 673 466

M_4 represents the total number of evaluations that would be required in the case of an upper-diagonal transition matrix; in this latter case, the number of outcomes for f(1) depends on the initial stock vectors n'. The process of enumeration suggests that there would be:

$$\tfrac{1}{2}\left\{(n_1'+1)(n_1'+2)(n_2'+n_3'+1)+(n_1'+1)n_2'(n_2'+2n_3'+1)\right\}$$

possible vectors v from $n'=(n_1',n_2',n_3')$.

In the case of systems of 3 grades and with upper-diagonal transition matrices, the computation times needed to complete the evaluation of all required probabilities P(f(1)=v/n') are approximately as follows:

N	TIME
18	145
24	840
25	1100

All the calculations were made in the CDC 7600 computer and times are expressed in CP seconds.

Additionally, i. the transition matrix is not upper-diagonal and the routine MAINT is not used, the execution times will be considerably higher than those given above. Nevertheless, since the total size is assumed to be constant and the possible n' at each step will be always the same, these evaluations have to made only once and the probabilities could be used as many times as required. If, in addition, the vectors R(T) are easily obtainable, the evaluation of $P(0 \leq f(h') \leq m/n)$ would be slightly longer than $P(0 \leq f(h) \leq m/n)$ for $h' > h \geq 3$.

If the total size under a recruitment strategy is not kept constant, the above difficulties would be dramatically amplified. It would be difficult to predict the outcomes of n(h-1),...,n(2). Also, for each step, we will have a set of new vectors n' for which the probabilities P(f(1)=v/n') have to be evaluated at this step.

3.2.2 Case of h=2

This is a special case, since the issue of the number of evaluations, as discussed above, is relatively much less serious than in the case of h $\geq$ 3 and the total size problem is irrelevant. In the case of h=2, equation (3.4) becomes:

$$P(0 \leq f(2) \leq m/n) = \sum_{v \in A} P(f(1)=v/n)\ P(0 \leq f(1) \leq m/n')$$

Since P(0 $\leq$ f(1) $\leq$ m/n') could be evaluated directly without a further use c relation (3.4), we only need to evaluate P(f(1)=v/n) for all possible vectors v, but only from the initial structure n. However, even in this special case, the number of vectors v could still be high and would affect the computation time of P(0 $\leq$ f(2) $\leq$ m/n). Table 6 gives the time required to compute the probability of maintaining a structure n in two steps. All calculations were made on the CDC 7600 and the times are expressed in CP seconds. The transition matrices P_3' and P_3'' are upper-diagonal and therefore the routine MAINT was used to evaluate the probabilities P(f(1)=v/n) and P(0 $\leq$ f(1) $\leq$ n/n').

Table 6
Computation times required to evaluate P(0 ≤ f(2) ≤ n/n)

Transition Matrix	Structure n	Recruitment Strategy F_1	F_2	S_2^1
$P_3' = \begin{pmatrix} 0.8 & 0.1 & \\ & 0.8 & 0.1 \\ & & 0.8 \end{pmatrix}$	(5, 5, 5)	0.83	1.29	0.66
	(10,10,10)	10.81	24.80	12.29
	(15,15,15)	97.01	162.47	79.25
	(20,20,20)	269.83	648.30	315.55
$P_3'' = \begin{pmatrix} 0.5 & 0.3 & \\ & 0.5 & 0.3 \\ & & 0.5 \end{pmatrix}$	(5, 5, 5)	2.29	1.43	0.67
	(10,10,10)	13.01	27.92	12.61
	(15,15,15)	293.93	181.91	80.52
	(20,20,20)	326.29	720.43	316.48

The computation times become very high if the recruitment vector, obtained by the rules of a recruitment policy, is real and has to be rounded to an integer vector. Thus, these times are relatively much higher in the case of F_2 than S_2^1; the rules of the policy S_2^1 lead always to an integer vector, while those of F_2 lead, for most outcomes of f(1), to a real vector. As to the policy F_1, since the recruitment vector is fixed in advance and does not depend on the outcomes of f(1), either of the equations (3.4) or (3.5) should always be used depending on whether R=n-nP is integer or not. If R is not integer, the computation times will still depend on the distribution of R obtained from the rounding process.

The computation times in table 6 are already high, even for small structures, and increase sharply with stock numbers. They would be much higher if the transition matrices were not upper-diagonal or if the number of grades was larger than three. There is obviously a need to look for an alternative method; the Multinormal approximation could be such a method.

3.3 NORMAL APPROXIMATION

As in the case of h=1, we will assume that the Multinormal distribution would approximate to the distribution of $f(h)=(\sum_{i=1}^{k} n_{ij}(h),\ j=1,2,...,k)$. But, unlike the case of h=1, the mean vector and the variance-covariance matrix of the distribution are not immediately available and will depend on the recruitment strategy over the period [0,h[. Thus, since it would be difficult, if not impossible, to investigate the Multinormal approximation issue in general, we will review in turn different recruitment strategies; these latter will be differentiated according to the dependence of recruitment vectors R(T) on the previous flows.

3.3.1 Recruitment Vectors that Do Not Depend on the Flows

In this case, the recruitment vectors R(T), (T=1,2,...,h) could be any vectors decided upon without any knowledge of the previous flows. They could be random or constant, equal or different. If a vector R(T) is random, its distribution should be given in advance. Under these assumptions, we can use the following recursive equations to evaluate the mean vectors $\bar{n}(T)$ and the variance-covariance matrices V(T) successively for the stock vectors n(T), (T=1,2,...,h):

$$\bar{n}(T)=\bar{n}(T-1)P+\bar{R}(T) \tag{3.6a}$$

$$V(T)=P'V(T-1)P+[\bar{n}(T-1)P]_d-P'[\bar{n}(T-1)]_dP+V(R(T)) \tag{3.6b}$$

$$n(0)=n, v(0)=0 \tag{3.6c}$$

where $[-]_d$ denotes a diagonal matrix with the elements of the vector contained within it on the principal diagonal, $\bar{R}(T)$ and V(R(T)) respectively the mean vector and the variance-covariance matrix of R(T).

The relations (3.6a) and (3.6b) were quoted in Bartholomew (1977) for slightly more restrictive assumptions than those considered in this section, but could be shown to hold for the latter assumptions.

The mean vector and the variance-covariance matrix of f(h) will be identical to those of n(h) if we put R(h)=0. Having now the possibility of evaluating the required moments of f(h) we will try to assess the quality of the multivariate Normal approximation to the probability of attaining a structure in many steps. We will restrict our comparison to the case of two steps and will consider the strategy F_1, i.e. R(1)=n(0)(I-P). If R(1) is not integer, we will use the procedure in section (3.1.3) to round it to an integer vector. This definition of R(1) satisfies the assumptions of this section and therefore the relations (3.6) will be valid for use. The results from table 7 suggest that the multivariate Normal approximation to $P(0 \leq f(h) \leq n/n)$ is still good for h=2 and we think it would be also good for $h \geq 3$.

Table 7

Comparison of the Normal approximation (NA) to $P(0 \leq f(2) \leq n/n)$ with the exact value (EV), in the case of the recruitment strategy F_1

Structure n	Transition Matrix P_3'			Transition Matrix P_3''		
	EV	NA	$\frac{NA-EV}{EV}x100$	EV	NA	$\frac{NA-EV}{EV}x100$
(5, 5, 5)	0.477	0.470	-1.5	0.639	0.631	-1
(10,10,10)	0.537	0.531	-1	0.716	0.717	0
(15,15,15)	0.579	0.573	-1	0.773	0.788	2
(20,20,20)	0.626	0.630	1	0.819	0.826	1

For all the examples in table 7, the computation times of NA were equal to a fraction of a second. And since the computation of NA requires the evaluation of the mean vector and the variance-covariance matrix of f(h), and only one multivariate Normal integral, independently of h, we expect the latter times to increase just slightly with h.

They would increase much more significantly with the number of grades but compared with those which would be required using the exact methods, they would still be negligible.

3.3.2 Recruitment Vectors that Depend on the Flows

Under such an assumption, there is no general method to evaluate the mean vector and the variance-covariance matrix of f(h). In some cases, this evaluation cannot be made unless the distribution of f(h) itself is known, and thus any approximation is irrelevant. An example of these strategies is the adaptive strategy S_2^1. Under this latter, the recruitment vector R(h) would depend on the flows to each grade by the end of the period [h-1,h[, but there is no analytical expression of R(h). This makes the task of evaluating the mean vector and the variance-covariance matrix of f(h) difficult, if not impossible. Another example is the fixed strategy F_2 under which the recruitment vectors R(T) depend only on the total losses during the periods [T-1,T[, (T=1,2,...,h). In the case of this example, results from Bartholomew (1975) are of interest; it was shown that if:

$$R(T)=\{\sum_{i=1}^{k} n_{i,k+1}(T)\}r, (T=1,2,\dots,h) \qquad (3.7)$$

Where r is a fixed vector, then:

$$\bar{n}(T) = \bar{n}(T-1)Q \qquad ,(T=1,2,\ldots,h) \quad (3.8a)$$

$$V(T) = Q'V(T-1)Q+[\bar{n}(T-1)Q]_d$$
$$-Q'[\bar{n}(T-1)]_dQ-\bar{n}(T-1)w'\{[r]_d-r'r\},(T=1,2,\ldots,h) \quad (3.8b)$$
$$V(0) = 0,\bar{n}(0)=n \qquad (3.8c)$$

where, in addition to the notations used in equations (3.6), w is the wastage vector and Q=P+w'r. A useful generalisation of these results is to replace r by r(T) in equation (3.7); if r(T), (T=1,2,...,h) are still fixed in advance, but not necessarily equal, we can prove that we need only to replace r by r(T) in (3.8a) and in the definition of Q. In the case of strategy F_2, we have:

$$r(T)=r=\{1/\sum_{i=1}^{k} n_i w\}n(I-P) \qquad , (T\geq 1)$$

But for the purpose of using equations (3.8), we will put r(h)=0; the mean vector and the variance-covariance matrix of f(h) will be respectively equal to $\bar{n}(h)$ and V(h). However, this development does not take into account the fact that R(T), as defined in equation (3.7), might need to be rounded to an integer vector. Fortunately, even with this omission, the results in table 8 show that the multivariate Normal distribution, with mean vector and variance-covariance matrix as defined by equation (3.8), does approximate quite well the distribution of f(h).

Table 8
Comparison of the Normal approximation (NA) to $P(0 \leq f(2) \leq n/n)$ with the exact value (EV), in the case of the recruitment strategy F_2

Structure n	Transition Matrix P_3'			Transition Matrix P_3''		
	EV	NA	$\frac{NA-EV}{EV}$x100	EV	NA	$\frac{NA-EV}{EV}$x100
(5, 5, 5)	0.475	0.484	2	0.642	0.643	0
(10,10,10)	0.531	0.533	0	0.725	0.732	1
(15,15,15)	0.585	0.585	0	0.785	0.794	1
(20,20,20)	0.632	0.633	0	0.831	0.841	1

As to the computation times, the same remarks as those concerning table 7 still apply.

CHAPTER IV

MAINTAINABILITY AND ATTAINABILITY IN A STOCHASTIC ENVIRONMENT

The stock numbers in a graded manpower system change over time as a result of wastage, promotion and recruitment flows. By acting upon the latter, we would direct the evolution of the system towards some desired situations. More precisely, the objective of control by recruitment, in a stochastic environment, would be minimising or maximising the expected value of some function of economic or social interest. Generally, this function would depend on the flows from and to different grades and on the importance given to each grade and to different steps. One particular function could be seen as a generalisation of deterministic theory; in the latter theory, there has been extensive work, by Bartholomew (1973), Grinold and Stanford (1974), Davies (1975) and Vajda (1978), on maintaining or attaining a structure in one or many steps. In a stochastic environment, such an aim would be impossible to achieve but one would try, instead, to get as close as possible to the goal structure in a specified number of steps. Our objective is then to minimise the expected distance between the structure reached in h steps and a target structure. We will refer to the problem as the maintainability problem if the initial and target structures are identical and as an attainability problem otherwise.

Given the complexity of the stochastic environment, and the lack of analytical methods to evaluate even simple probabilities that would be needed, we have to rely on numerical methods to assess different recruitment strategies.

In the following sections, after describing the basis of our numerical approach, we will concentrate on maintainability and attainability in a stochastic environment. We will describe the recruitment strategies and numerical examples that we will consider and present the main conclusions shown from such cases.

4.1 NUMERICAL APPROACH

The aim of this section is to explain how we propose to evaluate the expected value V(n,h) of an objective function over a period of h steps, starting at vector n. Let us suppose that at the end of each step, after wastage and promotion flows have taken place, we know what recruitment policy we should apply and we also know how to determine the recruitment vector R. At this stage it is not necessary to specify whether R depends on wastage and promotion flows or not. Our approach will be shown to deal with both cases in the same way.

Let us denote by A(v,R,T) a measure of the immediate consequences in taking R as the recruitment vector at time T, given that f(T)=v, $1 \leq T \leq h$. We will refer to the function

A as the return function. This function does not take into account any future consequences to the choice of R. For example, in a manpower context, we would be interested in the total salary bill and the recruiting costs over the period of h steps. Thus, if at time T, f(T)=v and R(T)=R we have:

$$A(v,R,T) = \sum_{i=1}^{k} b_i R_i + \sum_{i=1}^{k} c_i (v_i + R_i)$$

where b_i and c_i are respectively the average salary and recruiting cost in grade i; the costs b_i and c_i could depend on T. In this example, it is clear that whilst the function A takes into account the costs incurred at time T, it does not consider the effects of the decision upon R on future costs.

Thus, in general, if the return functions are additive, V(n,h) can be evaluated by using the following iterative equation:

$$V(n,h) = \sum_{v} P(f(1)=v \;/ n(0)=n) \left\{ A(v,R,1) + V(v+R,h-1) \right\} \quad (4.1)$$

where R is the recruitment vector at the end of the first period.

In addition, if we are interested in the minimal value $V^*(n,h)$ of V(n,h) with respect to {R(1),R(2),...,R(h)}, we would change the equation (4.1) to:

$$V^*(n,h) = \sum_{v} P(f(1)=v \;/ n(0)=n) \text{Min } A(v,R,1) + V^*(v+R,h-1) \qquad (4.2)$$

The equation (4.1) could be seen as a generalisation of the equation (3.4) in which:

$$A \equiv 0$$

$$V(n',T) \equiv P(0 \leq f(T) \leq m/n') \qquad , 1 \leq T \leq h$$

$$V(n',0) = \begin{cases} 1 & \text{if } n'=m \\ 0 & \text{if } n' \neq m \end{cases}$$

where n' is a possible stock vector at time T, $0 \leq T \leq h$.

As in the discussion of quation (3.4), if R is a random vector, equation (4.1) should be modified as follows:

$$V(n,h) = \sum_{v} P(f(1)=v/n(0)=n) \text{ x}$$
$$\left\{ \sum_{r \in E} P(R=r/f(1)=v) \text{x} \{A(v,R,1)+V(v+R,h-1)\} \right\} \qquad (4.3)$$

where E is the set of all possible outcomes of vector R, given f(1)=v.

The equations (4.1), (4.2) and (4.3) are very general in the sense that they could deal with a very wide range of objective functions and recruitment strategies. In the case of the attainability problem, for example, V(m,h) will represent the expected distance between n(h) and a goal structure m, under a specified recruitment strategy, and V*(m,h) will represent the minimal value of such distance over an acceptable set of recruitment strategies; in both cases, n is considered to be the initial vector. These values could be calculated respectively by using equation (4.1) and equation (4.2). We need, then, to put $A \equiv 0$ and $V(n',0)=V^*(n',0)=D(n',m)$, where n' is a possible outcome of n(h) and D is a measure of the distance between two vectors. However, since our approach is mainly a generalisation of equations (3.4) and (3.5), and for the same reasons given in section (3.2.1), it would be impracticable to use this approach for general recruitment strategies or for systems with either a high number of grades or high stock numbers; the amount of calculations would be too high. Therefore, we will consider only recruitment strategies which will not change the total size of the system. Also, we will concentrate only on structures with total sizes not greater than twenty four.

4.2 RECRUITMENT STRATEGIES

Besides very rare and special situations, there could be no certitude of attaining exactly a structure m from a structure n in one step or, indeed, in many steps. Thus for a fixed number h of steps, we would be interested to know how a recruitment strategy could bring us close to the target structure m.

Recruitment policies at the end of each step could be divided into two groups according to the timing of the decision on the number of recruits to different grades; policies which depend on internal transfer and losses, during an observed period, and policies which make no use of such flows. We will consider only policies from the former group; with this choice, we have the possibility of controlling exactly the total size of the system and taking into account the limitations of the total size discussed earlier. Many strategies could still be considered but we retain only three relevant strategies:

- Fixed strategy F: This, at each step, allocates new recruits to different grades proportionally to the fixed vector $\alpha = m'(I-P)$. This is the same strategy as F_2 from the previous chapter, where m was equal to n. The suffix "2" is dropped since it is the only fixed strategy that we will consider and therefore there is no need for such precision. This strategy cannot be used if m is not maintainable.

- Goal strategy A_h: This tries to get as close as possible to the target structure m in a fixed number h of steps, $h \geq 2$.

- Adaptive strategy A_1 This is a particular case of the latter strategy and tries, at each step, to get as close as possible to the target structure m. It is the same strategy as S_2^1 from the previous chapter.

The distance $D(u,v)$ between two vectors u and v could be measured in many ways, but we consider the squared distance function: $D(u,v)=\sum_{i=1}^{k}(u_i - v_i)^2$ to be the most appropriate. In the case of the fixed strategy, it may be necessary to round up or down the entries of a recruitment vector to integer values. There are many ways in which this could be done, but we will use the same techniques introduced in the previous chapter.

In the case of the fixed strategy, the choice of r as the vector with entries proportional to those of $m(I-P)$ was motivated by two considerations. Firstly, as shown in chapter one, the expected stock vector will converge in the long run to the goal vector m, independently of the initial structure. Secondly, if we consider the entries of r as probabilities and not as fixed proportions, such a recruitment vector will ensure, in comparison to other vectors, the best expected distance $\bar{D}(n(\infty),m)$ between $n(\infty)$ and m. A proof of this proposition is given below.

Given the new interpretation concerning the elements of the recruitment vector and the fixed total size assumption, a transition from grade i to grade j can take place either within the system or by loss from grade i and replacement to grade j with total probability $p_{ij}+w_i r_j$. It can then be shown that Q^t represents the probabilities of transition from one grade to another in t steps, with $Q=P+w'r$. The matrix Q^t was already proved to converge as

$t \to \infty$ to a stochastic matrix $\Pi = e\pi'$, where π is the only solution to the equation $\pi = \pi Q$ and e the row vector with all its elements equal to one. Thus, independently of n(0), the stock vector $n(\infty)$ has a Multinomial distribution with parameters (N, π), where N is the system total size. The limiting values of the expected stock numbers at grade i and the variance of the latter numbers are therefore as follows:

$$\bar{n}_i(\infty) = N\pi_i \qquad i=1,2,\ldots,k$$

$$V_i(\infty) = N\pi_i(1-\pi_i)$$

These results hold for any recruitment vector r, but in the case of $r^* = m(I-P)/mw'$ we have in addition:

$$\bar{n}^*_i(\infty) = m_i = N\pi^*_i \qquad i=1,2,\ldots,k$$

$$V^*_i(\infty) = N\pi^*_i\ (1-\pi^*_i)$$

It follows then:

$$D^* = \bar{D}(n^*(\infty), m) = N \sum_{i=1}^{k} \pi^*_i\ (1-\pi^*_i) + N(1-\sum_{i=1}^{k} \pi^{*2}_i)$$

For any other recruitment vector r, we have:

$$\bar{D}(n(\infty), m) = N \sum_{i=1}^{k} \pi_i(1-\pi_i) + N^2 \sum_{i=1}^{k} (\pi_i - \pi^*_i)^2$$

$$= N(1-\sum_{i=1}^{k} \pi_i^2) + N^2 \sum_{i=1}^{k} (\pi_i - \pi^*_i)^2$$

$$= N(1-\sum_{i=1}^{k} (\pi_i-\pi^*_i)^2 - \sum_{i=1}^{k} \pi^{*2}_i) + N^2 \sum_{i=1}^{k} (\pi_i-\pi^*_i)^2$$

$$= D^* + N(N-1) \sum_{i=1}^{k} (\pi_i-\pi^*_i)^2$$

Hence, $\qquad D^* \leq \bar{D}(n(\infty), m)$

4.3 NUMERICAL EXAMPLES

To see how the recruitment strategies affect the evolution of a system, we have to rely on numerical comparisons. In the choice of the transition matrices, we took into account two factors:

- The relative magnitudes of the off-diagonal elements in comparison to the diagonal elements.

- The wastage rates at all grades.

In his examples, Bartholomew (1977) considered only the first factor and used two transition matrices:

$$P_1 = \begin{pmatrix} 0.7 & 0.2 & \\ & 0.8 & 0.1 \\ & & 0.9 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

Both matrices present low wastage rates, but, the ratios between the diagonal and off-diagonal elements are much bigger in the case of P_1 than in the case of P_2. Thus, in order to allow a comparison with Bartholomew's work and to take into account the second factor, we considered in addition to P_1 and P_2 the following matrices:

$$P_3=\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.8 \\ & & 0.70 \end{pmatrix} \quad \text{and} \quad P_4=\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.63 \end{pmatrix}$$

The entries of P_3 and P_4 are in the same proportions as those of P_1 and P_2 respectively but the wastage rates in the case of the former matrices are much higher than those of the latter. As to the choice of the goal vector m, we considered in the case of each transition matrix and each fixed total size, two vectors, one that could be maintained by recruiting only at the lowest level and the other that could be maintained by equal recruitment at each level. When, for a given transition matrix and a fixed total size, there were no such vectors with integer entries, we considered neighbouring vectors. Finally, given the limitation of our numerical approach, we considered only systems with total sizes equal to 12 and 24. The examples discussed above are listed in table 9.

Table 9
Systems to be investigated by exact methods

Transition Matrix m	Total Size	Goal Vector m	m-mP
$P_1 = \begin{pmatrix} 0.7 & 0.2 & \\ & 0.8 & 0.1 \\ & & 0.9 \end{pmatrix}$	12	(4, 4, 4)	(1.20,0, 0)
		(1, 3, 8)	(0.30,0.40,0.50)
	24	(8, 8, 8)	(2.40,0, 0)
		(3, 7,14)	(0.90,0.80,0.70)
$P_3 = \begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7, 3, 2)	(3.22,0.02,0.36)
		(3, 4, 5)	(1.38,1.04,1.18)
	24	(15, 7, 2)	(6.90,0.26,0.04)
		(5, 9,10)	(2.30,2.62,2.28)
$P_2 = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$	12	(3, 3, 6)	(1.50,0.00,0.30)
		(1, 3, 8)	(0.50,0.80,0.70)
	24	(6, 7,11)	(0.30,0.40,0.10)
		(3, 6,15)	(1.50,1.20,1.20)
$P_4 = \begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5, 4, 3)	(3.05,0.57,0.28)
		(2, 4, 6)	(1.22,1.50,1.48)
	24	(11, 8, 5)	(6.71,0.83,0.16)
		(4, 8,12)	(2.44,3.00,2.96)

We notice that all matrices are upper-diagonal and the number of grades in all systems is equal to three. For other systems which do not fall within such specifications we have to rely on simulation methods. Examples of this latter type are listed in table 10.

Table 10

Systems to be investigated by simulation

Transition Matrix P	Total Size	Goal Vector m	m-mP
$P_5=\begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}$	12	(6, 3, 3)	(2.4,0, 0)
		(4, 4, 4)	(0.8,0.8,0.8)
	24	(12, 6, 6)	(4.8,0, 0)
		(8, 8, 8)	(1.6,1.6,1.6)
$P_6=\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3, 3, 3, 3)	(1.2,0, 0, 0)
		(1, 2, 3, 6)	(0.4,0.3,0.2,0.3)
	48	(12,12,12,12)	(4.8,0, 0, 0)
		(3, 7,13,25)	(1.2,1.2,1.2,1.2)

The first four examples in table 9 are treated by simulation and exact methods in order to allow a comparison of both methods.

All the examples in table 9 and table 10 will be used to survey the effects of different recruitment strategies upon the evolution of a graded system. This evolution will be observed over a quite long period in order to examine the behaviour of the system at its steady state. From our calculations, we found that a period of 10 steps meets that objective.

4.4 COMPARISON BETWEEN THE EXACT AND SIMULATION METHOD

The simulation cannot be applied in the case of the goal strategy; this latter requires a prior knowledge, at any step t, of smallest average distances, between all possible stock vectors and the goal vector in (h-t) steps. The evaluation of such distances by simulation would be too lengthy and of a very poor accuracy. By contrast, the use of simulation method in the case of fixed and adaptive strategies is much easier and presents more flexibility than our exact method. This is more true if we consider systems with general transition matrices or with a number of grades bigger than three.

As to the quality of the results obtained by means of simulation, we conducted a comparison of the latter results with those obtained by the exact method. This comparison was done in the case of two sets of examples; the first set consists of the first examples in table 9. In this case, we compared the results obtained by 10,000 simulations of a period of 10 steps. We were able in both methods, and in the case of fixed and adaptive strategies, to have information on the whereabouts of the system at each step. The results of such comparisons are given in table F_i and A_i (i=1,2,3,4) in Appendix B. They show the high quality of the simulation method. As to the second set, we considered four examples from Bartholomew (1977); the author observed the evolution of the systems over a sequence of 10,000 steps, using simulation. His results are therefore concerned with the whereabouts of the systems only when they reach their steady state behaviour. We found, as shown in tables 11.a and 11.b, the results obtained by the exact method, at the 10th step, to be very close to Bartholomew's.

Table 11.a
The average structures maintained in a sequence of 10,000 trials and the expected stock vector at the 10th step (initial and goal vectors are equal)

Transition Matrix	Goal Vector	Recruitment Strategy	Average Structures	Expected Stock Vector
	(8,8, 8)	F	*	(8.0,8.0,8.0)
$\begin{bmatrix} 0.7 & 0.2 & \\ & 0.8 & 0.1 \\ & & 0.9 \end{bmatrix}$		A	(6.16,8.21,9.63)	(6.5,8.24,9.26)
	(2,5,11)	F	*	(2.0,5.0,11.0)
		A_1	(1.75,4.95,11.30)	(1.84,4.94,11.22)
$\begin{bmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{bmatrix}$	(7,7,11)	F	*	(7.0,7.0,11.0)
		A_1	(5.58,7.11,12.31)	(5.8,7.06,12.14)
	(3,6,16)	F	*	(3.0,6.0,16.0)
		A_1	(2.78,5.96,16.27)	(2.86,5.94,16.20)

* Results for fixed strategies were not given in Bartholomew (1977).

Table 11.b
Variances of a sequence of 10,000 trials compared to the exact variances of stock vectors at the 10th step (Initial and goal vectors are equal)

Transition Matrix	Goal Vector	Recruit-ment Strategy	Variances from simulation method	Variances from exact method
$\begin{bmatrix} 0.7 & 0.2 & \\ & 0.8 & 0.1 \\ & & 0.9 \end{bmatrix}$	(8,8, 8)	F	(5.30,5.07,5.10)	(5.33,5.23,5.19)
		A_1	(2.03,2.34,3.31)	(1.92,2.71,3.09)
	(2,5,11)	F	(1.25,2.97,3.68)	(1.29,2.93,3.35)
		A_1	(0.31,0.59,0.71)	(0.16,0.58,0.64)
$\begin{bmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{bmatrix}$	(7,7,11)	F	(5.06,5.16,6.23)	(5.03,5.04,6.14)
		A_1	(1.66,1.98,3.08)	(1.60,2.36,3.21)
	(3,6,16)	F	(1.53,3.33,4.23)	(1.63,3.51,4.23)
		A_1	(0.29,0.52,0.76)	(0.16,0.53,0.71)

The comparison above, in the case of both sets of examples, shows that would be safe to rely on simulation whenever the use of exact methods is not practicable.

4.5 COMPARISON OF THE RECRUITMENT STRATEGIES

Let us denote by $E(n_i(t)-m_i)^2$ the expected distance between the number of individuals in the i^{th} grade of the stock vector n(t) reached after t steps and the number of those in the i^{th} grade of the structure m to be maintained. Then:

$E(n_i(t)-m_i)^2 = E(n_i(t)-\bar{n}_i(t))^2 + (\bar{n}_i(t)-m_i)^2$, where $E(n_i(t)-\bar{n}_i(t))^2$ is the variance of $n_i(t)$ and $\bar{n}_i(t)$ is the expected value of $n_i(t)$.

The difference $(\bar{n}_i(t)-m_i)$ will be called bias at the i^{th} grade. If under a recruitment strategy $\bar{n}_i(t)=m_i$, $(i=1,2,\dots,k)$, the strategy will be said to be unbiased. At the t^{th} step, the expected distance $E(n_i(t)-m_i)^2$ between the stock vector n(t) and the structure m is, thus, affected by two factors: variability and degree of bias of the variables $n_i(t)$, $i=1,2,\dots,k$. Following on from this discussion, let us now assess the performance of each recruitment strategy.

