"Necessary for"[1]

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ABSTRACT. A new language for quantified modal logic is presented in which the modal operators are indexed by terms : "it is necessary for $t_1, \ldots, t_n$". Systems of quantified modal logic are defined in that language and shown to be complete with respect to transition semantics. Formulas such as the Barcan formula, the Ghilardi formula, the necessity of identity can be expressed in a natural way in the new language and are shown to correspond to particular properties of the transition relation.

1 Introduction

Quantified modal logic, as usually understood, is the study of theories in a first-order language plus the box-operator. Typically, $\Box P(x)$ is read

'it is necessary that $P(x)$'.

A considerable variety of such theories have been studied from the pioneering work of Rudolf Carnap and Ruth Barcan to more recent publications such as [6] and [1]. Some dissatisfaction is still felt in particular when one tries to analyse natural language or to deal with semantical structures more general than Kripke frames. Attempts to build richer modal languages by modifying the underlying first-order language have been made in two directions:

- by adding the λ-abstraction operator so as to distinguish, e.g., between *de re vs de dicto* sentences, $\lambda x \Box P(x).i$ *vs* $\Box P(i)$, 'The first pilot was necessarily a pilot' *vs* 'Necessarily, the first pilot was a pilot'.
- by the introduction of a language with types. A wff $\Box A : n$ is of type n when the free variables occurring in it, either implicitely or explicitely, are $x_1, \ldots, x_n$. Moreover $\Box A : n$ is going to be satisfied or not satisfied by n-tuples of elements of the domain. See [2], [4] and [1].

We introduce a new language which combines features of languages with λ-abstraction operator and languages with types. $\Box P(x)$ is not a well-formed formula anymore since x is free in $P(x)$ and it has to be replaced by

$$|\,x\,|\,P(x)$$

to be read as

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$$\text{'it is necessary for } x \text{ to be } P(x)\text{'}.$$

$|\,x\,|$ is a box-operator indexed by x. A more complex form of the box-operator is the following one

$$|\,{}^{i}_{x}\,|\,P(x)$$

The notation has two roles:

- it binds the variable x
- it says that it is necessary for the individual i to have the property $\lambda x.P(x)$.

Dually,

$$\langle\,{}^{i}_{x}\,\rangle\,P(x)$$

says that it is possible for i to have the property $\lambda x.P(x)$. Again,

$$|\,{}^{i}_{x}\;{}^{j}_{y}\,|\,R(x,y)$$

says that 'it is necessary for i and j to stand in the relation $\lambda x\lambda y.R(x,y)$'. This reading emphasizes that the modal operator depends on i and j and it is alternative to 'i and j stand in the relation $\lambda x\lambda y.\Box R(x,y)$'.

Some examples:

$|x\,y|G(x)$: it is necessary for x and y that x gets a job.

$|x\,y\,z|G(x)$: it is necessary for x, y and z that x gets a job.

$|\,{}^{m}_{x}\;{}^{j}_{y}\,|G(x)$: it is necessary for Mary and John that she gets a job.

$|x,y|\langle y\rangle\exists wF(y,w)$: it is necessary for x and y that it is possible for y to have a friend.

$|x,y|\exists w\langle y,w\rangle F(y,w)$: it is necessary for x and y that there is someone of whom y is possibly a friend.

2 A language with indexed modalities

A language $\mathcal{L}$ with indexed modalities is a standard first-order language with identity whose logical symbols are $\bot$, $\to$, $\forall$, $|\,{}^{t_1}_{x_1}\ldots{}^{t_n}_{x_n}\,|$, $n\geq 0$, where $x_1,\ldots,x_n$ are pairwise distinct variables and $t_1,\ldots,t_n$ are terms. When $n=0$ we write $|\star|$.

Definition 1 Well-formed formulas and free variables occurring in a wff A, $fv(A)$.

- $\bot$ — $fv(\bot)=\emptyset$
- $P^n(t_1,\ldots,t_n)$ — $fv(P^n(t_1,\ldots,t_n))=fv(t_1)\cup\cdots\cup fv(t_n)$
- $A\to B$ — $fv(A\to B)=fv(A)\cup fv(B)$
- $|\,{}^{t_1}_{x_1}\ldots{}^{t_n}_{x_n}\,|A$, *where* $fv(A)\subseteq\{x_1,\ldots,x_n\}$ — $fv(|\,{}^{t_1}_{x_1}\ldots{}^{t_n}_{x_n}\,|A)=fv(t_1)\cup\cdots\cup fv(t_n)$
- $\forall xA$ — $fv(\forall xA)=fv(A)-\{x\}$

$\neg A$, $A \vee B$, $A \wedge B$, $A \leftrightarrow B$, $\exists x A$, $\langle {}^{t_1}_{x_1} \dots {}^{t_n}_{x_n} \rangle A$ are defined as usual, $|x_1 \dots x_n|A$ and $\langle x_1 \dots x_n \rangle A$ stand for $|{}^{x_1}_{x_1} \dots {}^{x_n}_{x_n}|A$ and $\langle {}^{x_1}_{x_1} \dots {}^{x_n}_{x_n} \rangle A$, respectively.

Advantages:

- *de re / de dicto* distinction

  $|{}^{i}_{x}|P(x)$ is a *de re* sentence, 'it is necessary for i to be $P(x)$', whereas $|\star|P(i)$ is a *de dicto* sentence, 'it is necessary that $P(i)$'.

- substitution

  As we shall see in a moment, $|{}^{t}_{x}|A$ is nothing but $(|x|A)[t/x]$; substitution is indicated inside the modality, it is not carried out in A. Substitution does not commute in general with modalities; actually, the modal operators prevent substitution from being performed in the formula that follows them.

- a richer language

  In a language with λ operator, $\lambda y(\lambda x \Box P(x).m).j$ is equivalent to $\lambda x \Box P(x).m$ by λ-conversion, whereas their corresponding wffs $|{}^{j}_{y}, {}^{m}_{x}|P(x)$ and $|{}^{m}_{x}|P(x)$ are not equivalent.

3 Transition semantics, t-semantics.

Given a *frame* $\mathcal{F} = \langle W, R \rangle$, where $W \neq \emptyset$ and $R \subseteq W^2$, a *system of domains* over $\mathcal{F}$[2] is a triple $\langle W, R, D \rangle$, where D is a function such that $D_w \neq \emptyset$, for each $w \in W$. D_w is said to be the *domain* of w. Domains are interrelated by the *transition relation*

$$\text{if} \quad wRv \quad \text{then} \quad \mathcal{T}_{\langle w,v \rangle} \subseteq D_w \times D_v$$

If $a\,\mathcal{T}_{\langle w,v \rangle}\, b$, then b is said to be an *inheritor* of a in v, or a *counterpart* of a in v.

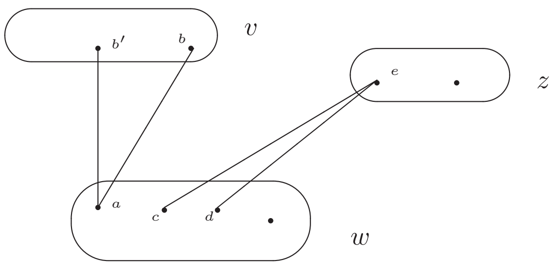

Figure 1:

Definition 2 *A* transition frame *or a* t-frame, $\mathcal{F}^t$, *is a quadruple* $\langle W, R, D, \mathcal{T} \rangle$ *where* $\langle W, R, D \rangle$ *is a system of domains and* $\mathcal{T} = \biguplus_{w,v \in W} \{\mathcal{T}_{\langle w,v \rangle}\}$, *where* $\mathcal{T}_{\langle w,v \rangle}$ *is defined as above.*

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[2] This terminology is taken from [6].

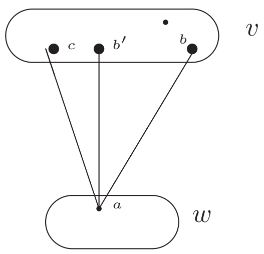
Particular cases of $\mathcal{T}$:

$\mathcal{T}$ is a	total	relation	Kripke bundles
	surjective	relation	
	partial	function	
	total	function	Kripke sheaves
	1-1	function	
	inclusion		Kripke frames with increasing domains

Definition 3 *A* t-model $\mathcal{M}$ *for* $\mathcal{L}$ *based on a t-frame* $\mathcal{F}^t = \langle W, R, D, \mathcal{T}\rangle$ *is a pair* $\langle \mathcal{F}^t, I\rangle$*, where* I *is a function such that for all* $w \in W$*,* I_w *is an* interpretation function relative to w *such that:*

- *for all relations* P^n*,* $I_w(P^n) \subseteq (D_w)^n$
- $I_w(=) = \{\langle a, a\rangle : a \in D_w\}$
- *for all constants* i*,* $I_w(i) \in D_w$
- *for all functions* f^n*,* $I_w(f^n) : (D_w)^n \rightarrow D_w$.

