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A UNIFIED COMPLETENESS THEOREM FOR QUANTIFIED MODAL LOGICS

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Abstract. A general strategy for proving completeness theorems for quantified modal logics is provided. Starting from free quantified modal logic K, with or without identity, extensions obtained either by adding the principle of universal instantiation or the converse of the Barcan formula or the Barcan formula are considered and proved complete in a uniform way. Completeness theorems are also shown for systems with the extended Barcan rule as well as for some quantified extensions of the modal logic B. The incompleteness of $Q^\circ.B{+}BF$ is also proved.

In this paper we consider all free and classical quantified extensions of the propositional modal logic K obtained by adding either the axioms of identity or the Converse of the Barcan Formula or the Barcan Formula or the Extended Barcan Rule. Quantified extensions of the propositional logic B are also examined.[1] The lack of "... a common completeness proof that can cover constant domains, varying domains, and models meeting other conditions..." has often been felt, see [3], p.132. In [4] and [5], p.273, we read "Ideally, we would like to find a completely general completeness proof." The production of such a proof is the aim of this paper. We proceed by presenting a completeness proof for the system $Q^\circ.K$, Kripke's original one[2] with the addition of individual constants, we then show that such a proof yields completeness results for extensions of $Q^\circ.K$ such as those characterized by models with increasing or constant domains, with or without non-existing objects, with or without identity. Our main goal is to offer a clear framework in which each completeness result considered, old or new, will find its natural place. Sometimes we will follow through the proof of a known result just to show how it fits into our framework. In the first part of the paper we will deal with the systems mentioned in the diagram below:

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[1]The first part of this paper was presented at the workshop "Methods for Modalities 2", Amsterdam, 29-30 November 2001.

[2]See [7].

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$$\begin{array}{ccc} & Q^\circ.K & \\ Q^\circ.K + CBF & & \\ & & Q^\circ.K + CBF + BF \\ Q.K & & \\ & Q.K + BF & \end{array}$$

In the second part we will consider systems containing the identity relation.
As we will see, $Q^\circ.K$ ($Q.K$) is obtained by adding to the normal propositional modal logic K the quantificational axioms and rules of free (classical) logic. The main feature of $Q^\circ.K$ is that the principle of Universal Instantiation, UI, is not a theorem, but only its universal closure, UI°, is. Semantically this fact has the important consequence that each world w of a model for $Q^\circ.K$ is endowed both with an *inner* domain, D_w, that represents the set of objects existing at w and coincides with the domain of variation of the quantifiers, and with an *outer* domain $U_w \supseteq D_w$ that also contains non-existing possible objects and coincides with the domain of interpretation at w of the variables, the predicates and the individual constants. Once the full axiom of Universal Instantiation is present, no distinction is made between existing and non-existing objects and only one domain is associated with each world. Here are the four formulas we shall be most concerned with:

UI°	$\forall y(\forall x A(x) \to A(y/x))$
UI	$\forall x A(x) \to A(t/x)$
CBF	$\Box\forall x A \to \forall x \Box A$
BF	$\forall x \Box A \to \Box\forall x A$

§1. Modal systems without identity.

First Order Modal Languages and Kripke Semantics. The *alphabet* of first-order modal language $\mathcal{L}$ (without identity) contains the unary connective $\Box$ (box) in addition to the Boolean connectives $\neg$ (not) and $\vee$ (or) and the quantifier $\exists$ (there is). Moreover $\mathcal{L}$ contains a countable set, *Var*, of variables, $x_1, x_2, x_3, \ldots$, the symbol of falsehood, $\perp$, and the following two sets, at most countable, of, respectively, individual constants $a, b, c, d, a_1, b_1, c_1, d_1, \ldots$, and predicate symbols, $P^n, Q^n, R^n, \ldots$ of arity n, $0 \leq n < \omega$.
A *term* is either a variable or an individual constant. s, s_1, s_2, ... t, t_1, t_2, ... are metavariables for terms.

Well formed formulas (wffs)
1. $\perp$ is a wff,
2. If P^n is an n-ary predicate symbol and $t_1, \ldots, t_n$ are n terms, then $P^n(t_1, \ldots, t_n)$ is a wff,
3. If A and B are wffs and x is a variable, then $\neg A$, $\Box A$, $A \vee B, \exists x A$ are wffs,
4. Nothing else is a well formed formula.

The formulas $A \wedge B, A \to B, \Diamond A, \forall x A$ are defined in the usual way. By $A(t/s)$ we denote the formula obtained from the wff $A(s)$ by replacing *all* free occurrences of s by t, changing the name of bound variables, if necessary, to avoid rendering the new occurrences of t bound in $A(t/s)$. $A(t//s)$ denotes that *some* (*all,*

none) free occurrences of s are replaced by t. $A(s_1, ..., t/s_i, ...s_n)$ stands for $A(s_1, ..., s_i, ...s_n)(t/s_i)$.

A *Kripke-frame*, *K-frame*, is a quadruple $\mathcal{F} = \langle W, R, D, U \rangle$ where
W is a non-empty set,
R is a binary relation on W, the *accessibility relation*,
D is a function which associates to each $w \in W$ a set D_w. D_w is the *inner* domain of w, and it can be empty,
U is a function which associates to each $w \in W$ a set U_w such that:
$U_w \neq \emptyset$ and if wRv than $U_w \subseteq U_v$. U_w is the *outer* domain of w.

The fact that $U_w \subseteq U_v$, if wRv, does not prevent D_w from being disjoint from D_v. In [7], Kripke stipulates that for all $v \in W$, $U_v = \bigcup_{w \in W} D_w$. We generalize Kripke's original semantics by allowing $U_w \subseteq U_v$ if wRv, and $\bigcup_{w \in W} U_w \supseteq \bigcup_{w \in W} D_w$. $\bigcup_{w \in W} U_w$ may contain individuals that never happen to come into existence.

When no condition is imposed on the domain function D, $\mathcal{F}$ is said to have *varying* domains, when wRv implies $D_w \subseteq D_v$ ($D_w \supseteq D_v$, $D_w = D_v$), $\mathcal{F}$ is said to have *increasing* (*decreasing*, *constant*) domains. The outer domains are always increasing.

A *K-model* $\mathcal{M}$ is given by a K-frame $\mathcal{F}$ plus a function I that together with every $w \in W$ determines an *interpretation* I_w of the descriptive symbols of the language. In particular,

$$I_w(P^n) \subseteq (U_w)^n \qquad \text{and} \qquad I_w(c) \in U_w.$$

Whenever $\mathcal{M} = \langle \mathcal{F}, I \rangle$, $\mathcal{M}$ is said to be *based on* $\mathcal{F}$. For each $w \in W$, a *w-assignment* is a function $\sigma : Var \to U_w$. Let σ and τ be two w-assignments. τ is said to be an *x-variant* of σ if σ and τ agree on all variables except possibly on the variable x. If σ is a w-assignment, it is also a v-assignment for any v such that wRv, because $U_w \subseteq U_v$. Given a w-assignment σ, we can interpret all terms of the language, by letting $I_w^\sigma(c) = I_w(c)$ and $I_w^\sigma(x) = \sigma(x)$.

The notion of a formula being *satisfied by a w-assignment* σ at w in a K-model $\mathcal{M}$ is defined so:

$\mathcal{M} \models_w^\sigma P^n(t_1, \ldots, t_n)$	iff	$\langle I_w^\sigma(t_1), \ldots, I_w^\sigma(t_n) \rangle \in I_w(P^n)$
$\mathcal{M} \not\models_w^\sigma \bot$		
$\mathcal{M} \models_w^\sigma \neg B$	iff	$\mathcal{M} \not\models_w^\sigma B$
$\mathcal{M} \models_w^\sigma B \vee C$	iff	$\mathcal{M} \models_w^\sigma B$ or $\mathcal{M} \models_w^\sigma C$
$\mathcal{M} \models_w^\sigma \exists x B$	iff	for some x-variant τ of σ, such that $\tau(x) \in D_w$, $\mathcal{M} \models_w^\tau B$
$\mathcal{M} \models_w^\sigma \Box B$	iff	for all v such that wRv, $\mathcal{M} \models_v^\sigma B$.

$\mathcal{M}$ *satisfies a set of formulas* Δ iff for some w and w-assignment σ, $\mathcal{M} \models_w^\sigma D$, for all $D \in \Delta$.
A formula B is *true in a K-model* $\mathcal{M}$ *at* w, $\mathcal{M} \models_w B$, iff for all w-assignments σ, $\mathcal{M} \models_w^\sigma B$.
A formula B is *valid on a K-model* $\mathcal{M}$, $\mathcal{M} \models B$, iff for all $w \in W$, $\mathcal{M} \models_w B$.

A formula B is *valid on a K-frame* $\mathcal{F}$, $\mathcal{F} \models B$, iff for all K-models $\mathcal{M}$ based on $\mathcal{F}$, $\mathcal{M} \models B$.
A formula B is *K-valid* iff for all K-frames $\mathcal{F}$, $\mathcal{F} \models B$.
$\mathcal{M}$ is a *model for a logic* L iff $\mathcal{M} \models A$, for all theorems A of L.

As is well-known, the following formulas are not K-valid:[3]

$\forall x \Box A \to \Box \forall x A$	(BF)	$\exists x \Box A \to \Box \exists x A$	(GF)	$\forall x A(x) \to A(t/x)$
$\Box \forall x A \to \forall x \Box A$	(CBF)	$\Box \exists x A \to \exists x \Box A$	(CGF)	$\forall x A(x) \to \exists x A(x)$

Denotation, existence and rigidity. In the K-semantics just introduced, every constant is *denoting*, in fact for all $w \in W$ and for all constants c, $I_w(c)$ is defined, but nothing is said about whether it denotes an *existing* or a *non-existing* individual, $I_w(c)$ can be in D_w as well as in (U_w - D_w). Moreover it is not assumed that constants are rigid designators, where an individual constant c is said to be a *rigid* designator iff

$$wRv \quad \text{implies} \quad I_w(c) = I_v(c).$$

In a language without identity no formula expresses that a constant is a rigid designator. At the semantical level, rigidity corresponds to the classical correlation between satisfaction and substitution as stated by the following lemma.

Lemma 1.1. *Let $\mathcal{M}$ be a K-model and σ a w-assignment. An individual constant c is a rigid designator iff ($\mathcal{M} \models^{\sigma}_{w} A(c/x)$ iff $\mathcal{M} \models^{\tau}_{w} A(x)$), for any w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c)$.*[4]

Proof. Suppose c is a rigid designator. The proof is by induction on A, we consider just one case. $\mathcal{M} \models^{\sigma}_{w} \Box B(c/x)$ iff for all $v.wRv$. $\mathcal{M} \models^{\sigma}_{v} B(c/x)$ iff, by induction hypothesis, $\mathcal{M} \models^{\tau}_{v} B(x)$, where τ is a v-assignment and an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator, $\tau(x) = I_w(c)$, whence τ is a w-assignment and so $\mathcal{M} \models^{\tau}_{w} \Box B(x)$.
Suppose c is not a rigid designator. Take a model $\mathcal{M}$ based on two worlds w and v such that wRv, moreover let $D_w = \{u_1\}$, $D_v = \{u_1, u_2\}$, $I_w(c) = u_1$, $I_v(c) = u_2$, $I_w(P) = I_v(P) = \{u_1\}$, where P is a unary predicate letter. Then $\mathcal{M} \not\models^{\sigma}_{w} \Box P(c/x)$ and $\mathcal{M} \models^{\tau}_{w} \Box P(x)$. $\dashv$

A particular case of Kripke semantics which has been widely studied in the literature is the one we will call Tarski-Kripke semantics, TK-semantics, in order to stress the fact that a Tarski-Kripke model is just a family of classical models interconnected by the accessibility relation. A TK-frame is a K-frame in which for all $w \in W$, $U_w = D_w$, so each world w is endowed with just *one* domain, D_w, which is both the domain of variation of the quantifiers, of the free variables and the domain of interpretation of the constant and predicate symbols. Of course $D_w \neq \emptyset$ and wRv implies that $D_w \subseteq D_v$. TK-models are defined exactly as K-models. Each world of a TK-model is a Tarskian model, and so classically valid formulas such as $\forall x A(x) \to A(x)$ or $\forall x A(x) \to A(c/x)$ or $\forall x A(x) \to \exists x A(x)$ turn

[3] As to the role of the Ghilardi formula (GF) in counterpart semantics, see [2].

[4] The equivalence stated in the lemma corresponds to the equivalence between *de dicto* and *de re* readings of substituted formulas, $\Box P(c)$ *versus* $\langle c \rangle \Box P(x)$, see [2].

out to be valid. Moreover CBF and GF are TK-valid too. On the contrary BF and CGF are not TK-valid.

A comparison with the semantics as presented in Kripke, 1963.

We will use the expression *original Kripke semantics*, OK-semantics, to refer to the semantics of Kripke, [7], 1963. An OK-model is a quadruple $\langle W, R, D, I\rangle$ where W, R and D are defined as in K-semantics. The interpretation function I, on the other hand, differs for now I is such that $I_w(P^n) \subseteq V$ and $I_w(c) \in V$, where $V = \bigcup_{w\in W} D_w$. Analogously, the codomain of any assignment function is V. At first sight, OK-models look more general than K-models because both the assignment and the interpretation functions are not *world-bound*, in the sense that the interpretation at w of, say, a unary predicate P need not be a subset of D_w, and the interpretation at w of a constant c need not be an element of D_w. A way of looking at this semantics is that each world has an 'inner' domain, D_w, the domain of variation of the quantifiers, that varies from world to world and can be empty, and an 'outer' domain, V, which remains fixed and is the domain of interpretation of the variables, the predicates and the individual constants. V is the global domain of discourse, the set of all things of which we are entitled to say at each world if a predicate is true or false of them at that world. Moreover each element of V is bound to exist in some world. Keeping the outer domain V fixed is a heavy limitation in building canonical models, for suppose that we want to define a model based on a frame with two worlds, w and v, and that we want to define first D_w and I_w and then, D_v and I_v. In defining the function I_w, we are bound to establish once and for all what the set V is like, so that there will be no way to add new individuals when we come to define either D_v or I_v. A further and most important advantage of K-semantics is that TK-models are particular cases of K-models, just let $U_w = D_w$. This is particularly relevant in the present context since we aim at a *unique* semantic framework that can accommodate both OK-models and TK-models.[5] This has induced us to generalize OK-semantics by allowing the outer domains to increase and at the same time to have *world-bound* interpretations and assignment functions. But this is no limitation because in K-semantics, as we have defined it, the codomain of I_w as well as of any w-assignment is U_w, the outer domain, and nothing prevents each U_w from including $\bigcup_{w\in W} D_w$.