4.5.1 Case of Maintainability

In this case the initial and goal vectors are identical. The results concerning the evolution of the twenty four examples discussed previously are given in tables 12 to 19 and in greater detail in Appendix B. They show that:

(a) Although the adaptive strategy is biased at all steps, the bias in most cases is not significant and does not show big fluctuations from one step to another. Furthermore, the bias tends to become relatively smaller when the total size of the system increases. In the case of systems with high wastage rates, the mentioned decrease in the bias was observed even in absolute terms. These observations hold better when the structure to be maintained is in the middle of the maintainable region than when it is at the border of this latter.

Table 12: The average bias of stock numbers $n_i(t)$ over a period of 10 steps, under the adaptive strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} E(n(t))-m$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} E(n(t))-m$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(-0.85,0.24,0.61)	(1,3,8)	(-0.09,-0.05,0.14)
	24	(8,8,8)	(-1.30,0.39,0.91)	(3,7,14)	(-0.18,-0.06,0.24)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(-0.54 0.57,-0.03)	(3,4,5)	(-0.05,0.05,0.00)
	24	(15,7,2)	(-0.97,0.51,0.46)	(5,9,10)	(-0.02,-0.01,0.03)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(-0.57,0.20,0.37)	(1,3,8)	(-0.07,-0.16,0.23)
	24	(6,7,11)	(-0.87,-0.09,0.96)	(3,6,15)	(-0.16,-0.09,0.25)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(-0.40,0.07,0.33)	(2,4,6)	(-0.03,-0.04,0.07)
	24	(11,8,5)	(-0.72,0.06,0.65)	(4,8,12)	(-0.02,-0.03,0.05)
$\begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{pmatrix}$	12	(6,3,3)	(-0.76,0.39,0.37)	(4,4,4)	(0.08,0.00,-0.08)
	24	(12,6,6)	(-1.19,0.59,0.60)	(8,8,8)	(0.08,0.00,-0.08)
$\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3,3,3,3)	(-0.80,0.01,0.39,0.40)	(1,2,3,6)	(-0.15,-0.06,0.14,0.06)
	48	(12,12,12,12)	(-1.92,0.06,0.92,0.95)	(3,7,13,25)	(-0.13,-0.05,0.06,0.11)

Table 13: The average variances of stock numbers $n_i(t)$ over a period of 10 steps, under the adaptive strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} V(n(t))$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} V(n(t)$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(0.86,1.31,1.22)	(1,3,8)	(0.08,0.38,0.38)
	24	(8,8,8)	(1.70,2.41,2.36)	(3,7,14)	(0.20,0.65,0.74)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(0.74,1.10,0.50)	(3,4,5)	(0.05,0.21,0.18)
	24	(15,7,2)	(1.71,2.03,1.09)	(5,9,10)	(0.02,0.10,0.11)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(0.59,1.05,1.14)	(1,3,8)	(0.07,0.39,0.49)
	24	(6,7,11)	(1.11,1.78,2.56)	(3,6,15)	(0.18,0.56,0.79)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(0.52,0.82,0.83)	(2,4,6)	(0.03,0.17,0.21)
	24	(11,8,5)	(1.18,1.59,1.72)	(4,8,12)	(0.02,0.09,0.14)
$\begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{pmatrix}$	12	(6,3,3)	(1.19,1.44,1.46)	(4,4,4)	(0.81,0.90,0.98)
	24	(12,6,6)	(2.15,2.62,2.66)	(8,8,8)	(0.99,1.12,1.22)
$\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3,3,3,3)	(0.70,1.14,1.41,1.15)	(1,2,3,6)	(0.13,0.43,0.73,0.68)
	48	(12,12,12,12)	(2.75,4.04,5.01,4.06)	(3,7,13,25)	(0.13,0.47,0.96,1.07)

Table 14: The average mean square errors of stock numbers $n_i(t)$ over a period of 10 steps, under the adaptive strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(1.61,1.37,1.64)	(1,3,8)	(0.09,0.38,0.40)
	24	(8,8,8)	(3.46,2.58,3.28)	(3,7,14)	(0.24,0.65,0.80)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(1.04,1.42,0.50)	(3,4,5)	(0.06,0.22,0.18)
	24	(15,7,2)	(2.66,2.29,1.31)	(5,9,10)	(0.02,0.10,0.12)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(0.92,1.09,1.29)	(1,3,8)	(0.08,0.41,0.55)
	24	(6,7,11)	(1.88,1.81,3.52)	(3,6,15)	(0.20,0.57,0.86)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(0.68,0.82,0.94)	(2,4,6)	(0.04,0.17,0.22)
	24	(11,8,5)	(1.69,1.60,2.16)	(4,8,12)	(0.02,0.10,0.14)
$\begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{pmatrix}$	12	(6,3,3)	(1.77,1.59,1.60)	(4,4,4)	(0.82,0.90,0.99)
	24	(12,6,6)	(3.56,2.97,3.02)	(8,8,8)	(1.00,1.12,1.23)
$\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3,3,3,3)	(1.36,1.16,1.57,1.35)	(1,2,3,6)	(0.15,0.44,0.76,0.68)
	48	(12,12,12,12)	(6.54,4.14,5.90,5.12)	(3,7,13,25)	(0.15,0.47,0.96,1.08)

(b) The variances of the stock numbers under the adaptive strategy, increase with the number of steps. This increase is more significant at the beginning of the period of observation and tends to disappear after six or seven steps when the systems start to present a steady state behaviour. These variances are small in general but are remarkably much smaller in the case of structures at the centre of the maintainable region. They increase almost in the same proportions as the total size. However, in the case of systems at the middle of the maintainable region or systems with very high wastage rates, the variances increase in lesser proportions than the total size.

(c) Although the fixed strategy F is unbiased, the variances of the stock numbers $n_i(t)$ $(i=1,2,3)$, $(t=1,2,...,10)$ are very high and are bigger than the mean square errors of $n_i(t)$ under the adaptive strategy A_1.

Table 15: The average mean square errors of stock numbers $n_i(t)$ over a period of 10 steps, under the fixed strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(2.41,2.27,2.02)	(1,3,8)	(0.69,1.56,1.74)
	24	(8,8,8)	(4.82,4.53,4.04)	(3,7,14)	(1.68,3.32,3.71)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(2.62 2.15,1.32)	(3,4,5)	(1.24,1.71,1.64)
	24	(15,7,2)	(5.28,4.72,1.67)	(5,9,10)	(1.91,3.21,3.15)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(2.10,2.07,2.60)	(1,3,8)	(0.63,1.71,2.01)
	24	(6,7,11)	(4.19,4.53,5.46)	(3,6,15)	(1.61,3.32,4.01)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(2.31,2.31,2.01)	(2,4,6)	(0.85,1.67,1.84)
	24	(11,8,5)	(5.06,4.64,3.69)	(4,8,12)	(1.49,3.14,3.45)
$\begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{pmatrix}$	12	(6,3,3)	(3.00,2.23,2.25)	(4,4,4)	(2.23,2.34,2.20)
	24	(12,6,6)	(5.97,4.47,4.48)	(8,8,8)	(4.34,4.46,4.31)
$\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3,3,3,3)	(2.13,1.97,1.92,1.60)	(1,2,3,6)	(0.72,1.37,1.84,1.81)
	48	(12,12,12,12)	(8.58,7.96,7.68,6.43)	(3,7,13,25)	(1.70,4.08,6.55,7.24)

(d) Under the fixed strategy F, the variances in all numerical examples increase in the same proportions as the total size, regardless of the position of the structure to be maintained in the maintainable region and of the wastage rates. This observation could be explained by the relation (3.8b) and (3.8c). As to the evolution of these variances over time, the same pattern as the adaptive strategy was observed: an increase in the first steps and a steady state behaviour later.

(e) In the early steps, the goal strategy takes the stock vectors n(t) very far from the goal structure m, presumably to where it would be easy to attain m, before bringing it back to the latter structure at the 10^{th} step. This behaviour introduces a very high bias at all steps except at the final step.

(f) The variances of the stock numbers under the goal strategy do not behave in the same way as under the fixed and adaptive strategies: high fluctuations are observed from one step to another and no kind of limit is reached. On average, they are smaller than those under the fixed strategy but are higher than those under the adaptive strategy. As to the other strategies, the goal strategy presents less variability in the case of structures in the middle of the maintainable region than in the case of those at the border of the latter region.

Table 16: The average bias of stock numbers $n_i(t)$ over a period of 10 steps, under the goal strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} E(n(t))-m$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} E(n(t))-m$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(-0.20,0.00,0.20)	(1,3,8)	(0.38,-0.25,-0.13)
	24	(8,8,8)	(-0.26,0.00,0.26)	(3,7,14)	(1.54,-0.80,-0.74)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(-1.89,-0.74,2.63)	(3,4,5)	(0.08,-1.83,1.75)
	24	(15,7,2)	(0.41,-0.32,-0.10)	(5,9,10)	(1.83,-3.33,1.50)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(-1.36,-1.22,2.59)	(1,3,8)	(0.15,-1.39,1.23)
	24	(6,7,11)	(-1.82,-2.54,4.36)	(3,6,15)	(-0.62,-3.01,3.63)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(-2.45,-2.32,4.77)	(2,4,6)	(-0.63,-2.62,3.25)
	24	(11,8,5)	(-4.78,-4.09,8.86)	(4,8,12)	(-1.28,-4.99,6.26)

Table 17: The average variances of stock numbers $n_i(t)$ over a period of 10 steps, under the goal strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} V(n(t))$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} V(n(t))$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(1.90,2.05,1.45)	(1,3,8)	(0.37,0.64,0.39)
	24	(8,8,8)	(3.89,4 12,2.94)	(3,7,14)	(1.41,1.85,0.85)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(1.61,1.49,1.13)	(3,4,5)	(0.60,0.82,0.61)
	24	(15,7,2)	(4.69,4.32,1.38)	(5,9,10)	(1.10,1.42,0.74)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(0.70,1.05,1.08)	(1,3,8)	(0.32,0.68,0.61)
	24	(6,7,11)	(2.05,2.66,2.92)	(3,6,15)	(0.54,1.29,1.18)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(0.80,1.01,1.11)	(2,4,6)	(0.18,0.47,0.51)
	24	(11,8,5)	(2.04,2.39,2.45)	(4,8,12)	(0.25,0.84,0.90)

Table 18: The average mean square errors of stock numbers $n_i(t)$ over a period of 10 steps, under the goal strategy

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$	Structure m maintained by equal recruitment at each level	$\frac{1}{10}\sum_{t=1}^{10} E(n(t)-m)^2$
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	(2.00,2.06,1.53)	(1,3,8)	(0.67,0.72,0.48)
	24	(8,8,8)	(4.06,[illegible].14,3.07)	(3,7,14)	(4.67,2.76,1.60)
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	(9.07,2.39,12.79)	(3,4,5)	(2.98,5.13,6.79)
	24	(15,7,2)	(5.01,4.50,1.41)	(5,9,10)	(13.14,17.24,11.60)
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	(3.90,3.26,10.30)	(1,3,8)	(1.07,3.25,3.50)
	24	(6,7,11)	(11.39,12.05,32.71)	(3,6,15)	(4.70,13.28,22.32)
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	(12.37,8.02,33.51)	(2,4,6)	(2.84,9.01,16.20)
	24	(11,8,5)	(50.72,25.68,124.53)	(4,8,12)	(9.78,33.11,60.73)

Table 19: The expected distances between the stock vectors at the 10th step and the goal vector, under different recruitment strategies

Transition matrix	Total Size	Structures m maintained by recruitment at the bottom	Distances under F	Distances under A_1	Distances under A_{10}	Structure m maintained by equal recruitment at each level	Distances under F	Distances under A_1	Distances under A_{10}
$\begin{pmatrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{pmatrix}$	12	(4,4,4)	7.88	5.72	4.77	(1,3,8)	4.78	0.95	0.93
	24	(8,8,8)	15.75	11.61	9.16	(3,7,14)	10.43	1.85	1.51
$\begin{pmatrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{pmatrix}$	12	(7,3,2)	6.41	3.17	2.41	(3,4,5)	4.87	0.46	0.41
	24	(15,7,2)	12.21	6.75	4.91	(5,9,10)	8.77	0.24	0.21
$\begin{pmatrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{pmatrix}$	12	(3,3,6)	7.19	3.47	2.90	(1,3,8)	4.59	1.07	0.98
	24	(6,7,11)	14.99	7.80	5.61	(3,6,15)	9.46	1.68	1.21
$\begin{pmatrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{pmatrix}$	12	(5,4,3)	6.79	2.50	1.77	(2,4,6)	4.47	0.42	0.38
	24	(11,8,5)	13.70	5.64	3.86	(4,8,12)	8.29	0.26	0.19
$\begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{pmatrix}$	12	(6,3,3)	7.65	5.07		(4,4,4)	6.59	2.71	
	24	(12,6,6)	14.75	9.56		(8,8,8)	13.00	3.37	
$\begin{pmatrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{pmatrix}$	12	(3,3,3,3)	8.86	6.59		(1,2,3,6)	6.81	2.28	
	48	(12,12,12,12)	35.40	26.32		(3,7,13,25)	23.28	2.79	

From the observations above, it is clear that there is no advantage in looking many steps ahead (goal strategy), rather than concentrating only on the next step (adaptive strategy). Indeed, as shown in table 19, expected distances between the stock vector at the 10th step and the goal vector m, under the adaptive strategy A_1, are not very far from those under the goal strategy A_{10}. Moreover, unlike the goal strategy, the adaptive strategy does not show bad behaviour in the intermediate steps (t=1,2,...,9) but always keeps n(t) very close to the goal vector m. It also gives better results than the fixed strategy and its implementation would not require more elaborate efforts.

These conclusions should not be considered to be specific to the observed numerical examples; indeed, these latter represent a very wide range of type of systems and take into consideration all relevant factors. It would be safe then to assume that for a graded system from the real world, the adaptative strategy would behave better than the goal and fixed strategies and would exert a tight control on its evolution at all steps, especially if the goal vector is not at the border of the maintainable region; the control of a graded manpower system in a stochastic environment could thus be well achieved by simple and practical strategies.

4.5.2 Case of Attainability

In this section, we propose to examine the behaviour of different recruitment strategies when the initial and goal vectors are not equal. This situation would occur if we were initially at a given structure and want to move to another one which is, according to some economic or social criteria, considered to be more desirable. We will, however, be concerned only with the periods of transition of short durations and in particular with three-step periods. Should a problem of attainability with a long period arise, one ought to rely on the conclusions from the previous section. Indeed, we observed in the case of long periods and under different recruitment strategies that the initial structure has little effect upon the behaviour of the system, especially at the final steps. This can be explained easily by the Markovian property of the different flows.

In addition to the three strategies that were considered in the previous section, two more strategies will be introduced. In that section, we considered only one deterministic strategy, that is F, under which the new recruits are always allocated to different grades proportionally to the elements of the vector m(I-P), where m is the goal structure. The initial and goal vectors were identical and equal to m, and consequently the strategy F was proved to maintain on average the structure m. In the present context, the initial and goal vectors are not equal and the latter property no longer holds. There is therefore a need to consider a new deterministic strategy which allows for attaining the goal vector. Such a requirement can be shown to be met by considering the strategy which allocates the new recruits to different grades proportionally to (m(1)-nP) in the first step, to (m(2)-m(1)P) in the second step and to (m-m(2)P) in the third and final step, {n,m(1),m(2),m} being a three-step attainable path from n to m. We will reserve the qualification "deterministic" for this strategy and will denote it by D_3. There would be as many strategies of type D_3 as attainable paths from n to m and henceforward the attainable path will always be given whenever the deterministic strategy is used.

In conjunction with an attainable path {n,m(1),m(2),m} from n to m, another strategy can be defined. It is similar to the adaptive strategy A_1 but tries, at each step t (t=1,2,3) to get as close as possible to m(t), m(3) being equal to m. Such strategy will be denoted by A_1'

It is clear that the fixed strategy F cannot be used if m is not maintainable and that the same applies for D_3 and A_1' if m is not attainable in three steps from n. The goal and adaptive strategies, however, can be used in all situations and do not require any prior knowledge concerning the maintainability or attainability of m. Thus, in order to compare these five strategies without restricting unnecessarily the numerical investigation to particular situations, we considered six different examples, which include all possible combinations with regard to the maintainability and the three-step attainability of the goal vector m.

Tables 20 to 25 give a comparison of different recruitment strategies in the case of the six examples. They show that at the third step, the adaptive strategy exerts a tight control on the evolution of the systems considered. Indeed, in tables 20, 22 and 23 the strategies A_1 and A_3 are almost identical and the differences between the two in the other tables are not very important. They are well compensated for by the better behaviour of the system under A_1 at the termediate steps.

The introduction of the strategy A_1' did not bring any major improvement over A_1. Its requirement of prior knowledge and the existence of an attainable path denies the strategy A_1' any real advantage on A_1. Furthermore, results from tables 20 and 22 show that A_1 can provide better control than A_1'.

As to the fixed and deterministic strategies, they both perform badly. Their main problem lies in the high variations of the stock numbers and makes the quality of their control doubtful, especially if the total system size happens to be high. In the latter case, the variations will be higher. Nevertheless, if much importance is attached to the unbiasedness of a strategy, D_3 is obviously of interest. Indeed, D_3 can be shown to allow for the attainability on average of m(t) at each step t (t=1,2,3). Results in tables 20, 21 and, much more markedly, in table 22 show that D_3 exerts tighter control than F.

Table 20

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	1.76	3.17	7.07	12.00	0.38	0.80	0.73	1.91
A_1	1.71	3.19	7.10	12.00	0.38	0.85	0.78	2.01
A_1'	1.61	3.29	7.10	12.00	0.51	1.01	0.93	2.45
F	1.98	3.62	6.40	12.00	1.26	2.58	2.76	6.60
D_3	2.00	3.00	7.00	12.00	1.50	2.01	2.59	6.10

with:

Initial vector n = (3, 5, 4)

Goal vector m = (2, 3, 7)

$$\text{Transition matrix} = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

Three-step attainable path from n to m:

m(0) = n
m(1) = (2, 5, 5)
m(2) = (1, 4, 7)
m(3) = m

Observation: m is maintainable and attainable in three steps from n.

Table 21

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	1.91	2.70	7.40	12.00	0.31	0.68	1.03	2.02
A_1	1.66	2.61	7.73	12.00	0.42	0.81	1.75	2.99
A_1'	1.89	2.59	7.59	12.00	0.31	0.76	1.24	2.31
F	2.00	1.93	8.07	12.00	1.19	2.51	3.27	6.96
D_3	2.00	3.00	7.00	12.00	1.45	1.92	2.79	6.15

with:

Initial vector n = (0, 0, 12)

Goal vector m = (2, 3, 7)

Transition matrix =
$$\begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

Three-step attainable path from n to m:

m(0) = n
m(1) = (2, 0, 10)
m(2) = (3, 1, 8)
m(3) = m

Observation: m is maintainable and attainable in three steps from n.

Table 22

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	1.14	3.01	7.85	12.00	0.34	0.50	0.53	1.36
A_1	1.13	3.08	7.79	12.00	0.34	0.54	0.58	1.46
A_1'	1.22	3.32	7.46	12.00	0.47	0.83	0.83	2.13
F	1.53	3.95	6.52	12.00	1.37	2.95	4.32	8.64
D_3	1.00	3.00	8.00	12.00	0.83	1.73	1.77	4.33

with:

Initial vector n = (6, 0, 6)

Goal vector m = (1, 3, 8)

Transition matrix = $\begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$

Three-step attainable path from n to m:

m(0) = n
m(1) = (3, 3, 6)
m(2) = (2, 3, 7)
m(3) = m

Observation: m is maintainable and attainable in three steps from n.

Table 23

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	1.85	4.49	5.65	12.00	0.92	4.46	3.46	8.84
A_1	1.85	4.50	5.65	12.00	0.92	4.47	3.47	8.86
A_1'	-	-	-	-	-	-	-	-
F	2.72	5.45	3.83	12.00	2.37	8.86	12.44	23.67
D_3	-	-	-	-	-	-	-	-

with:

Initial vector n = (12, 0, 0)

Goal vector m = (2, 3, 7)

$$\text{Transition matrix} = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

<u>Observation</u>: m is maintainable but it is not attainable in three steps from n.

Table 24

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	2.07	5.62	4.31	12.00	0.63	1.48	1.31	3.41
A_1	1.84	5.31	4.85	12.00	0.61	1.68	2.32	4.61
A_1'	2.15	5.50	4.34	12.00	0.67	1.54	1.31	3.51
F	–	–	–	–	–	–	–	–
D_3	2.00	6.00	4.00	12.00	1.62	2.87	2.40	6.89

with:

Initial vector n = (10, 0, 2)

Goal vector m = (2, 6, 4)

$$\text{Transition matrix} = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

Three-step attainable path from n to m:

$$\begin{aligned} m(0) &= n \\ m(1) &= (6, 4, 2) \\ m(2) &= (4, 5, 3) \\ m(3) &= m \end{aligned}$$

Observation: m is not maintainable but it is attainable in three steps from n.

Table 25

Expected stock vector and mean square errors of the stock numbers at the third step, under different recruitment strategies

Recruitment strategy	Expected stock vector				Mean square errors			
	Grade 1	Grade 2	Grade 3	Total	Grade 1	Grade 2	Grade 3	Total
A_3	2.06	5.81	4.13	12.00	0.65	1.42	1.19	3.26
A_1	1.92	5.49	4.60	12.00	0.66	1.55	1.92	4.13
A_1'	-	-	-	-	-	-	-	-
F	-	-	-	-	-	-	-	-
D_3	-	-	-	-	-	-	-	-

with:

Initial vector n = (11, 1, 0)

Goal vector m = (2, 6, 4)

$$\text{Transition matrix} = \begin{pmatrix} 0.5 & 0.4 & \\ & 0.6 & 0.3 \\ & & 0.8 \end{pmatrix}$$

Observation: m is neither maintainable nor attainable in three steps from n.

GENERAL CONCLUSIONS

In addition to other questions that arise in connection with attainability and maintainability in stochastic or deterministic environments, we addressed two major and important issues: the evaluation of probabilities related to the distribution of stock numbers at different steps, and a detailed comparison of a range of recruitment strategies on exact results basis.

So far as the computational aspects of our study are concerned, we believe that some progress has been achieved. An exact and efficient method has been developed to evaluate probabilities of maintaining or attaining a structure in one step. It was designed for a special but very important case of systems in which promotion is only possible to the next higher grade. Its generalisation to cover a wide range of systems and, especially, to deal with the maintainability and attainability in many steps, is still open and much needed. Nevertheless, as shown in chapters three and four, its efficiency has made possible the use of exact results in the comparison of different recruitment strategies and brought to completion the computation of the distribution of stock numbers which was formerly accomplished by means of simulation techniques only. The approach followed in the latter comparison imposed restrictions in the choice of numerical examples. Systems with a high number of grades or large total size, and strategies which allow for a variation in the total size of the system were not considered. These restrictions, however, were not so severe as to prevent us from covering a wide range of situations and from considering different factors which would have influenced the results of the numerical investigation.

From our comparison, the most practical and important conclusion to emerge is that the adaptive strategy is generally superior to all deterministic strategies and does achieve a tight control on the evolution of a manpower system. The assessment concerning the quality of the adaptive strategy was reinforced by the comparison of the latter with the newly defined goal strategy. This latter, however, cannot be recommended for implementation in practice given the complexity of its rules and, especially, its markedly bad behaviour in the early steps prior to the fixed time horizon. The fixed strategy in the context of maintainability and the deterministic strategy in the context of attainability, on the other hand, were proved to be unbiased. This could be weighted against the high variability they cause if much importance is attached to the unbiassedness.

The foregoing analysis focussed on models in which the objective is to bring the system as close as possible to a fixed goal structure. This objective is by no means the concern of every organisation but, as explained in chapter four, the approach can be modified easily to cope with other situations.

We made recourse to simulation techniques in the case of very limited number of examples. This was motivated by two main reasons. Firstly, having developed methods to obtain exact results, there was a good opportunity to test the quality of these techniques. We have agreed that these techniques offer a high flexibility with regard to the characteristics of the systems to be chosen. Indeed, they have been used successfully to compare different strategies in the case of systems that do not fall within the confines of limits imposed by the use of exact methods. This was precisely the second motivation for their use.

The determination of the number of simulations to be used in such techniques was based on experimentation only; we considered, successively, different numbers ranging from 10,000 to 100,000 and observed that 10,000 simulations leads to satisfactory results. Theoretically sound methods for such determination have yet to be developed.

REFERENCES

Bartholomew, D.J. (1963). "A multi-stage renewal process". J.R. Statist. Soc. **B**, **25**, pp. 150-168.

Bartholomew, D.J. (1967). Stochastic Models for Social Processes (1st ed.), Wiley, London.

Bartholomew, D.J. (1973). Stochastic Models for Social Processes (2nd ed.), Wiley, London.

Bartholomew, D.J. (1975). "A Stochastic control problem in the social science". Bull. Int. Statist. Inst. **46**, pp. 670-680.

Bartholomew, D.J. (Ed.) (1976). Manpower Planning, Penguin Modern Management Readings, Penguin Books, Harmondsworth, Middlesex, England.

Bartholomew, D.J. (1977). "Maintaining a grade or age structure in a stochastic environment". Adv. Appl. Prob., **9**, pp. 1-17.

Bartholomew, D.J., and A.F. Forbes (1979). Statistical Techniques for Manpower Planning, John Wiley, Chichester.

Bartholomew, D.J. (1979). "The control of a grade structure in a stochastic environment using promotion control". Adv. Appl. Prob., **11**, pp. 603-615.

Bartholomew, D.J. (1982). Stochastic Models for Social Processes (3rd ed.), Wiley, Chichester.

Bohrer, R.E. and M.J. Schervish (1981). "An error-bounded algorithm for normal probabilities of rectangular regions". Technometrics, **23**, pp. 297-300.

Charnes, A., W.W. Cooper and R.J. Niehaus (1972). Studies in Manpower Planning, Office of Civilian Manpower Management, Department of the Navy, Washington D.C.

Davies, G.S. (1973). "Structural control in a graded manpower system". Man. Sci., **20**, pp. 76-84.

Davies, G.S. (1975). "Maintainability of structures in Markov chains in models under recruitment control". J. Appl. Prob., **12**, pp. 376-382.

Davies, G.S. (1982). "Control of grade sizes in partially stochastic Markov manpower model". J. App. Prob., **19**, pp. 439-443.

Davies, G.S. (1983). "A note on the geometric/probabilistic relationship in a Markov manpower Model". J. Appl. Prob., **20**, pp. 423-428.

Finucan, H.M. (1964). "The mode of a multinomial distribution". Biometrika, **51**, pp. 513-517.

Forbes, A.F. (1971). "Promotions and recruitment policies for the control of quasi-stationary hierarchical systems". In A.R. Smith (1971), pp. 401-414.

Gani, J. (1963). "Formulae for projecting enrolments and degrees awarded in universities". J.R. Statist. Soc., **A**, **126**, pp. 400-409.

Grinold, R.C. and K.T. Marshall (1977). Manpower Planning Models, North-Holland, New York.

Grinold, R.C. and R.E. Stanford (1974). "Optimal control of a graded manpower system". Man. Sci., **20**, pp. 1201-1216.

Iosifiscu, M. (1980). Finite Markov Processes and their Applications, Wiley, Chichester.

Mehlmann, A. (1980). "An approach to optimal recruitment and transition strategies for manpower systems using dynamic programming". J. Operat. Res. Soc., **31**, pp. 1009-1015.

Milton, R.C. (1972). "Computer evaluation of the multivariate normal integral". Technometrics, **14**, pp. 881-889.

Morgan, R.W. (1971). "The use of a steady state model to obtain the recruitment and promotion policies of an expanding organisation". In D.J. Bartholomew and A.R. Smith (1971), pp. 283-291.

Schervish, M.J. (1984). Algorithm AS 195. "Multivariate normal probabilities with error bound". Appl. Statist., **33**, pp. 81-87.

Seal, H.L. (1945). "The mathematics of a population composed of k stationary strata each recruited from the stratum below and supported at the lowest level by a uniform annual number of entrants". Biometrika, **33**, pp. 226-230.

Smith, A.R. (Ed.) (1971). Models for Manpower Systems, English Universities Press, London.

Vajda, S. (1947). "The stratified semi-stationary population". Biometrika, **34**, pp. 243-254.

Vajda, S. (1975). "Mathematical aspects of manpower planning". Operat. Res. Quart., **26**, pp. 527-542.

Vajda, S. (1978). Mathematics of Manpower Planning, Wiley, Chichester.

Young, A. and G. Almond (1961). "Predicting distributions of staff". Comp. J., **3**, pp. 246-250.

APPENDIX A

RESOLUTION OF SOME QUADRATIC PROGRAMMING PROBLEMS

A.1 INTRODUCTION

Consider the problem

$$\text{Minimise } f(x) = \sum_{j=1}^{k} (a_j - x_j)^2$$

Subject to:

$$\sum_{j=1}^{k} x_j = S$$

$$x_j \geq 0 \qquad j=1,2,\ldots,k$$

For its solution we will consider two cases:

- x_j are reals
- x_j are integers

In the latter case we will refer to the problem by $P(a,S)$, where $a=(a_1, a_2, \ldots, a_k)$ and S an integer.