Rigid designators



In the context of a t-model, an individual constant i is a *rigid designator* iff if $I_w(i) = a$ and wRv, then $I_v(i)$ is one of the inheritors of a in v. In the above example, $I_v(i)$ is one among $\{c, b', b\}$.[3] So terms are rigid designators iff:

- if wRv then $I_w(i)\, \mathcal{T}_{\langle w,v\rangle}\, I_v(i)$
  and

- if $a_i\, \mathcal{T}_{\langle w,v\rangle}\, b_i, 1 \leq i \leq n,$ then $(I_w(f^n))(a_1, \ldots, a_n)\, \mathcal{T}_{\langle w,v\rangle}\, (I_v(f^n))(b_1, \ldots, b_n)$

Definition 4 Assignments *are world-relative functions* $\sigma : VAR \rightarrow D_w$*. Where* σ *is a* w*-assignment, by* $\sigma^{x \rhd d}$ *we denote the* w*-assignment that behaves exactly like* σ *except that the variable* x *is mapped to* $d \in D_w$*.*

Definition 5 Interpretation of terms. *Given a* w*-assignment* σ*, the* interpretation *of* t *in* w *under* σ*,* $I^\sigma_w(t)$*, is so defined*

- $I^\sigma_w(x) = \sigma(x)$
- $I^\sigma_w(i) = I_w(i)$
- $I^\sigma_w(f(t_1, \ldots, t_n)) = I_w(f)(I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n))$.

[3] Recall that in Kripke semantics for all constants i, if wRv then $I_w(i) = I_v(i)$.

When w and I are clear from the context, we write $\sigma(t)$ instead of $I_w^\sigma(t)$.

Definition 6 Simultaneous substitution for terms. *Given a term t containing the free variables $x_1, \ldots, x_k$, we define the term $t[s_1/x_1 \ldots s_k/x_k]$ where s_i is substituted for x_i, $1 \leq i \leq k$. Let $[\mathbf{s}/\mathbf{x}] =_{df} [s_1/x_1 \ldots s_k/x_k]$.*

- $t = y$

$$y[\mathbf{s}/\mathbf{x}] = \begin{cases} y & \text{if } y \neq x_i, \text{ for all } i, 1 \leq i \leq k \\ s_i & \text{if } y = x_i \text{ for some } i, 1 \leq i \leq k \end{cases}$$

- $t = i$

$$i[\mathbf{s}/\mathbf{x}] = i$$

- $t = f(t_1, \ldots, t_n)$

$$f(t_1, \ldots, t_n)[\mathbf{s}/\mathbf{x}] = f(t_1[\mathbf{s}/\mathbf{x}], \ldots, t_n[\mathbf{s}/\mathbf{x}])$$

Lemma 1 Interpretation and substitution for terms. *Let t and s be terms and σ be a w-assignment. Then*

$$\sigma(t[s/x]) \quad = \quad \sigma^{x \rhd \sigma(s)}(t)$$

If z doesn't not occur in t,

$$\sigma^{z \rhd a}(t[z/x]) \quad = \quad \sigma^{x \rhd a}(t)$$

Proof By induction on t.

$$\begin{array}{ll} t = x & \sigma^{x \rhd \sigma(s)}(x) = \sigma(s) = \sigma(x[s/x]) \\ t = z \neq x & \sigma^{x \rhd \sigma(s)}(z) = \sigma(z) = \sigma(z[s/x]) \\ t = i & \sigma^{x \rhd \sigma(s)}(i) = \sigma(i) = \sigma(i[s/x]) \\ t = f(t_1, \ldots, t_n) & \sigma^{x \rhd \sigma(s)}(f(t_1, \ldots, t_n)) \quad = \\ & (I_w(f))(\sigma^{x \rhd \sigma(s)}(t_1), \ldots, \sigma^{x \rhd \sigma(s)}(t_n))) \quad = \\ & (I_w(f))(\sigma(t_1[s/x]), \ldots, \sigma(t_n[s/x])) \quad = \\ & \sigma(f(t_1[s/x], \ldots, t_n[s/x])) = \sigma(f(t_1, \ldots, t_n)[s/x]) \end{array}$$

Let z not occur in t. $\sigma^{z \rhd a}(t[z/x]) = \sigma^{z \rhd a, x \rhd \sigma^{z \rhd a}(z)}(t) = \sigma^{z \rhd a, x \rhd a}(t) = \sigma^{x \rhd a}(t)$, since z doesn't occur in t. ♠

Definition 7 Satisfaction for formulas. *We define when a wff A is satisfied at w under σ in a t-model $\mathcal{M}$, $\sigma \models_w^{\mathcal{M}} A$.*

$\sigma \not\models^{\mathcal{M}}_{w} \bot$

$\sigma \models^{\mathcal{M}}_{w} P^k(t_1 \ldots t_k)$ *iff* $\langle \sigma(t_i), \ldots, \sigma(t_k) \rangle \in I_w(P^k)$

$\sigma \models^{\mathcal{M}}_{w} B \to G$ *iff* $\sigma \not\models^{\mathcal{M}}_{w} B$ *or* $\sigma \models^{\mathcal{M}}_{w} G$

$\sigma \models^{\mathcal{M}}_{w} \forall x G$ *iff* *for all* $d \in D_w$, $\sigma^{x \rhd d} \models^{\mathcal{M}}_{w} G$

$\sigma \models^{\mathcal{M}}_{w} |{}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n}| G$ *iff* *for all* v, wRv *and all* v*-assignments* τ*, such that* $\sigma(t_i)\, \mathcal{T}_{\langle w,v \rangle}\, \tau(x_i)$, $1 \leq i \leq n$, $\tau \models^{\mathcal{M}}_{v} G$

Consequently,

$\sigma \models^{\mathcal{M}}_{w} \langle {}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n} \rangle G$ *iff* *for some* v, wRv *and for some* v*-assignment* τ *such that* $\sigma(t_i)\, \mathcal{T}_{\langle w,v \rangle}\, \tau(x_i)$, $1 \leq i \leq n$, $\tau \models^{\mathcal{M}}_{v} G$

When no ambiguity can arise, we write $\sigma \models_w A$ instead of $\sigma \models^{\mathcal{M}}_{w} A$.

A is *true at* w *in* $\mathcal{M}$, $\models^{\mathcal{M}}_{w} A$, iff for all w-assignments σ, $\sigma \models^{\mathcal{M}}_{w} A$.
A is *true in* $\mathcal{M}$, $\models^{\mathcal{M}} A$, iff $\models^{\mathcal{M}}_{w} A$ for all $w \in W$.
A is *valid on a* t*-frame* $\mathcal{F}^t$, $\mathcal{F}^t \models A$, iff $\models^{\mathcal{M}} A$ for all models $\mathcal{M}$ based on $\mathcal{F}^t$.
A is t*-valid*, $t \models A$, iff $\mathcal{F}^t \models A$, for all t-frames $\mathcal{F}^t$.

An idea which is at the basis of the above definition of satisfaction is that only the worlds where an individual exist or its inheritors exist do matter in order to establish its modal properties, for

$$\sigma \models^{\mathcal{M}}_{w} |{}^{i}_{x}| P(x)$$

iff all the inheritors of $\sigma(i)$ in all related worlds satisfy $P(x)$. Worlds where there are no inheritors of $\sigma(i)$ are not taken into consideration. It turns out that inferences such as

$$\frac{\sigma \models_w |x\, y| Q(x,y) \qquad \sigma \models_w |x\, y| (Q(x,y) \to A(y))}{\sigma \models_w |y| A(y)}$$

are not valid. Let $\sigma(x) = a$ and $\sigma(y) = b$. Suppose it is true in w that "a always quarrels with b" and that "every time that a quarrels with b, then b gets angry", but from this it doesn't follows that "b is always angry", for b may not be angry in those worlds where a is absent.[4]

***de re* vs *de dicto* modalities**

There is an intuitive sense according to which the truth conditions for $|{}^{i}_{x}| P(x)$ are different from those for $|\star| P(i)$: in one case it is said that "it is necessary for i to have the property $P(x)$", in the other, the necessity of a sentence is asserted. In transition semantics we do justice to this difference in the following obvious way: in the first case, first we interpret i in the actual world (or the world we are in) and then we see if all its inheritors in all accessible worlds (where they exist) do satisfy the property $P(x)$, in the second case, we first consider all worlds accessible from the actual one and then check if the interpretation of i in those worlds satisfies

[4] See [2, p.12]

$P(x)$. This semantical analysis parallels that of Fitting, [5], p.114: "In short, there are two basic actions: letting i designate, and moving to an alternative world. These two actions commute only if i is a rigid designator. Ordinary first-order modal syntax has no machinery to distinguish the two alternative readings of $\Box P(i)$. Consequently when non-rigid designators have been treated at all, one of the readings has been disallowed, thus curtailing expressive power.". According to Fitting, if i is a rigid designator then $|^i_x|P(x) \leftrightarrow \Box P(i)$ holds, or, in his notation, the equivalence $\lambda x(\Box P(x))(i) \leftrightarrow \Box[\lambda x.P(x)(i)]$ holds. We are going to disagree on this point, for we shall show that the failure of the equivalence $|^i_x|P(x) \leftrightarrow |\star|P(i)$ does not depend on i being a non-rigid designator: in transition semantics this equivalence does not hold for rigid designators either.