The system $Q^\circ.K$ and some of its extensions.

The system $Q^\circ.K$ contains the following axioms and inference rules.[6]

Axiom schemata: truth-functional tautologies,

$\Box(A \to B) \to (\Box A \to \Box B)$ $\quad$ $\forall y(\forall x A(x) \to A(y/x))$ $\quad$ $\forall x \forall y A \leftrightarrow \forall y \forall x A$

$\forall x(A \to B) \to (\forall x A \to \forall x B)$ $\quad$ $A \to \forall x A$, x not free in A

Inference rules : Modus Ponens (from A and $A \to B$ infer B), Necessitation (from A infer $\Box A$), and Universal Generalization (from A infer $\forall x A$).

[5] Hughes and Cresswell's models with undefined formulas, [6] pp.277-280, are equivalent to TK-models.

[6] We follow Fitting and Mendelsohn, [3], for the choice of this axiom system.

The system $Q.K$ is just the system $Q^\circ.K$ with $\forall y(\forall x A(x) \to A(y/x))$ replaced by $\forall x A(x) \to A(t/x)$.[7]

DEFINITION 1.2. Let L be any quantified modal logic which extends $Q^\circ.K$, $L \supseteq Q^\circ.K$. A *proof* in L is a sequence of formulas such that each of them is either an axiom of L or it is obtained from preceding formulas in the sequence by application of an inference rule.
A wff A is a *theorem* of L, $\vdash_L A$,iff there is a proof in L whose last formula is A.
A wff A is *derivable* in L from a set Δ of formulas, $\Delta \vdash_L A$, iff for some finite number of formulas $A_1, ..., A_n$ in Δ, $\vdash_L A_1 \wedge ... \wedge A_n \to A$.

LEMMA 1.3. *Theorems of $Q^\circ.K$ that we will use in the sequel (often without mentioning them).*

(i)	$\forall y(A(y) \to \exists x A(x//y))$, *($EI^\circ$).*
(ii)	$\forall y_1 \ldots \forall y_n \forall y[\forall x A(y_1 \ldots y_n, x) \to A(y_1 \ldots y_n, y/x)]$, *($y$ may or may not occur in $A(y_1 \ldots y_n, x)$).*
(ii)*	$\forall w_1 \forall y_{i_1} \ldots \forall y \ldots \forall w_m \forall y_{i_n}[\forall x A(y_1 \ldots y_n, x) \to A(y_1 \ldots y_n, y/x)]$, *where $\{y_{i_1} \ldots y_{i_n}\} \subseteq \{y_1 \ldots y_n\}$ and $w_1 \ldots w_m$ do not occur in $A(y_1 \ldots y_n, x)$.*
(iii)	*If $\vdash_{Q^\circ.K} A_1 \wedge \cdots \wedge A_n \to B$, then $\vdash_{Q^\circ.K} \forall \vec{x} A_1 \wedge \cdots \wedge \forall \vec{x} A_n \to \forall \vec{x} B$, where $\forall \vec{x} = \forall x_1 \ldots \forall x_k$, for some $k \geq 0$.*
(iv)	$(A \vee \forall y B(y)) \leftrightarrow \forall y(A \vee B(y))$, *where y is not free in A.*
(v)	$(A \to \forall y B(y)) \leftrightarrow \forall y(A \to B(y))$, *where y is not free in A.*
(vi)	$\forall x(A(x) \to B) \leftrightarrow (\exists x A(x) \to B)$, *where y is not free in B.*
(vii)	$\forall x A \leftrightarrow \forall y A(y/x)$, *where y doesn't occur in $\forall x A$.*
(viii)	$\forall x A(x) \wedge \exists y B(y) \to \exists y(A(y/x) \wedge B(y))$, *where y doesn't occur in $\forall x A$.*

LEMMA 1.4. *Here is a list of well-known soundness results.*

q.m.l.	*is sound w.r.t. the class of K-frames with domains*	
	inner	outer
$Q^\circ.K$	*varying*	*increasing*
$Q^\circ.K+CBF$	*increasing*	*increasing*
$Q^\circ.K+BF$	*decreasing*	*increasing*
$Q^\circ.K+CBF+BF$	*constant*	*increasing*
$Q.K$	*increasing*	*= inner*
$Q.K+BF$	*constant*	*= inner*

COMPLETENESS RESULTS

The main idea behind the completeness proof we are going to present stems from a simple observation: the affinity of meaning between CBF and UI. Take an instance of UI, $\forall x P(x) \to P(x)$. The falsity at a world w of $\forall x P(x) \to P(x)$ under a w-assignment σ, implies that the individual $\sigma(x)$ does not belong to the domain of variation of the quantifiers, so $\sigma(x)$ does not exist at w. The falsity of an instance of CBF, $\Box\forall x P(x) \to \forall x \Box P(x)$, at a world w implies that $\forall x P(x)$ is true at some future moment v, whereas $P(x)$ is false at v under some w-assignment σ such that $\sigma(x) \in D_w$, and so at v under σ it is false that

[7]To rule out empty domains add to $Q^\circ.K$ the axiom $\forall x A \to A$, where x is not free in A.

$\forall x P(x) \to P(x)$. UI discriminates between existing and non-existing individuals in the current world, whereas CBF discriminates between existing and non-existing individuals in *future* worlds. Falsifying CBF has fatal consequences for some individual: any u which is a witness for $\exists x \Diamond \neg P(x)$ at a world w where $\Box \forall x P(x)$ is true, is bound to die in some subsequent world. This induces us to stipulate that

an individual constant c denotes an *existing* individual at w iff
for all sentences $\forall x A(x)$, $\forall x A(x) \to A(c/x)$ is true at w.

So, no wonder that validity of CBF yields that existing individuals never die. We would like to stress that in this way we are able to distinguish between existing and non-existing individuals without having recourse to the identity relation (or the existence predicate), as is usually the case,

c denotes an *existing* individual at w iff $\exists x(x = c)$ is true at w,

and so without becoming entangled in problems linked to identity, modalities and rigid designators.

Notational convention. By L we shall denote any q.m.l. which extends $Q^{\circ}.K$. If L is a q.m.l. with language $\mathcal{L}$ and C is a denumerable set of individual constants not occurring in $\mathcal{L}$, then $\mathcal{L}^C$ denotes the language obtained by adding all the constants of C to $\mathcal{L}$, and L^C denotes the logic L in the language $\mathcal{L}^C$. From now on we agree that $\mathcal{L}$ is the language of L and $\mathcal{L}^C$ is the language of L^C. Moreover, $Const(\mathcal{L})$ denotes the set of individual constants of $\mathcal{L}$.

Definition 1.5. A set Δ of formulas is *L-consistent* iff $\Delta \nvdash_L \bot$.

Note. It might well be that a set of sentences Δ is L-consistent and at the same time $\Delta \vdash_L \forall u_1 \ldots \forall u_k \bot$, for some $k \geq 1$. Take $\Delta = \{\forall x A \wedge \neg A\}$, where x does not occcur in A or $\Delta = \{\forall x \bot\}$.

Lemma 1.6. *on constants.*

(i) *If* $\vdash_{L^C} A(c_1, \ldots, c_n)$ *then* $\vdash_{L^C} A(w_1/c_1, \ldots, w_n/c_n)$, *where* $w_1, \ldots w_n$ *are variables not occurring in* $A(c_1, \ldots, c_n)$.

(ii) *If* $\vdash_{L^C} A$ *and no constant of* C *occurs in* A, *then* $\vdash_L A$.

(iii) *If* Δ *is an* L*-consistent set of sentences and no constant of* C *occurs in* Δ, *then* Δ *is* L^C*-consistent.*

Proof. (i) As for classical logic, by choosing variables $w_1, \ldots, w_n$ not occurring in the proof $\mathcal{D}$ of $A(c_1, \ldots, c_n)$ and by replacing uniformly in $\mathcal{D}$, c_i by w_i, $1 \leq i \leq n$. (ii) follows from (i), and (iii) from (ii). $\dashv$

Let C be a not-empty set of individual constants. Now we define a set of sentences, which if true, guarantee that C is a set of constants denoting 'existing' individuals.

Definition 1.7. $\mathcal{E}(C) =_{df} \{\forall x A(x) \to A(c/x) : c \in C$ and $\forall x A(x)$ is a *sentence* of $\mathcal{L}^C\}$.

Lemma 1.8. *Let* C *be a not-empty set of constants. If* Δ *is an* L*-consistent set of sentences and no constants of* C *occur in* Δ, *then either* $\Delta \vdash_{L^C} \forall z_1 ... \forall z_h \bot$, *for some* $h \geq 1$ *or* $\mathcal{E}(C) \cup \Delta$ *is* L^C*-consistent.*

Proof. Assume that $\Delta \nvdash_{L^C} \forall z_1 ... \forall z_h \bot$, for any $h \geq 1$ and suppose by *reductio* that $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \bot$. Then (*) $\vdash_{L^C} E_1 \wedge \cdots \wedge E_j \rightarrow [D_1 \wedge \cdots \wedge D_k \rightarrow \bot]$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(C)$ and $\{D_1, \ldots, D_k\} \subseteq \Delta$. Let $\vec{c} = c_1, \ldots, c_n$ be all the individual constants of C occurring in (*). Then $\vdash_{L^C} E_1(\vec{c}) \wedge \cdots \wedge E_j(\vec{c}) \rightarrow [D_1 \wedge \cdots \wedge D_k \rightarrow \bot]$. (By $E_i(\vec{c})$ we mean that the constants of C actually occuring in E_i are *among* $c_1, \ldots, c_n$.)
If $n = 0$ then $j = 0$ too, and so Δ would be L-inconsistent, contrary to the hypothesis.
If $n \geq 1$, let $\vec{z} = z_1 \ldots z_n$ be variables not occurring in (*), so by lemma 1.6(i), $\vdash_{L^C} E_1(\vec{z}/\vec{c}) \wedge \cdots \wedge E_j(\vec{z}/\vec{c}) \rightarrow [D_1 \wedge \cdots \wedge D_k \rightarrow \bot]$, where $(\vec{z}/\vec{c})$ stands for $(z_1/c_1 \ldots z_n/c_n, z/c)$. Then, by lemma 1.3($iii$),
$\vdash_{L^C} \forall \vec{z} E_1(\vec{z}) \wedge \cdots \wedge \forall \vec{z} E_j(\vec{z}) \rightarrow [\forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k \rightarrow \forall \vec{z} \bot]$.

Now, each $\forall \vec{z} E_i(\vec{z})$, $1 \leq i \leq j$, is of the form $\forall \vec{z}(\forall x A(\vec{z}, x) \rightarrow A(\vec{z}, z_l/x))$, for some $1 \leq l \leq n$ and wff $A(\vec{z}, x)$ (z_l may or may not occur in $\forall x A$), so by lemma 1.3(ii^*), $\forall \vec{z}(\forall x A(\vec{z}, x) \rightarrow A(\vec{z}, z_l/x))$ is a theorem of L^C. Consequently
$\vdash_{L^C} \forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k \rightarrow \forall \vec{z} \bot$. Since $D_1 \ldots D_k$ are sentences, $\Delta \vdash_{L^C} \forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k$, so $\Delta \vdash_{L^C} \forall \vec{z} \bot$, contrary to the assumption. ⊣

When the principle of Universal Instantiation is present, lemma 1.8 is nothing but lemma 1.6(iii).

Lemma 1.9. *Let Δ be a set of sentences of $\mathcal{L}^C$ not containing the individual constant $c \in C$. Then $\mathcal{E}(C) \cup \Delta \vdash_{L^C} A(c)$ only if $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z A(z/c)$, where z doesn't occur in $A(c)$.*

Proof. First observe that since $C \neq \emptyset$, all vacuous universal instantiations $\forall x B \rightarrow B$, where x doesn't occur free in B, are derivable from $\mathcal{E}(C)$. In fact let $\top(x)$ be any tautology containing the free variable x. Then $\forall x(B \wedge \top(x)) \rightarrow B \wedge \top(c/x) \in \mathcal{E}(C)$, hence $\mathcal{E}(C) \vdash \forall x(B \wedge \top(x)) \rightarrow B$, so $\mathcal{E}(C) \vdash \forall x B \rightarrow B$.

Now let $\mathcal{E}(C) \cup \Delta \vdash_{L^C} A(c)$. Then
(*) $\vdash_{L^C} E_1 \wedge \cdots \wedge E_j \rightarrow (D_1 \wedge \cdots \wedge D_k \rightarrow A(c))$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(C)$, $\{D_1, \ldots, D_k\} \subseteq \Delta$. Let $\vec{c} = c_1, \ldots, c_n, c$ be all the constants of C occurring in (*), then $\vdash_{L^C} E_1(\vec{c}) \wedge \cdots \wedge E_j(\vec{c}) \rightarrow (D_1(\vec{c}) \wedge \cdots \wedge D_k(\vec{c}) \rightarrow A(\vec{c}))$. Let $\vec{z} = z_1, \ldots, z_n, z$ be variables not occurring in (*), so by lemma 1.6(i),
$\vdash_{L^C} E_1(\vec{z}/\vec{c}) \wedge \cdots \wedge E_j(\vec{z}/\vec{c}) \rightarrow (D_1(\vec{z}/\vec{c}) \wedge \cdots \wedge D_k(\vec{z}/\vec{c}) \rightarrow A(\vec{z}/\vec{c}))$. Then
$\vdash_{L^C} \forall \vec{z} E_1(\vec{z}) \wedge \cdots \wedge \forall \vec{z} E_j(\vec{z}) \rightarrow \forall \vec{z}[D_1(\vec{z}) \wedge \cdots \wedge D_k(\vec{z}) \rightarrow A(\vec{z})]$.

