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SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY NEW GENERALISED DERIVATIVE OPERATOR

(Subkelas Fungsi Analisis yang Ditakrif oleh Pengoperasi Terbitan Teritlak)

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ABSTRACT

A new generalised derivative operator $D^{n,m}_{\lambda_1,\lambda_2,b}$ is introduced. This operator generalised some well-known operators studied earlier. New subclasses of analytic functions in the open unit disc which are defined using generalised derivative operator are introduced. Inclusion theorems are investigated. Furthermore, generalised Bernardi-Libera-Livington integral operator is shown to be preserved for these classes.

Keywords: analytic functions; univalent functions; starlike functions; convex functions; close-to-convex functions; subordination; Hadamard product; integral operator

ABSTRAK

Pengoperasi terbitan baharu teritlak $D^{n,m}_{\lambda_1,\lambda_2,b}$ diperkenalkan. Pengoperasi ini mengitlak beberapa pengoperasi terdahulu yang terkenal. Subkelas baharu fungsi analisis dalam cakera terbuka unit diperkenalkan yang ditakrif dengan menggunakan pengoperasi terbitan teritlak. Teorem rangkuman dikaji. Malah pengoperasi kamiran Bernardi-Libera-Livington ditunjukkan kekal untuk kelas tesebut.

Kata kunci: fungsi analisis; fungsi univalen; fungsi bak bintang; fungsi cembung; fungsi hampir cembung; subordinasi; hasil darab Hadamard; pengoperasi kamiran

1. Introduction

Let A denote the class of functions of the form

$$f(z) = \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

where a_k is a complex number, which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let S^*, C and K denote, respectively, the subclasses of A consisting of functions which are starlike, convex, and close to convex in U. An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, $(z \in U)$ if there exists an analytic function w in U, such that $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if g is univalent in U, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

The convolution of two analytic functions $\varphi(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $\psi(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is defined by

$$\varphi(z) * \psi(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = \psi(z) * \varphi(z). \tag{1.2}$$

In order to derive our new generalised derivative operator, we define the analytic function

$$F_{\lambda_1,\lambda_2,b}^{m}(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)+b}{1+\lambda_2(k-1)+b}\right]^m z^k, \tag{1.3}$$

where $m,b \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and $\lambda_2 \geq \lambda_1 \geq 0$. Now, we introduce the new generalised derivative operator $D_{\lambda_1,\lambda_2,b}^{n,m}$ as follows:

Definition 1.1. For $f \in \mathrm{A}$, the operator $D_{\lambda_1,\lambda_2,b}^{n,m}$ is defined by $D_{\lambda_1,\lambda_2,b}^{n,m} : \mathrm{A} \to \mathrm{A}$,

$$D_{\lambda_1,\lambda_2,b}^{n,m} f(z) = F_{\lambda_1,\lambda_2,b}^{m}(z) * R^n f(z), \qquad z \in U, \tag{1.4}$$

where $n,m,b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, and $R^n f(z)$ denotes the Ruscheweyh derivative operator (Ruscheweyh 1975), given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} C(n,k) a_k z^k, \qquad (n \in \mathbb{N}_0,\ z \in U), \tag{1.5}$$

where $C(n,k) = (n+1)_{k-1}/(1)_{k-1}$.

If f is given by (1.1), then we easily find from equality (1.4) that

$$D_{\lambda_1,\lambda_2,b}^{n,m} f(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+(\lambda_1+\lambda_2)(k-1)+b}{1+\lambda_2(k-1)+b}\right]^m C(n,k) a_k z^k, \ (z \in U), \tag{1.6}$$

where $n,m,b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, and $C(n,k) = \binom{n+k-1}{n} = (n+1)_{k-1}/(1)_{k-1}$.