Rigid designators
If i is a rigid designator, the implication $|^i_x|P(x) \to |\star|P(i)$ is t-valid, whereas $|\star|P(i) \to |^i_x|P(x)$ admits of countermodels. The t-validity of $|^i_x|P(x) \to |\star|P(i)$ is shown as follows. $\sigma \models_w |^i_x|P(x)$ iff for all v such that wRv and for all v-assignment τ such that $\tau(x)$ is an inheritor of $I_w(i)$ in D_v, $\tau \models_v P(x)$. Since i is a rigid designator, $I_v(i)$ is one of the inheritors of $I_w(i)$, therefore if τ is such that $\tau(x) = I_v(i)$, then $\tau(x) \in I_v(P)$, hence $I_v(i) \in I_v(P)$, so for all v, wRv and all v-assignment τ, $\tau \models_v P(i)$, consequently $\sigma \models_w |\star|P(i)$.

A countermodel for $|\star|P(i) \to |^i_x|P(x)$ can be readily constructed: assume that v is the only world related to w and that $I_v(i) \in I_v(P)$, so for all $v.wRv$. and all v-assignment τ, $\tau \models_v P(i)$, therefore $\sigma \models_w |\star|P(i)$. Assume moreover that $I_w(i)$ has two distinct inheritors in v, namely $I_v(i)$ and c and that $c \notin I_v(P)$, consequently there is a v-assignment τ such that $\tau(x) = c$ hence $\tau \not\models_v P(x)$, and so $\sigma \not\models_w |^i_x|P(x)$.

Variables are rigid designators so, in particular

$$|^{y}_{x_1} \dots {}^{y}_{x_n}|\, P(x_1, \dots, x_n) \to |y|(P(y, \dots, y)$$

is t-valid.

Let wRv and τ be a v-assignment. If $\sigma \models_w |^{y}_{x_1} \dots {}^{y}_{x_n}|\, P(x_1, \dots, x_n)$, then $P(x_1, \dots, x_n)$ is satisfied in v by any n-tuple of inheritors of $\sigma(y)$, therefore it is satisfied in v by the n-tuple $\langle \tau(y), \dots, \tau(y) \rangle$, for some particular inheritor $\tau(y)$.

Stable designators
The validity of $|\star|P(i) \to |^i_x|P(x)$ requires the assumption that $I_w(i)$ has at most one inheritor in any related world v and that the inheritor (if any) in v of $I_w(i)$ coincides with $I_v(i)$.

An individual constant i is *stable* iff

- if wRv and $I_w(i)\, \mathcal{T}_{\langle w,v \rangle}\, c$ then $I_v(i) = c$.

If an individual constant i is stable, then in particular

$$|x_1 \dots x_n|A(x_1, \dots, x_n, i) \to |x_1 \dots x_n {}^i_x|A(x_1, \dots, x_n, x)$$

is t-valid.

Definition 8 Simultaneous substitution for formulas. *Given a wff A containing the free variables $x_1, \dots, x_k$, we define the wff $A[s_1/x_1 \dots s_k/x_k]$ where s_i is substituted for x_i, $1 \le i \le k$. Let* $[\mathbf{s}/\mathbf{x}] =_{df} [s_1/x_1 \dots s_k/x_k]$.

- $\perp [\mathbf{s}/\mathbf{x}] = \perp$
- $(P^n t_1, \ldots, t_n)[\mathbf{s}/\mathbf{x}] = P^n t_1[\mathbf{s}/\mathbf{x}], \ldots, t_n[\mathbf{s}/\mathbf{x}]$
- $(A \to B)[\mathbf{s}/\mathbf{x}] = (A[\mathbf{s}/\mathbf{x}] \to B[\mathbf{s}/\mathbf{x}])$
- $(\forall y A)[\mathbf{s}/\mathbf{x}] =$

$$= \begin{cases} \forall yA & \textit{if } y \in \mathbf{x} \\ \forall z((A[z/y])[s/x]) & \textit{if } y \notin \mathbf{x} \textit{ and } y \in \mathbf{s} \\ & \textit{where } z \textit{ doesn't occur in } \forall yA \\ \forall y(A[s/x]) & \textit{if } y \notin \mathbf{x} \textit{ and } y \notin \mathbf{s} \end{cases}$$

- $(|\,{}^{t_1}_{y_1} \ldots {}^{t_n}_{y_n}\,|A)[\mathbf{s}/\mathbf{x}] = |\,{}^{t_1[\mathbf{s}/\mathbf{x}]}_{y_1} \ldots {}^{t_n[\mathbf{s}/\mathbf{x}]}_{y_n}\,|A$, *in particular*
- $(|\,x_1 \ldots x_k\,|A)[s_1/x_1 \ldots s_k/x_k] = |\,{}^{s_1}_{x_1} \ldots {}^{s_k}_{x_k}\,|A$

Lemma 2 *Let A be a wff and z a variable that doesn't occur in A. For all t-models $\mathcal{M}$ and w-assignments σ,*

$$\sigma^{x \rhd a} \models_w A \qquad iff \qquad \sigma^{z \rhd a} \models_w A[z/x].$$

Proof By induction on A.
$\sigma^{z \rhd a} \models_w P(t_1, \ldots, t_n)[z/x]$ iff $\sigma^{z \rhd a} \models_w P(t_1[z/x], \ldots, t_n[z/x])$ iff $\langle \sigma^{z \rhd a}(t_1[z/x]), \ldots, \sigma^{z \rhd a}(t_n[z/x]) \rangle \in I_w(P)$ iff by lemma 1, $\langle \sigma^{x \rhd a}(t_1), \ldots, \sigma^{x \rhd a}(t_n) \rangle \in I_w(P)$ iff $\sigma^{x \rhd a} \models_w P(t_1, \ldots, t_n)$.

$\sigma^{z \rhd a} \models_w (|\,{}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n}\,|A)[z/x]$ iff $\sigma^{z \rhd a} \models_w |\,{}^{t_1[z/x]}_{x_1} \ldots {}^{t_n[z/x]}_{x_n}\,|A$ iff $\tau \models_v A$, where $\sigma^{z \rhd a}(t_1[z/x])\,\mathcal{T}\,\tau(x_1) \ldots \sigma^{z \rhd a}(t_n[z/x])\,\mathcal{T}\,\tau(x_n)$, therefore, by lemma 1, $\sigma^{x \rhd a}(t_1)\,\mathcal{T}\,\tau(x_1) \ldots \sigma^{x \rhd a}(t_n)\,\mathcal{T}\,\tau(x_n)$, so $\sigma^{x \rhd a} \models_w |\,{}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n}\,|A$ ♠

Lemma 3 *(Alphabetic change of bound variables) Let A be a wff and z be a variable not occurring in A.*

$$\sigma \models_w \forall xA \qquad iff \qquad \sigma \models_w \forall z(A[z/x])$$

Proof $\sigma \models_w \forall xA$ iff $\sigma^{x \rhd a} \models_w A$ for all $a \in D_w$ iff (by lemma 2) $\sigma^{z \rhd a} \models_w A[z/x]$ for all $a \in D_w$ iff $\sigma \models_w \forall z(A[z/x])$. ♠

Lemma 4 Substitution and satisfaction for formulas. *Let σ be a w-assignment.*

$$\sigma \models_w A[s/x] \qquad iff \qquad \sigma^{x \rhd \sigma(s)} \models_w A$$

Proof By induction on A.

- $A = P^n(t_1, \ldots, t_n)$
  $\sigma^{x \rhd \sigma(s)} \models_w P^n(t_1, \ldots, t_n)$ iff $\langle \sigma^{x \rhd \sigma(s)}(t_1), \ldots, \sigma^{x \rhd \sigma(s)}(t_n) \rangle \in I_w(P^n)$ iff $\langle \sigma(t_1[s/x]), \ldots, \sigma(t_n[s/x]) \rangle \in I_w(P^n)$ iff $\sigma \models_w P^n(t_1[s/x], \ldots, t_n[s/x])$ iff $\sigma \models_w P^n(t_1, \ldots, t_n)[s/x]$.