Now, each $E_i(\vec{c})$, $1 \leq i \leq j$, is of the form $\forall x A(c_{i_1} \ldots c_{i_k}, x) \rightarrow A(c_{i_1} \ldots c_{i_k}, c_{i_{k+1}})$, where $\{c_{i_1} \ldots c_{i_k}, c_{i_{k+1}}\} \subseteq \{c_1, \ldots, c_n, c\}$, so $\forall \vec{z} E_i(\vec{z})$ is a theorem of L^C by lemma 1.3(ii^*), consequently $\vdash_{L^C} \forall \vec{z}[D_1(\vec{z}) \wedge \cdots \wedge D_k(\vec{z}) \rightarrow A(\vec{z})]$. Since z doesn't occur in $D_1(\vec{z}) \wedge \cdots \wedge D_k(\vec{z})$, $\vdash_{L^C} \forall z_1 \ldots \forall z_n[D_1(z_1, \ldots z_n) \wedge \cdots \wedge D_k(z_1, \ldots z_n) \rightarrow \forall z A(z_1, \ldots z_n, z)]$. Then, by Modus Ponens with sentences of $\mathcal{E}(C)$ or with vacuous universal instantiations (derivable from $\mathcal{E}(C)$), it obtains $\mathcal{E}(C) \vdash_{L^C} \forall z_2 \ldots \forall z_n[D_1(c_1/z_1, z_2 \ldots z_n) \wedge \cdots \wedge D_k(c_1/z_1, z_2 \ldots z_n) \rightarrow \forall z A(c_1/z_1, z_2 \ldots z_n, z)]$,
$\mathcal{E}(C) \vdash_{L^C} \forall z_3 \ldots \forall z_n[D_1(c_1/z_1, c_2/z_2, z_3 \ldots z_n) \wedge \cdots \wedge D_k(c_1/z_1, c_2/z_2, z_3 \ldots z_n) \rightarrow \forall z A(c_1/z_1, c_2/z_2, z_3 \ldots z_n, z)]$,

...

$\mathcal{E}(C) \vdash_{L^C} D_1(c_1 \ldots c_n) \wedge \cdots \wedge D_k(c_1 \ldots c_n) \rightarrow \forall z A(c_1 \ldots c_n, z)$, therefore
$\mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z A(z)$. ⊣

When the principle of Universal Instantiation is present, lemma 1.9 is an immediate corollary of lemma 1.6(i).

DEFINITION 1.10. Let Δ be a set of sentences of $\mathcal{L}$ and $Q \subseteq Const(\mathcal{L})$.

Δ is *L-deductively closed*	iff	for any sentence A of $\mathcal{L}$, $\Delta \vdash_L A$ iff $A \in \Delta$.
Δ is *L-complete*	iff	for any sentence A of $\mathcal{L}$, either $A \in \Delta$ or $\neg A \in \Delta$.
Δ is *L-maximal*	iff	Δ is L-consistent and L-complete.

Let $A(x)$ be any wff of $\mathcal{L}$ with one free variable, then

Δ is *Q-universal*	iff	if $\forall x A(x) \in \Delta$, then $A(c/x) \in \Delta$, for all individual constants $c \in Q$.
Δ is *Q-existential*	iff	if $A(c/x) \in \Delta$ for some constant $c \in Q$, $\exists x A(x) \in \Delta$.
Δ is *Q-inductive*	iff	if $A(c/x) \in \Delta$ for all constants $c \in Q$, $\forall x A(x) \in \Delta$.
Δ is *Q-rich*	iff	if $\exists x A(x) \in \Delta$, then $A(c/x) \in \Delta$, for some individual constant $c \in Q$.
Δ is *L-saturated*	iff	Δ is L-maximal and for some set $Q \subseteq Const(\mathcal{L})$, Δ is Q-universal and Q-rich.

LEMMA 1.11. *Let Δ be a set of sentences of $\mathcal{L}$ and $Q \subseteq Const(\mathcal{L})$.*

(i) If Δ is Q-universal and $Q^* \subseteq Q$, then Δ is Q^*-universal.

(ii) If Δ is Q-rich and $Q^* \supseteq Q$, then Δ is Q^*-rich.

If Δ is L-maximal, then

(iii) Δ is Q-universal iff Δ is Q-existential,

(iv) Δ is Q-inductive iff Δ is Q-rich.

DEFINITION 1.12. Let Δ be a set of sentences.

$Cl_L(\Delta) = \{\mathrm{A} : \Delta \vdash_L A\}$, $Cl_L(\Delta)$ is said to be *the L-deductive closure* of Δ. *When no confusion can possibly arise, we write $Cl(\Delta)$ instead of $Cl_L(\Delta)$.*

$\Box^-(\Delta) = \{A : \Box A \in \Delta\}$.

LEMMA 1.13. *Let Δ be a set of sentences.*

(i) Δ is L-consistent iff $Cl_L(\Delta)$ is L-consistent.

(ii) If Δ is L-consistent and $\Diamond B \in \Delta$, then $\Box^-(\Delta) \cup \{B\}$ is L-consistent.

(iii) If Δ is L-deductively closed, then $\Box^-(\Delta)$ is L-deductively closed.

(iv) If $\Box^-(\Delta) \vdash_L A$, then $\Delta \vdash_L \Box A$.

LEMMA 1.14. *For any L-consistent set of sentences Δ there is an L-maximal set Γ such that $\Gamma \supseteq \Delta$.*

LEMMA 1.15. *Let $\Delta \cup \{\exists y A\}$ be a set of sentences of $\mathcal{L}^C$ not containing the constant $c \in C$. If $\mathcal{E}(C) \cup \Delta \cup \{\exists y A\}$ is L^C-consistent, then $\mathcal{E}(C) \cup \Delta \cup \{A(c/y)\}$ is L^C-consistent.*

PROOF. Suppose by *reductio* that $\mathcal{E}(C) \cup \Delta \cup \{A(c/y)\} \vdash_{L^C} \bot$, then $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \neg A(c)$. Hence by lemma 1.9, $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \forall z \neg A(z/c)$, where z doesn't occur in $\neg A(c)$, so $\mathcal{E}(C) \cup \Delta \vdash_{L^C} \neg \exists z A$, contrary to the L^C-consistency of $\mathcal{E}(C) \cup \Delta \cup \{\exists y A\}$. $\dashv$

LEMMA 1.16. *Let Δ be an L-consistent set of sentences of $\mathcal{L}$. Then for some not-empty denumerable set C of new constants, there is a set Π of sentences of $\mathcal{L}^C$ such that $\Delta \subseteq \Pi$, Π is L^C-maximal, Π is Q-universal and Q-rich for some set $Q \subseteq Const(\mathcal{L}^C)$.*

PROOF. (a): $\Delta \vdash_L \forall z_1 ... \forall z_h \bot$, for some $h \geq 1$. Let Π be an L^C-maximal extension of Δ. By induction on h we see that if an existential sentence $\exists x A(x)$ is in Π, then $\bot \in \Pi$, therefore no existential sentence is in Π. Let $\exists x A(x) \in \Pi$, for some $A(x)$. Then by lemma 1.3$(viii)$, $\exists x(A(x) \wedge \forall z_2 ... \forall z_h \bot) \in \Pi$, $\exists x \forall z_2 ... \forall z_h \bot \in \Pi$, $\forall z_2 ... \forall z_h \bot \in \Pi$ (from axiom $A \to \forall x A$, where x does not occur in A), and so by induction hypothesis, $\bot \in w$ contrary to the L-consistency of Π. Let $Q = \emptyset$. Trivially Π is $\emptyset$-universal and $\emptyset$-rich.
(b): $\Delta \nvdash_L \forall z_1 ... \forall z_h \bot$, for any $h \geq 1$. Let $H_1, H_2, ...$ be an enumeration of all the existential sentences of $\mathcal{L}^C$. Define the following chain of sets of sentences of $\mathcal{L}^C$.

$\Gamma_0 = \Delta \cup \mathcal{E}(C)$.

Suppose the set Γ_n has already been defined and the constants of C occurring in Γ_n are $c_1, ..., c_k$. Choose the first sentence in the given enumeration (and cancel it) which from C contains at most the constants $c_1, ..., c_k$. Let it be $\exists x F(x)$.
$Case(1)$. $\Gamma_n \cup \{\exists x F(x)\}$ is L^C-consistent. Take a constant $c \in C$ not occurring in $\mathcal{L} \cup \{c_1, ..., c_k\}$ and define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$.
$Case(2)$. $\Gamma_n \cup \{\exists x F(x)\}$ is not L^C-consistent. Define $\Gamma_{n+1} = \Gamma_n$.

Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$.

Γ_0 is L^C-consistent in virtue of lemma 1.8 and so is $Cl(\Gamma_0)$. Each Γ_{n+1} is L^C-consistent in virtue of lemma 1.15, and so is $Cl(\Gamma_{n+1})$, consequently Γ is L^C-consistent. Γ is C-universal because it includes Γ_0, and C'-rich for some $C' \subseteq C$ by construction, therefore Γ is C-rich by lemma 1.11(ii). In virtue of lemma 1.14, Γ can be extended to a set Π which is L^C-maximal. Therefore Π is Q-universal and Q-rich for some $Q \subseteq Const(\mathcal{L}^C)$. $\dashv$

DEFINITION 1.17. Let a q.m.l. $L \supseteq Q^\circ.K$ be given with language $\mathcal{L}$. Let V be a set of constants of cardinality $\aleph_0$ such that $V \supset Const(\mathcal{L})$ and $|V - Const(\mathcal{L})| = \aleph_0$. A *canonical model* $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ *for* L is defined as follows:

- W is the class of all L_w-saturated sets of sentences w, where $\mathcal{L}_w = \mathcal{L}^C$, for some set C of constants such that $Const(\mathcal{L}^C) \neq \emptyset$, $C \subset V$ and $|V - Const(\mathcal{L}^C)| = \aleph_0$,
- wRv iff $\Box^-(w) \subseteq v$, for any $w, v \in W$,
- $D_w = \{c \in Const(\mathcal{L}_w) : \forall x A \to A(c/x) \in w$, for all sentences $\forall x A$ of $\mathcal{L}_w\}$,
- $U_w = Const(\mathcal{L}_w)$,
- $I_w(c) = c$,
- $I_w(P^n) = \{\langle c_1, ..., c_n \rangle : P^n(c_1, ..., c_n) \in w\}$.

Let us check that every canonical model is based on a K-frame. If a logic L is consistent, then the empty set of sentences is L-consistent, so by lemma 1.16 there is an L^C-saturated set of sentences for some set C of constants, therefore $W \neq \emptyset$. $Const(\mathcal{L}_w) \neq \emptyset$ by definition of W, so $U_w \neq \emptyset$. If wRv then $U_w \subseteq U_v$, for if $A(c_1 \dots c_n)$ is a tautology containing the constants $c_1 \dots c_n$, $\Box A(c_1 \dots c_n) \in w$ and so $A(c_1 \dots c_n) \in v$; therefore $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$. $D_w \subseteq U_w$ by definition.

FACT 1.18. (a) Every $w \in W$ is D_w-universal and D_w-rich. For, by the definition of D_w, w is D_w-universal and D_w is the greatest Q^* with respect to which w is Q^*-universal. Therefore w is Q-universal and Q-rich for some $Q \subseteq D_w$, whence by lemma 1.11(ii) w is D_w-rich.

(b) If $\forall z_1 ... z_h \bot \in w$, for some $h \geq 1$, then $D_w = \emptyset$. For, if $D_w \neq \emptyset$, then for some tautology $\top(x)$ and constant $c \in D_w$, $\forall x \neg \top(x) \to \neg \top(c/x) \in w$ and so $\exists x \top(x) \in w$, contrary to the L_w-consistency of w, as we saw in (a) of the proof of lemma 1.16.

LEMMA 1.19. *Let $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ be a canonical model for $L \supseteq Q^\circ.K$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$ and $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$.*

PROOF. By lemma 1.13(ii), $\Box^-(w) \cup \{A\}$ is L_w-consistent. Since, by definition of canonical model, $|V - Const(\mathcal{L}_w)| = \aleph_0$, there exists a countable set C of constants such that $(Const(\mathcal{L}_w) \cap C) = \emptyset$, $C \subset V$ and $|V - (Const(\mathcal{L}_w \cup C))| = \aleph_0$. Let $\mathcal{L}_v = \mathcal{L}_w^C$. By lemma 1.16, there is an L_v-saturated set of sentences v such that $v \supseteq (\Box^-(w) \cup \{A\})$. $\dashv$

LEMMA 1.20. *Let $\mathcal{M}^L$ be a canonical model for $L \supseteq Q^\circ.K$. For all formulas $A(x_1, \ldots, x_n)$ of $\mathcal{L}$ and for any w-assignment σ,*

$$\mathcal{M}^L \models_w^\sigma A(x_1, \ldots, x_n) \quad \textit{iff} \quad A(\sigma(x_1)/x_1, \ldots, \sigma(x_n)/x_n) \in w,$$

where $x_1, \ldots, x_n$ are all the variables occurring free in A.