Note that, $(n)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(n)_k = \begin{cases} 1 & \textit{for } k = 0, n \in \mathbb{C} \setminus \{0\} \\ n(n+1)(n+2)\cdots(n+k-1) & \textit{for } k \in \mathbb{N}, n \in \mathbb{C} \end{cases} \tag{1.7}$$

Remark 1.1. Special cases of the operator $D_{\lambda_1,\lambda_2,b}^{n,m}$ include the Ruscheweyh derivative operator in the case $D_{\lambda_1,\lambda_2,b}^{n,0}$ (Ruscheweyh 1975), the Salagean derivative operator in the case $D_{1,0,0}^{0,m} \equiv S^n$ (Salagean 1983), the generalised Salagean derivative operator introduced by Al-Oboudi in the case $D_{\lambda_1,0,0}^{0,m} \equiv D_{\lambda_1}^{m}$ (Al-Oboudi 2004), the generalised Ruscheweyh derivative operator in the case $D_{\lambda_1,0,0}^{n,1} \equiv D_{n}^{\lambda_1}$ (Al-Shaqsi & Darus 2009), the generalised Al-Shaqsi and Darus derivative operator in the case $D_{1,0,b}^{n,m} \equiv D_{n}^{m,b}$ (Darus & Al-Shaqsi 2008), the Uralegaddi and Somanatha derivative operator in the case $D_{1,0,1}^{0,m} \equiv D^m$ (Uralegaddi & Somanatha 1992), the Cho and

Srivastave derivative operator in the case $D_{1,0,b}^{0,m} \equiv D_b^m$ (Cho & Srivastava 2003), the Eljamal and Darus derivative operator in the case $D_{1,0,b}^{0,m} \equiv D_b^m$ (Eljamal & Darus 2011), and the Cătas derivative operator in the case $D_{\lambda_1,0,b}^{0,m} \equiv D_{\lambda_1,b}^m$ (Cătas 2008).

To prove our results, we need the following equations throughout the paper:

$$(1+b)D_{\lambda_1,\lambda_2,b}^{n,m+1} f(z) = (1-(\lambda_1+\lambda_2)+b)D_{\lambda_1,\lambda_2,b}^{n,m} f(z) + (\lambda_1+\lambda_2)z\left[D_{\lambda_1,\lambda_2,b}^{n,m} f(z)\right]', \quad (1.8)$$

$$nD_{\lambda_1,\lambda_2,b}^{n+1,m} f(z) = z\left[D_{\lambda_1,\lambda_2,b}^{n,m} f(z)\right]' + (n-1)D_{\lambda_1,\lambda_2,b}^{n,m} f(z). \quad (1.9)$$

Let N be the class of all analytic and univalent functions ϕ in U and for which $\phi(U)$ is convex with $\phi(0)=1$ and $\mathrm{Re}\{\phi(z)\}>0$, for $z \in U$. For $\phi, \psi \in$ N, Ma and Minda (1992) studied the subclasses $S^*(\phi), C(\phi)$, and $K(\phi,\psi)$ of the class A. These classes are defined using the principle of subordination as follows:

$$\begin{aligned} S^*(\phi) &:= \left\{ f : f \in A, \frac{zf'(z)}{f(z)} \prec \phi(z) \text{ in } U \right\}, \\ C(\phi) &:= \left\{ f : f \in A, 1+\frac{zf''(z)}{f'(z)} \prec \phi(z) \text{ in } U \right\}, \quad (1.10) \\ K(\phi,\psi) &:= \left\{ f : f \in A, \exists\, g \in S^*(\phi) \text{ such that } \frac{zf'(z)}{g(z)} \prec \psi(z) \text{ in } U \right\}. \end{aligned}$$

Obviously, we have the following relationships for special choices of ϕ and ψ

$$S^*\left(\frac{1+z}{1-z}\right) = S^*,\ C\left(\frac{1+z}{1-z}\right) = C, K\left(\frac{1+z}{1-z}, \frac{1+z}{1-z}\right) = K. \quad (1.11)$$