- $A = \forall yB$

  $\sigma^{x\rhd\sigma(s)} \models_w \forall yB$ iff for all $d \in D_w$, $\sigma^{x\rhd\sigma(s),y\rhd d} \models_w B$ iff for all $d \in D_w$, $\sigma^{y\rhd d,x\rhd\sigma(s)} \models_w B$ iff, by induction hypothesis, for all $d \in D_w$, $\sigma^{y\rhd d} \models_w B[s/x]$ iff $\sigma \models_w \forall y(B[s/x])$ iff by def. of substitution $\sigma \models_w (\forall yB)[s/x]$.

- $A = |{}^{t_1}_{y_1} \dots {}^{t_n}_{y_n}|B$

  $\sigma^{x\rhd\sigma(s)} \models_w |{}^{t_1}_{y_1} \dots {}^{t_n}_{y_n}|B$ iff for all v-assignment τ such that $\sigma^{x\rhd\sigma(s)}(t_i)\,\mathcal{T}_{\langle w,v\rangle}\,\tau(y_i)$, $1 \leq i \leq n$, $\tau \models_v B$ iff for all v-assignment τ such that $\sigma(t_i[s/x])\,\mathcal{T}_{\langle w,v\rangle}\,\tau(y_i)$, $1 \leq i \leq n$, $\sigma \models_v |{}^{t_1[s/x]}_{y_1} \dots {}^{t_n[s/x]}_{y_n}|B$ iff $\sigma \models_v (|{}^{t_1}_{y_1} \dots {}^{t_n}_{y_n}|B)[s/x]$.

♠

3.1 Relevant formulas

PRM (Permutation)

$$|x_1 \dots x_n|A \leftrightarrow |x_{i_1} \dots x_{i_n}|A$$

for any permutation $x_{i_1} \dots x_{i_n}$ of $x_1 \dots x_n$.

RG (Rigidity of terms)

$$|{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}|A \to |v_1 \dots v_k|(A[t_1/x_1 \dots t_n/x_n])$$

where $v_1 \dots v_k$ are all the variables occurring in $t_1 \dots t_n$.

RG^v (Rigidity of variables)

$$|{}^{y_1}_{x_1} \dots {}^{y_n}_{x_n}|A \to |y_1 \dots y_k|(A[y_1/x_1 \dots y_n/x_n])$$

where $y_1 \dots y_k$ are the variables $y_1 \dots y_n$ without repetitions.

RNM (Renaming)

$$|x_1 \dots x_n|A(x_1 \dots x_n) \to |{}^{x_1}_{y_1} \dots {}^{x_n}_{y_n}|(A[y_1/x_1 \dots y_n/x_n])$$

where $y_1 \dots y_n$ are pairwise distinct variables.

BF (Barcan Formula)

$$\forall z|x_1 \dots x_n, z|A \to |x_1 \dots x_n|\forall zA$$

CBF (Converse of Barcan Formula)

$$|x_1 \dots x_n|\forall zA \to \forall z|x_1 \dots x_n, z|A$$

GF (Ghilardi Formula)

$$\exists z|x_1 \dots x_n, z|A \to |x_1 \dots x_n|\exists zA$$

CGF (Converse of Ghilardi Formula)

$$|x_1 \dots x_n| \exists z A \to \exists z |x_1 \dots x_n, z| A$$

SHRT (Shortening)

$$|x_1 \dots x_n, z| A \to |x_1 \dots x_n|\, A$$

LNGT (Lenghtening)

$$|x_1 \dots x_n|\, A \to |x_1 \dots x_n, z| A$$

CRG (Converse of RG)

$$|v_1 \dots v_k|(A[t_1/x_1 \dots t_n/x_n]) \to |\,{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}\,| A$$

where $v_1, \dots, v_k$ are all the variables occurring in $t_1, \dots, t_n$.

CRG^v (Converse of RG^v)

$$|y_1 \dots y_k|(A[y_1/x_1 \dots y_n/x_n]) \to |\,{}^{y_1}_{x_1} \dots {}^{y_n}_{x_n}\,| A$$

where $y_1 \dots y_k$ are the variables $y_1 \dots y_n$ without repetitions.

SIV (Substitution that Identifies Variables)

$$|v_1 \dots v_k|(A[y/x_1, y/x_2, t_3/x_3 \dots t_n/x_n]) \to |\,{}^{y}_{x_1}\,{}^{y}_{x_2}\,{}^{t_3}_{x_3} \dots {}^{t_n}_{x_n}\,| A$$

where $v_1, \dots, v_k$ are all the variables occurring in $y, t_3, \dots, t_n$.

FCS (Full Commutativity of Substitution)

$$|v_1 \dots v_k|(A[t_1/x_1 \dots t_n/x_n]) \leftrightarrow |\,{}^{t_1}_{x_1} \dots {}^{t_n}_{x_n}\,| A$$

where $v_1, \dots, v_k$ *include* all the variables occurring in $t_1, \dots, t_n$.

NI (Necessity of Identity)

$$x = y \to |x, y|(x = y)$$

ND (Necessity of Distinction)

$$x \neq y \to |x, y|(x \neq y)$$

LBZ (Leibniz's law)

$$t = s \to (A[t/x] \to A[s/x])$$

4 A quantified modal logic with indexed modalities: $Q.K_{im}$.

Axioms

	Tautologies
PRM	$\|x_1 \dots x_n\|A \leftrightarrow \|x_{i_1} \dots x_{i_n}\|A$ for any permutation $x_{i_1} \dots x_{i_n}$ of $x_1 \dots x_n$
K	$\|x_1 \dots x_n\|(A \to B) \to (\|x_1 \dots x_n\|A \to \|x_1 \dots x_n\|B)$
UI	$\forall x A \to A$
LNGT	$\|x_1 \dots x_n\| A \to \|x_1 \dots x_n, z\|A$
RG^v	$\|{}^{y_1}_{x_1} \dots {}^{y_n}_{x_n}\|A \to \|y_1 \dots y_k\|(A[y_1/x_1 \dots y_n/x_n])$ where $y_1 \dots y_k$ are the variables $y_1 \dots y_n$ without repetitions.[5]
ID	$x = x$
LBZ	$t = s \to (A[t/x] \to A[s/x])$

Inference rules

$$\frac{A \quad A \to B}{B} \quad \textit{Modus Ponens}\,(\mathrm{MP})$$

$$\frac{A}{|x_1 \dots x_n|A} \quad \textit{Necessitation}\,(\mathrm{N}), \text{provided } \{x_1, \dots, x_n\} \supseteq fv(A).$$

$$\frac{A \to B}{A \to \forall x B} \quad \textit{Universal Generalization}\,(\mathrm{UG}), \text{provided } x \notin fv(A).$$

$$\frac{A}{A[s/x]} \quad \textit{Substitution for Free Variables}\,(\mathrm{SFV})$$

Theorem 1 *(Soundness.) Every theorem of $Q.K_{im}$ is t-valid. Every theorem of $R.K_{im} = Q.K_{im} + RG$ is true in all t-models with rigid designators based on any t-frame.*

Some derivations

[5]Axiom $\mathbf{RG}^v$ could be formulated in a more general form so as to imply axiom **LNGT**: $|{}^{y_1}_{x_1} \dots {}^{y_n}_{x_n}|A \to |v_1 \dots v_k|(A[y_1/x_1 \dots y_n/x_n])$, where $v_1 \dots v_k$ *include* all the different variables among $y_1 \dots y_n$.

$Q.K_{im} \vdash$ RNM

$|^{y_1}_{x_1} \dots {}^{y_n}_{x_n}|A \rightarrow |y_1 \dots y_n|(A[y_1/x_1 \dots y_n/x_n])$ RG^v

$(|^{y_1}_{x_1} \dots {}^{y_n}_{x_n}|A)[x_1 \dots x_n/y_1 \dots y_n] \rightarrow (|y_1 \dots y_n|(A[y_1/x_1 \dots y_n/x_n]))[x_1 \dots x_n/y_1 \dots y_n]$

$|x_1 \dots x_n|A \rightarrow |^{x_1}_{y_1} \dots {}^{x_n}_{y_n}|(A[y_1/x_1 \dots y_n/x_n])$
where $y_1 \dots y_n$ are pairwise distinct variables not occurring in A.