A.2 CASE 1: x_j ARE REALS

$$\text{Let } F = \sum_{j=1}^{k} (a_j - x_j)^2 + \lambda\left(S - \sum_{j=1}^{k} x_j\right)$$

$$\frac{\partial F}{\partial x_j} = -2\,(a_j - x_j) - \lambda$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 2 & \text{if} \quad i = j \end{cases}$$

Therefore, according to Kuhn-Tucker theory, any point $x^* = (x_1^*, x_2^*, \ldots, x_k^*)$ which satisfies the first-order necessary conditions is minimum. Moreover, since Ω, the set of feasible points, is convex and f is convex on Ω, any relative minimum of f is a global minimum.

The necessary conditions for a minimum are:

$$\frac{\partial F}{\partial x_j} = -2\,(a_j - x_j^*) - \lambda \geq 0 \qquad \text{if } x_j^* = 0$$

$$\frac{\partial F}{\partial x_j} = -2\,(a_j - x_j^*) - \lambda = 0 \qquad \text{if } x_j^* > 0$$

$$\sum_{j=1}^{k} x_j^* = S$$

In order to find a solution to this system we will look first at some properties of any solution.

Proposition 1

If $x^*_{j_o} = 0$ and $x^*_{j_+} > 0$, then $a_{j_o} < a_{j_+}$

Proof: From Kuhn-Tucker conditions we have

$$-2\,(a_{j_+} - x^*_{j_+}) = \lambda \qquad (1)$$

$$-2\,a_{j_o} \geq \lambda \qquad (2)$$

Hence $\quad -2a_{j_o} \geq -2a_{j_+} + 2x^*_{j_+} > -2a_{j_+}$

i.e. $\quad a_{j_o} < a_{j_+}$

Therefore, if we suppose, without loss of generality that $a_1 \geq a_2 \geq \quad \geq a_k$, there should be an integer L such that:

$$x^*_j > 0 \qquad j \leq L$$

$$x^*_j = 0 \qquad j > L$$

Proposition 2

(i) $\quad (S - \sum_{j=1}^{L} a_j) + La_L > 0$

(ii) $\quad (S - \sum_{j=1}^{L+1} a_j) + (L+1)a_{L+1} \leq 0$

(iii) L is the largest p such that

$$(S - \sum_{j=1}^{p} a_j) + pa_p > 0$$

Proof of (i)

By definition of L we have:

$$2\,(x_j^* - a_j) = \lambda \qquad j=1,2,\ldots,L \qquad (3)$$

$$-\,2\,a_j \geq \lambda \qquad j=L+1,\ldots,k \qquad (4)$$

from which we can deduce:

$$2\left(\sum_{j=1}^{L} x_j^* - \sum_{j=1}^{L} a_j\right) = L\,\lambda \qquad (5)$$

$$2\left(\sum_{j=1}^{L-1} x_j^* - \sum_{j=1}^{L-1} a_j\right) = (L-1)\,\lambda \qquad (6)$$

But given that $x_j^* = 0$ for $j > L$ and $x_L^* > 0$ we have

$$\sum_{j=1}^{L} x_j^* = S \quad \text{and} \quad \sum_{i=1}^{L-1} x_j^* < S$$

Thus, the previous system becomes:

$$2\,\left(S - \sum_{j=1}^{L} a_j\right) = L\,\lambda \qquad (7)$$

$$2\,\left(S - \sum_{j=1}^{L-1} a_j\right) > (L-1)\,\lambda \qquad (8)$$

Substituting in (8), using (7), we obtain:

$$2\,\left(S - \sum_{j=1}^{L-1} a_j\right) > (L-1) \times \frac{1}{L} \times 2\,\left(S - \sum_{j=1}^{L} a_j\right)$$

which leads to:

$$\left(S - \sum_{j=1}^{L} a_j \right) + La_L > 0$$

Proof of (ii):

From (4) and (7) we deduce that:

$$2 \left(S - \sum_{j=1}^{L+1} a_j \right) \geq (L+1) \lambda \tag{9}$$

Substituting into (9), using (7) we obtain:

$$2 \left(S - \sum_{j=1}^{L+1} a_j \right) \geq (L+1) \frac{2}{L} \left(S - \sum_{j=1}^{L} a_j \right)$$

which leads to:

$$\left(S - \sum_{j=1}^{L+1} a_j \right) + (L+1) \; a_{L+1} \leq 0$$

Proof of (iii):

Let $t(p) = \left(S - \sum_{j=1}^{p} a_j \right) + pa_p$. Then:

$t(p+1) - t(p)$

$$= \left(S - \sum_{j=1}^{p+1} a_j \right) + (p+1)a_{p+1} - \left(S - \sum_{j=1}^{p} a_j \right) - pa_p.$$

$$= - a_{p+1} + (p+1)a_{p+1} - pa_p$$

$$= p(a_{p+1} - a_p) \leq 0 \quad \text{(we assumed that } a_{i+1} \leq a_i\text{)}.$$

And since $t(L) > 0 \geq t(L+1)$, $L = \text{Max}\{p \in \mathbb{N} \,/\, t(p) > 0\}$.

This last property will allow us to find L easily; knowing L, the minimum to our problem will be:

$$x_j^* = \frac{\lambda}{2} + a_j \qquad j \leq L$$

$$x_j^* = 0 \qquad j > L$$

$$\lambda = 2 \, \frac{S - \sum_{j=1}^{L} a_j}{L}$$

A.3 CASE 2: x_j ARE INTEGERS

Our approach in finding a solution when S is integer, rests on two fundamental propositions:

Proposition 3

Let S_1 and S_2 be two integers such that

$$0 < S_1 \leq S \quad \text{and} \quad S_2 = S - S_1$$

If m^{*S_1} is an optimal solution to $P(a, S_1)$ and m^{*S_2} is an optimal solution to $P(a-m^{*S_1}, S_2)$ then $m^{*S} = m^{*S_1}+m^{*S_2}$ is an optimal solution to $P(a, S)$.

Proof:

Let $m = (m_1, m_2, \ldots, m_k)$ be an integer vector such that

$$\sum_{i=1}^{k} m_i = S \text{ and } m_i \geq 0, \qquad i=1,2,\ldots,k.$$

Given the definition of m^{*S_2}, we have:

$$\sum_{i=1}^{k} (a_i - m_i)^2 = \sum_{i=1}^{k} [(a_i - m_i^{*S_1}) - (m_i - m_i^{*S_1})]^2$$

$$\geq \sum_{i=1}^{k} [(a_i - m_i^{*S_1}) - m_i^{*S_2}]^2$$

Since $\sum_{i=1}^{k} (a_i - m_i)^2 \geq \sum_{i=1}^{k} [a_i - (m_i^{*S_1} + m_i^{*S_2})]^2$

for any m and since $\sum_{i=1}^{k} (m_i^{*S_1} + m_i^{*S_2}) = S$

$m^* = m^{*S_1} + m^{*S_2}$ is an optimal solution to P(a, S).

Proposition 4:

If S = 1 and j an integer such that $a_j \geq a_i$ for all $i=1,2,\ldots,k$, then $m^{*S} = e^j$, where e^j is the vector with one at the j^{th} position and zeros elsewhere.

Proof

Consider any solution m. Since S=1 there should be a j' such that $m_{j'} = 1$ and $m_i = 0$ for $i \neq j'$. In this case we will have:

$$\sum_{i=1}^{k} (a_i - m_i)^2 = \sum_{i=1}^{k} a_i^2 - 2a_{j'} + 1$$

$$\geq \sum_{i=1}^{k} a_i^2 - 2a_j + 1 = \sum_{i=1}^{k} (a_i - m^{*S}_i)^2$$

The combination of these two propositions suggests an iterative approach which solves successively the problems $P(a,1)$, $P(a,2)$, ..., $P(a,S)$ taking always $S_1=1$. More precisely the iterative method is as follows:

1° Set $m_j = 0$ $\quad j=1,2,\ldots,k$

2° If S is zero, go to 4

3° Find the maximum a_j. Put:

$m_j = m_j + 1$

$a_j = a_j - 1$

$S = S-1$

Go to 2

4° The solution of $P(a,S)$ is $m^{*S}=(m_1,m_2,\ldots,m_k)$.

STOP

However, when S is large, this method would be very long. Fortunately, the two following propositions will allow us to introduce an improvement;

Proposition 5

If: m^* is an optimal solution to P(a,S),

b is an integer vector such that $0 \leq b \leq m^*$,

$d = a-b$ and $S' = S - \sum_{i=1}^{k} b_i$ and

n^* is an optimal solution to P(d,S')

Then n^*+b is an optimal solution to P(a,S).

Proof

$$\sum_{i=1}^{k} (n_i^* + b_i) = S' + \sum_{i=1}^{k} b_i = S.$$

Since $n_i^* \geq 0$ and $b_i \geq 0$ then $n_i^* + b_i \geq 0$.

Moreover $\sum_{j=1}^{k} (a_i - m_i^*)^2 = \sum_{i=1}^{k} [(a_i-b_i) - (m_i^* - b_i)]^2$

i.e. $\sum_{i=1}^{k} (a_i - m_i^*)^2 = \sum_{i=1}^{k} [d_i - (m_i^* - b_i)]^2$

Thus, since $m_i^* - b_i \geq 0$ and $\sum_{i=1}^{k} (m_i^* - b_i) = S'$

and given the definition of n^*, we have

$$\sum_{i=1}^{k} (a_i - m_i^*)^2 \geq \sum_{i=1}^{k} (d_i - n_i^*)^2 = \sum_{i=1}^{k} [a_i - (n_i^* - b_i)]^2$$

Therefore $n^* + b$ is an optimal solution to $P(a,S)$.

Note that in this proof we did not use the fact that the different vectors are integers and therefore this proposition would apply to the real case.

Proposition 6

Let us consider the problem $P(a,S)$. If x^* is the real optimal solution and m^* is its integer optimal solution, then:

$$m_i^* \geq [x_i^*] \qquad\qquad i=1,2,\ldots,k$$

Proof

As we saw in the previous section, there exists an integer L such that $x_j^* = 0$ for any $j > L$.

Therefore if $i > L$, $[x_i^*] = 0$ and by definition of m^*, $m_i^* \geq 0 = [x_i^*]$.

Suppose that there exists an integer $j \leq L$ such that $m_j^* < [x_j^*]$. Then we should have at least one other $p \leq L$ such that:

$m_p^* > [x_p^*]$, otherwise $\sum_{i=1}^{L} m_i^* < \sum_{i=1}^{L} [x_i^*] \leq S$

Thus, since $m_j^* \leq x_j^* - 1$ we have

$$a_j - m_j^* \geq a_j - x_j^* + 1 = -\frac{\lambda}{2} + 1,$$ where λ is the same as defined in the previous section.

Moreover, since $m_p^* > [x_p^*]$ we have $m_p^* > x_p^*$ which gives

$$a_p - m_p^* < a_p - x_p^* = -\frac{\lambda}{2}$$

Therefore $a_p - m_p^* < a_j - m_j^* - 1$ (10)

Let $m_j' = m_j^* + 1$ and $m_p' = m_p^* - 1$. We have:

$$(a_j - m_j')^2 + (a_p - m_p')^2$$

$$= (a_j - m_j^* - 1)^2 + (a_p - m_p^* - 1)^2$$

$$= (a_j - m_j^*)^2 + (a_p - m_p^*)^2 - 2\left\{(a_j - m_j^*) - (a_p - m_p^*) - 1\right\}$$

$$\leq (a_j - m_j^*)^2 + (a_p - m_p^*)^2$$ because of inequation (10).

This result contradicts the definition of m^*. Thus, there is no $j \leq L$ such that $m_j^* < [x_j^*]$.

These two propositions suggest the following procedure:

1° Release the constraint of integrity of the variables and solve the problem as in Case 1.

2° Define $a_j' = a_j - [x_j^*]$ $\qquad j=1,2,\ldots,k$

$$S' = S - \sum_{j=1}^{L} [x_j^*]$$

3° Solve the problem P(a', S') using the previous iterative method.

4° Then, if n* is an optimal solution of P(a', S'), $m_i^* = n_i^* + [x_i^*]$, i=1,2,...,k is an optimal solution to P(a, S).

Proposition 7

(i) $S' \leq L$

(ii) There is an optimal solution n* of P(a', S') such that $n_i^* = 0$ $\qquad i=L+1, L+2, \ldots,k$

Proof of (i)

$$S' = S - \sum_{j=1}^{L} [x_j^*] \leq S - \sum_{j=1}^{L} (x_j^* - 1) = S-S + L = L$$

Proof of (ii)

Let $i > L$ and $j \leq L$

We have $a_j - [x^*_j] \geq a_j - x^*_j = - \frac{\lambda}{2} \geq a_i$

which is equivalent to $a'_j \geq a'_j$.

Since S' is at most equal to L and since there are L coefficients a'_j bigger than a_i, there should be an optimal solution with $n_i = 0$ for any $i > L$ because of propositions 3 and 4.

APPENDIX B

DETAILED RESULTS CONCERNING THE EVOLUTION OF 24 EXAMPLES UNDER DIFFERENT RECRUITMENT STRATEGIES

In this Appendix, tables are self-explanatory and their numbers consist of a letter X and a serial number, where:

$$X = \begin{cases} F & \text{in the case of fixed strategy} \\ A & \text{in the case of adaptive strategy} \\ G & \text{in the case of goal strategy.} \end{cases}$$

The examples considered are those given in tables 9 and 10; their serial numbers are as follows:

Transition matrix	Total Size	Structures maintained by recruitment at the bottom	Serial Number	Structures maintained by equal recruitment at each level	Serial Number
$P_1 = \begin{matrix} 0.70 & 0.20 & \\ & 0.80 & 0.10 \\ & & 0.90 \end{matrix}$	12	(4,4,4)	1	(1,3,8)	2
	24	(8,8,8)	3	(3,7,14)	4
$P_3 = \begin{matrix} 0.54 & 0.16 & \\ & 0.62 & 0.08 \\ & & 0.70 \end{matrix}$	12	(7,3,2)	5	(3,4,5)	6
	24	(15,7,2)	7	(5,9,10)	8
$P_2 = \begin{matrix} 0.50 & 0.40 & \\ & 0.60 & 0.30 \\ & & 0.80 \end{matrix}$	12	(3,3,6)	9	(1,3,8)	10
	24	(6,7,11)	11	(3,6,15)	12
$P_4 = \begin{matrix} 0.39 & 0.31 & \\ & 0.47 & 0.23 \\ & & 0.60 \end{matrix}$	12	(5,4,3)	13	(2,4,6)	14
	24	(11,8,5)	15	(4,8,12)	16
$P_5 = \begin{matrix} 0.40 & 0.20 & 0.20 \\ 0.20 & 0.40 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{matrix}$	12	(6,3,3)	17	(4,4,4)	18
	24	(12,6,6)	19	(8,8,8)	20
$P_6 = \begin{matrix} 0.6 & 0.3 & & \\ & 0.7 & 0.2 & \\ & & 0.8 & 0.1 \\ & & & 0.9 \end{matrix}$	12	(3,3,3,3)	21	(1,2,3,6)	22
	48	(12,12,12,12)	23	(3,7,13,25)	24

TABLE F 1: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 4 , 4) ;GOAL VECTOR M=(4 4 4) TRANSITION MATRIX P=
VECTOR M-M*P =(1.20 ,-0.00 , 0.00)

$$P = \begin{pmatrix} 0.700 & 0.200 & 0.000 \\ 0.000 & 0.800 & 0.100 \\ 0.000 & 0.000 & 0.900 \end{pmatrix}$$

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	4.00	4.00	4.00	12.00	1.36	1.28	0.72	3.36	1.36	1.28	0.72	3.36
2	4.00	4.00	4.00	12.00	2.03	1.85	1.26	5.13	2.03	1.85	1.26	5.13
3	4.00	4.00	4.00	12.00	2.35	2.12	1.66	6.14	2.35	2.12	1.66	6.14
4	4.00	4.00	4.00	12.00	2.51	2.28	1.96	6.75	2.51	2.28	1.96	6.75
5	4.00	4.00	4.00	12.00	2.59	2.39	2.17	7.15	2.59	2.39	2.17	7.15
6	4.00	4.00	4.00	12.00	2.63	2.46	2.33	7.42	2.63	2.46	2.33	7.42
7	4.00	4.00	4.00	12.00	2.65	2.52	2.44	7.60	2.65	2.52	2.44	7.60
8	4.00	4.00	4.00	12.00	2.66	2.56	2.51	7.73	2.66	2.56	2.51	7.73
9	4.00	4.00	4.00	12.00	2.66	2.59	2.56	7.82	2.66	2.59	2.56	7.82
10	4.00	4.00	4.00	12.00	2.67	2.61	2.60	7.88	2.67	2.61	2.60	7.88
AVERAGE	4.00	4.00	4.00	12.00	2.41	2.27	2.02	6.70	2.41	2.27	2.02	6.70

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.99	4.00	4.01	12.00	1.36	1.30	0.73	3.39	1.36	1.30	0.73	3.39
2	3.99	4.02	3.99	12.00	2.06	1.90	1.27	5.22	2.06	1.90	1.27	5.22
3	4.02	4.00	3.99	12.00	2.36	2.13	1.67	6.16	2.36	2.13	1.67	6.16
4	4.03	3.98	4.00	12.00	2.51	2.26	1.95	6.72	2.51	2.26	1.95	6.72
5	4.04	3.98	3.99	12.00	2.58	2.37	2.16	7.12	2.58	2.38	2.16	7.12
6	4.02	3.98	4.00	12.00	2.65	2.50	2.34	7.49	2.65	2.50	2.34	7.49
7	4.00	3.99	4.01	12.00	2.65	2.53	2.45	7.63	2.65	2.53	2.45	7.63
8	4.00	3.99	4.01	12.00	2.67	2.59	2.56	7.81	2.67	2.59	2.56	7.81
9	4.02	3.99	4.00	12.00	2.72	2.68	2.59	7.98	2.72	2.68	2.59	7.98
10	4.00	3.99	4.00	12.00	2.71	2.71	2.64	8.06	2.71	2.71	2.64	8.06
AVERAGE	4.01	3.99	4.00	12.00	2.43	2.30	2.03	6.76	2.43	2.30	2.03	6.76

TABLE A 1: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 4 , 4) ;GOAL VECTOR M=(4 4 4) TRANSITION MATRIX P=

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

VECTOR M-N*P =(1.20 ,-0.00 , 0.00)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.56	4.26	4.18	12.00	0.48	0.75	0.43	1.66	0.67	0.82	0.46	1.95
2	3.33	4.34	4.33	12.00	0.71	1.09	0.75	2.54	1.16	1.20	0.86	3.21
3	3.21	4.34	4.45	12.00	0.83	1.25	0.99	3.07	1.46	1.37	1.20	4.02
4	3.13	4.31	4.56	12.00	0.89	1.34	1.17	3.40	1.64	1.44	1.48	4.56
5	3.09	4.27	4.64	12.00	0.92	1.39	1.30	3.62	1.75	1.47	1.71	4.93
6	3.06	4.23	4.70	12.00	0.94	1.42	1.40	3.76	1.82	1.48	1.90	5.19
7	3.05	4.20	4.76	12.00	0.95	1.44	1.47	3.86	1.86	1.48	2.04	5.38
8	3.03	4.17	4.80	12.00	0.96	1.45	1.53	3.93	1.89	1.48	2.16	5.53
9	3.03	4.15	4.83	12.00	0.96	1.46	1.57	3.98	1.91	1.48	2.25	5.64
10	3.02	4.13	4.85	12.00	0.96	1.46	1.60	4.02	1.92	1.48	2.32	5.72
AVERAGE	3.15	4.24	4.61	12.00	0.86	1.31	1.22	3.38	1.61	1.37	1.64	4.61

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.56	4.26	4.19	12.00	0.48	0.75	0.44	1.66	0.67	0.81	0.47	1.95
2	3.34	4.33	4.33	12.00	0.68	1.08	0.76	2.52	1.11	1.19	0.87	3.17
3	3.21	4.33	4.46	12.00	0.83	1.28	1.00	3.11	1.45	1.40	1.20	4.05
4	3.13	4.31	4.56	12.00	0.90	1.34	1.16	3.40	1.65	1.44	1.48	4.56
5	3.09	4.27	4.64	12.00	0.92	1.38	1.30	3.60	1.75	1.45	1.71	4.92
6	3.07	4.23	4.70	12.00	0.94	1.44	1.43	3.81	1.82	1.49	1.92	5.23
7	3.05	4.20	4.76	12.00	0.96	1.44	1.51	3.91	1.87	1.48	2.08	5.43
8	3.04	4.16	4.80	12.00	0.94	1.47	1.55	3.96	1.86	1.50	2.19	5.55
9	3.03	4.14	4.83	12.00	0.97	1.46	1.59	4.02	1.91	1.48	2.28	5.66
10	3.03	4.12	4.86	12.00	0.95	1.47	1.59	4.01	1.89	1.49	2.32	5.70
AVERAGE	3.15	4.23	4.61	12.00	0.86	1.31	1.23	3.40	1.60	1.37	1.65	4.62

TABLE G 1: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 4 , 4) ;GOAL VECTOR M=(4 4 4) TRANSITION MATRIX P=
VECTOR M-M*P =(1.20 ,-0.00 , 0.00)

$$P = \begin{pmatrix} 0.700 & 0.200 & 0.000 \\ 0.000 & 0.800 & 0.100 \\ 0.000 & 0.000 & 0.900 \end{pmatrix}$$

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL
1	4.00	4.00	4.00	12.00	1.36	1.28	0.72	3.35	1.36	1.28	0.72	3.35
2	3.99	4.00	4.01	12.00	1.99	1.85	1.23	5.07	1.99	1.85	1.23	5.07
3	3.98	4.00	4.02	12.00	2.24	2.12	1.56	5.92	2.24	2.12	1.56	5.92
4	3.95	4.00	4.06	12.00	2.27	2.28	1.72	6.27	2.27	2.28	1.73	6.27
5	3.90	3.99	4.12	12.00	2.22	2.37	1.74	6.33	2.24	2.37	1.76	6.36
6	3.85	3.97	4.18	12.00	2.20	2.41	1.70	6.31	2.22	2.41	1.73	6.36
7	3.79	3.94	4.26	12.00	2.21	2.42	1.60	6.23	2.25	2.42	1.67	6.34
8	3.73	3.93	4.34	12.00	2.05	2.31	1.55	5.91	2.12	2.31	1.66	6.09
9	3.53	3.98	4.49	12.00	1.49	1.99	1.36	4.84	1.71	2.00	1.60	5.30
10	3.26	4.17	4.57	12.00	1.01	1.51	1.37	3.88	1.55	1.53	1.69	4.77
AVERAGE	3.80	4.00	4.20	12.00	1.90	2.05	1.45	5.41	1.99	2.06	1.53	5.58

TABLE F 2: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8) TRANSITION MATRIX P=
VECTOR M-M•P =(0.30 0.40 , 0.50)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.00	3.00	8.00	12.00	0.39	0.73	0.70	1.81	0.39	0.73	0.70	1.81
2	1.00	3.00	8.00	12.00	0.58	1.14	1.18	2.89	0.58	1.14	1.18	2.89
3	1.00	3.00	8.00	12.00	0.67	1.39	1.50	3.57	0.67	1.39	1.50	3.57
4	1.00	3.00	8.00	12.00	0.72	1.56	1.73	4.00	0.72	1.56	1.73	4.00
5	1.00	3.00	8.00	12.00	0.74	1.67	1.88	4.29	0.74	1.67	1.88	4.29
6	1.00	3.00	8.00	12.00	0.75	1.74	1.99	4.47	0.75	1.74	1.99	4.47
7	1.00	3.00	8.00	12.00	0.76	1.79	2.06	4.60	0.76	1.79	2.06	4.60
8	1.00	3.00	8.00	12.00	0.76	1.82	2.10	4.68	0.76	1.82	2.10	4.68
9	1.00	3.00	8.00	12.00	0.76	1.85	2.13	4.74	0.76	1.85	2.13	4.74
10	1.00	3.00	8.00	12.00	0.76	1.86	2.15	4.78	0.76	1.86	2.15	4.78
AVERAGE	1.00	3.00	8.00	12.00	0.69	1.55	1.74	3.98	0.69	1.55	1.74	3.98

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	0.95	3.07	7.99	12.00	0.37	0.79	0.67	1.83	0.37	0.80	0.67	1.84
2	0.91	3.12	7.98	12.00	0.54	1.24	1.11	2.89	0.55	1.26	1.11	2.92
3	0.88	3.15	7.97	12.00	0.62	1.50	1.39	3.52	0.64	1.53	1.39	3.55
4	0.87	3.15	7.98	12.00	0.67	1.66	1.62	3.96	0.69	1.69	1.62	3.99
5	0.86	3.15	7.99	12.00	0.69	1.76	1.75	4.20	0.70	1.79	1.75	4.25
6	0.84	3.16	8.00	12.00	0.69	1.81	1.82	4.31	0.71	1.83	1.82	4.36
7	0.83	3.16	8.01	12.00	0.69	1.88	1.89	4.46	0.72	1.90	1.89	4.51
8	0.83	3.15	8.02	12.00	0.68	1.92	1.96	4.56	0.71	1.94	1.96	4.61
9	0.83	3.14	8.03	12.00	0.68	1.94	1.98	4.60	0.71	1.96	1.98	4.65
10	0.82	3.15	8.03	12.00	0.67	1.93	1.98	4.58	0.70	1.95	1.98	4.63
AVERAGE	0.86	3.14	8.00	12.00	0.63	1.64	1.62	3.89	0.65	1.66	1.62	3.93

TABLE A 2: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8) TRANSITION MATRIX P=
VECTOR M-M•P =(0.30 , 0.40 , 0.50)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	0.94	2.98	8.08	12.00	0.06	0.24	0.22	0.52	0.06	0.24	0.23	0.53
2	0.92	2.97	8.12	12.00	0.08	0.34	0.33	0.74	0.08	0.34	0.34	0.76
3	0.91	2.96	8.13	12.00	0.08	0.38	0.37	0.83	0.09	0.38	0.39	0.86
4	0.91	2.95	8.14	12.00	0.09	0.39	0.40	0.88	0.10	0.40	0.42	0.91
5	0.90	2.95	8.15	12.00	0.09	0.40	0.41	0.90	0.10	0.40	0.43	0.93
6	0.90	2.95	8.15	12.00	0.09	0.41	0.42	0.91	0.10	0.41	0.44	0.94
7	0.90	2.95	8.15	12.00	0.09	0.41	0.42	0.91	0.10	0.41	0.44	0.95
8	0.90	2.95	8.15	12.00	0.09	0.41	0.42	0.92	0.10	0.41	0.44	0.95
9	0.90	2.95	8.15	12.00	0.09	0.41	0.42	0.92	0.10	0.41	0.44	0.95
10	0.90	2.95	8.15	12.00	0.09	0.41	0.42	0.92	0.10	0.41	0.44	0.95
AVERAGE	0.91	2.95	8.14	12.00	0.08	0.38	0.38	0.84	0.09	0.38	0.40	0.87

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	0.94	2.99	8.08	12.00	0.06	0.23	0.21	0.51	0.06	0.23	0.22	0.52
2	0.92	2.97	8.11	12.00	0.07	0.33	0.33	0.73	0.08	0.33	0.34	0.75
3	0.91	2.96	8.14	12.00	0.08	0.38	0.38	0.84	0.09	0.38	0.40	0.87
4	0.90	2.95	8.14	12.00	0.09	0.39	0.40	0.88	0.10	0.39	0.42	0.91
5	0.90	2.95	8.15	12.00	0.09	0.40	0.42	0.91	0.10	0.40	0.44	0.95
6	0.90	2.94	8.16	12.00	0.09	0.41	0.43	0.93	0.10	0.42	0.45	0.97
7	0.90	2.94	8.17	12.00	0.09	0.41	0.44	0.94	0.10	0.42	0.47	0.99
8	0.90	2.95	8.15	12.00	0.09	0.40	0.42	0.91	0.10	0.40	0.45	0.95
9	0.90	2.94	8.15	12.00	0.09	0.41	0.43	0.92	0.10	0.41	0.45	0.96
10	0.90	2.95	8.15	12.00	0.09	0.40	0.41	0.90	0.10	0.40	0.43	0.93
AVERAGE	0.91	2.95	8.14	12.00	0.08	0.38	0.39	0.85	0.09	0.38	0.41	0.88

TABLE G 2: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE <u>GOAL</u> RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8) TRANSITION MATRIX P=

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

VECTOR M-M•P =(0.30 , 0.40 , 0.50)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.70	2.62	7.68	12.00	0.68	0.58	0.53	1.79	1.17	0.73	0.63	2.52
2	1.97	2.51	7.52	12.00	0.77	0.81	0.56	2.13	1.71	1.05	0.79	3.55
3	1.81	2.73	7.45	12.00	0.47	0.80	0.50	1.78	1.13	0.88	0.80	2.81
4	1.71	2.66	7.63	12.00	0.38	0.85	0.45	1.69	0.89	0.97	0.59	2.44
5	1.52	2.60	7.88	12.00	0.38	0.82	0.35	1.55	0.65	0.98	0.36	1.99
6	1.25	2.75	8.00	12.00	0.31	0.70	0.30	1.31	0.38	0.76	0.30	1.44
7	1.08	2.85	8.07	12.00	0.22	0.54	0.32	1.07	0.22	0.56	0.32	1.10
8	0.94	2.90	8.16	12.00	0.21	0.47	0.27	0.95	0.21	0.48	0.30	0.99
9	0.87	2.91	8.21	12.00	0.18	0.41	0.27	0.87	0.20	0.42	0.32	0.94
10	0.91	2.92	8.17	12.00	0.12	0.38	0.39	0.89	0.12	0.38	0.42	0.93
AVERAGE	1.38	2.75	7.88	12.00	0.37	0.64	0.39	1.40	0.67	0.72	0.48	1.87