PROOF. For simplicity's sake we will write in the following $A(\sigma(x_1), \ldots, \sigma(x_n))$ instead of $A(\sigma(x_1)/x_1, \ldots, \sigma(x_n)/x_n)$.
$\mathcal{M}^L \models_w^\sigma P^k(x_{i_1}, \ldots, x_{i_n}, c_{i_{n+1}}, \ldots, c_{i_k})$ iff $\langle \sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), I_w(c_{i_{n+1}}), \ldots, I_w(c_{i_k}) \rangle \in I_w(P^k)$ iff by the definition of I_w in $\mathcal{M}^L$, $\langle \sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), c_{i_{n+1}}, \ldots, c_{i_k} \rangle \in I_w(P^k)$ iff, again by definition of I_w in $\mathcal{M}^L$, $P^k(\sigma(x_{i_1}), \ldots, \sigma(x_{i_n}), c_{i_{n+1}}, \ldots, c_{i_k}) \in w$.
$\bot \notin w$, since w is L_w-consistent.
If $\mathcal{M}^L \not\models_w^\sigma \Box B(x_1, \ldots, x_n)$, then there is a v, such that $\Box^-(w) \subseteq v$ and $\mathcal{M}^L \not\models_v^\sigma B(x_1, \ldots, x_n)$. Hence by induction hypothesis, $B(\sigma(x_1), \ldots, \sigma(x_n)) \notin v$, and so $\Box B(\sigma(x_1), \ldots, \sigma(x_n)) \notin w$.
If $\Box B(\sigma(x_1), \ldots, \sigma(x_n)) \notin w$, then, by the L_w-maximality of w, $\Diamond \neg B(\sigma(x_1), \ldots, \sigma(x_n)) \in w$. By lemma 1.19, there is a $v \in W$ such that $\Box^-(w) \subseteq v$ and $\neg B(\sigma(x_1), \ldots, \sigma(x_n)) \in v$, so $B(\sigma(x_1), \ldots, \sigma(x_n)) \notin v$. By induction hypothesis, $\mathcal{M}^L \not\models_v^\sigma B(x_1, \ldots, x_n)$. Moreover, by definition of R, wRv holds, so $\mathcal{M}^L \not\models_w^\sigma \Box B(x_1, \ldots, x_n)$.
Before examining the case of the quantifiers, let us recall that in canonical models individual constants are rigid designators, for $I_w(c) = c$, for all $w \in W$.
If $\mathcal{M}^L \models_w^\sigma \exists x B(x, x_1, \ldots, x_n)$ then $\mathcal{M}^L \models_w^\tau B(x, x_1, \ldots, x_n)$, for some w-assignment τ which is an x-variant of σ such that $\tau(x) = d$ for some $d \in D_w$. By lemma 1.1, $\mathcal{M}^L \models_w^\sigma B(d, x_1, \ldots, x_n)$, therefore by induction hypothesis, $B(d, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, consequently $\exists x B(x, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, since w is D_w-existential.
If $\exists x B(x, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, then $B(d, \sigma(x_1), \ldots, \sigma(x_n)) \in w$, for some constant $d \in D_w$, since w is D_w-rich. By induction hypothesis, $\mathcal{M}^L \models_w^\sigma B(d, x_1, \ldots, x_n)$ and by lemma 1.1, $\mathcal{M}^L \models_w^\tau B(x, x_1, \ldots, x_n)$, where τ is an x-variant of σ such that $\tau(x) = d$, therefore $\mathcal{M}^L \models_w^\sigma \exists x B(x, x_1, \ldots, x_n)$. $\dashv$

LEMMA 1.21. *Let $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ be a canonical model for $L \supseteq Q^\circ.K$.*

(i) *If Δ is an L-consistent set of formulas, then for some $w \in W$ and some w-assignment σ, $\mathcal{M}^L \models^\sigma_w D$, for all $D \in \Delta$.*
(ii) *If Δ is an L-consistent set of sentences, then for some $w \in W$, $\mathcal{M}^L \models_w D$, for all $D \in \Delta$.*
(iii) *$\mathcal{M}^L$ is a model for L.*
(iv) *If $\nvdash_L A$, then $\mathcal{M}^L \not\models A$.*

Proof. (i) Let $C = \{c_1, c_2, c_3, ...\}$ be a set of constants not occurring in $\mathcal{L}$ and $z_1, z_2, z_3, ...$ be all the variables occurring free in formulas of Δ. Then $\Delta^C = \{D(c_{i_1}/z_{i_1}, \ldots, c_{i_n}/z_{i_n}) : D(z_{i_1}, \ldots, z_{i_n}) \in \Delta$ and $c_{i_1} \ldots c_{i_n} \in C\}$ is L^C-consistent by lemma 1.6(i). Then by lemma 1.16 there is a set $\Pi \supseteq \Delta^C$ which is $L^{C \cup C^*}$-saturated, for some set C^* of new constants. Consider a canonical model $\mathcal{M}^L$ for L such that $V \supset Const(L^{C \cup C^*})$ and $|V - Const(L^{C \cup C^*})| = \aleph_0$. Then Π is a world, say w, of $\mathcal{M}^L$ and so $\Delta^C \subseteq w$. Given a w-assignment σ such that $\sigma(z_{i_j}) = c_{i_j}$, $\mathcal{M}^L \models^\sigma_w D(z_{i_1}, \ldots, z_{i_n})$ for any $D(z_{i_1}, \ldots, z_{i_n}) \in \Delta$, in virtue of lemma 1.20. $\dashv$

The standard pattern to show that a logic $L \supseteq Q^\circ.K$ is complete with respect to a class $\mathcal{H}$ of frames goes as follows. Take any wff A which is not a theorem of L, so $\{\neg A\}$ is L-consistent. By lemma 1.21(i), there is a world w of a canonical model $\mathcal{M}^L$ for L and a w-assignment σ, such that $\mathcal{M}^L \not\models^\sigma_w A$, therefore $\mathcal{M}^L \not\models A$. If $\mathcal{M}^L$ is based on a frame of $\mathcal{H}$, then L is complete with respect to $\mathcal{H}$.

Now, lemma 1.19 allows us to build canonical models of the most general kind: nothing is said about the inner domains and the outer domains are increasing. In order to prove that $\mathcal{M}^L$ is based on a frame of a given class $\mathcal{H}$, we need to prove *variations* of lemma 1.19 to the effect that the inner and outer domains fulfill the specific conditions of the frames of $\mathcal{H}$.

Actually, all the completeness proofs we shall present yield that the logics L under consideration are *strongly* complete, in fact we shall prove that every L-consistent set of wffs is satisfied on a model based on a frame for L.

$\boxed{Q^\circ.K}$

Since no condition is required on frames for $Q^\circ.K$, lemma 1.19 yields

Theorem 1.22. *$Q^\circ.K$ is strongly complete with respect to the class of all K-frames.*[8]

$\boxed{Q^\circ.K{+}CBF}$

The core fact to notice is that for any world w of a canonical model for $Q^\circ.K+CBF$, $\Box^-(w)$ is D_w-universal. So individuals 'existing' at w, are bound to exist in all accessible worlds. The following lemma elaborates this fact.

Lemma 1.23. *Let w be a world of a canonical model for $L \supseteq Q^\circ.K + CBF$.*

[8]Hughes and Cresswell in [6], pp.306-309, present, to our knowledge, the first completeness proof for a Kripke's style system, $LPCK$, without individual constants, characterized by the class of K-frames with varying not-empty domains (and outer increasing domains!). The present approach is more general and leads, as far as we can tell, to new completeness results such as those for $Q^\circ.K{+}CBF$, $Q^\circ.K{+}CBF{+}BF$, $Q^\circ_=.K$, $Q^\circ_=.K{+}CBF$, $Q^\circ_=.K{+}BF$, $Q^\circ_=.B$

(i) $\forall x A(x) \to A(d/x) \in \Box^-(w)$, *for all sentences* $\forall x A(x) \in \mathcal{L}_w$ *and* $d \in D_w$.
(ii) $\forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, d/x)) \in \Box^-(w)$, *for all for all wffs* $A(\vec{z}, x) \in \mathcal{L}_w$ *and* $d \in D_w$.

Proof. .
(ii) $\vdash_{Q^\circ.K+CBF}$ $\forall y \forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, y/x))$, by lemma 1.3($ii^*$),
$\vdash_{Q^\circ.K+CBF}$ $\Box \forall y \forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, y/x))$ by Necessitation,
$\vdash_{Q^\circ.K+CBF}$ $\forall y \Box \forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, y/x))$ by CBF, consequently $\forall y \Box \forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, y/x)) \in w$. Since w is D_w-universal, for all $d \in D_w$, $\Box \forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, d/x)) \in w$ and so $\forall \vec{z}(\forall x A(\vec{z}, x) \to A(\vec{z}, d/x)) \in \Box^-(w)$, for all $d \in D_w$. ⊣

Lemma 1.24. *Let* w *be a world of a canonical model for* $L \supseteq Q^\circ.K + CBF$ *and* C *be a set of constants disjoint from* $Const(\mathcal{L}_w)$. *If* $\forall z_1 ... z_h \bot \notin w$, *for any* $h \geq 1$, *and* $\Diamond B \in w$, *then* $\mathcal{E}(D_w \cup C) \cup \Box^-(w) \cup \{B\}$ *is* L_w^C*-consistent.*

Proof. We recall that $\mathcal{E}(D_w \cup C) = \{\forall x A(x) \to A(b/x) : \forall x A(x) \in \mathcal{L}_w^C$ and $b \in (D_w \cup C)\}$.
Suppose by *reductio* that $\mathcal{E}(D_w \cup C) \cup \Box^-(w) \cup \{B\}$ is not L_w^C-consistent, then
(*) $\vdash_{L_w^C} E_1 \wedge \cdots \wedge E_j \to [D_1 \wedge \cdots \wedge D_k \to \neg B]$, where $\{E_1, \ldots, E_j\} \subseteq \mathcal{E}(D_w \cup C)$ and $\{D_1, \ldots, D_k\} \subseteq \Box^-(w)$. Let $\vec{d} = d_1 \ldots d_m$ ($\vec{c} = c_1 \ldots c_n$) be all the individual constants of $\mathcal{L}_w$ (C) occurring in $E_1 \wedge \cdots \wedge E_j$. Then
$\vdash_{L_w^C} E_1(\vec{d}, \vec{c}) \wedge \cdots \wedge E_j(\vec{d}, \vec{c}) \to [D_1 \wedge \cdots \wedge D_k \to \neg B]$. Each $E_i(\vec{d}, \vec{c})$, $1 \leq i \leq j$, is of the form $\forall x A(\vec{d}, \vec{c}, x) \to A(\vec{d}, \vec{c}, b)$ with either $b \in \vec{d}$ or $b \in \vec{c}$.
If $n = 0$, then each $E_i(\vec{d}, \vec{c})$, $1 \leq i \leq j$ is of the form $\forall x A(\vec{d}, x) \to A(\vec{d}, b)$ with $b \in \vec{d}$ and so $b \in D_w$, therefore, as we saw in lemma 1.23(i), it is in $\Box^-(w)$. Consequently $\Box^-(w) \vdash_{L_w^C} \neg B$ contrary to the fact that $\Box^-(w) \cup \{B\}$ is L_w^C-consistent.
If $n \geq 1$, let $z_1 \ldots z_n$ be variables not occurring in (*), so by the lemma 1.6(i),
$\vdash_{L_w^C} E_1(\vec{d}, \vec{z}/\vec{c}) \wedge \cdots \wedge E_j(\vec{d}, \vec{z}/\vec{c}) \to [D_1 \wedge \cdots \wedge D_k \to \neg B]$,
$\vdash_{L_w^C} \forall \vec{z} E_1(\vec{d}, \vec{z}) \wedge \cdots \wedge \forall \vec{z} E_j(\vec{d}, \vec{z}) \to [\forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k \to \forall \vec{z} \neg B]$.
Now, each $\forall \vec{z} E_i(\vec{d}, \vec{z})$, $1 \leq i \leq j$, either is of the form $\forall \vec{z}[\forall x A(\vec{d}, \vec{z}, x) \to A(\vec{d}, \vec{z}, z_k/x)]$ for some $k, 1 \leq k \leq n$, (this is the case when $b \in \vec{c}$) and so it is a theorem of $Q^\circ.K$ by lemma 1.3(ii), or is of the form $\forall \vec{z}(\forall x A(\vec{d}, \vec{z}, x) \to A(\vec{d}, \vec{z}, d_h/x))$ for some $h, 1 \leq h \leq m$, (this is the case when $b \in \vec{d}$) and so it is in $\Box^-(w)$, by lemma 1.23(ii). Hence
$\Box^-(w) \vdash_{L_w^C} \forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k \to \forall \vec{z} \neg B$. Since $D_1, \ldots, D_k$ are sentences,
$\Box^-(w) \vdash_{L_w^C} \forall \vec{z} D_1 \wedge \cdots \wedge \forall \vec{z} D_k$, therefore
$\Box^-(w) \vdash_{L_w^C} \forall \vec{z} \neg B$. Then by lemma 1.13($iv$), $w \vdash_{L_w^C} \Box \forall \vec{z} \neg B$, $w \vdash_{L_w^C} \forall \vec{z} \Box \neg B$ by CBF. But $\Diamond B \in w$, hence $w \vdash_{L_w^C} \forall \vec{z} \Diamond B$, so $w \vdash_{L_w^C} \forall \vec{z} \bot$, contrary to the hypothesis of the lemma. ⊣

Lemma 1.25. (CBF-variation of lemma 1.19) *Let* $\mathcal{M}^L = \langle W, R, D, U, I \rangle$ *be a canonical model for* $L \supseteq Q^\circ.K + CBF$. *If* $w \in W$ *and* $\Diamond A \in w$, *then there is a* $v \in W$ *such that* $\Box^-(w) \subseteq v$, $A \in v$, $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$ *and* $D_w \subseteq D_v$.

Proof. As for lemma 1.19 provided that in lemma 1.16 at point (b), $\Gamma_0 = \mathcal{E}(D_w \cup C) \cup \Box^-(w) \cup \{A\}$. Γ_0 is L_w^C-consistent by lemma 1.24 and, trivially, $D_w \subseteq D_v$. ⊣

Theorem 1.26. *$Q^\circ.K + CBF$ is strongly complete with respect to the class of K-frames with increasing inner and outer domains.*

$\boxed{Q.K}$

Consider the system $Q.K$ obtained from $Q^\circ.K$ by adding the axiom of Universal Instantiation. As is well known, CBF is a theorem of $Q.K$,[9] so by lemma 1.25, if wRv, $D_w \subseteq D_v$. Moreover, because of axiom UI, each w is U_w-universal, consequently $U_w = D_w$, therefore

Theorem 1.27. *$Q.K$ is strongly complete with respect to the class of TK-frames with increasing domains.*

$\boxed{Q^\circ.K + CBF + BF}$

Let us now turn our attention to the Barcan Formula and consider canonical models for systems $L \supseteq Q^\circ.K{+}CBF{+}BF$. The core fact to notice is that for any world w of a canonical model for $Q^\circ.K + BF$ (CBF is not needed), $\Box^-(w)$ is D_w-inductive.