$Q.K_{im} + LNGT \vdash CBF$

$\forall x A(\vec{x}, x) \rightarrow A(\vec{x}, x)$ UI

$|\vec{x}, x|\forall x A(\vec{x}, x) \rightarrow |\vec{x}, x|A(\vec{x}, x)$ N

$|\vec{x}|\forall x A(\vec{x}, x) \rightarrow |\vec{x}, x|A(\vec{x}, x)$ LNGT

$|\vec{x}|\forall x A(\vec{x}, x) \rightarrow \forall x|\vec{x}, x|A(\vec{x}, x)$ UG

$Q.K_{im} + CBF \vdash LNGT$

$A(\vec{x}) \rightarrow A(\vec{x})$ ID

$A(\vec{x}) \rightarrow \forall x A(\vec{x}) \qquad x \notin A$ UG

$|\vec{x}|A(\vec{x}) \rightarrow |\vec{x}|\forall x A(\vec{x})$ N

$|\vec{x}|A(\vec{x}) \rightarrow \forall x|\vec{x}, x|A(\vec{x})$ CBF

$|\vec{x}|A(\vec{x}) \rightarrow |\vec{x}, x|A(\vec{x})$ UI

$Q.K_{im} + SHRT \vdash GF$

$A(\vec{x}, x) \rightarrow \exists x A(\vec{x}, x)$

$|\vec{x}, x|A(\vec{x}, x) \rightarrow |\vec{x}, x|\exists x A(\vec{x}, x)$ N

$|\vec{x}, x|A(\vec{x}, x) \rightarrow |\vec{x}|\exists x A(\vec{x}, x)$ SHRT

$\exists x|\vec{x}, x|A(\vec{x}, x) \rightarrow |\vec{x}|\exists x A(\vec{x}, x)$

$Q.K_{im} + GF \vdash SHRT$

$\neg A \rightarrow \neg A$

$\neg A \rightarrow \forall x \neg A \qquad x \notin A$

$\exists x A(\vec{x}) \rightarrow A(\vec{x})$

$|\vec{x}|\exists x A(\vec{x}) \rightarrow |\vec{x}|A(\vec{x})$ N

$\exists x|\vec{x}, x|A(\vec{x}) \rightarrow |\vec{x}|A(\vec{x})$ GF

$|\vec{x}, x|A(\vec{x}) \rightarrow \exists x|\vec{x}, x|A(\vec{x})$ from UI

$|\vec{x}, x|A(\vec{x}) \rightarrow |\vec{x}|A(\vec{x})$ trans.

$Q.K_{im} + SIV \vdash NI$

$x = x$ ID

$|x|(x = x)$ N

$|x|((x = y)[x/x, x/y]) \to |^{x,\, x}_{x,\, y}|(x = y)$ SIV

$|x|(x = x) \to |^{x,\, x}_{x,\, y}|(x = y)$

$|^{x,\, x}_{x,\, y}|(x = y)$ MP

$|^{x,\, x}_{x,\, y}|(x = y) \to (x = y \to |^{x,\, y}_{x,\, y}|(x = y))$ LBZ

$x = y \to |x, y|(x = y)$ MP

$Q.K_{im} + NI \vdash SIV$

Let $B(x, y)$ be given, and, for simplicity's sake, let us assume it to be atomic.

$x = y \to (B(x, x) \to B(x, y))$ LBZ

$|x\, y|(x = y) \to (|x\, y|B(x, x) \to |x\, y|B(x, y))$ N

$(x = y) \to |x\, y|(x = y)$ NI

$(x = y) \to (|x\, y|B(x, x) \to |x\, y|B(x, y))$

$(x = y)[x/x, x/y] \to (|x\, y|(B(x, x))[x/x, x/y] \to$ SFV

$(|x\, y|B(x, y))[x/x, x/y])$

$(x = x) \to (|^{x\ x}_{x\ y}|B(x, x) \to |^{x\ x}_{x\ y}|B(x, y))$

$(x = x)$ ID

$|^{x\ x}_{x\ y}|B(x, x) \to |^{x\ x}_{x\ y}|B(x, y)$ MP

$|x|B(x, x) \to |x\ y|B(x, x)$ LNGT

$|^{x}_{x}|B(x, x) \to |^{x\ x}_{x\ y}|B(x, x)$ SFV

$|^{x}_{x}|B(x, x) \to |^{x\ x}_{x\ y}|B(x, y)$ trans.

$Q.K_{im} + CRG \vdash SIV$

$|v_1 \dots v_k|(A[y/x_1, y/x_2, t_3/x_3 \dots t_n/x_n]) \to |\, {}^{y}_{x_1}\ {}^{y}_{x_2}\ {}^{t_3}_{x_3} \dots {}^{t_n}_{x_n}\, |A$ CRG

where $v_1, \dots, v_k$ are all the variables occurring in $y, t_3, \dots, t_n$

SIV is a particular case of CRG, exactly when $t_1 = t_2 = y$ so the same variable y is substituted for x_1 and x_2.

$Q.K_{im} + FCS \vdash LNGT$

$|^{x_1}_{x_1} \dots {}^{x_n}_{x_n}|A \to |x_1 \dots x_n, z|(A[x_1/x_1 \dots x_n/x_n])$ FCS

where $x_1, \dots, x_n, z$ *include* all the variables among $x_1, \dots, x_n$

$|x_1 \dots x_n|A \to |x_1 \dots x_n, z|A$

$Q.K_{im} + FCS \vdash SHRT$

$|x_1 \ldots x_n, z|(A[x_1/x_1 \ldots x_n/x_n]) \to |^{x_1}_{x_1} \ldots ^{x_n}_{x_n}|A$ FCS

where $x_1, \ldots, x_n, z$ *include* all the variables among $x_1, \ldots, x_n$

$|x_1 \ldots x_n, z|A \to |x_1 \ldots x_n|A$

Trivially, $Q.K_{im} + FCS \vdash RG, CRG, NI$. In the presence of the principle of full commutativity of substitution, indexed modalities are unnecessary, in fact every box-operator can be thought of as implicitely indexed by the variables of the formula that follows it. This yelds that the standard modal language will do, but, as we shall see, we are confined to t-frames where the transition relation is a totally defined function. See [3].

A Quinean sentence: 'Necessarily the number of planets is greater than 7.'

Let i denote 'the number of planets'. Then, according to Quine the following derivation:

1. $\Box(7 < 9)$
2. $i = 9$
3. $\Box(7 < i)$

transforms the truth $\Box(7 < 9)$ into the falsehood $\Box(7 < i)$. We want to point out that the conclusion is not obtained merely by an application of the substitution of identical terms, but rather it relies on the acceptance of strong principles about substitution. The above inference can be analyzed in a language with indexed modalities as follows:

$$\dfrac{i = 9 \qquad \dfrac{\dfrac{|\star|(7<9)}{|^{7}_{x}, ^{9}_{y}|(x<y)}\,CRG \qquad \dfrac{}{|^{7}_{x}, ^{9}_{y}|(x<y) \to (i = 9 \to |^{7}_{x}, ^{i}_{y}|(x<y))}\,LBZ}{i = 9 \to |^{7}_{x}, ^{i}_{y}|(x<y)}\,MP}{\dfrac{|^{7}_{x}, ^{i}_{y}|(x<y)}{|\star|(7<i)}\,RG}\,MP$$

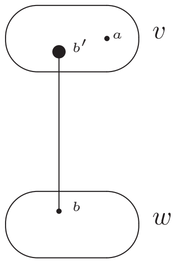
Even if we can accept that 9 and 7 are stable designators and so CRG holds for them, i can hardly be called a rigid designator.

5 Correspondence

$\boxed{BF}$

$$\mathcal{F}^t \models \forall x|x_1 \ldots x_n\, x|A \to |x_1 \ldots x_n|\forall x A \qquad \text{iff} \qquad \mathcal{T} \text{ is surjective.}$$

We show that if $\mathcal{T}$ is not surjective then $\mathcal{F}^t \not\models \forall x|x_1 \ldots x_n\, x|A \to |x_1 \ldots x_n|\forall x A$, where $\mathcal{T}$ is *surjective* iff for all w, v, if $b \in D_v$ then there is an $a \in D_w$ such that $a\mathcal{T}_{\langle w,v \rangle}b$.

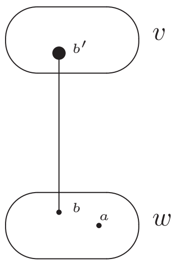


Let $a \notin I_v(P)$, $b' \in I_v(P)$ and $\sigma(x) = a$. Then $\sigma \models_w \forall x|x|P(x)$, $\tau \not\models_v \forall x P(x)$, $\sigma \not\models_w \forall x|x|P(x) \rightarrow |\star|\forall x P(x)$.

$\boxed{GF}$

$$\mathcal{F}^t \models \exists x|x_1 \dots x_n\, x|A \rightarrow |x_1 \dots x_n|\exists x A \qquad \text{iff} \qquad \mathcal{T} \text{ is totally defined.}$$

We show that if $\mathcal{T}$ is not totally defined then $\mathcal{F}^t \not\models \exists x|x_1 \dots x_n\, x|A \rightarrow |x_1 \dots x_n|\exists x A$, where $\mathcal{T}$ is *totally defined* iff for all w, v, if $a \in D_w$ then there is an $b \in D_v$ such that $a\mathcal{T}_{\langle w,v\rangle}b$.