TABLE F 3: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(8 , 8 , 8) ;GOAL VECTOR M=(8 8 8) TRANSITION MATRIX P=
VECTOR M-M•P =(2.40 , 0.00 , 0.00)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	Expected stock vector: Grade 1	Grade 2	Grade 3	Total	Variances of the stock vectors: Grade 1	Grade 2	Grade 3	Total	M.S.E. of the stock vectors: Grade 1	Grade 2	Grade 3	Total
1	8.00	8.00	8.00	24.00	2.72	2.56	1.44	6.72	2.72	2.56	1.44	6.72
2	8.00	8.00	8.00	24.00	4.05	3.69	2.52	10.26	4.05	3.69	2.52	10.26
3	8.00	8.00	8.00	24.00	4.71	4.25	3.32	12.28	4.71	4.25	3.32	12.28
4	8.00	8.00	8.00	24.00	5.03	4.57	3.92	13.51	5.03	4.57	3.92	13.51
5	8.00	8.00	8.00	24.00	5.18	4.78	4.35	14.30	5.18	4.78	4.35	14.30
6	8.00	8.00	8.00	24.00	5.26	4.93	4.65	14.84	5.26	4.93	4.65	14.84
7	8.00	8.00	8.00	24.00	5.30	5.04	4.87	15.21	5.30	5.04	4.87	15.21
8	8.00	8.00	8.00	24.00	5.32	5.12	5.02	15.46	5.32	5.12	5.02	15.46
9	8.00	8.00	8.00	24.00	5.33	5.18	5.12	15.63	5.33	5.18	5.12	15.63
10	8.00	8.00	8.00	24.00	5.33	5.23	5.19	15.75	5.33	5.23	5.19	15.75
AVERAGE	8.00	8.00	8.00	24.00	4.82	4.53	4.04	13.40	4.82	4.53	4.04	13.40

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	Expected stock vector: Grade 1	Grade 2	Grade 3	Total	Variances of the stock vectors: Grade 1	Grade 2	Grade 3	Total	M.S.E. of the stock vectors: Grade 1	Grade 2	Grade 3	Total
1	8.00	8.00	8.00	24.00	2.69	2.52	1.46	6.67	2.69	2.52	1.46	6.67
2	8.01	7.97	8.02	24.00	4.03	3.75	2.48	10.25	4.03	3.75	2.48	10.26
3	8.00	7.97	8.03	24.00	4.61	4.26	3.27	12.15	4.61	4.26	3.27	12.15
4	8.00	7.97	8.03	24.00	5.03	4.54	3.85	13.42	5.03	4.54	3.85	13.42
5	8.00	7.98	8.02	24.00	5.10	4.64	4.31	14.05	5.10	4.64	4.31	14.05
6	7.99	7.98	8.03	24.00	5.30	4.91	4.66	14.86	5.30	4.91	4.66	14.86
7	8.00	7.99	8.02	24.00	5.23	5.04	4.93	15.21	5.23	5.04	4.93	15.21
8	8.03	7.96	8.01	24.00	5.29	5.13	5.05	15.47	5.29	5.14	5.05	15.48
9	8.02	7.99	7.99	24.00	5.32	5.25	5.17	15.75	5.32	5.25	5.17	15.75
10	7.99	8.01	7.99	24.00	5.42	5.31	5.29	16.01	5.42	5.31	5.29	16.01
AVERAGE	8.00	7.98	8.01	24.00	4.80	4.53	4.05	13.38	4.80	4.54	4.05	13.38

TABLE A 3: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(8 , 8 , 8) ;GOAL VECTOR M=(8 8 8) TRANSITION MATRIX P=
VECTOR M-N*P =(2.40 , 0.00 , 0.00)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	7.31	8.42	8.27	24.00	0.91	1.35	0.83	3.10	1.39	1.52	0.91	3.81
2	6.97	8.54	8.50	24.00	1.38	1.98	1.45	4.80	2.44	2.27	1.69	6.40
3	6.78	8.54	8.68	24.00	1.6	2.31	1.91	5.84	3.11	2.60	2.37	8.08
4	6.67	8.50	8.83	24.00	1.76	2.48	2.25	6.49	3.52	2.73	2.94	9.19
5	6.61	8.44	8.95	24.00	1.83	2.58	2.51	6.92	3.77	2.77	3.41	9.95
6	6.57	8.39	9.05	24.00	1.87	2.64	2.70	7.21	3.92	2.79	3.80	10.50
7	6.54	8.34	9.12	24.00	1.89	2.67	2.84	7.41	4.02	2.78	4.10	10.90
8	6.52	8.30	9.18	24.00	1.91	2.69	2.95	7.55	4.09	2.78	4.34	11.20
9	6.51	8.26	9.23	24.00	1.92	2.70	3.03	7.65	4.13	2.77	4.53	11.43
10	6.50	8.24	9.26	24.00	1.92	2.71	3.09	7.72	4.17	2.76	4.68	11.61
AVERAGE	6.70	8.39	8.91	24.00	1.70	2.41	2.35	6.47	3.45	2.58	3.28	9.31

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	7.34	8.39	8.28	24.00	0.88	1.32	0.85	3.04	1.32	1.47	0.92	3.71
2	6.97	8.51	8.52	24.00	1.36	2.00	1.47	4.82	2.42	2.25	1.75	6.41
3	6.79	8.50	8.71	24.00	1.57	2.26	1.89	5.72	3.04	2.51	2.40	7.94
4	6.71	8.47	8.82	24.00	1.70	2.43	2.23	6.36	3.37	2.65	2.90	8.92
5	6.62	8.43	8.95	24.00	1.81	2.58	2.49	6.88	3.71	2.77	3.38	9.86
6	6.59	8.37	9.04	24.00	1.87	2.61	2.71	7.19	3.86	2.75	3.79	10.40
7	6.56	8.33	9.12	24.00	1.89	2.70	2.85	7.45	3.97	2.81	4.10	10.88
8	6.51	8.31	9.18	24.00	1.92	2.76	2.92	7.60	4.13	2.86	4.32	11.31
9	6.49	8.28	9.23	24.00	1.95	2.78	3.02	7.75	4.22	2.86	4.53	11.61
10	6.51	8.25	9.24	24.00	1.94	2.70	3.04	7.67	4.15	2.76	4.59	11.50
AVERAGE	6.71	8.38	8.91	24.00	1.69	2.41	2.35	6.45	3.42	2.57	3.27	9.25

TABLE G 3: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(8 , 8 , 8) ; GOAL VECTOR N=(8 8 8) TRANSITION MATRIX P=
VECTOR M-N*P =(2.40 , 0.00 , 0.00)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	8.00	8.00	8.00	24.00	2.72	2.56	1.44	6.71	2.72	2.56	1.44	6.71
2	7.99	8.00	8.01	24.00	3.99	3.69	2.46	10.14	3.99	3.69	2.46	10.14
3	7.97	8.00	8.03	24.00	4.50	4.25	3.13	11.88	4.51	4.25	3.13	11.89
4	7.94	7.99	8.07	24.00	4.64	4.56	3.48	12.67	4.64	4.56	3.49	12.68
5	7.88	7.98	8.14	24.00	4.55	4.74	3.53	12.82	4.56	4.74	3.55	12.85
6	7.82	7.96	8.22	24.00	4.56	4.83	3.48	12.87	4.60	4.83	3.53	12.95
7	7.74	7.93	8.33	24.00	4.42	4.85	3.32	12.59	4.48	4.86	3.43	12.77
8	7.65	7.90	8.45	24.00	4.24	4.81	3.07	12.13	4.36	4.83	3.28	12.46
9	7.47	7.94	8.60	24.00	3.41	4.22	2.84	10.47	3.70	4.22	3.20	11.12
10	6.90	8.33	8.77	24.00	1.88	2.74	2.63	7.24	3.09	2.85	3.22	9.16
AVERAGE	7.73	8.00	8.26	24.00	3.89	4.12	2.94	10.95	4.06	4.14	3.07	11.27

TABLE F 4: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE <u>FIXED</u> RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 7 ,14) ;GOAL VECTOR M=(3 7 ,14)
VECTOR M-M*P =(0.90 , 0.80 , 0.70)

TRANSITION MATRIX P=

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.00	7.00	14.00	24.00	0.95	1.58	1.47	3.99	0.95	1.58	1.47	3.99
2	3.00	7.00	14.00	24.00	1.41	2.46	2.49	6.35	1.41	2.46	2.49	6.35
3	3.00	7.00	14.00	24.00	1.64	2.99	3.19	7.81	1.64	2.99	3.19	7.81
4	3.00	7.00	14.00	24.00	1.75	3.32	3.67	8.74	1.75	3.32	3.67	8.74
5	3.00	7.00	14.00	24.00	1.80	3.55	4.00	9.36	1.80	3.55	4.00	9.36
6	3.00	7.00	14.00	24.00	1.83	3.71	4.23	9.76	1.83	3.71	4.23	9.76
7	3.00	7.00	14.00	24.00	1.84	3.81	4.38	10.04	1.84	3.81	4.38	10.04
8	3.00	7.00	14.00	24.00	1.85	3.89	4.48	10.22	1.85	3.89	4.48	10.22
9	3.00	7.00	14.00	24.00	1.85	3.94	4.55	10.34	1.85	3.94	4.55	10.34
10	3.00	7.00	14.00	24.00	1.85	3.98	4.60	10.43	1.85	3.98	4.60	10.43
AVERAGE	3.00	7.00	14.00	24.00	1.68	3.32	3.71	8.70	1.68	3.32	3.71	8.70

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.15	6.90	13.95	24.00	1.09	1.69	1.44	4.22	1.11	1.70	1.44	4.25
2	3.29	6.81	13.91	24.00	1.60	2.60	2.47	6.67	1.68	2.64	2.48	6.80
3	3.40	6.77	13.83	24.00	1.91	3.15	3.08	8.14	2.08	3.20	3.11	8.38
4	3.45	6.77	13.77	24.00	2.10	3.47	3.55	9.11	2.30	3.52	3.60	9.42
5	3.48	6.82	13.71	24.00	2.20	3.67	3.82	9.69	2.43	3.70	3.91	10.04
6	3.52	6.83	13.65	24.00	2.22	3.91	4.16	10.29	2.49	3.93	4.28	10.70
7	3.52	6.88	13.60	24.00	2.25	4.04	4.26	10.55	2.52	4.06	4.42	10.99
8	3.55	6.87	13.58	24.00	2.29	4.10	4.41	10.80	2.59	4.11	4.59	11.29
9	3.57	6.90	13.54	24.00	2.28	4.15	4.43	10.86	2.60	4.16	4.64	11.40
10	3.57	6.89	13.53	24.00	2.28	4.16	4.50	10.94	2.61	4.17	4.72	11.50
AVERAGE	3.45	6.84	13.71	24.00	2.02	3.49	3.61	9.12	2.24	3.52	3.72	9.48

TABLE A 4: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 7 ,14) ;GOAL VECTOR M=(3 7 ,14) TRANSITION MATRIX P=
VECTOR M-M•P =(0.90 , 0.80 , 0.70)

0.700	0.200	0.000
0.000	0.800	0.100
0.000	0.000	0.900

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	2.88	7.00	14.13	24.00	0.14	0.45	0.44	1.03	0.15	0.45	0.46	1.06
2	2.83	6.97	14.19	24.00	0.18	0.59	0.63	1.41	0.21	0.60	0.67	1.47
3	2.82	6.96	14.23	24.00	0.20	0.65	0.72	1.57	0.23	0.65	0.77	1.66
4	2.81	6.94	14.25	24.00	0.21	0.67	0.77	1.65	0.24	0.67	0.83	1.75
5	2.81	6.94	14.26	24.00	0.21	0.68	0.79	1.68	0.25	0.68	0.86	1.79
6	2.80	6.93	14.27	24.00	0.21	0.68	0.81	1.71	0.25	0.69	0.88	1.82
7	2.80	6.93	14.27	24.00	0.22	0.69	0.82	1.72	0.25	0.69	0.89	1.83
8	2.80	6.93	14.27	24.00	0.22	0.69	0.82	1.72	0.26	0.69	0.89	1.84
9	2.80	6.93	14.27	24.00	0.22	0.69	0.82	1.73	0.26	0.69	0.90	1.84
10	2.80	6.93	14.27	24.00	0.22	0.69	0.82	1.73	0.26	0.69	0.90	1.85
AVERAGE	2.81	6.95	14.24	24.00	0.20	0.65	0.74	1.59	0.24	0.65	0.80	1.69

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	2.87	7.00	14.13	24.00	0.14	0.46	0.44	1.05	0.16	0.46	0.46	1.08
2	2.84	6.97	14.18	24.00	0.18	0.59	0.62	1.39	0.20	0.59	0.66	1.45
3	2.82	6.96	14.22	24.00	0.19	0.63	0.69	1.51	0.23	0.63	0.74	1.60
4	2.81	6.95	14.23	24.00	0.20	0.68	0.75	1.63	0.24	0.68	0.81	1.72
5	2.82	6.94	14.25	24.00	0.20	0.68	0.80	1.68	0.23	0.69	0.86	1.78
6	2.81	6.93	14.27	24.00	0.21	0.69	0.81	1.71	0.25	0.70	0.88	1.83
7	2.81	6.92	14.27	24.00	0.20	0.67	0.79	1.66	0.24	0.67	0.87	1.78
8	2.80	6.93	14.27	24.00	0.22	0.69	0.82	1.72	0.26	0.69	0.89	1.84
9	2.80	6.94	14.27	24.00	0.22	0.70	0.82	1.75	0.27	0.71	0.89	1.87
10	2.81	6.94	14.25	24.00	0.20	0.67	0.79	1.67	0.24	0.68	0.86	1.77
AVERAGE	2.82	6.95	14.23	24.00	0.20	0.65	0.73	1.58	0.23	0.65	0.79	1.67

TABLE G 4: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 7 ,14) ;GOAL VECTOR M=(3 7 ,14) TRANSITION MATRIX P= 0.700 0.200 0.000 / 0.000 0.800 0.100 / 0.000 0.000 0.900
VECTOR F-M*P =(0.90 , 0.80 , 0.70)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	4.26	6.20	13.54	24.00	1.55	1.60	1.12	4.27	3.13	2.24	1.33	6.70
2	4.97	5.81	13.22	24.00	2.05	2.43	1.34	5.83	5.91	3.85	1.95	11.71
3	5.47	5.64	12.89	24.00	2.52	2.86	1.39	6.77	8.61	4.71	2.62	15.93
4	5.66	5.61	12.73	24.00	2.63	3.05	1.15	6.82	9.68	4.98	2.76	17.42
5	5.56	5.72	12.72	24.00	2.02	2.64	0.93	5.60	8.57	4.29	2.56	15.43
6	5.10	5.94	12.96	24.00	1.19	2.03	0.68	3.90	5.60	3.15	1.77	10.53
7	4.47	6.31	13.22	24.00	0.97	1.40	0.43	2.80	3.13	1.88	1.03	6.05
8	3.86	6.81	13.33	24.00	0.57	0.97	0.48	2.02	1.31	1.01	0.93	3.25
9	3.16	6.94	13.91	24.00	0.41	0.85	0.42	1.68	0.43	0.85	0.43	1.72
10	2.92	7.02	14.06	24.00	0.25	0.66	0.59	1.50	0.25	0.66	0.59	1.51
AVERAGE	4.54	6.20	13.26	24.00	1.41	1.85	0.85	4.12	4.66	2.76	1.60	9.02

TABLE F 5: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(7 , 3 , 2) ; GOAL VECTOR M=(7 3 2) TRANSITION MATRIX P=
VECTOR N-M·P =(3.22 , 0.02 , 0.36)

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	7.00	3.00	2.00	12.00	1.93	1.66	0.77	4.36	1.93	1.66	0.77	4.36
2	7.00	3.00	2.00	12.00	2.50	2.06	1.13	5.70	2.50	2.06	1.13	5.70
3	7.00	3.00	2.00	12.00	2.66	2.18	1.30	6.14	2.66	2.18	1.30	6.14
4	7.00	3.00	2.00	12.00	2.71	2.21	1.38	6.30	2.71	2.21	1.38	6.30
5	7.00	3.00	2.00	12.00	2.72	2.23	1.41	6.36	2.72	2.23	1.41	6.36
6	7.00	3.00	2.00	12.00	2.73	2.23	1.43	6.39	2.73	2.23	1.43	6.39
7	7.00	3.00	2.00	12.00	2.73	2.24	1.44	6.40	2.73	2.24	1.44	6.40
8	7.00	3.00	2.00	12.00	2.73	2.24	1.44	6.40	2.73	2.24	1.44	6.40
9	7.00	3.00	2.00	12.00	2.73	2.24	1.44	6.41	2.73	2.24	1.44	6.41
10	7.00	3.00	2.00	12.00	2.73	2.24	1.44	6.41	2.73	2.24	1.44	6.41
AVERAGE	7.00	3.00	2.00	12.00	2.62	2.15	1.32	6.09	2.62	2.15	1.32	6.09

TABLE A 5: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(7 , 3 , 2) ;GOAL VECTOR M=(7 3 2) TRANSITION MATRIX P=
VECTOR N-M*P =(3.22 , 0.02 , 0.36)

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	6.62	3.41	1.97	12.00	0.52	0.75	0.30	1.57	0.66	0.92	0.30	1.89
2	6.50	3.54	1.97	12.00	0.69	1.01	0.44	2.14	0.95	1.30	0.44	2.69
3	6.46	3.58	1.96	12.00	0.75	1.11	0.50	2.36	1.05	1.44	0.50	2.99
4	6.44	3.59	1.97	12.00	0.77	1.14	0.53	2.44	1.08	1.49	0.53	3.10
5	6.44	3.60	1.97	12.00	0.78	1.15	0.54	2.47	1.10	1.51	0.54	3.15
6	6.43	3.60	1.97	12.00	0.78	1.16	0.54	2.48	1.10	1.52	0.54	3.16
7	6.43	3.60	1.97	12.00	0.78	1.16	0.54	2.48	1.10	1.52	0.54	3.16
8	6.43	3.60	1.97	12.00	0.78	1.16	0.54	2.49	1.11	1.52	0.54	3.17
9	6.43	3.60	1.97	12.00	0.78	1.16	0.54	2.49	1.11	1.52	0.54	3.17
10	6.43	3.60	1.97	12.00	0.78	1.16	0.54	2.49	1.11	1.52	0.55	3.17
AVERAGE	6.46	3.57	1.97	12.00	0.74	1.09	0.50	2.34	1.04	1.42	0.50	2.96

TABLE G 5: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(7 , 3 , 2) ;GOAL VECTOR M=(7 3 2)
VECTOR M-M•P =(3.22 , 0.02 , 0.36)

TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.78	2.98	5.24	12.00	1.74	1.65	2.18	5.56	12.11	1.65	12.67	26.43
2	2.04	2.45	7.51	12.00	1.45	1.77	2.46	5.67	26.04	2.07	32.78	60.88
3	1.74	1.85	8.41	12.00	0.73	1.49	0.78	3.00	28.41	2.82	41.93	73.16
4	4.01	1.42	6.57	12.00	1.43	1.12	0.79	3.34	10.40	3.61	21.69	35.69
5	5.46	1.52	5.01	12.00	1.68	1.17	1.04	3.89	4.05	3.35	10.12	17.51
6	6.33	1.82	3.85	12.00	2.02	1.41	1.02	4.44	2.47	2.80	4.44	9.71
7	6.89	2.14	2.97	12.00	2.29	1.63	1.04	4.96	2.30	2.37	1.98	6.65
8	7.05	2.43	2.52	12.00	2.09	1.83	0.73	4.65	2.10	2.16	1.00	5.25
9	7.21	2.63	2.16	12.00	1.97	1.96	0.72	4.64	2.01	2.09	0.74	4.84
10	6.58	3.35	2.07	12.00	0.67	0.91	0.53	2.11	0.85	1.03	0.54	2.41
AVERAGE	5.11	2.26	4.63	12.00	1.61	1.49	1.13	4.23	9.07	2.39	12.79	24.25

TABLE F 6: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 4 , 5) ;GOAL VECTOR M=(3 4 5) TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

VECTOR M-M*P =(1.38 , 1.04 , 1.18)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.00	4.00	5.00	12.00	0.91	1.21	1.02	3.15	0.91	1.21	1.02	3.15
2	3.00	4.00	5.00	12.00	1.18	1.59	1.45	4.23	1.18	1.59	1.45	4.23
3	3.00	4.00	5.00	12.00	1.26	1.73	1.64	4.62	1.26	1.73	1.64	4.62
4	3.00	4.00	5.00	12.00	1.28	1.78	1.71	4.77	1.28	1.78	1.71	4.77
5	3.00	4.00	5.00	12.00	1.29	1.80	1.75	4.83	1.29	1.80	1.75	4.83
6	3.00	4.00	5.00	12.00	1.29	1.80	1.76	4.85	1.29	1.80	1.76	4.85
7	3.00	4.00	5.00	12.00	1.29	1.81	1.76	4.86	1.29	1.81	1.76	4.86
8	3.00	4.00	5.00	12.00	1.29	1.81	1.77	4.87	1.29	1.81	1.77	4.87
9	3.00	4.00	5.00	12.00	1.29	1.81	1.77	4.87	1.29	1.81	1.77	4.87
10	3.00	4.00	5.00	12.00	1.29	1.81	1.77	4.87	1.29	1.81	1.77	4.87
AVERAGE	3.00	4.00	5.00	12.00	1.24	1.71	1.64	4.59	1.24	1.71	1.64	4.59

TABLE A 6: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 4 , 5) ;GOAL VECTOR M=(3 4 5) TRANSITION MATRIX P=
VECTOR M-M*P =(1.38 1.04 , 1.18)

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: RADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL
1	2.96	4.04	5.00	12.00	0.05	0.18	0.15	0.38	0.05	0.18	0.15	0.38
2	2.95	4.05	5.00	12.00	0.05	0.21	0.18	0.44	0.06	0.21	0.18	0.45
3	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.45	0.06	0.22	0.18	0.46
4	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
5	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
6	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
7	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
8	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
9	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
10	2.95	4.05	5.00	12.00	0.06	0.22	0.18	0.46	0.06	0.22	0.18	0.46
AVERAGE	2.95	4.05	5.00	12.00	0.05	0.21	0.18	0.45	0.06	0.22	0.18	0.45

TABLE G 6: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 4 , 5) ;GOAL VECTOR M=(3 4 5) TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

VECTOR M-M•P =(1.38 , 1.04 , 1.18)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.62	2.96	7.42	12.00	0.75	1.35	1.57	3.66	2.65	2.43	7.43	12.51
2	0.88	2.09	9.03	12.00	0.62	1.40	1.70	3.71	5.14	5.03	17.94	28.11
3	0.58	1.44	9.98	12.00	0.39	1.13	1.07	2.59	6.26	7.69	25.90	39.85
4	2.75	0.98	8.27	12.00	0.95	0.82	0.37	2.14	1.02	9.91	11.06	21.98
5	4.01	1.05	6.94	12.00	0.65	0.80	0.40	1.85	1.67	9.51	4.17	15.34
6	5.10	1.29	5.61	12.00	1.02	1.01	0.53	2.55	5.42	8.34	0.90	14.66
7	5.12	1.75	5.13	12.00	0.96	0.93	0.16	2.05	5.46	5.98	0.17	11.61
8	3.98	2.97	5.05	12.00	0.28	0.32	0.06	0.65	1.24	1.37	0.06	2.67
9	3.74	3.21	5.06	12.00	0.31	0.29	0.06	0.67	0.85	0.92	0.07	1.83
10	3.04	3.98	4.97	12.00	0.10	0.15	0.16	0.41	0.10	0.15	0.16	0.41
AVERAGE	3.08	2.17	6.75	12.00	0.60	0.82	0.61	2.03	2.98	5.13	6.79	14.90

TABLE F 7: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(15 , 7 , 2) ;GOAL VECTOR M=(15 , 7 2)
VECTOR M-M*P =(6.90 , 0.26 , 0.04)

TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	15.00	7.00	2.00	24.00	3.90	3.71	0.97	8.58	3.90	3.71	0.97	8.58
2	15.00	7.00	2.00	24.00	5.04	4.58	1.42	11.04	5.04	4.58	1.42	11.04
3	15.00	7.00	2.00	24.00	5.37	4.79	1.64	11.79	5.37	4.79	1.64	11.79
4	15.00	7.00	2.00	24.00	5.46	4.85	1.74	12.05	5.46	4.85	1.74	12.05
5	15.00	7.00	2.00	24.00	5.49	4.86	1.79	12.14	5.49	4.86	1.79	12.14
6	15.00	7.00	2.00	24.00	5.50	4.87	1.81	12.18	5.50	4.87	1.81	12.18
7	15.00	7.00	2.00	24.00	5.50	4.88	1.82	12.20	5.50	4.88	1.82	12.20
8	15.00	7.00	2.00	24.00	5.50	4.88	1.82	12.20	5.50	4.88	1.82	12.20
9	15.00	7.00	2.00	24.00	5.50	4.88	1.82	12.21	5.50	4.88	1.82	12.21
10	15.00	7.00	2.00	24.00	5.50	4.88	1.83	12.21	5.50	4.88	1.83	12.21
AVERAGE	15.00	7.00	2.00	24.00	5.28	4.72	1.67	11.66	5.28	4.72	1.67	11.66

TABLE A 7: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(15 , 7 , 2) ;GOAL VECTOR M=(15 7 2)
VECTOR M-M*P =(6.90 , 0.26 , 0.04)

TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	14.34	7.47	2.20	24.00	1.15	1.48	0.57	3.21	1.60	1.70	0.61	3.91
2	14.11	7.56	2.33	24.00	1.57	1.93	0.88	4.38	2.36	2.24	0.99	5.59
3	14.03	7.56	2.42	24.00	1.72	2.07	1.04	4.83	2.67	2.37	1.21	6.25
4	13.99	7.54	2.47	24.00	1.78	2.11	1.12	5.01	2.79	2.39	1.34	6.53
5	13.98	7.52	2.50	24.00	1.80	2.12	1.17	5.09	2.84	2.39	1.42	6.65
6	13.97	7.51	2.52	24.00	1.81	2.12	1.20	5.12	2.86	2.37	1.47	6.70
7	13.97	7.50	2.53	24.00	1.81	2.12	1.21	5.14	2.87	2.37	1.49	6.73
8	13.97	7.50	2.54	24.00	1.81	2.12	1.22	5.15	2.88	2.36	1.50	6.74
9	13.97	7.49	2.54	24.00	1.81	2.11	1.22	5.15	2.88	2.36	1.51	6.75
10	13.97	7.49	2.54	24.00	1.82	2.11	1.22	5.15	2.88	2.36	1.52	6.75
AVERAGE	14.03	7.51	2.46	24.00	1.71	2.03	1.08	4.82	2.66	2.29	1.31	6.26

TABLE G 7: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(15 , 7 , 2) ;GOAL VECTOR M=(15 7 2) TRANSITION MATRIX P=

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

VECTOR M-M•P =(6.90 , 0.26 , 0.04)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	15.30	6.74	1.96	24.00	3.91	3.67	0.94	8.51	4.00	3.73	0.94	8.67
2	15.46	6.63	1.91	24.00	5.02	4.50	1.35	10.87	5.24	4.63	1.36	11.23
3	15.55	6.58	1.87	24.00	5.33	4.69	1.53	11.55	5.63	4.86	1.55	12.04
4	15.60	6.57	1.84	24.00	5.40	4.74	1.60	11.74	5.75	4.92	1.63	12.30
5	15.62	6.57	1.82	24.00	5.37	4.76	1.62	11.74	5.74	4.94	1.65	12.33
6	15.62	6.57	1.82	24.00	5.28	4.77	1.59	11.63	5.65	4.95	1.62	12.23
7	15.60	6.57	1.83	24.00	5.16	4.78	1.53	11.46	5.52	4.96	1.56	12.03
8	15.57	6.57	1.86	24.00	5.06	4.78	1.44	11.28	5.39	4.96	1.46	11.82
9	15.52	6.57	1.92	24.00	5.00	4.77	1.30	11.07	5.27	4.96	1.31	11.54
10	14.30	7.48	2.22	24.00	1.40	1.80	0.93	4.13	1.90	2.03	0.98	4.91
AVERAGE	15.41	6.68	1.90	24.00	4.69	4.32	1.38	10.40	5.01	4.50	1.40	10.91

TABLE F 8: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 9 ,10) ;GOAL VECTOR M=(5 9 ,10) TRANSITION MATRIX P=
VECTOR M-M*P =(2.30 , 2.62 , 2.28)