Lemma 1.28. *Let w be a world of a canonical model for $L \supseteq Q^\circ.K{+}BF$.*
(i) $\Box^-(w)$ is D_w-inductive.
(ii) If $\{B_1, \ldots, B_n\}$ is a finite set of sentences of $\mathcal{L}$ and $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_{L_w} A(c)$, for all $c \in D_w$, then $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_{L_w} \forall x A(x)$. Consequently, $Cl(\Box^-(w) \cup \{B_1, \ldots, B_n\})$ is D_w-inductive.

Proof. [10] (i) If $A(c) \in \Box^-(w)$ for all $c \in D_w$, then $\Box A(c) \in w$ for all $c \in D_w$, so, since w is D_w-inductive, $\forall x \Box A(x/c) \in w$, and by BF, $\Box \forall x A(x) \in w$, whence $\forall x A(x) \in \Box^-(w)$.
(ii) Suppose that $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_{L_w} A(c)$, for all $c \in D_w$, then where $B = B_1 \wedge \cdots \wedge B_n$, $\Box^-(w) \vdash_{L_w} B \to A(c)$, for all $c \in D_w$, hence $w \vdash_{L_w} \Box(B \to A(c))$, for all $c \in D_w$. So $\Box(B \to A(c)) \in w$, for all $c \in D_w$. (The constant c could occur also in B and D_w could be a finite set.) Take a variable y not occurring either in B or in $A(c)$ and consider the wff $\forall y \Box(B \to A(y/c))$. Since, $\Box(B \to A(c)) \in w$, for all $c \in D_w$ and w is D_w-inductive, then $\forall y \Box(B \to A(y/c) \in w$, and by BF, $\Box \forall y(B \to A(y/c)) \in w$. Therefore, $\Box(B \to \forall y A(y/c)) \in w$, $(B \to \forall y A(y/c)) \in \Box^-(w)$, $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_{L_w} \forall y A(y)$, and so $\Box^-(w) \cup \{B_1, \ldots, B_n\} \vdash_{L_w} \forall x A(x)$. ⊣

Now, if w is a world of a canonical model for $L \supseteq Q^\circ.K{+}CBF{+}BF$ and $\Diamond A \in w$, then $\mathrm{Cl}(\Box^-(w) \cup \{A\})$ is D_w-universal because of CBF (lemma 1.23), and D_w-inductive because of BF (lemma 1.28). This leads to the following lemma.

[9] $Q.K \vdash \forall x A(x) \to A(x)$, $Q.K \vdash \Box \forall x A(x) \to \Box A(x)$, $Q.K \vdash \Box \forall x A(x) \to \forall x \Box A(x)$.
[10] This proof is standard and it is due to Thomason [8].

LEMMA 1.29. (CBF+BF-variation of lemma 1.19) *Let* $\mathcal{M}^L = \langle W, R, D, U, I\rangle$ *be a canonical model for* $L \supseteq Q^\circ.K + CBF + BF$. *If* $w \in W$ *and* $\Diamond A \in w$, *then there is a* $v \in W$ *such that* $\Box^-(w) \subseteq v$, $A \in v$, $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$ *and moreover* v *is* D_w*-universal and* D_w*-rich, therefore* $D_w = D_v$.

PROOF. As for lemma 1.19 with $C = \emptyset$ and the set v constructed as follows. Let $H_1, H_2, H_3...$ be an enumeration of all the existential sentences of $\mathcal{L}_w$. Define the following chain of sets of sentences of $\mathcal{L}_w$.

$\Gamma_0 = \Box^-(w) \cup \{A\}$.

Suppose the set Γ_n has already been defined. Consider the sentence H_{n+1}. Let it be $\exists x F(x)$.

Case(1). $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. Define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$, where c is a constant of D_w such that $\Gamma_n \cup \{F(c/x)\}$ is L_w-consistent.

Case(2). $\Gamma_n \cup \{\exists x F(x)\}$ is not L_w-consistent. Define $\Gamma_{n+1} = \Gamma_n$.

Then let $\Gamma = \bigcup_{n\in N} Cl(\Gamma_n)$. Extend Γ to a set v which is L_w-maximal. The existence of a $c \in D_w$ such that $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\}$ is L_w-consistent is guaranteed by the fact that otherwise $\Gamma_n \vdash \neg F(c/x)$ for all $c \in D_w$. But Γ_n is $\Box^-(w)$ united with a finite set of sentences, say, $\{A, B_1, \dots B_k\}$, so by lemma 1.28, $\Gamma_n \vdash \forall x \neg F(x/c)$, contrary to the fact that $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. Therefore $F(c/x) \in Cl_L(\Gamma_n)$, for some $c \in D_w$ and so Γ is D_w-rich. Trivially $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$. Because of CBF, $\Box^-(w)$ is D_w-universal (lemma 1.23), therefore v is D_w-universal. ⊣

THEOREM 1.30. $Q^\circ.K + CBF + BF$ *is strongly complete with respect to the class of* K*-frames with constant inner and outer domains.*

$\boxed{Q.K + BF}$

Let $L \supseteq Q.K + BF$. Since $Q.K \vdash CBF$ and each w is U_w-universal thanks to UI, lemma 1.29 yields

THEOREM 1.31. $Q.K + BF$ *is strongly complete with respect to the class of* TK*-frames with constant domains.*

Theorems 1.22 and 1.26 can be improved to the effect that any model for $Q^\circ.K$ ($Q^\circ.K + CBF$) can be transformed into one with constant outer domains.

THEOREM 1.32. $Q^\circ.K$ *(*$Q^\circ.K + CBF$*) is strongly complete with respect to the class of K-frames with varying (increasing) inner domains and constant outer domains.*

PROOF. Take any K-model $\mathcal{M} = \langle W, R, D, U, I\rangle$ and build the model $\mathcal{M}^* = \langle W, R, D, U^*, I\rangle$, where for all $w \in W$, $U^*_w = \bigcup_{v\in W} U_v$. Then for any $w \in W$ and w-assignment σ of $\mathcal{M}$, $\mathcal{M} \models^\sigma_w A$ iff $\mathcal{M}^* \models^\sigma_w A$. In fact, σ is a w-assignment in $\mathcal{M}^*$ too, and moreover any x-variant of σ in $\mathcal{M}^*$ such that $\sigma(x) \in D_w$ is also an x-variant of σ in $\mathcal{M}$ since the inner domains of the two models are identical. Now, if $Q^\circ.K \nvdash A$, then, by lemma 1.22, for some $\mathcal{M}$, and w-assignment σ in $\mathcal{M}$, $\mathcal{M} \not\models^\sigma_w A$, and so by the construction above, for some $\mathcal{M}^*$ with constant outer domains, $\mathcal{M}^* \not\models^\sigma_w A$. ⊣

Note The fact that the outer domains are constant, say they are equal to V, doesn't imply that $V = \bigcup_w D_w$. Just consider a model for a set of sentences like $\{\Box\bot, \forall x P(x), \neg P(a)\}$. Therefore K-frames with constant outer domains differ, in general, from original Kripke frames.

The following table summarizes the completeness results obtained so far.

q.m.l.	is strongly complete w.r.t. the class of K-frames with domains	
	inner	*outer*
$Q^\circ.K$	varying	constant
$Q^\circ.K + CBF$	increasing	constant
$Q^\circ.K + CBF + BF$	constant	constant
$Q.K$	increasing	= inner
$Q.K{+}BF$	constant	= inner

§2. Modal systems with identity. We will start by examining the systems of the diagram below.

$$\begin{array}{ccc} & Q^\circ_=.K & \\ Q^\circ_=.K + CBF & & Q^\circ_=.K + BF \\ Q_=.K & & Q^\circ_=.K + CBF + BF \\ & Q.K_= + BF & \end{array}$$

Let us add to $Q^\circ.K$ the identity predicate ' = ' together with the following three axioms and let $Q^\circ_=.K$ be the resulting system.[11]

REF	$t = t$
$SUBS$	$s = t \to (A(s//x) \to A(t//x))$.
ND	$s \neq t \to \Box(s \neq t)$.

LEMMA 2.1. *Some theorems about identity.*

i	$\vdash_{Q^\circ_=.K}$	$s = t \to \Box(s = t)$, *Necessity of Identity, NI*
ii	$\vdash_{Q^\circ_=.K}$	$\forall x \exists y(x = y)$
iii	$\vdash_{Q^\circ_=.K}$	$\exists y(y = c) \to (\forall x A(x) \to A(c/x))$, *for all wffs* $A(x)$
iv	$\vdash_{Q^\circ_=.K}$	$\exists y(y = z) \wedge A(z/x) \to \exists x A(x)$
v	$\vdash_{Q^\circ_=.K}$	$\exists y(y = c)$ *if* $\vdash_{Q^\circ_=.K} \forall x A(x) \to A(c/x)$, *for all wffs* $A(x)$
vi	$\vdash_{Q^\circ_=.K+CBF}$	$\forall x \Box \exists y(x = y)$
vii	$\vdash_{Q^\circ_=.K+BF}$	$\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$
$viii$	$\vdash_{Q^\circ_=.K+BF}$	$\Diamond \exists y(x = y) \to \exists y(x = y)$

PROOF. .

(i)	$\vdash_{Q^\circ_=.K}$	$s = t \to (\Box(s = s)(s//s) \to \Box(s = s)(t//s))$
	$\vdash_{Q^\circ_=.K}$	$s = t \to (\Box(s = s) \to \Box(s = t))$
	$\vdash_{Q^\circ_=.K}$	$s = s$
	$\vdash_{Q^\circ_=.K}$	$\Box(s = s)$
	$\vdash_{Q^\circ_=.K}$	$s = t \to \Box(s = t)$.

[11] The system $Q^\circ_=.K$ is often called in the literature FK, free quantified K.

(ii) $\vdash_{Q^{\circ}_{=}.K}$ $\forall y[y = y \to \exists x((y = y)(x//y))]$ by 1.3(i)
$\vdash_{Q^{\circ}_{=}.K}$ $\forall y(y = y \to \exists x(x = y))$
$\vdash_{Q^{\circ}_{=}.K}$ $\forall y(y = y) \to \forall y \exists x(x = y)$
$\vdash_{Q^{\circ}_{=}.K}$ $\forall y \exists x(x = y)$.

(iii) $\vdash_{Q^{\circ}_{=}.K}$ $\exists y(y = c) \to [\exists y(y = c) \wedge \forall y(\forall x A \to A(y/x))]$
$\vdash_{Q^{\circ}_{=}.K}$ $\exists y(y = c) \to \exists y[(y = c) \wedge (\forall x A \to A(y/x))]$ by 1.3$(viii)$
$\vdash_{Q^{\circ}_{=}.K}$ $y = c \to [(\forall x A \to A)(y//x) \to (\forall x A \to A)(c//x)]$
$\vdash_{Q^{\circ}_{=}.K}$ $y = c \wedge (\forall x A \to A(y/x)) \to (\forall x A \to A(c/x))$
$\vdash_{Q^{\circ}_{=}.K}$ $\exists y[y = c \wedge (\forall x A \to A(y/x))] \to (\forall x A \to A(c/x))$
$\vdash_{Q^{\circ}_{=}.K}$ $\exists y(y = c) \to (\forall x A \to A(c/x))$.

(iv) $\vdash_{Q^{\circ}_{=}.K}$ $\exists y(y = z) \to (\forall x \neg A \to \neg A(z/x))$ by (iii)
$\vdash_{Q^{\circ}_{=}.K}$ $\exists y(y = z) \wedge A(z/x) \to \exists x A$.

(v) $\vdash_{Q^{\circ}_{=}.K}$ $\forall x \exists y(x = y) \to \exists y(c = y)$, let $A(x)$ be $\exists y(x = y)$
$\vdash_{Q^{\circ}_{=}.K}$ $\forall x \exists y(x = y)$ (ii)
$\vdash_{Q^{\circ}_{=}.K}$ $\exists y(c = y)$.

(vi) $\vdash_{Q^{\circ}_{=}.K+CBF}$ $\forall x \exists y(x = y)$ by (ii)
$\vdash_{Q^{\circ}_{=}.K+CBF}$ $\Box \forall x \exists y(x = y)$
$\vdash_{Q^{\circ}_{=}.K+CBF}$ $\forall x \Box \exists y(x = y)$ by CBF.

(vii) $\vdash_{Q^{\circ}_{=}.K+BF}$ $x = y \to x = y$
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\Diamond(x = y) \to \Diamond(x = y)$
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\exists y \Diamond(x = y) \to \exists y \Diamond(x = y)$
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$ by BF.

$(viii)$ $\vdash_{Q^{\circ}_{=}.K+BF}$ $\Diamond(x = y) \to (x = y)$ by ND
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\exists y \Diamond(x = y) \to \exists y(x = y)$
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\Diamond \exists y(x = y) \to \exists y \Diamond(x = y)$ (vii)
$\vdash_{Q^{\circ}_{=}.K+BF}$ $\Diamond \exists y(x = y) \to \exists y(x = y)$.

$\dashv$

Again on rigidity In a language with identity the fact that a constant c is a rigid designator is expressed by the formula $x = c \to \Box(x = c)$. Therefore, thanks to lemma 2.1(i), all the systems of q.m.l. with identity we are going to discuss are bound to be systems with rigid terms. Notice however that this is the case *given* general features of the K-semantics. The main feature is that universes of accessible worlds are related by the *inclusion function*: $U_w \subseteq U_v$. Therefore if we think of individuals of U_v as *counterparts* of individuals of U_w, each individual has one and only one counterpart in each related world (in fact it is the very same individual). It is because of this correlation that rigidity corresponds to NI or to the equivalence between *de dicto* and *de re* readings of substituted formulas, as pointed out in the footnote of lemma 1.1. For a more general semantics in which these notions are shown to be distinct from one another, see [2].

Definition 2.2. Let $\mathcal{M} = \langle W, R, D, U, I\rangle$ be a K-model. $\mathcal{M}$ is said to be *normal* iff

(a) for all $w \in W$, $I_w(=) = \{\langle d, d\rangle : d \in U_w\}$, and

(b) for all individual constants c, wRv implies $I_w(c) = I_v(c)$.