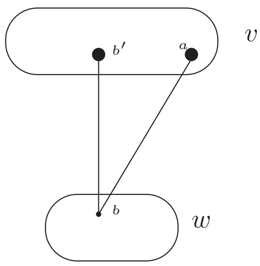


Let $b' \notin I_v(P)$ and $\sigma(x) = a$. Then $\sigma \models_w |x|P(x)$, so $\sigma \models_w \exists x|x|P(x)$, therefore $\sigma \not\models_w \exists x|x|P(x) \rightarrow |\star|\exists x P(x)$.

$\boxed{NI}$

$$\mathcal{F}^t \models x = y \rightarrow |x\, y|(x = y) \qquad \text{iff} \qquad \mathcal{T} \text{ is a partial function.}$$

We show that if $\mathcal{T}$ is not a partial function then $\mathcal{F}^t \not\models x = y \rightarrow |x\, y|(x = y)$, where $\mathcal{T}$ is a *partial function* if for all w, v, if $a\mathcal{T}_{\langle w,v\rangle}b$ and $a\mathcal{T}_{\langle w,v\rangle}c$ then $b = c$.

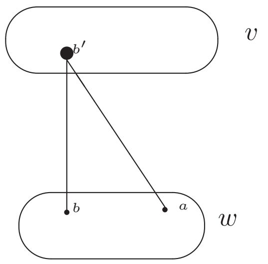


Let $\sigma(x) = \sigma(y) = b$. Then $\sigma \models_w x = y$, but $\sigma \not\models_w |x, y|(x = y)$, so $\sigma \not\models_w x = y \rightarrow |x\, y|(x = y)$.

$\boxed{ND}$

$$\mathcal{F}^t \models x \neq y \rightarrow |x\, y|(x \neq y) \qquad \text{iff} \qquad \mathcal{T} \text{ is not convergent.}$$

We show that if $\mathcal{T}$ is convergent then $\mathcal{F}^t \not\models x \neq y \rightarrow |x\, y|(x \neq y)$, where $\mathcal{T}$ is *not convergent* iff for all w, v, if $a\mathcal{T}_{\langle w,v\rangle}c$ and $b\mathcal{T}_{\langle w,v\rangle}c$ then $a = b$.



$\boxed{FCS}$

$$\mathcal{F}^t \models |v_1 \ldots v_k|(A[t_1/x_1 \ldots t_n/x_n]) \leftrightarrow |\,{}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n}\,|A$$
$$\text{iff}$$
$$\mathcal{T} \text{ is a totally defined function,}$$

where $v_1, \ldots, v_k$ *include* all the variables occurring in $t_1, \ldots, t_n$.

6 Completeness theorem for $R.K_{im}$

We start by considering the modal logic $R.K_{im} = Q.K_{im} + RG$, where RG is the axiom of rigidity of terms.

6.1 Preliminaries

First we define a classical first-order language $\mathcal{L}^c$ that mimics the modal language $\mathcal{L}$.[6]

- $\mathcal{L}^c$ contains all the predicate and function symbols of $\mathcal{L}$,
- for each wff of $\mathcal{L}$,

  $$|\, x_1 \ldots x_n\,|A$$

  $\mathcal{L}^c$ contains the n**-ary predicate symbol**

  $$P_{|\, x_1 \ldots x_n\,|A}$$

To every modal formula A of $\mathcal{L}$ we assign a classical formula $A^c \in \mathcal{L}^c$

$$\begin{array}{lcl}
(P^n t_1, \ldots, t_n)^c & = & P^n t_1, \ldots, t_n \\
(A \sharp B)^c & = & A^c \sharp B^c \\
(\forall x A)^c & = & \forall x (A^c) \\
(|\,{}^{t_1}_{x_1} \ldots {}^{t_n}_{x_n}\,|A)^c & = & P_{|\, x_1 \ldots x_n\,|A}(t_1 \ldots t_n)
\end{array}$$

[6]The proof we present here is based on Ghilardi’s completeness proof in [1]

We can easily see that if A contains no modal operators, then A^c is just A and that every formula B of $\mathcal{L}^c$ is equal to A^c for some $A \in \mathcal{L}$.

Second, we define the classical theory $C_{R.K_{im}}$ whose axioms are

$$\{A^c : R.K_{im} \vdash A\}$$

and whose inference rules are MP, UG and SFV.

Lemma 5 *$X \vdash_{R.K_{im}} A$ iff $X^c \vdash_{C_{R.K_{im}}} A^c$.*

Proof It is easy to see that $\vdash_{R.K_{im}} B_1 \wedge \cdots \wedge B_n \to A$ iff $\vdash_{C_{R.K_{im}}} B_1^c \wedge \cdots \wedge B_n^c \to A^c$, where $B_1, \ldots, B_n \in X$.

$\Rightarrow$ holds by definition of $C_{R.K_{im}}$.

$\Leftarrow$ holds because the axioms of $C_{R.K_{im}}$ are the c-translation of the theorems of $R.K_{im}$ and the inference rules of $C_{R.K_{im}}$ are also inference rules of $R.K_{im}$. ♠

$C_{R.K_{im}}$ is a first order theory, so models of $C_{R.K_{im}}$ are pairs $w = \langle D_w, I_w \rangle$ composed on a non-empty domain D_w and an interpretation function I_w such that the universal closure of all the theorems of $C_{R.K_{im}}$ is true in them.

We use the letters $w, v, \ldots$ to denote $C_{R.K_{im}}$ models. By $\langle \sigma, w \rangle \models A^c$ we denote that A^c is satisfied in the model $w = \langle D_w, I_w \rangle$ under the w-assignment σ.

An *admissible relation* $\mathcal{T}_{\langle w,v \rangle}$ among $C_{R.K_{im}}$-models w and v is a relation $\mathcal{T}_{\langle w,v \rangle} \subseteq D_w \times D_v$ satisfying the following two requirements

(A) for every term t, for every w-assignment π and for every v-assignment μ,

$$\text{if} \quad \pi(y_1)\, \mathcal{T}_{\langle w,v \rangle},\, \mu(y_1), \ldots, \pi(y_k)\, \mathcal{T}_{\langle w,v \rangle}\, \mu(y_k) \quad \text{then} \quad \pi(t)\, \mathcal{T}_{\langle w,v \rangle}\, \mu(t)$$

where t contains at most the variables $y_1, \ldots, y_k$.

(B) for every formula A of $\mathcal{L}$, for every w-assignment π and for every v-assignment μ,

$$\text{if} \quad \pi(y_1)\, \mathcal{T}_{\langle w,v \rangle}\, \mu(y_1), \ldots, \pi(y_k)\, \mathcal{T}_{\langle w,v \rangle}\, \mu(y_k),$$

then

$$\langle \pi, w \rangle \models P_{|y_1,\ldots,y_k|A}(y_1, \ldots, y_k) \quad \text{only if} \quad \langle \mu, v \rangle \models A^c,$$

where A contains at most the variables $y_1, \ldots, y_k$.

Lemma 6 *Let w, v be $C_{R.K_{im}}$-models and σ and τ be assignments in w and v, respectively. If for every formula A of $\mathcal{L}$ containing at most the variables $x_1, \ldots, x_n$,*

$$\langle \sigma, w \rangle \models P_{|x_1,\ldots,x_n|A}(x_1, \ldots, x_n) \quad \textit{only if} \quad \langle \tau, v \rangle \models A^c,$$

then there is an admissible relation $\mathcal{T}_{\langle w,v \rangle} \subseteq D_w \times D_v$ such that

$$\sigma(x_1)\mathcal{T}_{\langle w,v \rangle}\tau(x_1), \ldots, \sigma(x_n)\mathcal{T}_{\langle w,v \rangle}\tau(x_n).$$

Proof Define $\mathcal{T}_{\langle w,v\rangle}$ as follows:

$$e\,\mathcal{T}_{\langle w,v\rangle}e' \quad \text{iff} \quad \text{there is a term } s \text{ containing at most the variables } x_1,\dots,x_n, \text{ such that } \sigma(s)=e \text{ and } \tau(s)=e'.$$

Trivially $\sigma(x_1)\mathcal{T}_{\langle w,v\rangle}\tau(x_1)\,,\dots,\,\sigma(x_n)\mathcal{T}_{\langle w,v\rangle}\tau(x_n)$. We show that condition (A) holds. Let t be a term containing the variables $y_1,\dots,y_k$, and let π and μ be w and v assignments, respectively, such that

$$\pi(y_1)\,\mathcal{T}_{\langle w,v\rangle}\,\mu(y_1)\,,\dots,\,\pi(y_k)\,\mathcal{T}_{\langle w,v\rangle}\,\mu(y_k)$$

then we have to show that

$$\pi(t)\,\mathcal{T}_{\langle w,v\rangle}\,\mu(t).$$

This amounts to show that there is a term s contaning at most the variables $x_1,\dots,x_n$ such that

$$\sigma(s)=\pi(t) \quad \text{and} \quad \tau(s)=\mu(t).$$

By the definition of $\mathcal{T}_{\langle w,v\rangle}$ above, we know that for each i, $1\le i\le k$, there is a term s_i containing the variables $x_1,\dots,x_n$, such that

$$\sigma(s_i)=\pi(y_i) \quad \text{and} \quad \tau(s_i)=\mu(y_i)$$

Let $s=t[s_1/y_1,\dots,s_k/y_k]$. Then $\sigma(s)=\sigma(t[s_1/y_1,\dots,s_k/y_k])=$ (by lemma 1) $=\sigma^{y_1\rhd\sigma(s_1),\dots,y_k\rhd\sigma(s_k)}(t)=\pi(t)$, since $\sigma(s_i)=\pi(y_i)$, $1\le i\le k$.