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	5.00	9.00	10.00	24.00	1.41	2.24	1.97	5.62	1.41	2.24	1.97	5.62
2	5.00	9.00	10.00	24.00	1.82	2.97	2.80	7.58	1.82	2.97	2.80	7.58
3	5.00	9.00	10.00	24.00	1.94	3.23	3.14	8.31	1.94	3.23	3.14	8.31
4	5.00	9.00	10.00	24.00	1.98	3.33	3.29	8.59	1.98	3.33	3.29	8.59
5	5.00	9.00	10.00	24.00	1.99	3.37	3.35	8.70	1.99	3.37	3.35	8.70
6	5.00	9.00	10.00	24.00	1.99	3.38	3.37	8.74	1.99	3.38	3.37	8.74
7	5.00	9.00	10.00	24.00	1.99	3.39	3.38	8.76	1.99	3.39	3.38	8.76
8	5.00	9.00	10.00	24.00	1.99	3.39	3.38	8.76	1.99	3.39	3.38	8.76
9	5.00	9.00	10.00	24.00	1.99	3.39	3.39	8.77	1.99	3.39	3.39	8.77
10	5.00	9.00	10.00	24.00	1.99	3.39	3.39	8.77	1.99	3.39	3.39	8.77
AVERAGE	5.00	9.00	10.00	24.00	1.91	3.21	3.14	8.26	1.91	3.21	3.14	8.26

TABLE A 8: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE <u>ADAPTIVE</u> RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 9 ,10) ;GOAL VECTOR M=(5 , 9 ,10)   TRANSITION MATRIX P=
VECTOR M-M*P   =(2.30 , 2.62 , 2.28)

0.540	0.160	0.000
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	4.98	8.99	10.03	24.00	0.02	0.10	0.11	0.22	0.02	0.10	0.11	0.22
2	4.98	8.99	10.03	24.00	0.02	0.10	0.11	0.24	0.02	0.10	0.12	0.24
3	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
4	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
5	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
6	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
7	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
8	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
9	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
10	4.98	8.99	10.03	24.00	0.02	0.10	0.12	0.24	0.02	0.10	0.12	0.24
AVERAGE	4.98	8.99	10.03	24.00	0.02	0.10	0.11	0.23	0.02	0.10	0.11	0.24

TABLE G 8: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 9 ,10) ;GOAL VECTOR M=(5 9 ,10) TRANSITION MATRIX P=
VECTOR M-M*P =(2.30 , 2.62 , 2.28)

0.540	0.160	C.CCO
0.000	0.620	0.080
0.000	0.000	0.700

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.F. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	2.70	6.38	14.92	24.00	1.24	2.79	3.17	7.20	6.53	9.66	27.38	43.57
2	1.87	4.39	17.75	24.00	1.03	2.89	1.84	5.75	10.85	24.16	61.86	96.87
3	6.60	3.02	14.39	24.00	2.30	2.21	0.68	5.19	4.84	37.99	19.91	62.74
4	9.64	2.93	11.44	24.00	2.64	2.13	0.74	5.51	24.14	39.01	2.81	65.95
5	11.21	3.36	9.44	24.00	2.47	2.47	0.59	5.53	41.01	34.32	0.91	76.24
6	10.37	4.57	9.06	24.00	1.07	1.09	0.09	2.25	29.91	20.68	0.98	51.57
7	8.01	6.97	9.02	24.00	0.14	0.20	0.05	0.38	9.18	4.32	1.00	14.50
8	6.95	8.00	9.05	24.00	0.09	0.15	0.06	0.31	3.91	1.16	0.97	6.04
9	5.99	8.09	9.93	24.00	0.06	0.17	0.10	0.32	1.03	1.00	0.10	2.13
10	5.02	8.98	10.00	24.00	0.04	0.07	0.10	0.21	0.04	0.07	0.10	0.21
AVERAGE	6.83	5.67	11.50	24.00	1.11	1.42	0.74	3.27	13.14	17.24	11.60	41.98

TABLE F 9: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 3 , 6) ;GOAL VECTOR M=(3 3 6) TRANSITION MATRIX P=
VECTOR M-M•P =(1.50 , 0.00 , 0.30)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	3.00	3.00	6.00	12.00	1.71	1.44	1.45	4.60	1.71	1.44	1.45	4.60
2	3.00	3.00	6.00	12.00	2.08	1.82	2.26	6.16	2.08	1.82	2.26	6.16
3	3.00	3.00	6.00	12.00	2.14	2.04	2.63	6.80	2.14	2.04	2.63	6.80
4	3.00	3.00	6.00	12.00	2.15	2.14	2.77	7.06	2.15	2.14	2.77	7.06
5	3.00	3.00	6.00	12.00	2.15	2.19	2.81	7.15	2.15	2.19	2.81	7.15
6	3.00	3.00	6.00	12.00	2.15	2.21	2.82	7.18	2.15	2.21	2.82	7.18
7	3.00	3.00	6.00	12.00	2.15	2.21	2.83	7.19	2.15	2.21	2.83	7.19
8	3.00	3.00	6.00	12.00	2.15	2.21	2.83	7.19	2.15	2.21	2.83	7.19
9	3.00	3.00	6.00	12.00	2.15	2.21	2.83	7.19	2.15	2.21	2.83	7.19
10	3.00	3.00	6.00	12.00	2.15	2.21	2.83	7.19	2.15	2.21	2.83	7.19
AVERAGE	3.00	3.00	6.00	12.00	2.10	2.07	2.60	6.77	2.10	2.07	2.60	6.77

TABLE A 9: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 3 , 6) ;GOAL VECTOR M=(3 3 6)

VECTOR M-M*P =(1.50 , 0.00 , 0.30)

TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	2.57	3.30	6.14	12.00	0.47	0.82	0.69	1.99	0.66	0.91	0.71	2.28
2	2.45	3.28	6.27	12.00	0.58	1.02	1.01	2.61	0.89	1.10	1.09	3.07
3	2.42	3.23	6.35	12.00	0.60	1.07	1.15	2.82	0.95	1.12	1.28	3.35
4	2.41	3.20	6.39	12.00	0.61	1.08	1.20	2.89	0.96	1.12	1.36	3.44
5	2.41	3.18	6.41	12.00	0.61	1.08	1.22	2.91	0.96	1.11	1.39	3.46
6	2.41	3.17	6.42	12.00	0.61	1.08	1.23	2.91	0.96	1.11	1.40	3.47
7	2.41	3.17	6.42	12.00	0.61	1.08	1.23	2.91	0.96	1.11	1.41	3.47
8	2.41	3.17	6.43	12.00	0.61	1.08	1.23	2.91	0.96	1.10	1.41	3.47
9	2.41	3.17	6.43	12.00	0.61	1.08	1.23	2.91	0.96	1.10	1.41	3.47
10	2.41	3.17	6.43	12.00	0.61	1.08	1.23	2.91	0.96	1.10	1.41	3.47
AVERAGE	2.43	3.20	6.37	12.00	0.59	1.04	1.14	2.78	0.92	1.09	1.29	3.30

TABLE G 9: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 3 , 6) ;GOAL VECTOR M=(3 3 6) TRANSITION MATRIX P=
VECTOR M-M*P =(1.50 , 0.00 , 0.30)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.50	3.00	7.50	12.00	0.75	1.44	0.99	3.18	3.00	1.44	3.24	7.68
2	0.75	2.40	8.85	12.00	0.56	1.43	1.33	3.33	5.63	1.79	9.46	16.87
3	0.38	1.74	9.89	12.00	0.33	1.20	1.26	2.79	7.22	2.79	16.35	26.36
4	0.19	1.19	10.62	12.00	0.18	0.93	1.00	2.11	8.09	4.19	22.33	34.61
5	0.09	0.79	11.12	12.00	0.09	0.67	0.73	1.49	8.54	5.55	26.89	40.97
6	1.91	0.51	9.58	12.00	0.76	0.46	0.88	2.10	1.95	6.65	13.67	22.27
7	2.81	1.07	8.12	12.00	1.16	0.79	1.19	3.13	1.19	4.50	5.70	11.40
8	3.07	1.77	7.17	12.00	1.22	1.22	1.22	3.66	1.23	2.74	2.58	6.55
9	3.08	2.29	6.63	12.00	1.35	1.50	1.12	3.97	1.36	2.01	1.52	4.89
10	2.59	3.03	6.38	12.00	0.60	0.90	1.08	2.58	0.77	0.90	1.23	2.90
AVERAGE	1.64	1.78	8.58	12.00	0.70	1.05	1.08	2.83	3.90	3.26	10.30	17.45

TABLE F10: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8)
VECTOR N-M*P =(0.50 0.80 , 0.70)

TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.00	3.00	8.00	12.00	0.50	1.21	1.31	3.02	0.50	1.21	1.31	3.02
2	1.00	3.00	8.00	12.00	0.61	1.58	1.83	4.02	0.61	1.58	1.83	4.02
3	1.00	3.00	8.00	12.00	0.64	1.71	2.03	4.38	0.64	1.71	2.03	4.38
4	1.00	3.00	8.00	12.00	0.65	1.77	2.10	4.52	0.65	1.77	2.10	4.52
5	1.00	3.00	8.00	12.00	0.65	1.79	2.13	4.57	0.65	1.79	2.13	4.57
6	1.00	3.00	8.00	12.00	0.65	1.80	2.14	4.58	0.65	1.80	2.14	4.58
7	1.00	3.00	8.00	12.00	0.65	1.80	2.14	4.59	0.65	1.80	2.14	4.59
8	1.00	3.00	8.00	12.00	0.65	1.80	2.14	4.59	0.65	1.80	2.14	4.59
9	1.00	3.00	8.00	12.00	0.65	1.80	2.14	4.59	0.65	1.80	2.14	4.59
10	1.00	3.00	8.00	12.00	0.65	1.80	2.14	4.59	0.65	1.80	2.14	4.59
AVERAGE	1.00	3.00	8.00	12.00	0.63	1.71	2.01	4.34	0.63	1.71	2.01	4.34

TABLE A10: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8) TRANSITION MATRIX P=
VECTOR M-M*P =(0.50 , 0.80 , 0.70)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	0.94	2.89	8.17	12.00	0.06	0.33	0.39	0.78	0.06	0.34	0.42	0.82
2	0.93	2.85	8.22	12.00	0.07	0.38	0.48	0.93	0.07	0.41	0.52	1.00
3	0.92	2.84	8.24	12.00	0.07	0.39	0.50	0.96	0.08	0.42	0.55	1.05
4	0.92	2.84	8.24	12.00	0.07	0.39	0.50	0.97	0.08	0.42	0.56	1.06
5	0.92	2.83	8.24	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
6	0.92	2.83	8.25	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
7	0.92	2.83	8.25	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
8	0.92	2.83	8.25	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
9	0.92	2.83	8.25	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
10	0.92	2.83	8.25	12.00	0.07	0.39	0.51	0.97	0.08	0.42	0.57	1.07
AVERAGE	0.92	2.84	8.23	12.00	0.07	0.39	0.49	0.95	0.08	0.41	0.55	1.03

TABLE G10: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 , 3 , 8) ;GOAL VECTOR M=(1 3 8) TRANSITION MATRIX P=

VECTOR M-M*P =(0.50 , 0.80 , 0.70)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	0.50	2.20	9.30	12.00	0.25	0.96	0.81	2.02	0.50	1.60	2.50	4.60
2	0.25	1.52	10.23	12.00	0.19	0.94	0.91	2.03	0.75	3.13	5.88	9.76
3	0.13	1.01	10.86	12.00	0.11	0.74	0.76	1.61	0.88	4.69	8.96	14.52
4	0.06	0.66	11.28	12.00	0.06	0.54	0.56	1.15	0.94	6.02	11.32	18.28
5	1.90	0.42	9.68	12.00	0.82	0.37	0.78	1.96	1.63	7.03	3.60	12.25
6	2.36	1.01	8.62	12.00	0.60	0.72	0.67	1.99	2.46	4.67	1.06	8.19
7	2.27	1.55	8.18	12.00	0.52	1.01	0.51	2.05	2.13	3.10	0.55	5.78
8	1.87	2.19	7.94	12.00	0.27	0.59	0.41	1.26	1.02	1.25	0.41	2.69
9	1.20	2.73	8.07	12.00	0.24	0.53	0.29	1.06	0.28	0.60	0.30	1.18
10	1.00	2.87	8.13	12.00	0.12	0.38	0.45	0.95	0.12	0.40	0.46	0.98
AVERAGE	1.15	1.62	9.23	12.00	0.32	0.68	0.61	1.61	1.07	3.25	3.50	7.82

TABLE F11: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(6 , 7 ,11) ;GOAL VECTOR M=(6 7 ,11) TRANSITION MATRIX P=
VECTOR M-M*P =(3.00 , 0.40 , 0.10)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	6.00	7.00	11.00	24.00	3.41	3.22	3.21	9.83	3.41	3.22	3.21	9.83
2	6.00	7.00	11.00	24.00	4.14	4.07	4.83	13.04	4.14	4.07	4.83	13.04
3	6.00	7.00	11.00	24.00	4.27	4.49	5.53	14.29	4.27	4.49	5.53	14.29
4	6.00	7.00	11.00	24.00	4.30	4.69	5.78	14.77	4.30	4.69	5.78	14.77
5	6.00	7.00	11.00	24.00	4.30	4.78	5.85	14.93	4.30	4.78	5.85	14.93
6	6.00	7.00	11.00	24.00	4.30	4.80	5.87	14.98	4.30	4.80	5.87	14.98
7	6.00	7.00	11.00	24.00	4.30	4.81	5.88	14.99	4.30	4.81	5.88	14.99
8	6.00	7.00	11.00	24.00	4.30	4.81	5.88	14.99	4.30	4.81	5.88	14.99
9	6.00	7.00	11.00	24.00	4.30	4.81	5.88	14.99	4.30	4.81	5.88	14.99
10	6.00	7.00	11.00	24.00	4.30	4.81	5.88	14.99	4.30	4.81	5.88	14.99
AVERAGE	6.00	7.00	11.00	24.00	4.19	4.53	5.46	14.18	4.19	4.53	5.46	14.18

TABLE A11: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(6 , 7 ,11) ;GOAL VECTOR M=(6 7 ,11)

VECTOR M-M*P =(3.00 , 0.40 , 0.10)

TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	5.34	7.21	11.45	24.00	0.88	1.44	1.49	3.80	1.31	1.48	1.69	4.48
2	5.17	7.07	11.76	24.00	1.08	1.75	2.21	5.04	1.78	1.75	2.79	6.31
3	5.12	6.96	11.93	24.00	1.13	1.82	2.54	5.49	1.91	1.82	3.40	7.13
4	5.10	6.89	12.01	24.00	1.14	1.83	2.68	5.66	1.95	1.85	3.70	7.50
5	5.09	6.86	12.05	24.00	1.14	1.83	2.75	5.72	1.97	1.86	3.85	7.67
6	5.09	6.84	12.07	24.00	1.15	1.83	2.77	5.75	1.97	1.86	3.91	7.74
7	5.09	6.83	12.08	24.00	1.15	1.83	2.79	5.76	1.97	1.86	3.95	7.78
8	5.09	6.83	12.08	24.00	1.15	1.83	2.79	5.77	1.97	1.86	3.96	7.79
9	5.09	6.83	12.08	24.00	1.15	1.83	2.79	5.77	1.97	1.86	3.97	7.80
10	5.09	6.83	12.08	24.00	1.15	1.83	2.79	5.77	1.97	1.86	3.97	7.80
AVERAGE	5.13	6.91	11.96	24.00	1.11	1.78	2.56	5.45	1.88	1.81	3.52	7.20

TABLE G11: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(6 , 7 ,11) ;GOAL VECTOR M=(6 7 ,11) TRANSITION MATRIX P=
VECTOR M-M*P =(3.00 , 0.40 , 0.10)

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.00	6.60	14.40	24.00	1.50	3.12	2.22	6.84	10.50	3.28	13.78	27.56
2	1.50	5.16	17.34	24.00	1.13	3.09	2.90	7.11	21.38	6.48	43.09	70.94
3	0.75	3.70	19.55	24.00	0.66	2.57	2.69	5.92	28.22	13.49	75.86	117.57
4	0.38	2.52	21.11	24.00	0.35	1.97	2.12	4.44	31.99	22.06	104.28	158.33
5	4.34	1.66	18.00	24.00	2.08	1.41	2.85	6.34	4.84	29.92	51.83	86.59
6	6.25	2.73	15.02	24.00	3.57	2.13	3.90	9.60	3.63	20.34	20.07	44.03
7	6.87	4.14	13.00	24.00	3.76	3.06	4.15	10.97	4.51	11.25	8.13	23.89
8	6.84	5.23	11.94	24.00	3.40	3.70	3.56	10.66	4.10	6.84	4.43	15.37
9	6.53	5.88	11.60	24.00	3.09	3.90	2.69	9.68	3.37	5.16	3.05	11.58
10	5.38	6.99	11.64	24.00	0.98	1.66	2.17	4.82	1.38	1.66	2.58	5.61
AVERAGE	4.18	4.46	15.36	24.00	2.05	2.66	2.92	7.64	11.39	12.05	32.71	56.15

TABLE F12: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 6 ,15) ;GOAL VECTOR M=(3 , 6 ,15)
VECTOR M-M*P =(1.50 , 1.20 , 1.20)

TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	3.00	6.00	15.00	24.00	1.28	2.34	2.55	6.16	1.28	2.34	2.55	6.16
2	3.00	6.00	15.00	24.00	1.58	3.05	3.62	8.24	1.58	3.05	3.62	8.24
3	3.00	6.00	15.00	24.00	1.64	3.33	4.05	9.01	1.64	3.33	4.05	9.01
4	3.00	6.00	15.00	24.00	1.65	3.45	4.21	9.31	1.65	3.45	4.21	9.31
5	3.00	6.00	15.00	24.00	1.65	3.49	4.26	9.41	1.65	3.49	4.26	9.41
6	3.00	6.00	15.00	24.00	1.65	3.51	4.28	9.44	1.65	3.51	4.28	9.44
7	3.00	6.00	15.00	24.00	1.65	3.52	4.29	9.46	1.65	3.52	4.29	9.46
8	3.00	6.00	15.00	24.00	1.65	3.52	4.29	9.46	1.65	3.52	4.29	9.46
9	3.00	6.00	15.00	24.00	1.65	3.52	4.29	9.46	1.65	3.52	4.29	9.46
10	3.00	6.00	15.00	24.00	1.65	3.52	4.29	9.46	1.65	3.52	4.29	9.46
AVERAGE	3.00	6.00	15.00	24.00	1.61	3.32	4.01	8.94	1.61	3.32	4.01	8.94

TABLE A12: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 6 ,15) ;GOAL VECTOR M=(3 6 ,15) TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

VECTOR M-M*P =(1.50 , 1.20 , 1.20)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	2.87	5.96	15.18	24.00	0.15	0.49	0.63	1.26	0.16	0.49	0.66	1.32
2	2.84	5.92	15.24	24.00	0.17	0.56	0.77	1.50	0.20	0.57	0.82	1.59
3	2.84	5.91	15.25	24.00	0.18	0.57	0.81	1.55	0.20	0.58	0.87	1.65
4	2.84	5.91	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.88	1.67
5	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
6	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
7	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
8	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
9	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
10	2.84	5.90	15.26	24.00	0.18	0.57	0.82	1.57	0.21	0.58	0.89	1.68
AVERAGE	2.84	5.91	15.25	24.00	0.17	0.56	0.79	1.53	0.20	0.57	0.86	1.63

TABLE G12: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 6 ,15) ;GOAL VECTOR M=(3 6 ,15) TRANSITION MATRIX P=

0.500	0.400	0.000
0.000	0.600	0.300
0.000	0.000	0.800

VECTOR M-M*P =(1.50 , 1.20 , 1.20)

RESULTS OBTAINED BY EXACT METHODS

STEP	Expected stock vector: Grade 1	Grade 2	Grade 3	Total	Variances of the stock vectors: Grade 1	Grade 2	Grade 3	Total	M.S.E. of the stock vectors: Grade 1	Grade 2	Grade 3	Total
1	1.50	4.80	17.70	24.00	0.75	2.16	1.71	4.62	3.00	3.60	9.00	15.60
2	0.75	3.48	19.77	24.00	0.56	2.12	2.02	4.71	5.63	8.47	24.78	38.87
3	0.38	2.39	21.24	24.00	0.33	1.71	1.77	3.81	7.22	14.76	40.67	62.64
4	0.19	1.58	22.23	24.00	0.18	1.27	1.34	2.78	8.09	20.78	53.61	82.47
5	0.09	1.03	22.88	24.00	0.09	0.89	0.94	1.92	8.54	25.64	63.06	97.24
6	3.47	0.65	19.87	24.00	0.67	0.59	0.90	2.16	0.89	29.19	24.66	54.74
7	5.21	1.78	17.01	24.00	1.43	1.22	1.14	3.79	6.30	19.03	5.18	30.51
8	5.25	3.26	15.49	24.00	0.91	1.59	0.94	3.44	5.97	9.11	1.19	16.27
9	3.95	5.03	15.03	24.00	0.31	0.82	0.48	1.60	1.21	1.77	0.48	3.46
10	2.98	5.95	15.07	24.00	0.18	0.49	0.54	1.21	0.18	0.49	0.54	1.21
AVERAGE	2.38	2.99	18.63	24.00	0.54	1.29	1.18	3.00	4.70	13.28	22.32	40.30

TABLE F13: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 4 , 3) ;GOAL VECTOR M=(5 , 4 3) TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

VECTOR M-M*P =(3.05 , 0.57 , 0.28)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	5.00	4.00	3.00	12.00	2.04	2.02	1.49	5.55	2.04	2.02	1.49	5.55
2	5.00	4.00	3.00	12.00	2.31	2.27	1.94	6.52	2.31	2.27	1.94	6.52
3	5.00	4.00	3.00	12.00	2.34	2.34	2.06	6.74	2.34	2.34	2.06	6.74
4	5.00	4.00	3.00	12.00	2.35	2.35	2.08	6.78	2.35	2.35	2.08	6.78
5	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
6	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
7	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
8	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
9	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
10	5.00	4.00	3.00	12.00	2.35	2.36	2.09	6.79	2.35	2.36	2.09	6.79
AVERAGE	5.00	4.00	3.00	12.00	2.31	2.31	2.01	6.63	2.31	2.31	2.01	6.63

TABLE A13: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 4 , 3) ;GOAL VECTOR M=(5 4 3) TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

VECTOR M-M*P =(3.05 0.57 , 0.28)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	4.66	4.14	3.20	12.00	0.45	0.73	0.61	1.79	0.56	0.75	0.65	1.97
2	4.61	4.10	3.30	12.00	0.52	0.82	0.79	2.13	0.67	0.83	0.88	2.38
3	4.60	4.07	3.33	12.00	0.53	0.83	0.84	2.20	0.69	0.83	0.96	2.48
4	4.60	4.06	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.98	2.50
5	4.60	4.06	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.98	2.50
6	4.60	4.05	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.99	2.51
7	4.60	4.05	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.99	2.51
8	4.60	4.05	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.99	2.51
9	4.60	4.05	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.99	2.50
10	4.60	4.05	3.35	12.00	0.53	0.83	0.86	2.21	0.69	0.83	0.99	2.50
AVERAGE	4.60	4.07	3.33	12.00	0.52	0.82	0.83	2.16	0.68	0.82	0.94	2.44

TABLE G13: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(5 , 4 , 3) ;GOAL VECTOR M=(5 4 3) TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

VECTOR M-M*P =(3.05 , 0.57 , 0.28)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.95	3.43	6.62	12.00	1.19	2.07	2.05	5.30	10.49	2.39	15.15	28.03
2	0.76	2.22	9.02	12.00	0.65	1.67	1.91	4.22	18.62	4.85	38.18	61.65
3	0.30	1.28	10.43	12.00	0.28	1.09	1.26	2.63	22.40	8.50	56.41	87.30
4	0.12	0.69	11.19	12.00	0.11	0.63	0.72	1.47	23.97	11.57	67.83	103.38
5	0.05	0.36	11.59	12.00	0.05	0.35	0.38	0.77	24.60	13.59	74.24	112.42
6	0.99	0.18	10.83	12.00	0.02	0.18	0.15	0.34	16.14	14.74	61.48	92.36
7	4.77	0.39	6.84	12.00	1.64	0.29	1.56	3.49	1.70	13.31	16.31	31.32
8	5.87	1.66	4.47	12.00	1.86	1.29	1.45	4.60	2.62	6.75	3.61	12.98
9	5.95	2.60	3.45	12.00	1.81	1.89	1.00	4.70	2.71	3.84	1.21	7.75
10	4.75	4.03	3.23	12.00	0.38	0.62	0.65	1.66	0.44	0.62	0.71	1.77
AVERAGE	2.55	1.68	7.77	12.00	0.80	1.01	1.11	2.92	12.37	8.02	33.51	53.90

TABLE F14: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(2 , 4 , 6) ; GOAL VECTOR M=(2 4 6)

VECTOR M-M*P =(1.22 , 1.50 , 1.48)

TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	2.00	4.00	6.00	12.00	0.74	1.40	1.44	3.58	0.74	1.40	1.44	3.58
2	2.00	4.00	6.00	12.00	0.84	1.64	1.79	4.27	0.84	1.64	1.79	4.27
3	2.00	4.00	6.00	12.00	0.86	1.69	1.87	4.42	0.86	1.69	1.87	4.42
4	2.00	4.00	6.00	12.00	0.86	1.71	1.89	4.46	0.86	1.71	1.89	4.46
5	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.46	0.86	1.71	1.90	4.46
6	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.46	0.86	1.71	1.90	4.46
7	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.47	0.86	1.71	1.90	4.47
8	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.47	0.86	1.71	1.90	4.47
9	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.47	0.86	1.71	1.90	4.47
10	2.00	4.00	6.00	12.00	0.86	1.71	1.90	4.47	0.86	1.71	1.90	4.47
AVERAGE	2.00	4.00	6.00	12.00	0.85	1.67	1.84	4.35	0.85	1.67	1.84	4.35

TABLE A14: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(2 , 4 , 6) ;GOAL VECTOR M=(2 , 4 6)
VECTOR M-M*P =(1.22 , 1.50 , 1.48)

TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.97	3.97	6.07	12.00	0.03	0.16	0.20	0.38	0.03	0.16	0.20	0.39
2	1.97	3.96	6.07	12.00	0.03	0.17	0.21	0.41	0.04	0.17	0.22	0.42
3	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
4	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
5	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
6	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
7	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
8	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
9	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
10	1.97	3.96	6.08	12.00	0.03	0.17	0.21	0.42	0.04	0.17	0.22	0.42
AVERAGE	1.97	3.96	6.07	12.00	0.03	0.17	0.21	0.41	0.03	0.17	0.22	0.42

TABLE G14: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(2 , 4 , 6) ;GOAL VECTOR M=(2 4 6) TRANSITION MATRIX P=
VECTOR M-M*P =(1.22 , 1.50 , 1.48)

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	0.78	2.50	8.72	12.00	0.48	1.42	1.42	3.32	1.96	3.67	8.82	14.45
2	0.30	1.42	10.28	12.00	0.26	1.08	1.18	2.51	3.13	7.75	19.49	30.37
3	0.12	0.76	11.12	12.00	0.11	0.66	0.73	1.50	3.65	11.15	26.96	41.76
4	0.05	0.39	11.56	12.00	0.05	0.37	0.40	0.81	3.86	13.37	31.31	48.54
5	0.02	0.20	11.78	12.00	0.02	0.19	0.21	0.42	3.95	14.64	33.64	52.22
6	0.01	0.10	11.89	12.00	0.01	0.10	0.10	0.21	3.98	15.31	34.84	54.13
7	3.63	0.05	8.32	12.00	0.49	0.05	0.46	1.01	3.16	15.66	5.84	24.66
8	3.85	1.40	6.76	12.00	0.18	0.46	0.31	0.95	3.60	7.24	0.88	11.71
9	2.93	3.01	6.06	12.00	0.11	0.20	0.09	0.40	0.98	1.18	0.09	2.25
10	2.04	3.97	5.99	12.00	0.07	0.13	0.17	0.37	0.07	0.13	0.17	0.38
AVERAGE	1.37	1.38	9.25	12.00	0.18	0.46	0.51	1.15	2.83	9.01	16.20	28.05

TABLE F15: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(11 , 8 , 5) ;GOAL VECTOR M=(11 8 , 5) TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

VECTOR M-M*P =(6.71 , 0.83 , 0.16)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	11.00	8.00	5.00	24.00	4.46	4.10	2.68	11.24	4.46	4.10	2.68	11.24
2	11.00	8.00	5.00	24.00	5.06	4.55	3.55	13.16	5.06	4.55	3.55	13.16
3	11.00	8.00	5.00	24.00	5.12	4.68	3.78	13.58	5.12	4.68	3.78	13.58
4	11.00	8.00	5.00	24.00	5.13	4.72	3.84	13.68	5.13	4.72	3.84	13.68
5	11.00	8.00	5.00	24.00	5.13	4.73	3.84	13.70	5.13	4.73	3.84	13.70
6	11.00	8.00	5.00	24.00	5.13	4.73	3.85	13.70	5.13	4.73	3.85	13.70
7	11.00	8.00	5.00	24.00	5.13	4.73	3.85	13.70	5.13	4.73	3.85	13.70
8	11.00	8.00	5.00	24.00	5.13	4.73	3.85	13.70	5.13	4.73	3.85	13.70
9	11.00	8.00	5.00	24.00	5.13	4.73	3.85	13.70	5.13	4.73	3.85	13.70
10	11.00	8.00	5.00	24.00	5.13	4.73	3.85	13.70	5.13	4.73	3.85	13.70
AVERAGE	11.00	8.00	5.00	24.00	5.06	4.64	3.69	13.39	5.06	4.64	3.69	13.39