Lemma 2.3. *Each of the logics mentioned in the diagram at the beginning of this section is sound with respect to the class of normal K-models based on frames with respect to which the corresponding system without identity is sound.*

As to canonical models for systems $L \supseteq Q^{\circ}_{=}.K$, if w is an L-saturated set of sentences, the relation

$$a \sim b \quad \text{iff} \quad (a = b) \in w$$

is an equivalence relation and hence divides $Const(\mathcal{L}_w)$ into disjoint partitions. A problem presents itself immediately: the standard canonical model tecnique does not, in general, satisfy both the conditions (a) and (b) of definition 2.2. In fact, one way of matching condition (a) is to interpret each constant c in w on its equivalence classs $[c]_w$, and to define U_w as the set of equivalence classes of the constants mentioned in w. But by so doing we, in general, violate condition (b), because it might well happen that wRv, $[c]_w \in U_w$ and that a new constant c^* belongs to $[c]_v$, i.e. $(c = c^*) \in v$, with the consequence that $[c]_w \neq [c]_v$. We show how to overcome this difficulty by constructing canonical models where W is a class of sets satisfying the following condition:

(#) if wRv then $[c]_w = [c]_v$, for all constants $c \in Const(\mathcal{L}_w)$.

This can be achieved because each time we need to introduce a new constant c, we add that c is different from all the constants present so far.

Fact 2.4. If $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$, then condition (#) always holds. For, if $b \in [c]_w$, then$(b = c) \in w$, so $\Box(b = c) \in w$ by NI, hence $(b = c) \in v$, therefore $b \in [c]_v$. If $b \in [c]_v$, then $(b = c) \in v$, so $\Diamond(b = c) \in w$ since $b \in Const(\mathcal{L}_w)$, so by ND, $(b = c) \in w$, hence $b \in [c]_w$.

Definition 2.5. Let $L \supseteq Q^{\circ}_{=}.K$ be given with language $\mathcal{L}$. Let V be a set of constants of cardinality $\aleph_0$ such that $V \supset Const(\mathcal{L})$ and $|V - Const(\mathcal{L})| = \aleph_0$. A *normal canonical model for L is a quintuple* $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ such that

- W is the class of all L_w-saturated sets w, where $\mathcal{L}_w = \mathcal{L}^C$ for some set C of constants such that $Const(\mathcal{L}^C) \neq \emptyset$, $C \subset V$ and $|V - Const(\mathcal{L}^C)| = \aleph_0$,
- wRv iff $\Box^-(w) \subseteq v$ and for all $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$, where $[c]_v = \{b \in Const(\mathcal{L}_v) : (c = b) \in v\}$,
- $D_w = \{[c]_w : \exists y(y = c) \in w\}$,
- $U_w = \{[c]_w : c \in Const(\mathcal{L}_w)\}$,
- $I_w(c) = [c]_w$,
- $I_w(P^n) = \{\langle [c_1]_w, ..., [c_n]_w\rangle : P^n(c_1, ..., c_n) \in w\}$.

It is easy to see that canonical models so defined are normal K-models. That $W \neq \emptyset$ is due, as before, to lemma 1.16. Each w is D_w-universal thanks to lemma 2.1(iii). Condition (#) is trivially satisfied, so wRv implies that $U_w \subseteq U_v$ and

$I_w(c) = I_v(c)$ for all constants $c \in Const(\mathcal{L}_w)$. Moreover $I_w(=) = \{\langle u, u\rangle : u \in U_w\rangle\}$.

$Q^\circ_=.K$	$Q^\circ_=.K{+}CBF$	$Q_=.K$

To make sure that (#) holds also when $Const(\mathcal{L}_w) \neq Const(\mathcal{L}_v)$, we prove the following lemma:

LEMMA 2.6. (Variation of lemma 1.19 for systems with identity) *Let $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ be a normal canonical model for $L \supseteq Q^\circ_=.K$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $A \in v$, $\Box^-(w) \subseteq v$, $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$ and for all $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$.*

PROOF. As for lemma 1.19 with the set v constructed as follows. Let C be a countable set of new constants and let $H_1, H_2, ...$ be an enumeration of all the existential sentences of $\mathcal{L}^C_w$. Define the following chain of sets of sentences of $\mathcal{L}^C_w$.

$\Gamma_0 = \Box^-(w) \cup \{A\}$.

Suppose the set Γ_n has already been defined and the constants of C occurring in Γ_n are $c_1, ..., c_k$. Choose the first sentence in the given enumeration (and cancel it) which from C contains at most the constants $c_1, ..., c_k$. Let it be $\exists x F(x)$.

$Case(1)$. $\Gamma_n \cup \{\exists x F(x)\}$ is L^C_w-consistent.

$Case(1.1)$ For some constant b of $\mathcal{L}_w \cup \{c_1, ..., c_k\}$, $\Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$ is L^C_w-consistent. Define $\Gamma_{n+1} = \Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$.

$Case(1.2)$ For all constants b of $\mathcal{L}_w \cup \{c_1, ..., c_k\}$, $\Gamma_n \cup \{F(b/x)\} \cup \{\exists y(y = b)\}$ is not L^C_w-consistent. Take a constant $c \in C$ not occurring in $\mathcal{L}_w \cup \{c_1, ..., c_k\}$ and define $\Gamma_{n+1} = \Gamma_n \cup \{F(c/x)\} \cup \{\exists y(c = y)\} \cup \{c \neq b :$ for all constants b of $\mathcal{L}_w \cup \{c_1, ..., c_k\}\}$.

$Case(2)$. $\Gamma_n \cup \{\exists x F(x)\}$ is not L^C_w-consistent. Define $\Gamma_{n+1} = \Gamma_n$.

Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$ and $Q = \{c : \exists y(y = c) \in \Gamma\}$. Extend Γ to a set v which is L^Q_w-maximal.

Γ is Q-rich by construction and Q-universal because of lemma 2.1(iii), so is v. Let us show that condition (#) holds. In virtue of the way in which Γ has been defined, every constant occuring in formulas of Γ_{n+1} either belongs to $Const(\mathcal{L}_w)$ or has been introduced as a witness for an existential sentence containing no variables of C other than those occurring already in Γ_n. In addition, v is L^Q_w-maximal, so no constants occur in v which do not occur also in Γ. It follows that for all $b \in Const(\mathcal{L}_v)$, either $b \in Const(\mathcal{L}_w)$ or $(b \neq c) \in v$ for all $c \in Const(\mathcal{L}_w)$. Consequently, $b \in (Const(\mathcal{L}_v) - Const(\mathcal{L}_w))$ only if $b \notin [c]_v$. Moreover, if $b \in Const(\mathcal{L}_w)$, then, as we saw in fact 2.4, $b \in [c]_w$ iff $b \in [c]_v$. So (#) is proved.

What remains to be proved is that Γ_{n+1} as defined in case (1.2) is L^C_w-consistent. First we prove that $\Gamma_n \cup \{F(c/x)\} \cup \{\exists y(c = y)\}$ is L^C_w-consistent, where c doesn't occur in Γ_n. Suppose not, then $\Gamma_n \vdash_{L^C_w} \exists y(y = c) \to \neg F(c)$. So for some variable z not occurring in $\neg F(c)$, $\Gamma_n \vdash_{L^C_w} \exists y(y = z) \to \neg F(z/c)$. Since Γ_n is a set of sentences $\Gamma_n \vdash_{L^C_w} \forall z(\exists y(y = z) \to \neg F(z))$, so $\Gamma_n \vdash_{L^C_w} \forall z \exists y(y = z) \to \forall z \neg F(z)$, then by lemma 2.1($ii$) $\Gamma_n \vdash_{L^C_w} \forall z \neg F(z)$, contrary to the L^C_w-consistency of $\Gamma_n \cup \{\exists x F(x)\}$. Suppose now that Γ_{n+1} is not L^C_w-consistent,

then $\Gamma_n \vdash_{L_w^C} \exists y(y=c) \wedge F(c) \to [(c \neq b_1 \wedge ... \wedge c \neq b_h) \to \bot]$, for some constants $b_1 \dots b_h \in \mathcal{L}_w \cup \{c_1, ..., c_k\}$, so $\Gamma_n \vdash_{L_w^C} \exists y(y=c) \wedge F(c) \to (c = b_1 \vee ... \vee c = b_h)$. Therefore $\Gamma_n \cup \{\exists y(y=c)\} \cup \{F(c)\} \cup \{c = b_j\}$ is L_w^C-consistent for some j, $1 \leq j \leq h$, consequently $\Gamma_n \cup \{\exists y(y = b_j)\} \cup \{F(b_j/c)\}$ is L_w^C-consistent, contrary to the fact that for no $b \in \mathcal{L}_w \cup \{c_1, \dots, c_k\}$, $\Gamma_n \cup \{F(b)\} \cup \{\exists y(y=b)\}$ is L_w^C-consistent. ⊣

Definition 2.7. For each equivalence class $[c]_w \in U_w$, $f[c]_w$ is the canonical representative of $[c]_w$. When no confusion can possibly occur, we write $f[c]$.

Lemma 2.8. *Let $\mathcal{N}^L$ be a normal canonical model for $L \supseteq Q^\circ_=.K$. For all formulas $A(x_1, \dots, x_n)$ of $\mathcal{L}$ and for any w-assignment σ,*

$$\mathcal{N}^L \models_w^\sigma A(x_1, \dots, x_n) \qquad \textit{iff} \qquad A(f\sigma(x_1)/x_1, \dots, f\sigma(x_n)/x_n) \in w,$$

where $x_1, \dots, x_n$ are all the variables occurring free in A.

Proof. $\mathcal{N}^L \models_w^\sigma P^k(x_{i_1}, ..., x_{i_n}, c_{i_{n+1}}, ..., c_{i_k})$ iff $\langle \sigma(x_{i_1}), ..., \sigma(x_{i_n}), I_w(c_{i_{n+1}}), ..., I_w(c_{i_k})\rangle \in I_w(P^k)$ iff $\langle \sigma(x_{i_1}), ..., \sigma(x_{i_n}), [c_{i_{n+1}}], ..., [c_{i_k}]\rangle \in I_w(P^k)$ iff since $\sim$ is a congruence relation, $P^k(f\sigma(x_{i_1}), ..., f\sigma(x_{i_n}), c_{i_{n+1}}, ..., c_{i_k}) \in w$. The other cases are as in lemma 1.20. ⊣

Theorem 2.9. *(i) $Q^\circ_=.K$ is strongly complete with respect to the class of normal K-models based on all K-frames with constant outer domains.*
(ii) $Q^\circ_=.K{+}CBF$ is strongly complete with respect to the class of normal K-models based on all K-frames with increasing inner domains and constant outer domains.
(iii) $Q_=.K$ is strongly complete with respect to the class of normal K-models based on TK-frames with increasing domains.

Proof. (i) Lemmas 2.6 and 2.8 yield that $Q^\circ_=.K$ is complete w.r.to the class of all K-frames and by the construction of theorem 1.32 it obtains that $Q^\circ_=.K$ is complete w.r.to the class of all K-frames with constant outer domains. (ii) $D_w \subseteq D_v$. For if $c \in D_w$, then $\exists y(y=c) \in w$, so by lemma 2.1(vi), $\Box\exists y(y=c) \in w$, therefore $\exists y(y=c) \in v$, so $c \in D_v$. Whence $Q^\circ_=.K{+}CBF$ is complete w.r.to the class of all K-frames with increasing inner domains and, by the construction of theorem 1.32, with constant outer domains. (iii) follows from (ii) and the fact that $Q_=.K \vdash \exists y(y=c)$, for all constants c. ⊣

$Q^\circ_=.K{+}BF$	$Q^\circ_=.K{+}CBF{+}BF$	$Q_=.K{+}BF$

Lemma 2.10. (BF-variation of lemma 1.19) *Let $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ be a normal canonical model for $L \supseteq Q^\circ_=.K{+}BF$. If $w \in W$ and $\Diamond A \in w$, then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$, $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$ and moreover v is Q-universal and Q-rich for some $Q \subseteq D_w$.*

Proof. The proof is the same as that of lemma 1.29 except that,
Case(1.) $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. Define $\Gamma_{n+1} = \Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c/x)\}$ for some constant $c \in D_w$ such that $\Gamma_n \cup \{\exists y(y=c)\} \cup \{F(c/x)\}$ is L_w-consistent.

We show that such a constant c is always available. Suppose not, then $\Gamma_n \vdash_{L_w} \exists y(y = c) \to \neg F(c)$, for all $c \in D_w$. But $Cl(\Gamma_n)$ is D_w-inductive by lemma 1.28, so $\Gamma_n \vdash_{L_w} \forall z(\exists y(y = z) \to \neg F(z/c))$, where z does not occur in $\neg F(c)$, then $\Gamma_n \vdash_{L_w} \forall z \exists y(y = z) \to \forall z \neg F(z)$, $\Gamma_n \vdash_{L_w} \forall z \neg F(z))$, contrary to the L_w-consistency of $\Gamma_n \cup \{\exists x F(x)\}$. ⊣

Since $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$, it always holds that for all $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$, see fact 2.4.

Let us show that, in general, $D_v \subseteq D_w$ if wRv. Since CBF is not a theorem of $Q^\circ_=.K + BF$, the set $\{\Box\forall x P(x), \exists x \Diamond \neg P(x)\}$ is $Q^\circ_=.K + BF$-consistent, therefore there is a world w of the canonical model for $Q^\circ_=.K + BF$ such that $\{\Box\forall x P(x), \exists x \Diamond \neg P(x)\} \subseteq w$. Then for some $[d] \in D_w$, $\Diamond\neg P(d) \in w$. But then there is a v, related to w, such that $\{\forall x P(x), \neg P(d)\} \subseteq v$, so $[d] \notin D_v$ consequently v is bound to be D_v-universal and D_v-rich for some $D_v \subset D_w$.