As to condition (B), let A be a formula of $\mathcal{L}$ and let us assume that its free variables are among $y_1,\dots,y_k$. Let π and μ be assignments in w and v, respectively, such that

- $\pi(y_1)\,\mathcal{T}_{\langle w,v\rangle}\,\mu(y_1)\,,\dots,\,\pi(y_k)\,\mathcal{T}_{\langle w,v\rangle},\mu(y_k)$, $1\le i\le k$
- $\langle\pi,w\rangle\models P_{|y_1,\dots,y_k|A}(y_1,\dots,y_k)$

We have to show that $\langle\mu,v\rangle\models A^c$. By the definition of $\mathcal{T}_{\langle w,v\rangle}$, there are terms s_i containing at most the variables $x_1,\dots,x_n$, such that $\sigma(s_i)=\pi(y_i)$ and $\tau(s_i)=\mu(y_i)$, $1\le i\le k$. So

$$\langle\pi^{y_1\rhd\sigma(s_1),\dots,y_k\rhd\sigma(s_k)},w\rangle\models P_{|y_1\dots y_k|A}(y_1,\dots,y_k),$$

and consequently

$$\langle\sigma^{y_1\rhd\sigma(s_1),\dots,y_k\rhd\sigma(s_k)},w\rangle\models P_{|y_1\dots y_k|A}(y_1,\dots,y_k),$$

since all the free variables are among $y_1,\dots,y_k$. Then by lemma 4,

$$\langle\sigma,w\rangle\models P_{|y_1\dots y_k|A}(s_1,\dots,s_k).$$

Since

$$R.K_{im}\vdash |^{s_1\dots s_k}_{y_1\dots y_k}|A\to|x_1\dots x_n|(A[s_1/y_1,\dots,s_k/y_k]),(\text{axiom } RG),$$

then

$$C_{R.K_{im}} \vdash P_{|y_1 \dots y_k|A}(s_1, \dots, s_k) \to P_{|x_1 \dots x_n|(A[s_1/y_1, \dots, s_k/y_k])}(x_1, \dots, x_n),$$

so

$$\langle \sigma, w \rangle \models P_{|x_1 \dots x_n|(A[s_1/y_1, \dots, s_k/y_k])}(x_1, \dots, x_n).$$

By the hypothesis of the lemma

$$\langle \tau, v \rangle \models (A[s_1/y_1, \dots, s_k/y_k])^c,$$

i.e.

$$\langle \tau, v \rangle \models A^c[s_1/y_1, \dots, s_k/y_k],$$

therefore by lemma 4

$$\langle \tau^{y_1 \rhd \tau(s_1), \dots, y_k \rhd \tau(s_k)}, v \rangle \models A^c,$$

then

$$\langle \mu, v \rangle \models A^c,$$

since

$$\tau(s_1) = \mu(y_1), \dots, \tau(s_k) = \mu(y_k).$$

♠

Lemma 7 *Let w be a $C_{R.K_{im}}$-model and $|x_1 \dots x_m|A$ be a formula of $\mathcal{L}$ such that $\langle \sigma, w \rangle \not\models P_{|x_1 \dots x_m|A}(x_1, \dots, x_m)$. Then*

1. *the set of classical formulas*

$$\Gamma = \{B^c : \langle \sigma, w \rangle \models P_{|x_1 \dots x_m|B}(x_1 \dots x_m)\} \cup \{\neg A^c\}$$

   is $C_{R.K_{im}}$-consistent, where B contains at most the variables $x_1 \dots x_m$,
2. *there is a classical model v of Γ and a v-assignment τ such that*

$$\langle \tau, v \rangle \models \Gamma,$$

3. *there is an admissible relation $\mathcal{T}_{\langle w,v \rangle}$ such that*

$$\sigma(x_1)\, \mathcal{T}_{\langle w,v \rangle}\, \tau(x_1)\,, \dots,\, \sigma(x_m) \mathcal{T}_{\langle w,v \rangle}\, \tau(x_m).$$

Proof

1. Assume by *reductio* that

$$C_{R.K_{im}} \vdash B_1^c \wedge \dots \wedge B_r^c \to A^c$$

   Then

$$R.K_{im} \vdash B_1 \wedge \dots \wedge B_r \to A$$

$$R.K_{im} \vdash |x_1 \dots x_m|B_1 \wedge \dots \wedge |x_1 \dots x_m|B_r \to |x_1 \dots x_m|A \qquad \text{by N}$$

$$C_{R.K_{im}} \vdash P_{|x_1 \dots x_m|B_1}(x_1 \dots x_m) \wedge \dots \wedge P_{|x_1 \dots x_m|B_r}(x_1 \dots x_m) \rightarrow P_{|x_1 \dots x_m|A}(x_1, \dots, x_m).$$

Therefore

$$\langle \sigma, w \rangle \models P_{|x_1 \dots x_m|A}(x_1, \dots, x_m)$$

contrary to the fact that

$$\langle \sigma, w \rangle \not\models P_{|x_1 \dots x_m|A}(x_1, \dots, x_m).$$

2. By classical model theory.
3. By lemma 6.

♠

Subordination model. A *subordination model* is a tree $\langle S, \Sigma \rangle$ each node of which is (associated to) a classical model $w = \langle D_w, I_w \rangle$ together with an assignment $\sigma : VAR \rightarrow D_w$, so any element of S (any node of the tree) is a triple $\langle \sigma, D_w, I_w \rangle$. Given the node $\langle \sigma, D_w, I_w \rangle$ an *immediate subordinate* node $\langle \tau, D_v, I_v \rangle$, i.e. one for which the relation $\langle \sigma, D_w, I_w \rangle \Sigma \langle \tau, D_v, I_v \rangle$ holds, is defined according to the following procedure.

1. For each formula $\exists x A \in \mathcal{L}$ such that $\langle \sigma, D_w, I_w \rangle \models \exists x A^c$, consider a triple $\langle \sigma^{x \rhd a}, D_w, I_w \rangle$ such that $\langle \sigma^{x \rhd a}, D_w, I_w \rangle \models A^c$, for some $a \in D_w$.

   We say that $\langle \sigma, D_w, I_w \rangle \Sigma \langle \sigma^{x \rhd a}, D_w, I_w \rangle$.
2. For each formula $\exists x A \in \mathcal{L}$ such that $\langle \sigma, D_w, I_w \rangle \not\models \exists x A^c$, consider all the triples $\langle \sigma^{x \rhd a}, D_w, I_w \rangle$ such that $\langle \sigma^{x \rhd a}, D_w, I_w \rangle \not\models A^c$, for any $a \in D_w$.

   We say that $\langle \sigma, D_w, I_w \rangle \Sigma \langle \sigma^{x \rhd a}, D_w, I_w \rangle$, for all $a \in D_w$.
3. For each formula $|\, x_1 \dots x_m \,|A \in \mathcal{L}$ such that $\langle \sigma, D_w, I_w \rangle \not\models P_{|\, x_1 \dots x_m \,|A}$, consider a triple $\langle \tau, D_v, I_v \rangle$ such that
   $\langle \tau, D_v, I_v \rangle \models \{B^c : \langle \sigma, D_w, I_w \rangle \models P_{|x_1 \dots x_n|B}(x_1, \dots, x_n)\} \cup \{\neg A^c\}$.

   We say that $\langle \sigma, D_w, I_w \rangle \Sigma \langle \tau, D_v, I_v \rangle$ and that $\sigma(x_1)\, \mathcal{T}_{\langle w,v \rangle}\, \tau(x_1)\,, \dots,$ $\sigma(x_m) \mathcal{T}_{\langle w,v \rangle}\, \tau(x_m)$.

Steps 1 and 2 are feasible thanks to classical model theory, step 3 thanks to lemma 6.