TABLE A15: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(11 , 8 , 5) ;GOAL VECTOR M=(11 8 , 5) TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

VECTOR M-M*P =(6.71 , 0.83 , 0.16)

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	10.40	8.21	5.39	24.00	1.01	1.45	1.22	3.67	1.36	1.49	1.37	4.22
2	10.29	8.12	5.59	24.00	1.17	1.61	1.62	4.40	1.67	1.62	1.97	5.26
3	10.27	8.06	5.66	24.00	1.20	1.62	1.75	4.56	1.72	1.62	2.19	5.53
4	10.27	8.04	5.69	24.00	1.20	1.61	1.79	4.60	1.73	1.61	2.27	5.61
5	10.27	8.03	5.70	24.00	1.20	1.61	1.80	4.61	1.73	1.61	2.29	5.63
6	10.27	8.03	5.70	24.00	1.20	1.61	1.80	4.61	1.73	1.61	2.30	5.64
7	10.27	8.03	5.70	24.00	1.20	1.61	1.81	4.61	1.73	1.61	2.30	5.64
8	10.27	8.03	5.70	24.00	1.20	1.61	1.81	4.61	1.73	1.61	2.30	5.64
9	10.27	8.03	5.70	24.00	1.20	1.61	1.81	4.61	1.73	1.61	2.30	5.64
10	10.27	8.03	5.70	24.00	1.20	1.61	1.81	4.61	1.73	1.61	2.30	5.64
AVERAGE	10.28	8.06	5.65	24.00	1.18	1.59	1.72	4.49	1.69	1.60	2.16	5.45

TABLE G15: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(11 , 8 , 5) ;GOAL VECTOR M=(11 8 5) TRANSITION MATRIX P=
VECTOR M-M*P =(6.71 , 0.83 , 0.16)

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	4.29	7.17	12.54	24.00	2.62	4.35	4.30	11.27	47.64	5.04	61.15	113.83
2	1.67	4.70	17.63	24.00	1.42	3.53	4.05	9.00	88.41	14.42	163.50	266.33
3	0.65	2.73	20.62	24.00	0.61	2.31	2.70	5.63	107.68	30.11	246.69	384.48
4	0.25	1.48	22.26	24.00	0.25	1.36	1.55	3.16	115.72	43.81	299.51	459.03
5	0.10	0.78	23.12	24.00	0.10	0.74	0.83	1.67	118.93	52.92	329.32	501.16
6	8.10	0.40	15.50	24.00	1.30	0.39	1.24	2.92	9.68	58.21	111.49	179.39
7	11.72	2.70	9.58	24.00	4.37	1.98	3.18	9.53	4.89	30.09	24.17	59.14
8	12.50	4.90	6.60	24.00	4.41	3.58	2.97	10.95	6.66	13.18	5.53	25.36
9	12.44	6.18	5.39	24.00	4.48	4.38	2.31	11.17	6.55	7.70	2.46	16.70
10	10.52	8.10	5.38	24.00	0.83	1.30	1.34	3.47	1.07	1.31	1.49	3.86
AVERAGE	6.22	3.91	13.86	24.00	2.04	2.39	2.45	6.88	50.72	25.68	124.53	200.93

TABLE F16: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 8 ,12) ;GOAL VECTOR M=(4 8 ,12) TRANSITION MATRIX P=
VECTOR M-M*P =(2.44 , 3.00 , 2.96)

$$P = \begin{pmatrix} 0.390 & 0.310 & 0.000 \\ 0.000 & 0.470 & 0.230 \\ 0.000 & 0.000 & 0.600 \end{pmatrix}$$

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL
1	4.00	8.00	12.00	24.00	1.30	2.64	2.72	6.66	1.30	2.64	2.72	6.66
2	4.00	8.00	12.00	24.00	1.49	3.09	3.37	7.94	1.49	3.09	3.37	7.94
3	4.00	8.00	12.00	24.00	1.51	3.19	3.52	8.22	1.51	3.19	3.52	8.22
4	4.00	8.00	12.00	24.00	1.51	3.21	3.55	8.28	1.51	3.21	3.55	8.28
5	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
6	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
7	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
8	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
9	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
10	4.00	8.00	12.00	24.00	1.51	3.22	3.56	8.29	1.51	3.22	3.56	8.29
AVERAGE	4.00	8.00	12.00	24.00	1.49	3.14	3.45	8.08	1.49	3.14	3.45	8.08

TABLE A16: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 8 ,12) ;GOAL VECTOR M=(4 8 ,12) TRANSITION MATRIX P=
VECTOR M-M*P =(2.44 , 3.00 , 2.96)

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	3.98	7.97	12.05	24.00	0.02	0.09	0.13	0.25	0.02	0.09	0.14	0.25
2	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
3	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
4	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
5	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
6	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
7	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
8	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
9	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
10	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.10	0.14	0.26
AVERAGE	3.98	7.97	12.05	24.00	0.02	0.09	0.14	0.26	0.02	0.09	0.14	0.26

TABLE G16: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE GOAL RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 8 ,12) ;GOAL VECTOR M=(4 , 8 ,12)
VECTOR M-M*P =(2.44 , 3.00 , 2.96)

TRANSITION MATRIX P=

0.390	0.310	0.000
0.000	0.470	0.230
0.000	0.000	0.600

RESULTS OBTAINED BY EXACT METHODS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	1.56	5.00	17.44	24.00	0.95	2.85	2.83	6.63	6.91	11.85	32.43	51.18
2	0.61	2.83	20.56	24.00	0.52	2.16	2.35	5.03	12.02	28.85	75.59	116.46
3	0.24	1.52	22.24	24.00	0.22	1.32	1.46	2.99	14.38	43.30	106.36	164.04
4	0.09	0.79	23.12	24.00	0.09	0.73	0.80	1.62	15.36	52.74	124.44	192.54
5	0.04	0.40	23.57	24.00	0.04	0.38	0.42	0.83	15.75	58.16	134.16	208.07
6	0.98	0.20	22.82	24.00	0.02	0.20	0.15	0.36	9.11	61.05	117.16	187.32
7	7.70	0.40	15.90	24.00	0.57	0.30	0.68	1.55	14.25	58.08	15.92	88.24
8	6.98	4.07	12.95	24.00	0.06	0.24	0.14	0.43	8.94	15.71	1.05	25.69
9	5.03	6.95	12.01	24.00	0.05	0.17	0.06	0.28	1.12	1.26	0.06	2.44
10	4.00	7.98	12.02	24.00	0.02	0.07	0.10	0.19	0.02	0.07	0.10	0.19
AVERAGE	2.72	3.01	18.26	24.00	0.25	0.84	0.90	1.99	9.78	33.11	60.73	103.62

TABLE F17: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE <u>FIXED</u> PECRUITMENT STRATEGY.

INITIAL VECTOR N=(6 , 3 , 3) ;GOAL VECTOR M=(6 3 3) TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

VECTOR M-M*P =(2.40 , 0.00 ,-0.00)

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	5.98	3.00	3.03	12.00	2.88	2.15	2.20	7.22	2.88	2.15	2.20	7.22
2	6.01	2.97	3.03	12.00	3.02	2.23	2.27	7.52	3.02	2.23	2.27	7.52
3	6.02	2.97	3.01	12.00	2.97	2.24	2.22	7.43	2.97	2.24	2.22	7.43
4	6.01	2.97	3.02	12.00	3.00	2.20	2.17	7.37	3.00	2.20	2.17	7.37
5	5.99	3.01	2.99	12.00	3.04	2.25	2.26	7.55	3.04	2.25	2.26	7.55
6	5.99	3.02	2.99	12.00	3.05	2.25	2.28	7.58	3.05	2.25	2.28	7.58
7	6.01	3.02	2.98	12.00	3.00	2.23	2.26	7.49	3.00	2.23	2.26	7.49
8	5.98	3.04	2.98	12.00	3.01	2.25	2.24	7.49	3.01	2.25	2.24	7.49
9	6.03	3.00	2.98	12.00	2.93	2.25	2.24	7.42	2.93	2.25	2.24	7.42
10	5.99	3.01	3.00	12.00	3.06	2.27	2.33	7.65	3.06	2.27	2.33	7.65
AVERAGE	6.00	3.00	3.00	12.00	2.99	2.23	2.25	7.47	2.99	2.23	2.25	7.47

TABLE A17: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(6 , 3 , 3) ;GOAL VECTOR M=(6 , 3 , 3)   TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

VECTOR M-M*P =(2.40 , 0.00 ,-0.00)

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	5.33	3.34	3.33	12.00	1.09	1.31	1.35	3.75	1.54	1.42	1.46	4.42
2	5.25	3.39	3.36	12.00	1.20	1.45	1.45	4.10	1.76	1.60	1.58	4.94
3	5.23	3.39	3.38	12.00	1.20	1.45	1.46	4.11	1.80	1.61	1.60	5.00
4	5.24	3.38	3.38	12.00	1.18	1.44	1.50	4.12	1.75	1.59	1.64	4.98
5	5.22	3.41	3.38	12.00	1.22	1.46	1.48	4.16	1.83	1.63	1.63	5.09
6	5.22	3.39	3.39	12.00	1.20	1.46	1.50	4.16	1.81	1.61	1.66	5.07
7	5.24	3.39	3.36	12.00	1.20	1.46	1.49	4.15	1.77	1.61	1.62	5.01
8	5.25	3.39	3.36	12.00	1.20	1.44	1.45	4.08	1.76	1.59	1.58	4.93
9	5.23	3.39	3.38	12.00	1.20	1.44	1.49	4.13	1.80	1.60	1.64	5.03
10	5.23	3.39	3.38	12.00	1.23	1.47	1.47	4.17	1.83	1.62	1.62	5.07
AVERAGE	5.24	3.39	3.37	12.00	1.19	1.44	1.46	4.09	1.77	1.59	1.60	4.95

TABLE F18: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 4 , 4) ;GOAL VECTOR M=(4 4 4) TRANSITION MATRIX P=
VECTOR M-M*P =(0.80 , 0.80 , 0.80)

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	2	3	TOTAL
1	4.13	3.82	4.05	12.00	2.12	2.18	2.09	6.38	2.13	2.21	2.09	6.44
2	4.13	3.79	4.08	12.00	2.22	2.32	2.22	6.76	2.24	2.37	2.23	6.83
3	4.13	3.79	4.08	12.00	2.27	2.31	2.15	6.72	2.29	2.35	2.15	6.79
4	4.15	3.75	4.10	12.00	2.22	2.31	2.16	6.68	2.24	2.37	2.17	6.78
5	4.15	3.78	4.07	12.00	2.22	2.31	2.23	6.75	2.24	2.36	2.23	6.83
6	4.13	3.78	4.09	12.00	2.25	2.32	2.22	6.78	2.26	2.37	2.23	6.85
7	4.14	3.75	4.11	12.00	2.21	2.30	2.17	6.68	2.23	2.36	2.18	6.77
8	4.14	3.76	4.11	12.00	2.22	2.24	2.23	6.70	2.24	2.30	2.25	6.78
9	4.14	3.76	4.10	12.00	2.21	2.30	2.25	6.76	2.23	2.36	2.26	6.85
10	4.14	3.75	4.11	12.00	2.22	2.30	2.25	6.77	2.24	2.36	2.26	6.86
AVERAGE	4.14	3.77	4.09	12.00	2.21	2.29	2.19	6.70	2.23	2.34	2.20	6.78

TABLE A18: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(4 , 4 , 4) ;GOAL VECTOR M=(4 4 4) TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

VECTOR M-M*P =(0.80 , 0.80 , 0.80)

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	4.08	4.00	3.92	12.00	0.81	0.87	0.98	2.67	0.82	0.87	0.99	2.68
2	4.09	3.99	3.92	12.00	0.80	0.89	0.98	2.67	0.81	0.89	0.98	2.68
3	4.09	4.01	3.90	12.00	0.82	0.92	0.95	2.69	0.82	0.92	0.96	2.71
4	4.09	4.00	3.91	12.00	0.80	0.91	0.97	2.68	0.81	0.91	0.98	2.70
5	4.08	4.01	3.91	12.00	0.82	0.90	0.95	2.68	0.83	0.90	0.96	2.69
6	4.07	4.01	3.93	12.00	0.80	0.91	0.98	2.70	0.81	0.91	0.99	2.71
7	4.09	3.99	3.92	12.00	0.82	0.88	0.97	2.67	0.83	0.88	0.98	2.68
8	4.09	3.99	3.91	12.00	0.85	0.95	1.04	2.84	0.86	0.95	1.05	2.86
9	4.07	4.00	3.94	12.00	0.80	0.91	0.99	2.70	0.81	0.91	1.00	2.71
10	4.09	3.99	3.92	12.00	0.82	0.90	1.00	2.71	0.82	0.90	1.00	2.73
AVERAGE	4.08	4.00	3.92	12.00	0.81	0.90	0.98	2.70	0.82	0.90	0.99	2.71

TABLE F19: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(12 , 6 , 6) ;GOAL VECTOR M=(12 , 6 6) TRANSITION MATRIX P=

VECTOR M-M*P =(4.80 , 0.00 , 0.00)

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR: GRADE 1	GRADE 2	GRADE 3	TOTAL	VARIANCES OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL	M.S.E. OF THE STOCK VECTORS: GRADE 1	GRADE 2	GRADE 3	TOTAL
1	11.95	6.04	6.00	24.00	5.79	4.34	4.28	14.41	5.79	4.35	4.28	14.41
2	11.99	6.02	5.98	24.00	5.96	4.54	4.49	15.00	5.96	4.54	4.49	15.00
3	11.96	6.01	6.03	24.00	6.00	4.50	4.50	14.99	6.00	4.50	4.50	14.99
4	12.01	5.96	6.03	24.00	6.02	4.43	4.46	14.92	6.02	4.43	4.46	14.92
5	11.99	6.00	6.02	24.00	5.88	4.49	4.51	14.88	5.88	4.49	4.51	14.88
6	12.03	6.00	5.97	24.00	5.92	4.41	4.49	14.81	5.92	4.41	4.49	14.81
7	12.03	5.97	6.01	24.00	6.08	4.50	4.49	15.06	6.08	4.50	4.49	15.06
8	12.01	6.00	5.99	24.00	6.17	4.50	4.61	15.27	6.17	4.50	4.61	15.27
9	12.01	5.99	6.01	24.00	5.98	4.60	4.54	15.12	5.98	4.60	4.54	15.12
10	12.00	6.00	6.00	24.00	5.93	4.43	4.39	14.75	5.93	4.43	4.39	14.75
AVERAGE	12.00	6.00	6.00	24.00	5.97	4.47	4.48	14.92	5.97	4.47	4.48	14.92

TABLE A19: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(12 , 6 , 6) ;GOAL VECTOR M=(12 6 , 6)
VECTOR M-M*P =(4.80 , 0.00 , 0.00)

TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR GRADE 1	2	3	TOTAL	VARIANCES OF THE STOCK VECTORS GRADE 1	2	3	TOTAL	M.S.E. OF THE STOCK VECTORS GRADE 1	2	3	TOTAL
1	10.95	6.53	6.53	24.00	1.91	2.38	2.39	6.68	3.02	2.65	2.67	8.35
2	10.82	6.56	6.62	24.00	2.14	2.58	2.70	7.42	3.54	2.90	3.08	9.53
3	10.82	6.59	6.59	24.00	2.10	2.61	2.66	7.37	3.48	2.95	3.01	9.44
4	10.80	6.60	6.61	24.00	2.21	2.62	2.67	7.49	3.66	2.98	3.03	9.67
5	10.77	6.59	6.64	24.00	2.25	2.74	2.81	7.79	3.76	3.08	3.21	10.05
6	10.79	6.62	6.59	24.00	2.17	2.67	2.72	7.55	3.63	3.05	3.07	9.75
7	10.78	6.61	6.62	24.00	2.18	2.70	2.76	7.65	3.68	3.07	3.15	9.90
8	10.80	6.60	6.60	24.00	2.14	2.64	2.64	7.42	3.57	3.00	3.00	9.57
9	10.79	6.61	6.60	24.00	2.23	2.61	2.64	7.48	3.69	2.98	3.00	9.67
10	10.81	6.60	6.58	24.00	2.13	2.67	2.64	7.44	3.54	3.03	2.98	9.56
AVERAGE	10.81	6.59	6.60	24.00	2.15	2.62	2.66	7.43	3.56	2.97	3.02	9.55

TABLE F20: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(8 , 8 , 8) ;GOAL VECTOR M=(8 , 8 , 8)

VECTOR M-M*P =(1.60 , 1.60 , 1.60)

TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	8.14	7.82	8.04	24.00	4.10	4.28	4.07	12.45	4.12	4.31	4.07	12.50
2	8.15	7.78	8.07	24.00	4.37	4.55	4.30	13.23	4.40	4.60	4.31	13.30
3	8.13	7.81	8.07	24.00	4.28	4.45	4.36	13.09	4.30	4.49	4.36	13.15
4	8.16	7.74	8.10	24.00	4.35	4.46	4.38	13.19	4.37	4.53	4.39	13.29
5	8.13	7.77	8.10	24.00	4.32	4.40	4.32	13.04	4.33	4.45	4.33	13.12
6	8.14	7.78	8.08	24.00	4.32	4.32	4.23	12.87	4.34	4.37	4.24	12.95
7	8.13	7.77	8.11	24.00	4.46	4.48	4.38	13.32	4.48	4.54	4.39	13.40
8	8.14	7.73	8.13	24.00	4.32	4.39	4.37	13.08	4.34	4.46	4.38	13.18
9	8.12	7.77	8.11	24.00	4.35	4.40	4.36	13.11	4.36	4.45	4.38	13.19
10	8.13	7.78	8.09	24.00	4.30	4.37	4.26	12.93	4.31	4.42	4.27	13.00
AVERAGE	8.14	7.78	8.09	24.00	4.32	4.41	4.30	13.03	4.34	4.46	4.31	13.11

TABLE A20: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS,UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(8 , 8 , 8) ;GOAL VECTOR M=(8 8 8) TRANSITION MATRIX P=

0.400	0.200	0.200
0.200	0.400	0.200
0.200	0.200	0.400

VECTOR M-M*P =(1.60 , 1.60 , 1.60)

RESULTS OBTAINED FROM 10000 SIMULATIONS

STEP	EXPECTED STOCK VECTOR				VARIANCES OF THE STOCK VECTORS				M.S.E. OF THE STOCK VECTORS			
	GRADE			TOTAL	GRADE			TOTAL	GRADE			TOTAL
	1	2	3		1	2	3		1	2	3	
1	8.07	8.01	7.92	24.00	0.92	1.07	1.19	3.18	0.93	1.07	1.20	3.20
2	8.07	7.98	7.95	24.00	1.03	1.10	1.24	3.37	1.04	1.10	1.24	3.38
3	8.08	8.00	7.92	24.00	0.99	1.13	1.22	3.34	1.00	1.13	1.23	3.36
4	8.08	8.02	7.90	24.00	0.94	1.10	1.19	3.23	0.95	1.10	1.20	3.25
5	8.08	8.01	7.91	24.00	0.99	1.12	1.18	3.29	1.00	1.12	1.19	3.31
6	8.08	8.01	7.92	24.00	0.98	1.12	1.18	3.27	0.98	1.12	1.19	3.29
7	8.09	7.99	7.92	24.00	0.98	1.13	1.25	3.35	0.99	1.13	1.25	3.37
8	8.07	8.00	7.93	24.00	1.03	1.15	1.25	3.42	1.03	1.15	1.25	3.43
9	8.10	8.00	7.91	24.00	1.05	1.16	1.25	3.45	1.06	1.16	1.26	3.47
10	8.10	7.98	7.93	24.00	1.02	1.09	1.24	3.35	1.03	1.09	1.25	3.37
AVERAGE	8.08	8.00	7.92	24.00	0.99	1.12	1.22	3.33	1.00	1.12	1.23	3.34

TABLE F21: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 , 3 , 3 , 3) ;GOAL VECTOR M=(3 , 3 3 , 3)

VECTOR M-M*P =(1.20 , 0.00 ,-0.00 ,-0.00)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	3.02	2.99	3.00	3.00	12.00
2	3.01	2.99	3.00	2.99	12.00
3	3.01	3.00	3.00	2.99	12.00
4	3.00	3.02	3.01	2.99	12.00
5	3.00	3.01	3.01	2.98	12.00
6	3.00	3.01	3.02	2.98	12.00
7	3.00	3.00	3.02	2.98	12.00
8	3.02	2.99	3.01	2.98	12.00
9	3.02	2.99	3.01	2.99	12.00
10	3.04	2.99	2.98	2.99	12.00
AVERAGE	3.01	3.00	3.01	2.99	12.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	1.43	1.23	0.98	0.56	4.20
2	1.97	1.63	1.49	0.97	6.07
3	2.16	1.86	1.82	1.26	7.10
4	2.21	1.97	1.98	1.52	7.68
5	2.20	2.08	2.06	1.71	8.05
6	2.25	2.11	2.13	1.84	8.32
7	2.24	2.15	2.16	1.92	8.47
8	2.32	2.22	2.17	2.01	8.72
9	2.25	2.20	2.20	2.08	8.74
10	2.28	2.23	2.22	2.13	8.85
AVERAGE	2.13	1.97	1.92	1.60	7.62

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	1.43	1.23	0.98	0.56	4.21
2	1.97	1.63	1.49	0.97	6.07
3	2.16	1.86	1.82	1.26	7.10
4	2.21	1.97	1.98	1.52	7.68
5	2.20	2.08	2.06	1.71	6.05
6	2.25	2.11	2.13	1.84	8.32
7	2.24	2.15	2.16	1.92	8.47
8	2.32	2.22	2.17	2.01	8.72
9	2.26	2.20	2.20	2.08	8.74
10	2.28	2.23	2.22	2.13	8.86
AVERAGE	2.13	1.97	1.92	1.60	7.62

TABLE A21: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3   3 , 3   3) ;GOAL VECTOR M=(3 , 3   3 , 3)

VECTOR M-M*P   =(1.20 , 0.00 ,-0.00 ,-0.00)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	2	3	4	TOTAL
1	2.53	3.24	3.15	3.08	12.00
2	2.33	3.24	3.28	3.15	12.00
3	2.23	3.16	3.38	3.24	12.00
4	2.17	3.07	3.44	3.32	12.00
5	2.15	3.00	3.45	3.40	12.00
6	2.12	2.95	3.45	3.48	12.00
7	2.12	2.90	3.45	3.53	12.00
8	2.12	2.86	3.44	3.58	12.00
9	2.10	2.86	3.41	3.62	12.00
10	2.10	2.84	3.41	3.65	12.00
AVERAGE	2.20	3.01	3.39	3.41	12.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	0.47	0.78	0.66	0.40	2.32
2	0.64	1.03	1.08	0.67	3.40
3	0.71	1.17	1.33	0.87	4.08
4	0.73	1.16	1.46	1.04	4.38
5	0.73	1.19	1.52	1.20	4.64
6	0.74	1.21	1.56	1.33	4.84
7	0.73	1.21	1.58	1.39	4.91
8	0.74	1.19	1.60	1.48	5.01
9	0.75	1.23	1.64	1.52	5.13
10	0.74	1.21	1.66	1.56	5.17
AVERAGE	0.70	1.14	1.41	1.15	4.39

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	0.69	0.84	0.69	0.41	2.63
2	1.08	1.08	1.16	0.69	4.01
3	1.30	1.19	1.47	0.93	4.90
4	1.41	1.16	1.65	1.14	5.36
5	1.45	1.19	1.72	1.36	5.72
6	1.51	1.21	1.77	1.55	6.04
7	1.51	1.22	1.78	1.67	6.18
8	1.52	1.21	1.79	1.82	6.34
9	1.56	1.25	1.81	1.91	6.52
10	1.55	1.23	1.83	1.98	6.59
AVERAGE	1.36	1.16	1.56	1.35	5.43

TABLE F22: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 2 3 , 6) ;GOAL VECTOR M=(1 , 2 3 , 6)

VECTOR M-M*P =(0.40 , 0.30 0.20 , 0.30)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	1.02	2.02	3.00	5.96	12.00
2	1.04	2.06	2.99	5.91	12.00
3	1.05	2.10	2.97	5.88	12.00
4	1.07	2.12	2.96	5.85	12.00
5	1.06	2.14	2.97	5.83	12.00
6	1.06	2.14	2.99	5.80	12.00
7	1.07	2.14	2.99	5.80	12.00
8	1.07	2.13	3.03	5.77	12.00
9	1.07	2.13	3.04	5.75	12.00
10	1.07	2.13	3.06	5.74	12.00
AVERAGE	1.06	2.11	3.00	5.83	12.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	0.48	0.79	0.87	0.70	2.84
2	0.66	1.13	1.37	1.16	4.32
3	0.71	1.30	1.69	1.51	5.22
4	0.74	1.37	1.85	1.72	5.68
5	0.75	1.42	1.94	1.90	6.01
6	0.75	1.47	2.06	2.05	6.33
7	0.77	1.49	2.09	2.14	6.48
8	0.77	1.51	2.11	2.13	6.51
9	0.77	1.53	2.20	2.19	6.69
10	0.77	1.54	2.19	2.22	6.71
AVERAGE	0.72	1.36	1.84	1.77	5.68

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	0.48	0.79	0.87	0.70	2.84
2	0.66	1.13	1.37	1.17	4.34
3	0.72	1.31	1.69	1.53	5.25
4	0.75	1.38	1.85	1.75	5.73
5	0.75	1.44	1.94	1.93	6.06
6	0.76	1.50	2.06	2.09	6.40
7	0.77	1.51	2.09	2.18	6.55
8	0.77	1.52	2.11	2.19	6.59
9	0.77	1.55	2.20	2.25	6.77
10	0.77	1.56	2.19	2.29	6.81
AVERAGE	0.72	1.37	1.84	1.81	5.73

TABLE A22: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(1 2 3 6) ;GOAL VECTOR M=(1 2 3 , 6) TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

VECTOR M-N*P =(0.40 , 0.30 , 0.20 , 0.30)

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	2	3	4	TOTAL
1	0.90	2.01	3.07	6.02	12.00
2	0.86	2.00	3.11	6.03	12.00
3	0.85	1.97	3.13	6.05	12.00
4	0.85	1.95	3.15	6.05	12.00
5	0.84	1.93	3.16	6.07	12.00
6	0.84	1.91	3.18	6.07	12.00
7	0.84	1.92	3.16	6.09	12.00
8	0.84	1.93	3.16	6.07	12.00
9	0.84	1.93	3.15	6.08	12.00
10	0.83	1.92	3.15	6.10	12.00
AVERAGE	0.85	1.95	3.14	6.06	12.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	0.09	0.30	0.42	0.34	1.14
2	0.12	0.40	0.61	0.54	1.66
3	0.13	0.43	0.71	0.64	1.91
4	0.13	0.45	0.75	0.68	2.01
5	0.13	0.46	0.78	0.72	2.10
6	0.14	0.45	0.82	0.74	2.15
7	0.14	0.46	0.82	0.78	2.20
8	0.13	0.45	0.81	0.77	2.16
9	0.14	0.46	0.81	0.78	2.18
10	0.14	0.46	0.83	0.79	2.22
AVERAGE	0.13	0.43	0.73	0.68	1.97

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	0.10	0.30	0.42	0.34	1.16
2	0.14	0.40	0.62	0.54	1.69
3	0.15	0.43	0.72	0.65	1.95
4	0.15	0.46	0.77	0.68	2.06
5	0.16	0.47	0.81	0.73	2.16
6	0.16	0.46	0.85	0.75	2.22
7	0.17	0.47	0.85	0.79	2.26
8	0.16	0.45	0.83	0.78	2.22
9	0.16	0.47	0.83	0.78	2.24
10	0.17	0.47	0.85	0.80	2.28
AVERAGE	0.15	0.44	0.76	0.68	2.02

TABLE F23: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(12 ,12 ,12 ,12) ;GOAL VECTOR M=(12 ,12 ,12 ,12)

VECTOR M-M*P =(4.80 ,-0.00 0.00 , 0.00)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	2	3	4	TOTAL
1	12.02	12.01	11.99	11.98	48.00
2	12.04	11.98	12.01	11.96	48.00
3	12.04	11.98	12.02	11.97	48.00
4	12.10	11.94	12.01	11.96	48.00
5	12.05	12.02	12.00	11.93	48.00
6	12.04	12.03	12.02	11.92	48.00
7	12.06	12.02	12.02	11.91	48.00
8	12.05	12.01	12.03	11.91	48.00
9	12.05	12.00	12.04	11.91	48.00
10	12.02	12.02	12.02	11.94	48.00
AVERAGE	12.05	12.00	12.02	11.94	48.00

VARIANCES OF THE STOCK VECTORS

GRADE 1	2	3	4	TOTAL
5.81	4.98	3.96	2.18	16.93
7.95	6.64	6.04	3.88	24.50
8.67	7.30	7.28	5.09	28.33
8.91	7.82	8.07	6.07	30.87
9.03	8.42	8.17	6.77	32.38
9.11	8.72	8.42	7.50	33.75
9.12	8.84	8.68	7.78	34.42
8.82	8.99	8.69	8.05	34.54
9.16	8.98	8.75	8.38	35.27
9.18	8.88	8.72	8.61	35.39
8.58	7.96	7.68	6.43	30.64