Using the generalised differential operator $D_{\lambda_1,\lambda_2,b}^{n,m} f$, new classes $S_{\lambda_1,\lambda_2,b}^{n,m}(\phi), C_{\lambda_1,\lambda_2,b}^{n,m}(\phi)$ and $K_{\lambda_1,\lambda_2,b}^{n,m}(\phi,\psi)$, are introduced and defined as follows:

$$\begin{aligned} S_{\lambda_1,\lambda_2,b}^{n,m}(\phi) &:= \left\{ f \in A : D_{\lambda_1,\lambda_2,b}^{n,m} f(z) \in S^*(\phi) \right\}, \\ C_{\lambda_1,\lambda_2,b}^{n,m}(\phi) &:= \left\{ f \in A : D_{\lambda_1,\lambda_2,b}^{n,m} f(z) \in C(\phi) \right\}, \quad (1.12) \\ K_{\lambda_1,\lambda_2,b}^{n,m}(\phi,\psi) &:= \left\{ f \in A : D_{\lambda_1,\lambda_2,b}^{n,m} f(z) \in K(\phi,\psi) \right\}. \end{aligned}$$

It can be shown easily that

$$f(z) \in C_{\lambda_1,\lambda_2,b}^{n,m}(\phi) \Leftrightarrow zf'(z) \in S_{\lambda_1,\lambda_2,b}^{n,m}(\phi). \quad (1.13)$$

Janowski (1973) introduced the class $S^*[A,B] = S^*((1+Az)/(1+Bz))$, and in particular for $\phi(z) = (1 + Az) / (1 + Bz)$, we set

$$S^{n,m}_{\lambda_1,\lambda_2,b}\left(\frac{1+Az}{1+Bz}\right)=S^{*}_{n,m,\lambda_1,\lambda_2,b}[A,B],\ (1\geq A>B\geq -1). \tag{1.14}$$

In (Omar and Halim 2012), the authors studied the inclusion properties for classes defined using Dziok-Srivastava operator. This paper investigates similar properties for analytic functions in the classes defined by the generalized differential operator $D^{n,m}_{\lambda_1,\lambda_2,b} f$. Furthermore, applications of other families of integral operators are considered involving these classes.

2. Inclusion Properties Involving $D^{n,m}_{\lambda_1,\lambda_2,b} f$

To prove our results, we need the following lemmas:

Lemma 2.1 (see Eenigenburg *et al.* 1983). *Let* ϕ *be convex univalent in U, with* $\phi(0)=1$ *and* $\operatorname{Re}\{k\phi(z)+\eta\}>0,\ (k,\eta\in\mathbb{C})$. *If p is analytic in U with p*(0)=1 *then*

$$p(z)+\frac{zp'(z)}{kp(z)+\eta}\prec\phi(z)\Rightarrow p(z)\prec\phi(z). \tag{2.1}$$

Lemma 2.2 (see Miller and Mocanu (1981)). *Let* ϕ *be convex univalent in U and w be analytic in U with* $\operatorname{Re}\{w(z)\}\geq 0$. *If p is analytic in U and p*(0)= $\phi(0)$ *then*

$$p(z)+w(z)zp'(z)\prec\phi(z)\Rightarrow p(z)\prec\phi(z). \tag{2.2}$$

Theorem 2.3. *For any real numbers* m, λ_1 *and* λ_2, *where* $m\geq 0,\ \lambda_2\geq\lambda_1\geq 0$ *and* $b\geq 0$. *Let* $\phi\in\mathrm{N}$ *and* $\operatorname{Re}\{\phi(z)+(1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)\}>0$, *then* $S^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi)\subset S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$, $(n\geq 0)$.

Proof. Let $f\in S^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi)$, and set $p(z)=(z[D^{n,m}_{\lambda_1,\lambda_2,b}f(z)]')/(D^{n,m}_{\lambda_1,\lambda_2,b}f(z))$, where p is analytic in U, with $p(0)=1$. Rearranging (1.8), we have

$$\frac{(1+b)D^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)}{D^{n,m}_{\lambda_1,\lambda_2,b}f(z)}=(1-(\lambda_1+\lambda_2)+b)+\frac{(\lambda_1+\lambda_2)z[D^{n,m}_{\lambda_1,\lambda_2,b}f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b}f(z)}. \tag{2.3}$$