Theorem 2.11. *(i) $Q^\circ_=.K+BF$ is strongly complete with respect to the class of normal models based on K-frames with decreasing inner domains and constant outer domains.*
(ii) $Q^\circ_=.K+CBF+BF$ is strongly complete with respect to the class of normal models based on K-frames with constant inner domains and constant outer domains.
(iii) $Q_=.K+BF$ is strongly complete with respect to the class of normal models based on TK-frames with constant domains.

Systems with the Extended Barcan Rule

$$BR(n{+}1) \quad \frac{A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1})\dots)}{A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1})\dots)} \quad \text{where } x \text{ is not free in } A_0, ..., A_n$$

By EBR, Extended Barcan Rule, we denote the set of all rules $BR(n+1), n \geq 0$. The rule EBR was first introduced by R.Thomason in [8] and since then discussed at various points in the literature.[12]

Lemma 2.12. *EBR is valid on K-frames with constant outer domains, i.e. for any K-model $\mathcal{M}$ with constant outer domains, if the premise of EBR is valid on $\mathcal{M}$, then the conclusion is also valid on $\mathcal{M}$.*

Proof. Suppose that for some $\mathcal{M}$, w and w-assignment σ, $\mathcal{M} \not\models^\sigma_w A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1}(x))...)$. Therefore for some v, $wR^{n+1}v$, $\mathcal{M} \not\models^\sigma_v \forall x A_{n+1}(x)$, hence for some x-variant τ of σ such that $\tau(x) \in D_v \subseteq U_v$, $\mathcal{M} \not\models^\tau_v A_{n+1}(x)$. Since $U_w = U_v$, this is impossible because τ is also a w-assignment and, by hypothesis, $\mathcal{M} \models^\tau_w A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(x))\dots)$, consequently, $\mathcal{M} \models^\tau_v A_{n+1}(x)$. ⊣

If the outer domains of a given K-frame $\mathcal{F}$ are not constant, then it might well be that, e.g., $BR(1)$ is valid on $\mathcal{F}$ (just take K-frames in which wRv implies $D_v \subseteq U_w$) and $BR(2)$ is not. Here is an instance in case. Let $\mathcal{F} = \langle W, R, D, U\rangle$,

[12] Our treatment of $Q^\circ_=.K + EBR$ has similarities with [6], pp.296-302.

where $W = \{w, v, z\}$, $R = \{\langle w, v\rangle, \langle v, z\rangle\}$, $D_w = D_v = \{u\}$, $D_z = \{u, u^*\}$, $U_w = \{u\}$, $U_v = U_z = \{u, u^*\}$. We can easily see that $BR(1)$, i.e.

$$\frac{A_0 \to \Box A_1(x)}{A_0 \to \Box\forall x A_1(x)}$$

is valid on $\mathcal{F}$. On the contrary, if $\mathcal{M} = \langle \mathcal{F}, I\rangle$ is such that $I_w(P) = I_v(Q) = I_z(R) = \{u\}$, $I_w(a) = I_v(a) = I_z(a) = u$, then
$\mathcal{M} \models_w P(a)$,
$\mathcal{M} \not\models_v P(a)$, $\mathcal{M} \models_v Q(a)$,
$\mathcal{M} \not\models_z P(a)$, $\mathcal{M} \models_z R(a)$, $\mathcal{M} \not\models_z \forall x R(x)$,

therefore the following instance of $BR(2)$ is not valid on $\mathcal{M}$:

$$\frac{P(a) \to \Box(Q(a) \to \Box R(x))}{P(a) \to \Box(Q(a) \to \Box\forall x R(x))}$$

Let us add EBR to the systems considered so far.

Lemma 2.13. (a)

$Q^\circ.K$	*is equivalent to*	$Q^\circ.K{+}EBR$,
$Q^\circ.K{+}CBF$	*is equivalent to*	$Q^\circ.K{+}EBR{+}CBF$,
$Q^\circ_=.K$	*is equivalent to*	$Q^\circ_=.K{+}EBR$,
$Q^\circ_=.K{+}CBF$	*is equivalent to*	$Q^\circ_=.K{+}EBR{+}CBF$.

(b)

$$Q.K{+}BF \quad \textit{is equivalent to} \quad Q.K{+}EBR.$$

Proof. (a) If $Q^\circ.K \nvdash A$, then by theorem 1.32 for some K-model $\mathcal{M}$ with constant outer domains, $\mathcal{M} \not\models A$, consequently, by lemma 2.12, $Q^\circ.K{+}EBR \nvdash A$.
(b) $Q.K + EBR \vdash \forall x\Box A(x) \to \Box A(x)$, so $Q.K + EBR \vdash \forall x\Box A(x) \to \Box\forall x A(x)$, therefore adding EBR to $Q.K$ gives a stronger system: $Q.K{+}BF$. Consequently $Q.K{+}EBR$ is complete with respect to TK-frames with constant domains. $\dashv$

The following table summarizes the results obtained so far for systems with identity.

q.m.l.	is strongly complete w.r.t. the class of normal models based on K-frames with domains	
	inner	*outer*
$Q^\circ_=.K$	varying	constant
$Q^\circ_=.K{+}EBR$	varying	constant
$Q^\circ_=.K{+}CBF$	increasing	constant
$Q^\circ_=.K{+}CBF{+}EBR$	increasing	constant
$Q^\circ_=.K{+}BF$	decreasing	constant

$Q^\circ_=.K{+}CBF{+}BF$	constant	constant
$Q_=.K$	increasing	= inner
$Q_=.K{+}BF$	constant	= inner

As a matter of fact, the construction of canonical models for $Q^\circ.K$ as described in the first part of this paper, allows us to build models for $Q^\circ.K$ with increasing outer domains and we do not know of any way of building canonical models for $Q^\circ.K$ or $Q^\circ.K{+}EBR$ with constant outer domains. When identity is present, a second strategy, to be described below, is at our disposal and it provides us with canonical models with constant outer domains. Both strategies are needed since, for example, the completeness proof of $Q_=.K$ requires the first, whereas the completeness proof of $Q^\circ_=.B$ (see lemma 2.21(b)) requires the second.

Building canonical models with constant outer domains requires going through lemmas analogous to 1.16, 1.28 and 2.10.

LEMMA 2.14. (EBR-counterpart of lemma 1.16) *Let $L \supseteq Q^\circ_=.K{+}EBR$, and let Δ be an L-consistent set of sentences. Then for some not-empty denumerable set C of new constants there is an extension Π of Δ which is L^C-maximal, Q-rich and Q-universal, for some $Q \subseteq Const(\mathcal{L}^C)$ and moreover Π is $\Diamond$-$\mathcal{L}^C$-rich, where Π is $\Diamond$-$\mathcal{L}^C$-rich iff*

if $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots) \in \Pi$, *where x is not free in $A_1, \dots, A_n$, then for some $c \in \mathcal{L}^C$, $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots) \in \Pi$.*

PROOF. Let $H_1, H_2, H_3, \dots$ be an enumeration of all the sentences of $\mathcal{L}^C$ which are either of the form $\exists x F(x)$, for some wff $F(x)$ or of the form $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots)$ for some wffs $A_0 \dots A_{n+1}$ such that x is not free in $A_0 \dots A_n$. Define the following chain of sets of sentences of $\mathcal{L}^C$.

$\Gamma_0 = \Delta$.

Suppose the set $\Gamma_n = \Delta \cup \{B_1, \dots, B_k\}$ has already been defined. Consider the sentence H_{n+1} in the given enumeration.

Case(1) $\Gamma_n \cup \{H_{n+1}\}$ is L^C-consistent.

Case(1.1) H_{n+1} is $\exists x F(x)$. Define $\Gamma_{n+1} = \Gamma_n \cup \{\exists y(c = y)\} \cup \{F(c/x)\}$, for some $c \in \mathcal{L}^C$ such that $\Gamma_n \cup \{\exists y(c = y)\} \cup \{F(c/x)\}$ is L^C-consistent.

Case(1.2) H_{n+1} is $A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots)$. Define $\Gamma_{n+1} = \Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots)\}$, for some $c \in \mathcal{L}^C$ such that $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots)\}$ is L^C-consistent.

Case(2) $\Gamma_n \cup \{H_{n+1}\}$ is not L^C-consistent. Define $\Gamma_{n+1} = \Gamma_n$. Then let $\Gamma = \bigcup_{n \in N} Cl(\Gamma_n)$ and let Π be an L^C-maximal extension of Γ. It is easy to see that Π is $\Diamond$-rich, Q-rich and Q-universal, where $Q = \{c \in \mathcal{L}^C : \exists y(y = c) \in \Pi\}$. Now we show that in cases (1.1) and (1.2) the appropriate constant c is always available.

Case(1.1) Suppose that $\Gamma_n \cup \{\exists y(c = y)\} \cup \{F(c/x)\}$ is not L^C-consistent for all $c \in \mathcal{L}^C$. Then, for any $c \in \mathcal{L}^C$ not occurring in Γ_n, $\Gamma_n \vdash_{L^C} \exists y(y = c) \to \neg F(c)$, therefore for some variable z not occurring in $\neg F(c)$, $\Gamma_n \vdash_{L^C} \exists y(y = z) \to \neg F(z/c)$, hence, since Γ_n is a set of sentences, $\Gamma_n \vdash_{L^C} \forall z(\exists y(y = z) \to \neg F(z))$, so $\Gamma_n \vdash_{L^C} \forall z \exists y(y = z) \to \forall z \neg F(z)$, then by lemma 2.1($ii$) $\Gamma_n \vdash_{L^C} \forall z \neg F(z)$, contrary to the L^C-consistency of $\Gamma_n \cup \{\exists x F(x)\}$.

Case(1.2) Suppose that Γ_{n+1} is not L^C-consistent, then $\Gamma_n \vdash A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots) \to \bot$, for all $c \in \mathcal{L}^C$. Therefore for some c which doesn't occur either in $B = (B_1 \wedge \cdots \wedge B_k)$ or in $A_0 \dots A_n$, $\Delta \vdash_{L^C} B \wedge A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots) \to \bot$. Therefore for some conjunction D of sentences of Δ, $\vdash_{L^C} D \wedge B \wedge A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(c/x))\dots) \to \bot$. Let z be a variable not occurring in this last formula, then $\vdash_{L^C} D \wedge B \wedge A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond A_{n+1}(z/c))\dots) \to \bot$. Whence $\vdash_{L^C} D \wedge B \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\neg A_{n+1}(z))\dots)$, $\vdash_{L^C} D \wedge B \wedge A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x\neg A_{n+1}(x/z))\dots)$, via *EBR*. But then $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots)\} \vdash \Diamond\dots\Diamond\Diamond\bot$, therefore $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots)\} \vdash \bot$, contrary to the L^C-consistency of $\Gamma_n \cup \{A_0 \wedge \Diamond(A_1 \wedge \cdots \wedge \Diamond(A_n \wedge \Diamond\exists x A_{n+1}(x))\dots)\}$. $\dashv$

LEMMA 2.15. *Let $L \supseteq Q^\circ_=.K{+}EBR$. If Δ is a set of sentences which is L^C-maximal and $\Diamond$-$\mathcal{L}^C$-rich, then it is $\Box$-$\mathcal{L}^C$-inductive, where Δ is said to be $\Box$-$\mathcal{L}^C$-inductive iff if*

$A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(c))\dots) \in \Delta$, *for all $c \in Const(\mathcal{L}^C)$, then*

$A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1}(x))\dots) \in \Delta$.

DEFINITION 2.16. Let $L \supseteq Q^\circ_=.K{+}EBR$. Let C be a set of constants such that $C \supset Const(\mathcal{L})$ and $|C - Const(\mathcal{L})| = \aleph_0$. A *normal canonical model* $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ *for* L is defined as follows:

- W is the class of all L^C-saturated and $\Box$-$\mathcal{L}^C$-inductive set of sentences,
- wRv iff $\Box^-(w) \subseteq v$, for any $w, v \in W$,
- $D_w = \{[c] : \exists y(y = c) \in w\}$,
- $U_w = \{[c] : c \in Const(\mathcal{L}^C)\}$,
- $I_w(c) = [c]$,
- $I_w(P^n) = \{\langle [c_1], ..., [c_n]\rangle : P^n(c_1, ..., c_n) \in w\}$.

Lemmas 2.14 and 2.15 guarantee that $W \neq \emptyset$.

LEMMA 2.17. *Let $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ be a normal canonical model for $L \supseteq Q^\circ_=.K{+}EBR$. For all $w \in W$, $Cl(\Box^-(w) \cup \{B_1, \dots, B_k\})$ is $\Box$-$\mathcal{L}^C$-inductive.*

PROOF. Let $B = B_1 \wedge \cdots \wedge B_k$. Suppose that for all $c \in Const(\mathcal{L}_w)$, $\Box^-(w) \cup \{B\} \vdash A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(c))\dots)$, then $\Box^-(w) \vdash (B \wedge A_0) \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(c))\dots)$, for all $c \in Const(\mathcal{L}_w)$, $w \vdash \Box[(B \wedge A_0) \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box A_{n+1}(c))\dots)]$, but w is $\Box$-$\mathcal{L}^C$-inductive,[13] so $w \vdash \Box[(B \wedge A_0) \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall A_{n+1}(x))\dots)]$, $[(B \wedge A_0) \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall A_{n+1}(x))\dots)] \in \Box^-(w)$, $\Box^-(w) \cup \{B_1, \dots, B_k\} \vdash A_0 \to \Box(A_1 \to \cdots \to \Box(A_n \to \Box\forall x A_{n+1}(x))\dots)$. $\dashv$

LEMMA 2.18. (EBR-variation of lemma 1.19) *Let $\mathcal{N}^L = \langle W, R, D, U, I\rangle$ be a normal canonical model for $L \supseteq Q^\circ_=.K{+}EBR$. If $w \in W$ and $\Diamond A \in w$ then there is a $v \in W$ such that $\Box^-(w) \subseteq v$, $A \in v$ and $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$.*

[13] Being $\Box$-$\mathcal{L}^C$-inductive can be paraphrased as being *EBR*-closed. Now, in order to show that $\Box^-(w) \cup \{B_1, \dots, B_k\}$ is $BR(n+1)$ closed, we need to make use of the fact that w is $BR(n+2)$ closed. Therefore we cannot limit ourselves to any finite set $BR(1)$ .... $BR(n+1)$ of rules.