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	5.81	4.98	3.96	2.18	16.93
2	7.95	6.64	6.04	3.88	24.51
3	8.67	7.30	7.28	5.09	28.34
4	8.92	7.83	8.07	6.07	30.89
5	9.03	8.42	8.17	6.77	32.39
6	9.11	8.72	8.42	7.51	33.76
7	9.13	8.84	8.68	7.78	34.44
8	8.82	8.99	8.69	8.06	34.55
9	9.16	8.98	8.75	8.39	35.28
10	9.18	8.89	8.72	8.62	35.40
AVERAGE	8.58	7.96	7.68	6.43	30.65

TABLE A23: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(12 ,12 ,12 ,12) ;GOAL VECTOR M=(12 ,12 ,12 ,12)

VECTOR M-M*P =(4.80 ,-0.00 , 0.00 , 0.00)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	2	3	4	TOTAL
1	10.84	12.58	12.35	12.23	48.00
2	10.37	12.51	12.69	12.43	48.00
3	10.15	12.33	12.92	12.60	48.00
4	10.05	12.18	13.00	12.77	48.00
5	9.96	12.03	13.06	12.95	48.00
6	9.94	11.91	13.07	13.09	48.00
7	9.90	11.84	13.03	13.23	48.00
8	9.86	11.78	13.04	13.32	48.00
9	9.83	11.74	13.04	13.38	48.00
10	9.85	11.72	12.96	13.47	48.00
AVERAGE	10.08	12.06	12.92	12.95	48.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	1.78	2.72	2.45	1.43	8.39
2	2.53	3.73	3.97	2.43	12.65
3	2.75	4.03	4.75	3.18	14.71
4	2.83	4.19	5.00	3.77	15.79
5	2.91	4.29	5.38	4.21	16.78
6	2.94	4.28	5.64	4.59	17.44
7	2.96	4.20	5.67	4.89	17.72
8	2.99	4.33	5.64	5.13	18.09
9	2.89	4.36	5.83	5.40	18.47
10	2.93	4.29	5.80	5.53	18.54
AVERAGE	2.75	4.04	5.01	4.06	15.86

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	3.12	3.06	2.57	1.48	10.23
2	5.18	3.99	4.45	2.62	16.24
3	6.16	4.14	5.60	3.54	19.45
4	6.63	4.22	6.00	4.37	21.22
5	7.06	4.29	6.51	5.11	22.96
6	7.18	4.29	6.77	5.77	24.01
7	7.35	4.23	6.74	6.39	24.71
8	7.57	4.38	6.72	6.88	25.55
9	7.58	4.42	6.91	7.31	26.22
10	7.55	4.37	6.72	7.69	26.32
AVERAGE	6.54	4.14	5.90	5.12	21.69

TABLE F24: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS,UNDER THE FIXED RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 7 ,13 ,25) ;GOAL VECTOR M=(3 , 7 ,13 ,25) TRANSITION MATRIX P=

VECTOR M-M*P =(1.20 , 1.20 , 1.20 , 1.20)

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	2	3	4	TOT
1	2.87	7.11	12.89	25.13	48.00
2	2.81	7.15	12.81	25.23	48.00
3	2.78	7.18	12.74	25.30	48.00
4	2.75	7.19	12.71	25.36	48.00
5	2.75	7.16	12.73	25.37	48.00
6	2.73	7.14	12.69	25.44	48.00
7	2.73	7.15	12.66	25.46	48.00
8	2.73	7.16	12.63	25.47	48.00
9	2.72	7.15	12.61	25.53	48.00
10	2.72	7.12	12.59	25.58	48.00
AVERAGE	2.76	7.15	12.71	25.39	48.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	1.14	2.35	3.11	2.66	9.25
2	1.56	3.42	4.88	4.66	14.52
3	1.68	3.95	5.85	6.01	17.50
4	1.71	4.15	6.43	7.01	19.30
5	1.73	4.38	6.93	7.66	20.70
6	1.72	4.46	7.14	8.03	21.34
7	1.73	4.48	7.34	8.38	21.93
8	1.71	4.48	7.50	8.71	22.40
9	1.73	4.47	7.65	8.78	22.63
10	1.68	4.46	7.70	8.84	22.68
AVERAGE	1.64	4.06	6.45	7.07	19.23

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	2	3	4	TOTAL
1	1.16	2.36	3.12	2.68	9.31
2	1.60	3.45	4.92	4.71	14.67
3	1.73	3.98	5.92	6.11	17.74
4	1.78	4.18	6.52	7.14	19.61
5	1.80	4.40	7.00	7.80	21.00
6	1.79	4.48	7.23	8.22	21.72
7	1.80	4.50	7.46	8.59	22.35
8	1.79	4.51	7.64	8.93	22.86
9	1.81	4.49	7.81	9.07	23.18
10	1.77	4.47	7.87	9.17	23.28
AVERAGE	1.70	4.08	6.55	7.24	19.57

TABLE A24: EVOLUTION OF A GRADED SYSTEM OVER A PERIOD OF 10 STEPS, UNDER THE ADAPTIVE RECRUITMENT STRATEGY.

INITIAL VECTOR N=(3 7 ,13 ,25) ;GOAL VECTOR M=(3 , 7 ,13 ,25)

VECTOR M-M*P =(1.20 1.20 1.20 , 1.20)

TRANSITION MATRIX P=

0.600	0.300	0.000	0.000
0.000	0.700	0.200	0.000
0.000	0.000	0.800	0.100
0.000	0.000	0.000	0.900

RESULTS OBTAINED FROM 10000 SIMULATIONS

EXPECTED STOCK VECTOR

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	2.91	7.00	13.04	25.04	48.00
2	2.88	6.97	13.07	25.08	48.00
3	2.87	6.96	13.07	25.10	48.00
4	2.87	6.95	13.06	25.12	48.00
5	2.87	6.96	13.05	25.12	48.00
6	2.86	6.96	13.08	25.11	48.00
7	2.86	6.94	13.08	25.13	48.00
8	2.87	6.93	13.06	25.15	48.00
9	2.86	6.94	13.05	25.15	48.00
10	2.87	6.94	13.07	25.11	48.00
AVERAGE	2.87	6.95	13.06	25.11	48.00

VARIANCES OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	0.09	0.37	0.69	0.71	1.86
2	0.12	0.43	0.92	0.94	2.42
3	0.13	0.49	0.98	1.06	2.66
4	0.14	0.49	0.97	1.10	2.70
5	0.14	0.50	0.98	1.10	2.72
6	0.15	0.50	1.00	1.12	2.77
7	0.15	0.50	1.02	1.16	2.82
8	0.14	0.49	0.99	1.19	2.81
9	0.15	0.49	0.99	1.17	2.80
10	0.14	0.46	1.00	1.15	2.75
AVERAGE	0.13	0.47	0.96	1.07	2.63

M.S.E. OF THE STOCK VECTORS

STEP	GRADE 1	GRADE 2	GRADE 3	GRADE 4	TOTAL
1	0.10	0.37	0.69	0.71	1.87
2	0.14	0.43	0.93	0.95	2.44
3	0.15	0.49	0.99	1.07	2.69
4	0.16	0.49	0.98	1.12	2.74
5	0.15	0.50	0.99	1.11	2.75
6	0.17	0.50	1.01	1.13	2.81
7	0.16	0.50	1.02	1.17	2.86
8	0.16	0.50	0.99	1.21	2.85
9	0.17	0.49	0.99	1.19	2.84
10	0.15	0.47	1.01	1.16	2.79
AVERAGE	0.15	0.47	0.96	1.08	2.67

'PENDIX C: THE ROUTINE "MAINT"

The purpose of this routine is to evaluate the probabilities PR = $P(a \leq f(1) \leq b/n(0) = n)$, in which a, b and n are given non-negative vectors.

The routine is written in standard Fortran IV. It can be called in any other routine by the use of the statement "CALL MAINT (PR)". The input parameters should, then, be passed to it by the use of the two following statements:

```
COMMON/C1/NLOW(30), NHIGH(30), NSTART(30), P(30,31),N
COMMON/C2/SLMT
```

where:

N	=	number of grades	
P(i,j)	=	transition probabilities between grades i and j	(i,j=1,2,...,N)
NLOW(i)	=	a_i	(i =1,2,...,N)
NHIGH(i)	=	b_i	(i =1,2,...,N)
NSTART(i)	=	n_i	(i =1,2,...,N)
SLMT	=	$1-\alpha/N$, α being the absolute error acceptable to the user.	

The routine, as listed, can be used for any hierarchical system with a maximum of 30 grades and a maximum of 1000 members at any grade. If the need arises to alter such limits, all the 30s and 1000s in the dimension statements have to be modified in accordance with the new requirements.

LISTING

```
      SUBROUTINE MAINT(PR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/C1/NLOW(30),NHIGH(30),NSTART(30),P(30,31),N
      COMMON/C2/SLMT
      COMMON/C3/MTOT,INDEX,SUM,KANUSE,NOTUSE
      COMMON/C4/B(2,1001)
      KANUSE=1
      NOTUSE=2
      NMIN1=N-1
      IF(NLOW(1).GT.0.OR.NHIGH(1).LT.NSTART(1)) GO TO 3
      CALL BINOM(NSTART(1),P(1,2),0)
      IDEBUT=2
      GO TO 8
    3 IDEBUT=1
    8 DO 5 ID=IDEBUT,NMIN1
      KEEP=0
      INTERM=KANUSE
      KANUSE=NOTUSE
      NOTUSE=INTERM
      NZ=NSTART(ID)
      NZZ=NZ+1
      DO 7 J=1,NZZ
    7 B(KANUSE,J)=0.0
      CALL ENUMER(NZ,P(ID ID),P(ID,ID+1),ID,KEEP)
    5 CONTINUE
      INTERM=KANUSE
      KANUSE=NOTUSE
      NOTUSE=INTERM
      CALL BINOM(NSTART(N),P(N,N),1)
      PR=0.0
      JUP=MINO(NSTART(N-1),NHIGH(N))+1
      DO 6 JX=1,JUP
      J=JX-1
      JUN=MINO(NHIGH(N)-J,NSTART(N))+1
      JDEUX=MINO(NLOW(N)-J-1,NSTART(N))+1
      PR1=B(KANUSE,JUN)
      PR2=0.0
      IF(JDEUX.GE.1)  PR2=B(KANUSE,JDEUX)
    6 PR=PR+B(NOTUSE,JX)*(PR1-PR2)
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE BINOM(N,P,KCUMUL)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/C2/GRAND
      COMMON/C3/MTOT,INDEX,SUM,KANUSE,NOTUSE
      COMMON/C4/B(2,1001)
      PP=P
      NR=N+1
      DO 273 I=1,NR
```

```
 273 B(KANUSE,I)=0.
     IF(N.EQ.0) GO TO 210
     IND=0
     IF(PP.LE..5)   GO TO 201
     IND=1
     PP=1.-PP
 201 M=INT(DFLOAT(N)*PP)+1
     Z=1.
     L=N-M
     Q=1.-PP
     LL=L
     DO 203 I=1,M
     J=M-I+1
     Z=Z*(DFLOAT(L+J)/DFLOAT(J))*PP
   3 IF(LL.EQ.0.OR.Z.LT.1.)  GO TO 203
     Z=Z*Q
     LL=LL-1
     GO TO 3
 203 CONTINUE
     IF(LL.NE.0)  Z=Z*(Q**LL)
     B(KANUSE,M+1)=Z
     SUM=Z
     YG=Q/PP
     YD=PP/Q
     ZG=Z
     ZD=Z
     J=0
  10 J=J+1
     I=M-J
     IF(I)20,30,30
  30 ZG=ZG*YG*DFLOAT(I+1)/DFLOAT(N-I)
     B(KANUSE,I+1)=ZG
     SUM=SUM+ZG
     IF(SUM.GT.GRAND)  GO TO 209
  20 I=M+J
     IF(I-N) 40,40,50
  40 ZD=ZD*YD*DFLOAT(NR-I)/DFLOAT(I)
     B(KANUSE,I+1)=ZD
     SUM=SUM+ZD
     IF(SUM.GT.GRAND) GO TO 209
  50 IF(I.GT.N.AND.J.GT.M) GO TO 209
     GO TO 10
 209 CONTINUE
     IF(IND.NE.1) GO TO 220
     NN=INT(0.5*DFLOAT(N))+1
     DO 207 J=1,NN
     I=NN-J
     Z=B(KANUSE,N-I+1)
     B(KANUSE,N-I+1)=B(KANUSE,I+1)
     B(KANUSE,I+1)=Z
 207 CONTINUE
     GO TO 220
```

```
  210 IF(N.EQ.0) B(KANUSE,1)=1.
      IF(N.EQ.0) RETURN
      B(KANUSE,1)=1.-P
      B(KANUSE,2)=P
  220 IF(KCUMUL.EQ.0) RETURN
      DO 300 I=2,NR
      B(KANUSE,I)=B(KANUSE,I)+B(KANUSE,I-1)
  300 CONTINUE
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE ENUMER(N,P1,P2,ID,KEEP)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION AX(3),MODE(3),P(3)
      COMMON/C2/SLMT
      COMMON/C3/MTOT,INDEX,SUM,KANUSE,NOTUSE
      P(1)=P1
      P(2)=P2
      P(3)=1.-P(1)-P(2)
      SUM=0.
      MTOT=(N+1)*(N+2)/2
      INDEX=0
      IF(N.EQ.0) GO TO 96
      NSUM=0
      DO 41 I=1,3
      AX(I)=DFLOAT(N+1)*P(I)
      MODE(I)=INT(AX(I))
   41 NSUM=NSUM+MODE(I)
      ISG=0
      JM=1
      IF(NSUM-N) 42,43,44
   42 XM=(1.-AX(1)+DFLOAT(MODE(1)))/P(1)
      ISG=1
      DO 46 I=2,3
      X=(1.-AX(I)+DFLOAT(MODE(I)))/P(I)
      IF(X.GE.XM) GO TO 46
      XM=X
      JM=I
   46 CONTINUE
      GO TO 43
   44 XM=(1.+AX(1)-DFLOAT(MODE(1)))/P(1)
      ISG=-1
      DO 47 I=2,3
      X=(1.+AX(I)-DFLOAT(MODE(I)))/P(I)
      IF(X.GE.XM) GO TO 47
      XM=X
      JM=I
   47 CONTINUE
   43 MODE(JM)=MODE(JM)+ISG
      K1=MODE(1)
```

```
      K2=MODE(2)
      K3=MODE(3)
      IF(K1+K2.NE.N) GO TO 49
      IF(K2.EQ.0) K1=K1-1
      IF(K2.NE.0) K2=K2-1
      K3=K3+1
   49 CONTINUE
      D=TRINOM(N,K1,K2,K3,P)
      IS=K1
      JS=K2
      JOLD=-1
      IOLD=-1
      IF(JS.NE.0) GO TO 170
      IOLD=IS
      OLDI=D
  170 IF(IS.NE.0) GO TO 171
      JOLD=JS
      OLDJ=D
  171 CONTINUE
      INC=1
      ISGN=1
      JJ=(N-K1-K2-1)/2+K2
      II=N-1-JJ
      IF(IS.EQ.II.AND.JS.EQ.JJ) OLDIJ=D
      JLMT=K2+INC*ISGN
      ILMT=K1
      GO TO 97
   96 IS=0
      JS=0
      D=1.
   97 CONTINUE
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
    3 JS=JS+ISGN
      IF(JS*ISGN.GT.JLMT*ISGN) GO TO 1
      IF(JS.GE.0) GO TO 4
      ILMT=ILMT-INC
      ISGN=1
      GOTO 2
    4 CONTINUE
      D=RECENT(D,2,ISGN,IS,JS,N,P)
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
      IF(IS.EQ.II.AND.JS.EQ.JJ) OLDIJ=D
      IF(IS.NE.0) GO TO 71
      JOLD=JS
      OLDJ=D
   71 CONTINUE
      IF(IS+JS.NE.N) GO TO 3
    8 INC=INC+1
      JLMT=JLMT-INC
      IF(JJ.EQ.-1.AND.IOLD.EQ.0.AND.JOLD.EQ.N) GO TO 50
```

```
      IF(JJ.EQ.-1) GO TO 2
      IS=II+1
      JS=JJ
      ILMT=II+1
      D=RECENT(OLDIJ,1,1,IS,JS,N,P)
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
      II=IS
      JJ=JS-1
      OLDIJ=D
      ISGN=-1
      GOTO 3
    1 JS=JS-ISGN
      ILMT=ILMT+ISGN*INC
      GOTO 6
    7 IS=IS-ISGN
      ISGN=-ISGN
      INC=INC+1
      JLMT=JLMT+ISGN*INC
      GO TO 3
    6 IS=IS+ISGN
      IF(IS*ISGN.GT.ILMT*ISGN) GO TO 7
      D=RECENT(D,1,ISGN,IS,JS,N,P)
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
      IF(IS.EQ.II.AND.JS.EQ.JJ) OLDIJ=D
      IF(JS.NE.0) GO TO 72
      IOLD=IS
      OLDI=D
   72 CONTINUE
      IF(IS.EQ.ILMT) GO TO 6
      IF(IS.NE.0.AND.IS+JS.NE.N) GO TO 6
      IF(IS+JS.EQ.N) GO TO 8
      INC=INC+1
      JLMT=JLMT+INC
      ILMT=ILMT+INC
      IF(JOLD.EQ.-1) GO TO 3
      GO TO 14
    2 CONTINUE
      IF(IOLD.EQ.-1) GO TO 3
      INC=INC+1
      JLMT=JLMT+INC
      IF(IOLD.EQ.0) GO TO 14
      IS=IOLD-1
      JS=0
      IOLD=IS
      D=RECENT(OLDI,1,-1,IS,JS,N,P)
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
      ILMT=IS
      OLDI=D
      GO TO 3
```

```
   14 IF(JOLD.EQ.N) GO TO 8
      JS=JOLD+1
      IS=0
      JOLD=JS
      D=RECENT(OLDJ,2,1,IS,JS,N,P)
      CALL MANIP(KEEP,ID,IS,JS,D)
      IF(KEEP.EQ.-1) RETURN
      OLDJ=D
      ILMT=ILMT+INC
      ISGN=1
      IF(JS.NE.N) GO TO 6
      GO TO 8
   50 ISGN=1
      ILMT=1
      JS=0
      IS=II
      GO TO 6
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE MANIP(KEEP,ID,IX,JX,DX)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/C1/NLOW(30),NHIGH(30),NSTART(30),P(30,31),N
      COMMON/C2/ SLMT
      COMMON/C3/MTOT,INDEX,SUM,KANUSE,NOTUSE
      COMMON/C4/B(2,1001)
      INTEGER T,TMIN,TMAX,ALMI,BLMI
      J=JX+1
      IF(IX.GT.NHIGH(ID).OR.JX.GT.NHIGH(ID+1)) GO TO 90
      IF(ID.NE.1)  GO TO 10
      IF(IX.LT.NLOW(ID)) GO TO 90
      B(KANUSE,J)=B(KANUSE,J)+DX
      GO TO 90
   10 NLM1=NSTART(ID-1)
      IF(IX.LT.NLOW(ID)-NLM1)  GO TO 90
      ALMI=NLOW(ID)-IX
      BLMI=NHIGH(ID)-IX
      TMIN=MAX0(0,ALMI)+1
      TMAX=MIN0(NLM1,BLMI)+1
      DO 1 T=TMIN,TMAX
    1 B(KANUSE,J)=B(KANUSE,J)+B(NOTUSE,T)*DX
   90 INDEX=INDEX+1
      SUM=SUM+DX
      IF(SUM.LT.SLMT.AND.INDEX.LT.MTOT) RETURN
      KEEP=-1
      RETURN
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
```

```
      DOUBLE PRECISION FUNCTION RECENT(X,IV,ISGNX,ISX,JSX,N,P)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION P(3)
      IF(IV.EQ.2) GO TO 2
      IF(ISGNX.GT.0) GO TO 3
      RECENT=X*DFLOAT(ISX+1)*P(3)/(DFLOAT(N-ISX-JSX)*P(1))
      RETURN
    3 RECENT=X*DFLOAT(N-ISX-JSX+1)*P(1)/(P(3)*DFLOAT(ISX))
      RETURN
    2 IF(ISGNX.GT.0) GO TO 5
      RECENT=X*DFLOAT(JSX+1)*P(3)/(DFLOAT(N-ISX-JSX)*P(2))
      RETURN
    5 RECENT=X*DFLOAT(N-ISX-JSX+1)*P(2)/(P(3)*DFLOAT(JSX))
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      DOUBLE PRECISION FUNCTION TRINOM(N,K1,K2,K3,P)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION K(3),P(3),Q(3)
      K(1)=K1
      K(2)=K2
      K(3)=K3
      Q(1)=P(1)
      Q(2)=P(2)
      Q(3)=P(3)
      DO 2 I=1,2
      II=I+1
      DO 2 J=II,3
      IF(K(I)-K(J)) 2,2,3
    3 IK=K(I)
      K(I)=K(J)
      K(J)=IK
      X=Q(I)
      Q(I)=Q(J)
      Q(J)=X
    2 CONTINUE
      M=N
      D=1.
      LL=K(3)
      IF(K(2).EQ.0)  GO TO 8
      KK2=K(2)
      DO 4 I=1,KK2
      J=K(2)-I
      D=D*Q(2)*(DFLOAT(M-J)/DFLOAT(K(2)-J))
    5 IF(D.LE.1..OR.LL.EQ.0)  GO TO 4
      D=D*Q(3)
      LL=LL-1
      GO TO 5
    4 CONTINUE
      M=M-K(2)
```

```
      IF(K(1).EQ.0) GO TO 8
      KK1=K(1)
      DO 6 I=1,KK1
      J=I-1
      D=D*Q(1)*(DFLOAT(M-J)/DFLOAT(K(1)-J))
    7 IF(D.LT.1..OR.LL.EQ.0)  GO TO 6
      D=D*Q(3)
      LL=LL-1
      GO TO 7
    6 CONTINUE
    8 IF(LL.NE.0)  D=D*(Q(3)**LL)
      TRINOM=D
      RETURN
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
```

APPENDIX D: THE ROUTINE "GENERAL"

This routine is designed to obtain the exact value of the probability PR = P(0a $\leq$ f(1) $\leq$ m/n(0) = n) for a three-grade system with general transition matrix, m and n being two fixed non-negative vectors.

The routine GENERAL can be called in any other routine by the use of the statement "CALL GENERAL (PR)". The input parameters should, then, be passed to it by the use of the following statement:

COMMON/GN/ NHIGH(3), NSTART(3), P(3,4)

where:

P(i,j)	=	transition probability between grades i and j	(i,j=1,2,3)
NHIGH(i)	=	m_i	(i =1,2,3)
NSTART(i)	=	n_i	(i =1,2,3)

The routine, as listed, can be used for any three-grade system in which:

$(m_1+1)\ (m_2+1)\ (m_3+1) \leq 10000$

$(n_i+1)\ (n_i+2)\ (n_i+3)/6 \leq 3000 \quad (i = 1,2,3)$

If the system does not fall within such limits, all the 10000s and 3000s in the dimension statements have to be replaced with the values of the left-hand sides of the above inequalities.

LISTING

```
      SUBROUTINE GENERAL(PR)
      COMMON/GN/ NSTART(3),NHIGH(3),P(3,4)
      DIMENSION TAB(10000),MV(3)
      DIMENSION PP(3000,3),IVEC(3000,3,3),KKK(3)
      N=3
      CALL QUADR(PP,IVEC,KKK)
      MY=(NHIGH(3)+1)
      MX=MY*(NHIGH(2)+1)
      KK4=MX*(NHIGH(1)+1)
      DO 1 I=1,KK4
      TAB(I)=0.0
    1 CONTINUE
      KK1=KKK(1)
      KK2=KKK(2)
      KK3=KKK(3)
      DO 2 K1=1,KK1
      IF(PP(K1,1).EQ.0.0) GO TO 2
      DO 3 K2=1,KK2
      DO 5 J=1,3
      MV(J)=IVEC(K1,J,1)+IVEC(K2,J,2)
    5 CONTINUE
      DO 6 J=1,3
      IF(MV(J).GT.NHIGH(J)) GO TO 3
    6 CONTINUE
      LL=MV(1)*MX+MV(2)*MY+MV(3)+1
      TAB(LL)=TAB(LL)+PP(K1,1)*PP(K2,2)
    3 CONTINUE
    2 CONTINUE
      PR=0.0
      DO 8 LL=1,KK4
      IF(TAB(LL).EQ.0.0)  GO TO 8
      K4=LL-1
      L2=MOD(K4,MX)
      L3=MOD(L2,MY)
      L1=K4/MX
      L2=L2/MY
      DO 7 K3=1,KK3
      MV(1)=IVEC(K3,1,3)+L1
      MV(2)=IVEC(K3,2,3)+L2
      MV(3)=IVEC(K3,3,3)+L3
      DO 9 J=1,3
      IF(MV(J).GT.NHIGH(J)) GO TO 7
    9 CONTINUE
      PR=PR+TAB(LL)*PP(K3,3)
    7 CONTINUE
    8 CONTINUE
      RETURN
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
```

```
      SUBROUTINE QUADR(PP,IVEC,KKK)
      COMMON/GN/ NHIGH(3),NSTART(3),P(3,4)
      DIMENSION Q(4)
      DIMENSION PP(3000,3),IVEC(3000,3,3),KKK(3)
      DO 10 IX=1,3
      Q(4)=1.0
      DO 4 J=1,3
      Q(J)=P(IX,J)
      Q(4)=Q(4)-Q(J)
    4 CONTINUE
      N1=NSTART(IX)+1
      SUM=0.0
      KK=0
      PI=1.0
      LI=NSTART(IX)
      DO 1 I=1,N1
      PJ=PI
      JLMT=N1-I+1
      LJ=N1-I
      DO 2 J=1,JLMT
      PK=PJ
      KLMT=JLMT-J+1
      DO 3 K=1,KLMT
      L=N1-I-J-K+2
      KK=KK+1
      PP(KK,IX)=PK
      SUM=SUM+PK
      IVEC(KK,1,IX)=I-1
      IVEC(KK,2,IX)=J-1
      IVEC(KK,3,IX)=K-1
      PK=(Q(3)*PK*L)/(Q(4)*K)
    3 CONTINUE
      PJ=(PJ*Q(2)*LJ)/(Q(4)*J)
      LJ=LJ-1
    2 CONTINUE
      PI=(PI*Q(1)*LI)/(Q(4)*I)
      LI=LI-1
    1 CONTINUE
      DO 5 I=1,KK
      PP(I,IX)=PP(I,IX)/SUM
    5 CONTINUE
      KKK(IX)=KK
   10 CONTINUE
      RETURN
      END

C++++++++++++C++++++++++++C++++++++++++C++++++++++++C++++++++++++C
```

APPENDIX E: FORTRAN PROGRAMS USED IN INVESTIGATING DIFFERENT RECRUITMENT STRATEGIES

E.1 GENERAL

The examination of the behaviour of a system of total size N, under a given recruitment strategy, is achieved in three stages:

1° Firstly, the probabilities P(f(1)=v/n(0)=n) for all structures n of total size N and all possible structure v of total size less than or equal to N, are evaluated. This requires lengthy computations and is achieved by the program FLOWBR (flows before recruitment).

2° Secondly, the program FLOWAR (flows after recruitment) uses the above probabilities and follows the rules of the chosen recruitment policy to evaluate the probabilities P(n(1)=n'/n(0)=n) for all possible vectors n' and n of total size N.

3° Finally, the program TABLES uses the probabilities evaluated by the program FLOWAR at different steps to produce, for a given recruitment strategy, tables identical to those in Appendix B.

The probabilities evaluated by the program FLOWAR will be written in a file assigned to the logical unit 9. However, if the recruitment policy is "GOAL" and the horizon time is ITER, ITER files assigned to the logical units 9,18,...,9*ITER are generated.

The logical unit 9*t (t=1,2,...,ITER) will hold the probabilities related to the recruitment policy at the step (ITER-t+1).

The files generated by the program FLOWBR and FLOWAR can be saved for subsequent use. Our three-stage approach allows the maximum flexibility in combining a wide range of recruitment policies and benefits largely from filing facilities offered by the Fortran.

E.2 Program FLOWBR

E.2.1 Parameters

Input of the parameters is from logical unit 5 (cards); output is to logical unit 15. Description and format of each input card are given below.

Card	Parameters name and description		Format
1	P(1,1), P(1,2)	Transition probabilities from grade 1	2F10.5
2	P(2,2), P(2,3)	Transition probabilities from grade 2	2F10.5
3	P(3,3), P(3,4)	Transition probabilities from grade 3	2F10.5
4	NTOT	System total size	I5

E.2.2 Auxiliary Routines

MAINT

E.3 Program FLOWAR

E.3.1 Parameters

Input of the parameters is from logical unit 5 (cards); output is to logical unit(s) 9 (,18,...).