Lemma 8 *Let $R.K_{im} \not\vdash A$. Then there is a t-model $\mathcal{M} = \langle W, R, D, \mathcal{T}, I \rangle$ with rigid terms such that $\mathcal{M} \not\models A$.*

Proof Let us first build a subordination model $\langle S, \Sigma \rangle$ having at its root a node $\langle \sigma, D_w, I_w \rangle$ such that $\langle \sigma, D_w, I_w \rangle \models \neg A^c$. Then we define a transition model $\mathcal{M} = \langle W, D, R, \mathcal{T}, I \rangle$ as follows:

- $W = \{\langle D_w, I_w \rangle : \, for\, some\, \sigma, \langle \sigma, D_w, I_w \rangle \in S\}$
- D is such that $D(\langle D_w, I_w \rangle) = D_w$
- $R \subseteq W^2$ is such that $\langle D_w, I_w \rangle R \langle D_v, I_v \rangle$ iff $\langle \sigma, D_w, I_w \rangle \Sigma \langle \tau, D_v, I_v \rangle$ for some σ and τ

- $\mathcal{T} = \{\langle a, b\rangle :$ for some $\langle \sigma, D_w, I_w\rangle$ and $\langle \tau, D_v, I_v\rangle$, $a \in D_w$, $b \in D_v$, $\langle \sigma, D_w, I_w\rangle \Sigma \langle \tau, D_v, I_v\rangle$, $a = \sigma(x), b = \tau(x)$, and $\sigma(x)\mathcal{T}_{\langle w,v\rangle}\tau(x)\}$
- I is such that $I(\langle D_w, I_w\rangle) = I_w$

In the following, we write w instead of $\langle D_w, I_w\rangle$ and $\langle \sigma, w\rangle \models D^c$ instead of $\langle \sigma, D_w, I_w\rangle \models D^c$. It remains to show that

$$\sigma \models^{\mathcal{M}}_w D \quad iff \quad \langle \sigma, w\rangle \models D^c$$

for all $w \in W$ and all formulas $D \in \mathcal{L}$.

By induction on D. We examine just one case.

$$D = |^{t_1}_{y_1} \dots ^{t_n}_{y_n}|A$$

where $(fv(t_1) \cup \dots \cup fv(t_n)) = \{x_1, \dots, x_m\}$.

If

$$\sigma \not\models^{\mathcal{M}}_w |^{t_1}_{y_1} \dots\dots ^{t_n}_{y_n}|A$$

then by lemma 4

$$\pi \not\models^{\mathcal{M}}_w |y_1 \dots y_n|A$$

where $\pi = \sigma^{y_1 \triangleright \sigma(t_1), \dots, y_n \triangleright \sigma(t_n)}$. Then by definition of satisfaction there is a v and a v-assignment τ, such that $\tau \not\models^{\mathcal{M}}_v A$, and $\sigma(t_i)\, \mathcal{T}\, \tau(y_i)$, $1 \le i \le n$. By induction hypothesis $\langle \tau, v\rangle \not\models A^c$, whence $\langle \pi, w\rangle \not\models P_{|y_1 \dots y_n|A}(y_1, \dots, y_n)$, because of condition (B). Consequently $\langle \sigma, w\rangle \not\models P_{|y_1 \dots y_n|A}(t_1, \dots, t_n)$.

If

$$\langle \sigma, w\rangle \not\models P_{|y_1 \dots y_n|A}(t_1, \dots, t_n),$$

then by lemma 4

$$\langle \pi, w\rangle \not\models P_{|y_1 \dots y_n|A}(y_1, \dots, y_n)$$

where $\pi = \sigma^{y_1 \triangleright \sigma(t_1) \dots\dots y_n \triangleright \sigma(t_n)}$. Then by lemma 7 there is a model v of $\Gamma = \{B^c : \langle \pi, w\rangle \models P_{|y_1 \dots y_n|B}(y_1, \dots, y_n)\} \cup \{\neg A^c\}$ and a v-assignment τ such that $\langle \tau, v\rangle \models \Gamma$ and $\sigma(t_i)\, \mathcal{T}_{\langle w,v\rangle}\, \tau(y_i)$, $1 \le i \le n$. Hence

$$\langle \tau, v\rangle \not\models A^c,$$

therefore by induction hypothesis $\tau \not\models^{\mathcal{M}}_v A$, so

$$\pi \not\models^{\mathcal{M}}_w |y_1 \dots y_n|A.$$

Consequently

$$\sigma \not\models^{\mathcal{M}}_w |^{t_1}_{y_1} \dots\dots ^{t_n}_{y_n}|A.$$

♠

7 Completeness theorem for $Q.K_{im}$

The completenes theorem for $Q.K_{im}$ is easily obtained from the corresponding theorem for $R.K_{im}$. A relation $\mathcal{T}_{\langle w,v\rangle}$ among $C_{Q.K_{im}}$-models w, v is an *admissible relation* iff condition (B) is satisfied. In the proof of Lemma 6, define $\mathcal{T}_{\langle w,v\rangle}$ as follows:

$$e\,\mathcal{T}_{\langle w,v\rangle}e'$$

iff there is a variable $x_i \in \{x_1, \ldots, x_n\}$, such that $\sigma(x_i) = e$ and $\tau(x_i) = e'$.
Trivially $\sigma(x_i)\,\mathcal{T}_{\langle w,v\rangle}\,\tau(x_i)$, $1 \leq i \leq n$.
As to condition (B), take any modal formula A of $\mathcal{L}$ containing at most the free variables $y_1, \ldots, y_k$, and a pair of assignments π and μ in w and v, respectively, such that

- $\pi(y_1)\mathcal{T}_{\langle w,v\rangle}\mu(y_1), \ldots, \pi(y_k)\mathcal{T}_{\langle w,v\rangle}\mu(y_k)$
- $\langle \pi, w\rangle \models P_{|y_1,\ldots,y_k|A}(y_1, \ldots, y_k)$

We have to show that $\langle \mu, v\rangle \models A^c$. By the definition of $\mathcal{T}_{\langle w,v\rangle}$, there are variables $x_1^\star, \ldots x_k^\star$ among $x_1, \ldots, x_n$ such that $\sigma(x_i^\star) = \pi(y_i)$ and $\tau(x_i^\star) = (\mu(y_i)$, $1 \leq i \leq k$. So

$$\langle \pi^{y_1 \rhd \sigma(x_1^\star), \ldots, y_k \rhd \sigma(x_k^\star)}, w\rangle \models P_{|y_1\ldots y_k|A}(y_1, \ldots, y_k)$$

and consequently

$$\langle \sigma^{y_1 \rhd \sigma(x_1^\star), \ldots, y_k \rhd \sigma(x_k^\star)}, w\rangle \models P_{|y_1\ldots y_k|A}(y_1, \ldots, y_k)$$

since all the free variables are among $y_1, \ldots, y_k$. Then by lemma 4

$$\langle \sigma, w\rangle \models P_{|y_1\ldots y_k|A}(x_1^\star, \ldots, x_k^\star).$$

Since

$$Q.K_{im} \vdash |^{x_1^\star \ldots x_k^\star}_{y_1 \ldots y_k}|A \to |x_1 \ldots x_n|(A[x_1^\star/y_1, \ldots, x_k^\star/y_k]) \text{ (axiom } RG^v\text{)},$$

then

$$C_{Q.K_{im}} \vdash P_{|y_1\ldots y_k|A}(x_1^\star, \ldots, x_k^\star) \to P_{|x_1\ldots x_n|(A[x_1^\star/y_1,\ldots,x_k^\star/y_k])}(x_1, \ldots, x_n),$$

so

$$\langle \sigma, w\rangle \models P_{|x_1\ldots x_n|(A[x_1^\star/y_1,\ldots,x_k^\star/y_k])}(x_1, \ldots, x_n).$$

By the hypothesis of the lemma

$$\langle \tau, v\rangle \models (A[x_1^\star/y_1, \ldots, x_k^\star/y_k])^c,$$

i.e.

$$\langle \tau, v\rangle \models A^c[x_1^\star/y_1, \ldots, x_k^\star/y_k],$$

therefore by lemma 4

$$\langle \tau^{y_1 \rhd \tau(x_1^\star), \ldots, y_k \rhd \tau(x_k^\star)}, v\rangle \models A^c,$$

therefore since $\tau(x_1^\star) = d_1 , \ldots , \tau(x_k^\star) = d_k$,

$$\langle \tau^{y_1 \rhd d_1, \ldots, y_k \rhd d_k}, v \rangle \models A^c,$$

whence

$$\langle \mu, v \rangle \models A^c(y_1, \ldots, y_k)$$

since

$$\mu(y_1) = d_1, \ldots, \mu(y_k) = d_k.$$

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