Next, differentiating (2.3) logarithmically with respect to *z* and multiplying by *z,* we obtain

$$\frac{z[D^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)]'}{D^{n,m+1}_{\lambda_1,\lambda_2,b}f(z)}=\frac{z[D^{n,m}_{\lambda_1,\lambda_2,b}f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b}f(z)}+\frac{z\left((z[D^{n,m}_{\lambda_1,\lambda_2,b}f(z)]')/(D^{n,m}_{\lambda_1,\lambda_2,b}f(z))\right)'}{(z[D^{n,m}_{\lambda_1,\lambda_2,b}f(z)]')/(D^{n,m}_{\lambda_1,\lambda_2,b}f(z))+(1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)} \tag{2.4}$$

$$=p(z)+\frac{zp'(z)}{p(z)+(1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)}.$$

Since $(z[D^{n,m}_{\lambda_1,\lambda_2,b}\, f(z)]')\big/(D^{n,m}_{\lambda_1,\lambda_2,b}\, f(z)) \prec \phi(z)$ and applying Lemma 2.1, it follows that $p \prec \phi$. Thus $f \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$.

Theorem 2.4. *Let* $m, \lambda_1, \lambda_2 \in °$, *where* $m \geq 0$, $\lambda_2 \geq \lambda_1 \geq 0$ *and* $n \geq 0$. *Then* $S^{n+1,m}_{\lambda_1,\lambda_2,b}(\phi) \subset S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$, $(b \geq 0, \phi \in \mathrm{N})$.

Proof. Let $f \in S^{n+1,m}_{\lambda_1,\lambda_2,b}(\phi)$, and from (1.9), we obtain that

$$\frac{nD^{n+1,m}_{\lambda_1,\lambda_2,b} f(z)}{D^{n,m}_{\lambda_1,\lambda_2,b} f(z)} = \frac{z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b} f(z)} + (n-1). \tag{2.5}$$

Making use of the differentiating (2.5) logarithmically with multiplying by *z* and setting

$$p(z) = (z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]')\big/(D^{n,m}_{\lambda_1,\lambda_2,b} f(z)),$$

we get the following:

$$\frac{z[D^{n+1,m}_{\lambda_1,\lambda_2,b}\, f(z)]'}{D^{n+1,m}_{\lambda_1,\lambda_2,b}\, f(z)} = p(z) + \frac{zp'(z)}{p(z)+(n-1)} \prec \phi(z). \tag{2.6}$$

Since $n \geq 0.$ and $\mathrm{Re}\{\phi(z)+(n-1)\} > 0,$ using Lemma 2.1, we conclude that $f \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$,

Corollary 2.5. *Let* $\lambda_2 \geq \lambda_1 \geq 0, n \geq 0,$ *and* $1 \geq A > B \geq -1.$ Then $S^*_{m+1,\lambda_1,\lambda_2,b}[n;A,B] \subset S^*_{m,\lambda_1,\lambda_2,b}[n;A,B]$ and $S^*_{m,\lambda_1,\lambda_2,b}[n+1;A,B] \subset S^*_{m,\lambda_1,\lambda_2,b}[n;\ A,B]$.

Theorem 2.6. *Let* $\lambda_2 \geq \lambda_1 \geq 0,$ *and* $n \geq 0.$ *Then* $C^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi) \subset C^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$ *and* $C^{n+1,m}_{\lambda_1,\lambda_2,b}(\phi) \subset C^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$.

Proof. Using (1.12) and Theorem 2.3, we observe that

$$\begin{aligned} f(z) \in C^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi) &\Leftrightarrow zf'(z) \in S^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi) \\ &\Rightarrow zf'(z) \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi) \\ &\Leftrightarrow D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z) \in S^*(\phi) \\ &\Leftrightarrow z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]' \in S^*(\phi) \\ &\Leftrightarrow D^{n,m}_{\lambda_1,\lambda_2,b} f(z) \in C(\phi) \\ &\Leftrightarrow f(z) \in C^{n,m}_{\lambda_1,\lambda_2,b}(\phi). \end{aligned} \tag{2.7}$$

To prove the second part of the theorem, we use similar steps and apply Theorem 2.4, the result is obtained.