PROOF. The proof is exactly as that of lemmas 1.19 and 1.16 except that, *Case*(1.) $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. Define $\Gamma_{n+1} = \Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c/x)\}$ for some constant $c \in \mathcal{L}_w$ such that $\Gamma_n \cup \{\exists y(y = c)\} \cup \{F(c/x)\}$ is L_w-consistent.
We show that such a constant c is always available. Suppose not, then for all $c \in \mathcal{L}_w$, $\Box^-(w) \vdash G \wedge \exists y(y = c) \to \neg F(c)$, where G is the conjunction of all the sentences of $(\Gamma_n - \Box^-(w))$, so $w \vdash \Box(G \wedge \exists y(y = c) \to \neg F(c))$. Since w is $\Box$-$\mathcal{L}^C$-inductive, $w \vdash \Box\forall z(G \wedge \exists y(y = z) \to \neg F(z/c))$, $w \vdash \Box(G \wedge \forall z \exists y(y = z) \to \forall z \neg F(z))$, $w \vdash \Box(G \to \forall z \neg F(z))$, $(G \to \forall z \neg F(z)) \in \Box^-(w)$, $\Gamma_n \vdash \forall z \neg F(z)$, contrary to the fact that $\Gamma_n \cup \{\exists x F(x)\}$ is L_w-consistent. $\dashv$

Again, since $Const(\mathcal{L}_w) = Const(\mathcal{L}_v)$, it always holds that for all $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$. Lemmas 2.14-2.18 yield that

THEOREM 2.19. *$Q^\circ_=.K + EBR$ is strongly complete with respect to the class of normal models based on K-frames with constant outer domains.*

QUANTIFIED EXTENSIONS OF THE PROPOSITIONAL MODAL LOGIC B.

By $Q^\circ.B$, ($Q.B$, $Q^\circ_=.B$) we denote the logic $Q^\circ.K$, ($Q.K$, $Q^\circ_=.K$) plus the propositional axiom B: $A \to \Box\Diamond A$, i.e. the axiom characteristic of frames whose accessibility relation is symmetric. The rule EBR is derivable in $Q^\circ.B$. For the reader's sake, here is the proof of EBR as given in [6], p.295, for $n = 2$.

$\vdash_{Q^\circ.B}$	$A_0 \to \Box(A_1 \to \Box(A_2 \to \Box A_3(x)))$	premise of $BR(3)$,
$\vdash_{Q^\circ.B}$	$\Diamond A_0 \to (A_1 \to \Box(A_2 \to \Box A_3(x)))$	
$\vdash_{Q^\circ.B}$	$\Diamond A_0 \wedge A_1 \to \Box(A_2 \to \Box A_3(x))$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \to \Box A_3(x))$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2 \to \Box A_3(x)$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to A_3(x)$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2) \to \forall x A_3(x)$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond A_0 \wedge A_1) \wedge A_2 \to \Box\forall x A_3(x)$	
$\vdash_{Q^\circ.B}$	$\Diamond(\Diamond A_0 \wedge A_1) \to (A_2 \to \Box\forall x A_3(x))$	
$\vdash_{Q^\circ.B}$	$(\Diamond A_0 \to A_1) \to \Box(A_2 \to \Box\forall x A_3(x))$	
$\vdash_{Q^\circ.B}$	$\Diamond A_0 \to (A_1 \to \Box(A_2 \to \Box\forall x A_3(x)))$	
$\vdash_{Q^\circ.B}$	$A_0 \to \Box(A_1 \to \Box(A_2 \to \Box\forall x A_3(x)))$	

LEMMA 2.20. $\vdash_{Q^\circ_=.B-ND} ND$ *and* $\vdash_{Q^\circ_=.B+BF} \exists y(x = y) \to \Box\exists y(x = y)$.

PROOF. .

$\vdash_{Q^\circ_=.B}$	$x = y \to \Box(x = y)$	NI
$\vdash_{Q^\circ_=.B}$	$\Diamond(x \neq y) \to (x \neq y)$	
$\vdash_{Q^\circ_=.B}$	$\Box\Diamond(x \neq y) \to \Box(x \neq y)$	
$\vdash_{Q^\circ_=.B}$	$(x \neq y) \to \Box(x \neq y)$	by B

$\vdash_{Q^\circ_=.B+BF}$	$x = y \to \Box(x = y)$	NI
$\vdash_{Q^\circ_=.B+BF}$	$\Diamond(x = y) \to (x = y)$	via B
$\vdash_{Q^\circ_=.B+BF}$	$\exists y\Diamond(x = y) \to \exists y(x = y)$,	
$\vdash_{Q^\circ_=.B+BF}$	$\Diamond\exists y(x = y) \to \exists y(x = y)$	via BF
$\vdash_{Q^\circ_=.B+BF}$	$\Box\Diamond\exists y(x = y) \to \Box\exists y(x = y)$,	
$\vdash_{Q^\circ_=.B+BF}$	$\exists y(x = y) \to \Box\exists y(x = y)$	via B.

⊣

LEMMA 2.21. .

	q.m.l.	*is strongly complete w.r.t. the class of K-frames where R is symmetric and the domains are* inner	outer
(a)	$Q^\circ.B + CBF$	*constant*	*constant*
(b)	$Q^\circ_=.B$	*varying*	*constant*
(c)	$Q^\circ_=.B + CBF$	*constant*	*constant*
(d)	$Q^\circ_=.B + BF$	*constant*	*constant*
(e)	$Q.B$	*constant*	*= inner*
(f)	$Q_=.B$	*constant*	*= inner*

PROOF. That the accessibility relation R is symmetric is easily seen as for the propositional case because the languages of all the worlds of a given canonical model are equal.

(a) Since $Q^\circ.B{+}CBF \vdash BF$, the completeness proof of $Q^\circ.B + CBF$ is analogous to that for $Q^\circ.K{+}CBF{+}BF$. For the reader's sake here is the proof of BF as given in [3], p.138.

$\vdash_{Q^\circ.B+CBF}$	$\forall x[\forall x\Box A(x) \to \Box A(x)]$	
$\vdash_{Q^\circ.B+CBF}$	$\Box\forall x[\forall x\Box A(x) \to \Box A(x)]$	
$\vdash_{Q^\circ.B+CBF}$	$\forall x\Box[\forall x\Box A(x) \to \Box A(x)]$	by CBF,
$\vdash_{Q^\circ.B+CBF}$	$\forall x[\Diamond\forall x\Box A(x) \to \Diamond\Box A(x)]$	
$\vdash_{Q^\circ.B+CBF}$	$\forall x[\Diamond\forall x\Box A(x) \to A(x)]$	via B
$\vdash_{Q^\circ.B+CBF}$	$\forall x\Diamond\forall x\Box A(x) \to \forall xA(x)$	
$\vdash_{Q^\circ.B+CBF}$	$\Diamond\forall x\Box A(x) \to \forall xA(x)$	
$\vdash_{Q^\circ.B+CBF}$	$\Box\Diamond\forall x\Box A(x) \to \Box\forall xA(x)$	
$\vdash_{Q^\circ.B+CBF}$	$\forall x\Box A(x) \to \Box\forall xA(x)$,	via B.

(b) Since, as we have seen, EBR is derivable in $Q^\circ.B$, the completeness proof of $Q^\circ_=.B$ is obtained through lemmas 2.14-2.18 as for $Q^\circ_=.K{+}EBR$, see theorem 2.19.

(c) from (a).

(d) Since CBF is a theorem of $Q^\circ_=.B{+}BF$, the completeness proof of $Q^\circ_=.B{+}BF$ is analogous to that for $Q^\circ_=.K{+}BF{+}CBF$. Here is a proof of CBF in $Q^\circ_=.B{+}BF$.

$\vdash_{Q^\circ_=.B+BF}$	$\exists y(x = y) \to \Box\exists y(x = y)$	lemma 2.20,
$\vdash_{Q^\circ_=.B+BF}$	$\neg[\exists y(x = y) \wedge \Diamond\forall y\neg(x = y)]$.	

Let A be any wff and let $B = A(x/y)$, where x is a variable not occurring in A. Trivially y doesn't occur free in B. Then

$\vdash_{Q^\circ_=.B+BF}$ $\neg[\exists y(x = y) \wedge \Diamond[B \wedge (B \rightarrow \forall y \neg(x = y))]]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg[\exists y(x = y) \wedge [\Diamond B \wedge \Box(B \rightarrow \forall y \neg(x = y))]]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg[\exists y(x = y) \wedge \Diamond B \wedge \Box \forall y(x = y \rightarrow \neg B)]$
$\vdash_{Q^\circ_=.B+BF}$ $\neg[\exists y(x = y) \wedge \Diamond B \wedge \Box \forall y \neg A(y)]$ since $\neg A(y) \rightarrow (x = y \rightarrow \neg A(x/y))$,
$\vdash_{Q^\circ_=.B+BF}$ $\neg \exists y(x = y) \vee \neg \Diamond B \vee \Diamond \exists y A(y)$
$\vdash_{Q^\circ_=.B+BF}$ $\neg(\exists y(x = y) \wedge \Diamond B) \vee \Diamond \exists y A(y)$
$\vdash_{Q^\circ_=.B+BF}$ $\neg \exists y \Diamond A(y) \vee \Diamond \exists y A(y)$ since $\exists y \Diamond A(y) \rightarrow (\exists y(x = y) \wedge \Diamond A(x/y))$,
$\vdash_{Q^\circ_=.B+BF}$ $\exists y \Diamond A(y) \rightarrow \Diamond \exists y A(y)$.

(e) Since CBF, BF and UI are all theorems of $Q.B$, $Q.B$ is complete with respect to the class of TK-frames with constant domains. $\dashv$

(f) from (e).

Q°.B+BF is K-incomplete

Here is a model for $Q^\circ.B{+}BF$ in which CBF fails.[14] The model is based on a counterpart Kripke frame. For details about counterpart semantics, see [2]. A counterpart Kripke frame, $\mathfrak{C}K$-frame, is a quintuple $\mathcal{F} = \langle W, R, D, U, \mathfrak{C}\rangle$, where $\langle W, R, D, U\rangle$ is a K-frame and $\mathfrak{C}$, the counterpart relation, is such that

$\mathfrak{C} =_{df} \biguplus_{w,v \in W} \{\mathfrak{C}_{\langle w,v\rangle}\}$, where for any $w, v \in W$ such that wRv, $\mathfrak{C}_{\langle w,v\rangle} \subseteq (U_w \times U_v)$.

It can be easily shown that $Q^\circ.K$ formulated in a language with types is valid with respect to the class of all $\mathfrak{C}K$-models, where the notion of satisfaction is defined thus:

$\langle a_1, ..., a_n\rangle \models_w P^n(n : x_1, \ldots, x_n)$	iff	$\langle a_1, ..., a_n\rangle \in I_w(P^n)$
$\langle a_1, ..., a_n\rangle \models_w \langle n : s_1, ..., s_k\rangle B$	iff	$\langle a_1, ..., a_n\rangle[n : s_1, ..., s_k]_w \models_w B$
$\langle a_1, ..., a_n\rangle \models_w \neg C$	iff	$\langle a_1, ..., a_n\rangle \not\models_w C$
$\langle a_1, ..., a_n\rangle \models_w C \vee D$	iff	$\langle a_1, ..., a_n\rangle \models_w C$ or $\langle a_1, ..., a_n\rangle \models_w D$
$\langle a_1, ..., a_n\rangle \models_w \exists x_{n+1} G$	iff	for some $b \in D_w$, $\langle a_1, ..., a_n, b\rangle \models_w G$
$\langle a_1, ..., a_n\rangle \models_w \Box C$	iff	for all v such that wRv and for all counterparts $a_1^*, \ldots, a_n^*$ in D_v of $a_1, \ldots, a_n$, respectively, $\langle a_1^*, ..., a_n^*\rangle \models_v C$.

A counterpart frame is said to be *symmetric* iff both R and $\mathfrak{C}$ are symmetric. A counterpart relation is said to be *surijective* iff if wRv, then for all $b \in U_v$ there is an $a \in U_w$ such that $a\mathfrak{C}b$ holds. From [2] we know that BF is valid on a counterpart K-frame iff the counterpart relation is surijective.
Consider the following counterpart K-frame $\mathcal{F} = \langle W, R, D, U, \mathfrak{C}\rangle$, where

$W = \{w, v\}$,
$R = \{\langle w, v\rangle, \langle v, w\rangle\}$
$D_w = \{a, b\}$, $D_v = \{a^*\}$
$U_w = \{a, b\}$, $U_v = \{a^*, b^*\}$
$\mathfrak{C} = \{\langle a, a^*\rangle, \langle b, a^*\rangle, \langle b, b^*\rangle, \langle a^*, a\rangle, \langle a^*, b\rangle, \langle b^*, b\rangle\}$

[14] This answers a question raised in [3], p.138, whether CBF is a theorem of $Q^\circ.B{+}BF$.

Both R and $\mathfrak{C}$ are symmetric relations and $\mathfrak{C}$ is surijective, so $\mathcal{F}$ is a frame for $Q^\circ.B{+}BF$. Consider now a model $\mathcal{M} = \langle \mathcal{F}, I\rangle$ such that $I_w(P) = \{a, b\}$ and $I_v(P) = \{a^*\}$. Then $\mathcal{M} \models_w \Box\forall x P(x)$ because $a^* \in I_v(P)$ and $\mathcal{M} \not\models_w \forall x \Box P(x)$ because $b\mathfrak{C}b^*$ and $b^* \notin I_v(P)$, so $\mathcal{M} \not\models_w \Box\forall x P(x) \to \forall x \Box P(x)$. Therefore $Q^\circ.B{+}BF \not\vdash CBF$. But CBF is valid on all K-frames for $Q^\circ.B{+}BF$ since each of them is bound to have inner constant domains, whence

THEOREM 2.22. *$Q^\circ.B{+}BF$ is not characterized by any class of K-frames.*

OPEN PROBLEMS Completeness property of $Q^\circ.K{+}BF$ and $Q^\circ.B$.

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