Description and format of each input card are given below:

Card	Parameters name and description		Format
1	P(1,1), P(1,2)	Transition probabilities from grade 1	2F10.5
2	P(2,2), P(2,3)	Transition probabilities from grade 2	2F10.5
3	P(3,3), P(3,4)	Transition probabilities from grade 3	2F10.5
4	NPO	Recruitment policy	A4
5	ITER	Number of steps (=1 if NPO is Fixed or ADAPTIVE)	I5
6	NGOAL	GOAL vector m	3I5
7	MR	Vector with integer entries proportional to m-mP (for fixed policy only)	3I5

E.3.2 Auxiliary Routines

INITF

INITG

MAJITR

EXAREC

EXAFIX

EXAGL

AFFECT

VERFT

E.3.3 Observation

This program cannot be run unless the probabilities P(f(1)=v/n(0)=n) are already available in a file assigned to logical unit 15.

E.4 Program TABLES

E.4.1 Parameters

Input of the parameters is from logical unit 5 (cards); output to logical unit 6 (printer).

Description and format of each input card are given below:

Card	Parameters name and description		Format
1	P(1,1), P(1,2)	Transition probabilities from grade 1	2F10.5
2	P(2,2), P(2,3)	Transition probabilities from grade 2	2F10.5
3	P(3,3), P(3,4)	Transition probabilities from grade 3	2F10.5
4	ITER	Number of steps	I5
5	NSTART	Initial vector n	3I5
6	NGOAL	GOAL vector m	3I5
7	ITAPE	Logical units assigned to files holding the probabilities P(n(t+1)=n'/n(t)=n); (ITAPE(t) corresponds to step t)	ITER*I5
8	NPOLI	Recruitment strategy	A4
9	MR	Vector with integer entries proportional to m-mP (for fixed policy only)	3I5

E.4.2 Auxiliary Routines

ATNCR1

MAINT

VERFT

PRTTAB

SUBPRT

E.4.3 Observation

This program cannot be run unless the probabilities P(n(t+1)=n'/n(t)=n) are already in files assigned to the logical units ITAPE(t) (t=1,2,...,ITER).

E.5 RESTRICTIONS

These programs, as listed, can be used for any three-grade hierarchical system with a total size less than or equal to 25. The observation period to consider for any recruitment strategy should not exceed 10 steps. These limits are those discussed in chapter four.

E.6 LISTINGS

```
      PROGRAM FLOWBR
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4),PRECI
      CHARACTER NCTRL*4
      COMMON/C1/NLOW(30),NHIGH(30),NSTART(30),PP(30,31),N
      COMMON/C2/SLMT
      DIMENSION PRMOVE(3000)
      DATA NCTRL/'PFLW'/
      NSTEP=1
      N=3
      DO 2 I=1,N
      DO 3 J=1,N
      P(I,J)=0.0
    3 CONTINUE
      READ(5,200) P(I,I),P(I,I+1)
    2 CONTINUE
      PRECI=0.0
      SLMT=1.-PRECI/DFLOAT(N)
      NN=N+1
      DO 1 I=1,N
      DO 1 J=1,NN
      PP(I,J)=P(I,J)
    1 CONTINUE
      READ(5,100) NTOT
      IENLT=NTOT+1
      DO 9 IEN=1,IENLT
      NSTART(1)=IEN-1
      JENLT=IENLT-IEN+1
      DO 9 JEN=1,JENLT
      NSTART(2)=JEN-1
      NSTART(3)=NTOT-NSTART(1)-NSTART(2)
      KK=0
      N1=NSTART(1)+1
      DO 4 I=1,N1
      N2=NSTART(1)-I+2
      N3=NSTART(2)+NSTART(3)+1
      NHIGH(1)=I-1
      NLOW(1)=NHIGH(1)
      DO 6 J=1,N2
      NHIGH(2)=J-1
      NLOW(2)=NHIGH(2)
      DO 6 K=1,N3
      NHIGH(3)=K-1
      NLOW(3)=NHIGH(3)
      CALL MAINT(PR)
      KK=KK+1
      PRMOVE(KK)=PR
    6 CONTINUE
      IF(NSTART(2).EQ.0) GO TO 4
      N2=N2+1
      N4=NSTART(1)+NSTART(2)+2-I
      N3=NTOT-I+3
```

```
      DO 7 J=N2,N4
      N5=N3-J
      NHIGH(2)=J-1
      NLOW(2)=NHIGH(2)
      DO 7 K=1,N5
      NHIGH(3)=K-1
      NLOW(3)=NHIGH(3)
      CALL MAINT(PR)
      KK=KK+1
      PRMOVE(KK)=PR
    7 CONTINUE
    4 CONTINUE
      WRITE(15) (NSTART(J),J=1,N),((P(I,J),J=1,N),I=1,N),
     +          NSTEP,KK,NCTRL
      WRITE(15) (PRMOVE(I),I=1,KK)
    9 CONTINUE
  100 FORMAT(16I5)
  200 FORMAT(8F10.5)
      STOP
      END


C++++++++++++C++++++++++++C++++++++++++C++++++++++++C++++++++++++C
```

```
      PROGRAM FLOWAR
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4)
      COMMON/AD/NREC(8,3),PREC(8),NR,MTOT
      COMMON/AD5/MR(3),NTRE
      COMMON/GNRL1/PRMOVE(3000),ATN(351)
      COMMON/GNRL2/N,MT,NTOT
      COMMON/GNRL3/COMP(351),VALITR(351,10),IFROM(351,3)
      COMMON/GNRL4/P,PRECI
      COMMON/GNRL5/NGOAL(3)
      CHARACTER*4 NPO
      N=3
      DO 1 I=1,N
      DO 2 J=1,N
      P(I,J)=0.0
    2 CONTINUE
      READ(5,200) P(I,I),P(I,I+1)
    1 CONTINUE
      READ(5,400) NPO
      READ(5,100) ITER
      READ(5,100) (NGOAL(I),I=1,N)
      NTOT=0
      DO 41 I=1,N
      NTOT=NTOT+NGOAL(I)
   41 CONTINUE
      CALL INITG
      MTOT=NTOT
      IF(NPO.EQ.'GOAL'.OR.NPO.EQ.'ADAP') GO TO 4
      IF(NPO.EQ.'FIXE') GO TO 3
      WRITE(6,300) NPO
      STOP
    3 CALL INITF
    4 CONTINUE
      DO 20 ITR=1,ITER
      CALL MAJITR(ITR,NPO)
   20 CONTINUE
      STOP
  100 FORMAT(16I5)
  200 FORMAT(8F10.5)
  300 FORMAT(1X,'NO ROUTINE FOR THE RECRUITMENT POLICY ',A4)
  400 FORMAT(A4)
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE INITF
      COMMON/AD5/MR(3),NTRE
      COMMON/GNRL2/N,MT,NTOT
      READ(5,100) (MR(J),J=1,N)
      NTRE=0
      DO 7 I=1,N
      NTRE=NTRE+MR(I)
```

```
    7 CONTINUE
  100 FORMAT(16I5)
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE INITG
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/GNRL2/N,K,NTOT
      COMMON/GNRL3/COMP(351),VALITR(351,10),IFROM(351,3)
      COMMON/GNRL5/NGOAL(3)
      K=0
      NTOTT=NTOT+1
      DO 1 I=1,NTOTT
      NJ=NTOTT-I+1
      DO 1 J=1,NJ
      K=K+1
      IFROM(K,1)=I-1
      IFROM(K,2)=J-1
      IFROM(K,3)=NTOTT-I-J+1
      COMP(K)=0.0
      DO 2 L=1,N
      COMP(K)=COMP(K)+(NGOAL(L)-IFROM(K,L))*(NGOAL(L)-IFROM(K,L))
    2 CONTINUE
    1 CONTINUE
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE MAJITR(ITR,NPO)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4),PRECI
      CHARACTER *4 NPO
      COMMON/GNRL1/PRMOVE(3000),ATN(351)
      COMMON/GNRL2/N,MT,NTOT
      COMMON/GNRL3/COMP(351),VALITR(351,10),IFROM(351,3)
      COMMON/GNRL4/P,PRECI
      COMMON/GNRL5/NGOAL(3)
      COMMON/MAJ1/NFLOW(3),LLL,KK
      COMMON/AD5/MR(3),NTRE
      DIMENSION NS(3)
      ITP=9*ITR
      NN=N+1
      DO 3 I=1,MT
      VALITR(I,ITR)=0.0
    3 CONTINUE
      REWIND 15
      DO 9 LX=1,MT
      LLL=LX
      DO 1 J=1,N
```

```
      NS(J)=IFROM(LLL,J)
    1 CONTINUE
      DO 5 I=1,MT
      ATN(I)=0.0
    5 CONTINUE
      CALL VERFT(NS,P,1,N,15,MEPS,'PFLW',NFLOW)
      READ(15) (PRMOVE(L),L=1,MEPS)
      KK=0
      N1=NS(1)+1
      DO 4 I=1,N1
      N2=NS(1)-I+2
      N3=NS(2)+NS(3)+1
      NFLOW(1)=I-1
      DO 6 J=1,N2
      NFLOW(2)=J-1
      DO 6 K=1,N3
      NFLOW(3)=K-1
      KK=KK+1
      CALL EXAREC(ITR,NPO)
    6 CONTINUE
      IF(NS(2).EQ.0) GO TO 4
      N2=N2+1
      N4=NS(1)+NS(2)-I+2
      N3=NTOT-I+3
      DO 7 J=N2,N4
      N5=N3-J
      NFLOW(2)=J-1
      DO 7 K=1,N5
      NFLOW(3)=K-1
      KK=KK+1
      CALL EXAREC(ITR,NPO)
    7 CONTINUE
    4 CONTINUE
      WRITE(ITP)(NS(I),I=1,N),((P(I,J),J=1,N),I=1,N),ITR,MT,NPO
      IF(NPO.EQ.'FIXE') WRITE(ITP) (MR(J),J=1,N)
      IF(NPO.NE.'FIXE') WRITE(ITP) (NGOAL(J),J=1,N)
      WRITE(ITP) (ATN(I),I=1,MT)
    9 CONTINUE
      REWIND ITP
      IF(NPO.NE.'GOAL') RETURN
      DO 2 I=1,MT
      COMP(I)=VALITR(I,ITR)
    2 CONTINUE
      RETURN
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE EXAREC(ITR,NPO)
      CHARACTER*4 NPO
      IF(NPO.NE.'FIXE') GO TO 1
      CALL EXAFIX(ITR)
```

```
      RETURN
    1 IF(NPO.NE.'GOAL'.AND.NPO.NE.'ADAP') GO TO 2
      CALL EXAGL(ITR)
      RETURN
    2 WRITE(6,100) NPO
  100 FORMAT(2X,'NO ROUTINE FOR THE RECRUITMENT POLICY ',A4)
      STOP
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE EXAFIX(ITR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/MAJ1/NFLOW(3),LLL,KK
      COMMON/AD/NREC(8,3),PREC(8),NR,MTOT
      COMMON/GNRL1/PRMOVE(3000),ATN(351)
      COMMON/GNRL2/N,MT,NTOT
      DIMENSION MX(3)
      KR=NTOT
      NT=NTOT+1
      DO 1 I=1,N
      KR=KR-NFLOW(I)
    1 CONTINUE
      X=1.0
      NR=0
      CALL AFFECT(KR,X)
      DO 2 I=1,NR
      DO 3 J=1,N
      MX(J)=NFLOW(J)+NREC(I,J)
    3 CONTINUE
      IJ=NT*MX(1)+MX(2)+1-MX(1)*(MX(1)-1)/2
      ATN(IJ)=ATN(IJ)+PRMOVE(KK)*PREC(I)
    2 CONTINUE
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE EXAGL(ITR)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/GNRL1/PRMOVE(3000),ATN(351)
      COMMON/GNRL2/N,MT,NTOT
      COMMON/GNRL3/COMP(351),VALITR(351,10),IFROM(351,3)
      COMMON/MAJ1/NFLOW(3),LLL,KK
      PMIN=N*NTOT*NTOT
      NTOTT=NTOT+1
      II=NFLOW(1)+1
      INT=NTOTT-NFLOW(3)
      IFI=INT-NFLOW(2)
      JI=NFLOW(2)+1
      DO 2 I=II,IFI
      IS=NTOTT*(I-1)-(I-1)*(I-2)/2
```

```
      JF=INT-I+1
      DO 2 J=JI,JF
      L=IS+J
      IF(PMIN-COMP(L)) 2,3,3
    3 PMIN=COMP(L)
      IJ=L
    2 CONTINUE
      VALITR(LLL,ITR)=VALITR(LLL,ITR)+PRMOVE(KK)*COMP(IJ)
      ATN(IJ)=ATN(IJ)+PRMOVE(KK)
      RETURN
      END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE AFFECT(KR,P)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/AD/NREC(8,3),PREC(8),NR,NTOT
      COMMON/AD5/ MR(3),NTRE
      DIMENSION MX(3)
      NI=NR
      II=NI+1
      IF=NI+3
      DO 1 J=1,3
      KY=KR*MR(J)
      M=KY/NTRE
      DO 2 I=II,IF
      NREC(I,J)=M
    2 CONTINUE
      MX(J)=MOD(KY,NTRE)
    1 CONTINUE
      MSUM=0
      DO 3 J=1,3
      MSUM=MSUM+MX(J)
    3 CONTINUE
      MSUM=MSUM/NTRE
      IF(MSUM-1) 4,5,6
    4 NR=1
      I=NI+1
      PREC(I)=1.0
      GO TO 10
    5 NR=3
      K=0
      DO 7 I=II,IF
      K=K+1
      NREC(I,K)=NREC(I,K)+1
      PREC(I)=DFLOAT(MX(K))/DFLOAT(NTRE)
    7 CONTINUE
      GO TO 10
    6 NR=3
      K=0
      DO 8 I=II,IF
      DO 9 J=1,3
```

```
      NREC(I,J)=NREC(I,J)+1
    9 CONTINUE
      K=K+1
      NREC(I,K)=NREC(I,K)-1
      PREC(I)=1.-DFLOAT(MX(K))/DFLOAT(NTRE)
    8 CONTINUE
   10 NR=NI+NR
      DO 11 I=II,NR
      PREC(I)=PREC(I)*P
   11 CONTINUE
      RETURN
      END

C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
```

```
      PROGRAM TABLES
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4),PD(3,4),PRECI,EPS
      DIMENSION TAB(351,10),PR(351),NSTART(3),MV(3)
      DIMENSION NPO(10),ITAPE(10)
      COMMON/PRT/AV(10,3),AAV(10,3),DDIS(10,3)
      DIMENSION MR(3),NGOAL(3),NS(3)
      COMMON/AD7/IFROM(351,3),ATN1(351)
      DIMENSION LET(12),PRATN(10),KSTEP(10)
      DIMENSION NVEC(10,3),NDUMY(10)
      CHARACTER NPOLI*4,LET*3,NCTX*4
      DATA LET/'F  ','A  ','G 1','G 2','G 3','G 4',
     +         'G 5','G 6','G 7','G 8','G 9','G10'/

      N=3
      NN=N+1
      DO 45 I=1,N
      DO 46 J=1,N
      P(I,J)=0.0
   46 CONTINUE
      READ(5,200) P(I,I),P(I,I+1)
   45 CONTINUE
      PRECI=0.00
      READ(5,100) ITER
      READ(5,100) (NSTART(J),J=1,N)
      READ(5,100) (NGOAL(J) ,J=1,N)
      READ(5,100) (ITAPE(I), I=1,ITER)
       IDX=0
      READ(5,500) NPOLI,IFX
      IF(IFX.EQ.0) GO TO 18
      IF(NPOLI.NE.'GOAL') GO TO 51
      READ(5,100) (KSTEP(I),I=1,ITER)
      GO TO 23
   51 CONTINUE
      DO 19 I=1,ITER
      READ(5,100) (NVEC(I,J),J=1,N)
      KSTEP(I)=1
   19 CONTINUE
      GO TO 20
   18 IF(NPOLI.EQ.'GOAL') IDX=1
      DO 31 I=1,ITER
      KSTEP(I)=(ITER-I)*IDX+1
   31 CONTINUE
      IF(NPOLI.NE.'FIXE') GO TO 23
      READ(5,100) (MR(I),I=1,N)
      DO 22 I=1,ITER
      DO 22 J=1,N
      NVEC(I,J)=MR(J)
   22 CONTINUE
      GO TO 20
   23 DO 24 I=1,ITER
      DO 24 J=1,N
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      NVEC(I,J)=NGOAL(J)
   24 CONTINUE
   20 IF(NPOLI.EQ.'FIXE') IDX=1
      IF(NPOLI.EQ.'ADAP') IDX=2
      IF(NPOLI.EQ.'GOAL') IDX=3
      DO 35 I=1,ITER
      PRATN(I)=0.0
   35 CONTINUE
      CALL ATNCR1(NGOAL,P,PRECI,N,NTOT,MT)
      ITR=1
      NT=NTOT+1
      IJ=NT*NSTART(1)+NSTART(2)-NSTART(1)*(NSTART(1)-1)/2
      IDG=IJ+1
      ITP=ITAPE(1)
      IF(IJ.EQ.0) GO TO 3
      DO 4 I=1,IJ
      READ(ITP) (NS(J),J=1,N),((PD(L,J),J=1,N),L=1,N),
     +           NSTEP,MT,EPS,NCTX
      READ(ITP) (NDUMY(J),J=1,N)
      READ(ITP) (PR(J),J=1,MT)
    4 CONTINUE
    3 CONTINUE
      NSTEP=KSTEP(ITR)
      DO 71 J=1,N
      NDUMY(J)=NVEC(1,J)
   71 CONTINUE
      CALL VERFT(NSTART,P,NSTEP,N,ITP,MT,NPOLI,NDUMY)
      READ(ITP) (PR(I),I=1,MT)
      DO 10 I=1,MT
      TAB(I,1)=PR(I)
      DO 10 ITR=2,ITER
      TAB(I,ITR)=0.0
   10 CONTINUE
      PRATN(1)=ATN1(IDG)
      DO 11 ITR=2,ITER
      ITP=ITAPE(ITR)
      NSTEP=KSTEP(ITR)
      DO 72 J=1,N
      NDUMY(J)=NVEC(ITR,J)
   72 CONTINUE
      REWIND ITP
      IT=ITR-1
      DO 8 I=1,MT
      DO 1 J=1,N
      NS(J)=IFROM(I,J)
    1 CONTINUE
      PRATN(ITR)=PRATN(ITR)+TAB(I,IT)*ATN1(I)
      CALL VERFT(NS,P,NSTEP,N,ITP,MT,NPOLI,NDUMY)
      READ(ITP) (PR(L),L=1,MT)
      DO 9 L=1,MT
      TAB(L,ITR)=TAB(L,ITR)+TAB(I,IT)*PR(L)
    9 CONTINUE
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    8 CONTINUE
   11 CONTINUE
      DO 14 I=1,ITER
      DO 14 J=1,N
      AV(I,J)=0.0
      AAV(I,J)=0.0
      DDIS(I,J)=0.0
   14 CONTINUE
      DO 15 IJ=1,MT
      DO 21 J=1,N
      MV(J)=IFROM(IJ,J)
   21 CONTINUE
      DO 15 I=1,ITER
      DO 15 J=1,N
      AV(I,J)=AV(I,J)+TAB(IJ,I)*MV(J)
      AAV(I,J)=AAV(I,J)+TAB(IJ,I)*MV(J)*MV(J)
      X=FLOAT(MV(J)-NGOAL(J))
      DDIS(I,J)=DDIS(I,J)+TAB(IJ,I)*X*X
   15 CONTINUE
      DO 41 I=1,ITER
      DO 41 J=1,N
      AAV(I,J)=AAV(I,J)-AV(I,J)*AV(I,J)
   41 CONTINUE
      CALL PRTTAB(N,P,NSTART,NGOAL,NPOLI,IDX,ITER)
      WRITE(6,1400)
      WRITE(6,400) (I,I=1,ITER)
      WRITE(6,150) (PRATN(J),J=1,ITER)
      IF(IFX.EQ.0) STOP
      WRITE(6,1500)
      DO 16 I=1,ITER
      ID=IDX+KSTEP(I)-1
      WRITE(6,1300) I,LET(ID),(NVEC(I,J),J=1,N)
   16 CONTINUE
  100 FORMAT(16I5)
  150 FORMAT(4X,10(F8.4,3X))
  200 FORMAT(8F10.5)
  400 FORMAT(/,4X,10(3X,1HP,I2,5X))
  500 FORMAT(A4,6X,I1)
 1300 FORMAT(6X,I2,5X,A3,3X,'(',3(I3,1X),')')
 1400 FORMAT(//,3X,'PROBABILITY PI OF ATTINING THE VECTOR',
     +      ' M FROM THE VECTOR N IN I STEPS')
 1500 FORMAT(//,3X,'DETAILED INFORMATION ABOUT THE RECRUI',
     +       'TMENT STRATEGY',//,5X,'STEP',2X,'POLICY',2X,
     +       'CORRESPONDING VECTOR')
      STOP
      END

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      SUBROUTINE ATNCR1(NGOAL,P,PRECI,NNN,NTOT,K)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4),PRECI
      COMMON/C1/NLOW(30),NHIGH(30),NSTART(30),PP(30,31),N
      COMMON/C2/SLMT
      DIMENSION NGOAL(3)
      COMMON/AD7/ IFROM(351,3),ATN(351)
      N=NNN
      SLMT=1.-PRECI/FLOAT(N)
      N1=N+1
      NTOT=0
      DO 1 I=1,N
      NTOT=NTOT+NGOAL(I)
      NHIGH(I)=NGOAL(I)
      NLOW(I)=NGOAL(I)
      NLOW(I)=0
      DO 1 J=1,N1
      PP(I,J)=P(I,J)
    1 CONTINUE
      K=0
      NI=NTOT+1
      DO 2 I=1,NI
      NSTART(1)=I-1
      NJ=NI-I+1
      DO 2 J=1,NJ
      NSTART(2)=J-1
      NSTART(3)=NI-I-J+1
      CALL MAINT(PR)
      K=K+1
      DO 3 L=1,N
    3 IFROM(K,L)=NSTART(L)
      ATN(K)=PR
    2 CONTINUE
      RETURN
      END

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      SUBROUTINE PRTTAB(K,P,N,M,NPOLI,L,ITER)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      REAL P(3,4)
      DIMENSION N(3),M(3),V(3),LET(3),STRAT(3),TIRET(3)
      CHARACTER STRAT*10,LET*1
      CHARACTER TIRET*10,NPOLI*4
      DATA TIRET/'  -----   ',' -------- ','   ----   '/
      DATA LET/'F','A','G'/,STRAT/'  FIXED   ',' ADAPTIVE ',
     +  '   GOAL   '/
      DO 2 J=1,K
      V(J)=M(J)
      DO 2 I=1,K
      V(J)=V(J)-M(I)*P(I,J)
    2 CONTINUE
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```
      WRITE(6,100) LET(L),ITER,STRAT(L)
      WRITE(6,600) TIRET(L)
      WRITE(6,300) (P(1,J),J=1,K)
      WRITE(6,200) (N(I),I=1,K),(M(I),I=1,K),(P(2,J),J=1,K)
      WRITE(6,500) (V(I),I=1,K),(P(3,J),J=1,K)
      CALL SUBPRT(ITER)
  100 FORMAT(1H1,///,2X,'TABLE ',A1,2X,': EVOLUTION OF A ',
     +       'GRADED SYSTEM OVER A PERIOD OF ',I2,' STEPS,',
     +       'UNDER THE',A10,'RECRUITMENT STRATEGY.')
  200 FORMAT(3X,'INITIAL VECTOR N=(',2(I2,' ,'),I2,') ;',
     +       'GOAL VECTOR M=(',2(I2,' ,'),I2,') ; ',
     +       'TRANSITION MATRIX P=',3(2X,F5.3))
  300 FORMAT(83X,3(2X,F5.3))
  500 FORMAT(3X,'VECTOR M-M*P    =(',2(F5.2,' ,'),
     + F5.2,' )',41X,3(2X,F5.3))
  600 FORMAT(78X,A10,/)
      RETURN
      END


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      SUBROUTINE SUBPRT(ITER)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      COMMON/PRT/XAV(10,3),XAAV(10,3),XDDIS(10,3)
      DIMENSION TAB(12),AV(12)
      WRITE(6,400)
      WRITE(6,1000) ((I,I=1,3),J=1,3)
      DO 2 J=1,12
      AV(J)=0.0
    2 CONTINUE
      DO 1 I=1,ITER
      DO 6 J=1,3
      JJ=J+4
      JJJ=J+8
      TAB(J)=XAV(I,J)
      TAB(JJ)=XAAV(I,J)
      TAB(JJJ)=XDDIS(I,J)
    6 CONTINUE
      TAB(4)=TAB(1)+TAB(2)+TAB(3)
      TAB(8)=TAB(5)+TAB(6)+TAB(7)
      TAB(12)=TAB(9)+TAB(10)+TAB(11)
      DO 3 J=1,12
      AV(J)=AV(J)+TAB(J)
    3 CONTINUE
      WRITE(6,1100) I,(TAB(J),J=1,12)
    1 CONTINUE
      WRITE(6,1200)
      DO 4 J=1,12
      AV(J)=AV(J)/FLOAT(ITER)
    4 CONTINUE
      WRITE(6,1300) (AV(J),J=1,12)
      WRITE(6,1200)
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 400 FORMAT(10X,'...   RESULTS OBTAINED BY EXACT METHODS',
    +      '   ...')
1100 FORMAT(2X,1HI,3X,I2,4X,1HI,3(2X,4(1HI,F7.2,1X),1HI))
1200 FORMAT(2X,11(1H-),3(2X,37(1H-)))
1300 FORMAT(2X,11HI AVERAGE I,3(2X,4(1HI,F7.2,1X),1HI))
1000 FORMAT(//,22X,'EXPECTED STOCK VECTOR',15X,
    +  'VARIANCES OF THE STOCK VECTORS',9X,
    +  'M.S.E. OF THE STOCK VECTORS',/,
    + 2X,11(1H-),3(2X,37(1H-)),/,
    + 2X,1HI,9X,1HI,3(2X,1HI,10X,'GRADE',11X,1HI,8X,1HI),/,
    + 2X,11HI   STEP  I,3(2X,1HI,26(1H-),10HI  TOTAL I),/,
    + 2X,1HI,9X,1HI,3(2X,3(1HI,3X,I1,4X),1HI,8X,1HI),/,
    + 2X,11(1H-),3(2X,37(1H-)))
     RETURN
     END


C+++++++++++C+++++++++++C+++++++++++C+++++++++++C+++++++++++C
      SUBROUTINE VERFT(NG,P,ITX,N,ITP,MT,NCT,MR)
      DIMENSION MRR(3),MR(3)
      CHARACTER *4 NCT,NCTT
      DIMENSION NG(3),ND(3)
      REAL P(3,4),PD(3,4)
      SMALL=1.E-6
      READ(ITP) (ND(J),J=1,N),((PD(I,J),J=1,N),I=1,N),
     +          NT,MT,NCTT
      IF(NCT.NE.'PFLW') READ(ITP) (MRR(I),I=1,N)
      IFLG=3
      IF(NT.NE.ITX) GO TO 3
      DO 1 I=1,N
      IFLG=1
      IF(ND(I).NE.NG(I)) GO TO 3
      IFLG=2
      DO 1 J=1,N
      IF(ABS(PD(I,J)-P(I,J)).GT.SMALL)  GO TO 3
    1 CONTINUE
      IFLG=7
      IF(NCT.EQ.'ADAP'.AND.NCTT.EQ.'GOAL') GO TO 5
      IF(NCTT.NE.NCT) GO TO 3
      IF(NCT.EQ.'PFLW') RETURN
    5 IFLG=8
      DO 4 I=1,N
      IF(MR(I).NE.MRR(I)) GO TO 3
    4 CONTINUE
      RETURN
    3 WRITE(6,100) ITP
      WRITE(6,200) IFLG
  200 FORMAT(1H0,'CONFLICT IN ARGUMENT  ',I2)
      WRITE(6,1000)
      IARG=1
      WRITE(6,1100) IARG,(NG(I),I=1,N),(ND(I),I=1,N)
      IARG=2
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      WRITE(6,1200) IARG,(P(1,J),J=1,N1),(PD(1,J),J=1,N1)
      DO 2 I=2,N
      WRITE(6,1300) (P(I,J),J=1,N1),(PD(I,J),J=1,N1)
    2 CONTINUE
      IARG=3
      WRITE(6,1500) IARG,ITX,NT
      IARG=7
      WRITE(6,1600) IARG,NCT,NCTT
      IARG=8
      IF(NCT.NE.'PFLW') WRITE(6,1100) IARG,(MR(I),I=1,N),
     +                                  (MRR(I),I=1,N)
 1000 FORMAT(3X,'ARGUMENT',22X,'SUPPLIED',42X,'READ IN',/)
 1100 FORMAT(6X,I2,4X,3(I3,2X),35X,3(I3,2X))
 1200 FORMAT(6X,I2,4X,4F10.3,10X,4F10.3)
 1300 FORMAT(12X,4F10.3,10X,4F10.3)
 1500 FORMAT(6X,I2,4X,I3,47X,I3)
 1600 FORMAT(6X,I2,4X,A4,46X,A4)
      STOP
  100 FORMAT(1H0,'THE DATA IN FILE ',I3,2X,'IS INAPROPRIATE')
      END

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