Theorem 2.7. *Let* $\lambda_2 \geq \lambda_1 \geq 0,\ b \geq 0$ *and* $\operatorname{Re}\{\phi(z)+(1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)\}>0.$ *Then* $K^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi,\psi) \subset K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi)$ *and* $K^{n+1,m}_{\lambda_1,\lambda_2,b}(\phi,\psi) \subset K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi),\ (\phi,\psi)\in N$.

Proof. Let $f \in K^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi,\psi).$ In view of the definition of the class $K^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi,\psi),$ there is a function

$g \in S^{n,m+1}_{\lambda_1,\lambda_2,b}(\phi),$ such that

$$\frac{z\,[D^{n,m+1}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m+1}_{\lambda_1,\lambda_2,b}\, g(z)} \prec \psi(z) \tag{2.8}$$

Applying Theorem 2.3, then $g \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi),$ and let $q(z)=(z[D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)]')\big/(D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)) \prec \phi(z).$

Let the analytic function p with $p(0)=1$, as follows:

$$p(z)=\frac{z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)}. \tag{2.9}$$

Thus, rearranging and differentiating (2.9), we have

$$\frac{[D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)}=\frac{p(z)[D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)}+p'(z). \tag{2.10}$$

Making use of (1.8), (2.9), (2.10), and $q(z)$, we obtain that

$$\frac{z\,[D^{n,m+1}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m+1}_{\lambda_1,\lambda_2,b}\, g(z)}=\frac{[D^{n,m+1}_{\lambda_1,\lambda_2,b}\, zf'(z)]}{D^{n,m+1}_{\lambda_1,\lambda_2,b}\, g(z)}$$

$$=\frac{(1-(\lambda_1+\lambda_2)+b)D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z)+(\lambda_1+\lambda_2)z\,[D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z)]'}{(1-(\lambda_1+\lambda_2)+b)D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)+(\lambda_1+\lambda_2)z\,[D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)]'} \tag{2.11}$$

$$=\frac{\big((1-(\lambda_1+\lambda_2)+b)D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z)\big)\big/\big(D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)\big)+\big((\lambda_1+\lambda_2)z\,[D^{n,m}_{\lambda_1,\lambda_2,b}\, zf'(z)]'\big)\big/\big(D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)\big)}{(1-(\lambda_1+\lambda_2)+b)+\big((\lambda_1+\lambda_2)z\,[D^{n,m}_{\lambda_1,\lambda_2,b}\, g(z)]'\big)\big/D^{n,m}_{\lambda_1,\lambda_2,b}\,\big(g(z)\big)}$$

$$=\frac{(1-(\lambda_1+\lambda_2)+b)p(z)+(\lambda_1+\lambda_2)[p(z)q(z)+p'(z)]}{(1-(\lambda_1+\lambda_2)+b)+(\lambda_1+\lambda_2)q(z)}$$

$$= p(z) + \frac{zp'(z)}{q(z) + (1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)} \prec \psi(z).$$

Since $q(z) \prec \phi(z)$ and $\mathrm{Re}\{(1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)\} > 0,$ then

$$\mathrm{Re}\{q(z) + (1-(\lambda_1+\lambda_2)+b)/(\lambda_1+\lambda_2)\} > 0.$$

Using Lemma 2.2, we conclude that $p(z) \prec \psi(z)$ and thus $f \in K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi).$

By using similar manner and (1.9), we obtain the second result.

In summary, by using subordination technique, inclusion properties have been established for certain analytic functions defined via the generalised differential operator.

3. Inclusion Properties Involving $F_c f$

In this section, we determine properties of generalised Bernardi-Libera-Livington integral operator defined by (Bernardi 1969; Jung *et al.* 1993; Libera 1965; Livington 1966).

$$F_c[f(z)] = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\,dt \qquad (f \in \mathrm{A},\ c > -1).$$

$$= z + \sum_{n=2}^{\infty} \frac{c+1}{n+c} a_n z^n, \tag{3.1}$$

and satisfies the following:

$$cD^{n,m}_{\lambda_1,\lambda_2,b} F_c[f(z)] + z\left[D^{n,m}_{\lambda_1,\lambda_2,b} F_c[f(z)]\right]' = (c+1)D^{n,m}_{\lambda_1,\lambda_2,b} f(z). \tag{3.2}$$

Theorem 3.1. *If* $f \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi),$ *then* $F_c f \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi).$

Proof. Let $f \in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi),$ then $\left(z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]'\right)/\left(D^{n,m}_{\lambda_1,\lambda_2,b} f(z)\right) \prec \phi(z).$ Taking the differentiation on both sides of (3.2) and multiplying by z, we obtain

$$\frac{z\left[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)\right]'}{D^{n,m}_{\lambda_1,\lambda_2,b} f(z)} = \frac{z\left[D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]\right]'}{D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]} + \frac{z\left(\left(z\left[D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]\right]'\right)\Big/\left(D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]\right)\right)'}{\left(z\left[D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]\right]'\right)\Big/\left(D^{n,m}_{\lambda_1,\lambda_2,b} F_C[f(z)]\right) + c} \tag{3.3}$$

Setting

$$p(z)=\left(z\left[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[f(z)]\right]'\right)\Big/\left(D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[f(z)]\right),$$

we have

$$\frac{z\,[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b} f(z)}=p(z)+\frac{zp'(z)}{p(z)+c}. \tag{3.4}$$

Lemma 2.1 implies $\left(z\left[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[f(z)]\right]'\right)\Big/\left(D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[f(z)]\right)\prec\phi(z)$. Hence $F_c f\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$.

Theorem 3.2. *Let* $f\in C^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$, *then* $F_c f\in C^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$.
Proof. By using (1.12) and Theorem 3.1, it follows that

$$\begin{aligned} f\in C^{n,m}_{\lambda_1,\lambda_2,b}(\phi) &\Leftrightarrow zf'(z)\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)\\ &\Rightarrow F_c[zf'(z)]\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)\\ &\Leftrightarrow z[F_c[f(z)]]'\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)\Leftrightarrow F_c[f(z)]\in C^{n,m}_{\lambda_1,\lambda_2,b}(\phi). \end{aligned} \tag{3.5}$$

Theorem 3.3. *Let* $\phi,\psi\in N$ *and* $f\in K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi)$, *then* $F_c f\in K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi)$.

Proof. Let $f\in K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi)$, then there exists a function $g\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$, such that $\left(z[D^{n,m}_{\lambda_1,\lambda_2,b}\ f(z)]'\right)\Big/\left(D^{n,m}_{\lambda_1,\lambda_2,b}\ g(z)\right)\prec\psi(z)$. Since $g\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$, therefore from Theorem 3.1, $F_c[g(z)]\in S^{n,m}_{\lambda_1,\lambda_2,b}(\phi)$. Then let

$$q(z)=\frac{z\left[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]\right]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]}\prec\phi(z). \tag{3.6}$$

Set

$$p(z)=\frac{z\left[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[f(z)]\right]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]}. \tag{3.7}$$

By rearranging and differentiating (3.7), we obtain that

$$\frac{[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[zf'(z)]]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]}=\frac{p(z)[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]]'}{D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]}+\frac{p'(z)[D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]]}{D^{n,m}_{\lambda_1,\lambda_2,b}\ F_c[g(z)]}. \tag{3.8}$$

Making use of (3.2), (3.7), and (3.6), it can be derived that

$$\frac{z[D^{n,m}_{\lambda_1,\lambda_2,b} f(z)]'}{D^{n,m}_{\lambda_1,\lambda_2,b} g(z)} = p(z) + \frac{zp'(z)}{q(z)+c}. \tag{3.9}$$

Hence, applying Lemma 2.2, we conclude that $p(z) \prec \psi(z)$, and it follows that $F_c[f(z)] \in K^{n,m}_{\lambda_1,\lambda_2,b}(\phi,\psi)$. For analytic functions in the classes defined by generalised differential operator, the generalised Bernardi-Libera-Livington integral operator has been shown to be preserved in these classes.

4. Conclusion

Results involving functions defined using the generalised differential operator, namely, inclusion properties and the Bernardi-Libera-Livington integral operator were obtained using subordination principles. In Omar and Halim (2012), similar results were discussed for functions defined using the Dziok-Srivastava operator.

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