Allowing for Non-Additively Separable and Flexible Utility Forms in Multiple Discrete-Continuous Models

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ABSTRACT

Many consumer choice situations are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes for one another, along with a continuous quantity dimension for each chosen alternative. To model such multiple discrete-continuous choices, most multiple discrete-continuous models in the literature use an additively-separable utility function, with the assumption that the marginal utility of one good is independent of the consumption of another good. In this paper, we develop model formulations for multiple discrete-continuous choices that allow a non-additive utility structure, and accommodate rich substitution structures and complementarity effects in the consumption patterns. Specifically, three different non-additive utility formulations are proposed based on alternative specifications and interpretations of stochasticity: (1) The deterministic utility random maximization (DU-RM) formulation, which considers stochasticity due to the random mistakes consumers make during utility maximization; (2) The random utility deterministic maximization (RU-DM) formulation, which considers stochasticity due to the analyst's errors in characterizing the consumer's utility function; and (3) The random utility random maximization (RU-RM) formulation, which considers both analyst's errors and consumer's mistakes within a unified framework. When applied to the consumer expenditure survey data in the United States, the proposed non-additively separable utility formulations perform better than the additively separable counterparts, and suggest the presence of substitution and complementarity patterns in consumption.

Keywords: Discrete-continuous system, multiple discreteness, Karush-Kuhn-Tucker demand systems, mixed discrete choice, random utility maximization, non-additively separable utility form, transportation expenditure.

1. INTRODUCTION

Multiple discrete-continuous (MDC) choice situations are quite ubiquitous in consumer decision-making, and constitute a generalization of the more classical single discrete-continuous choice situation. Examples of MDC contexts include the participation decision of individuals in different types of activities over the course of a day and the duration in the chosen activity types (see Bhat, 2005, Chikaraishi *et al.*, 2010, and Wang and Li, 2011), household holdings of multiple vehicle body/fuel types and the annual vehicle miles of travel on each vehicle (Ahn *et al.*, 2008), and consumer purchase of multiple brands within a product category and the quantity of purchase (Kim et al., 2002), .

There are several differences between the traditional single discrete choice (SDC) and MDC utility frameworks, primarily originating in the functional form of the utility function. MDC models typically assume imperfect substitution among alternatives based on a more general utility function than the SDC case, which assumes perfect substitution among alternatives. But, at a basic level, the choice process faced by the consumer in both the SDC and MDC situations may be formulated from a microeconomic consumer utility maximization theory perspective as follows:

$$\underset{\boldsymbol{x}}{Max}\, U(\boldsymbol{x}) \text{ subject to } \sum_{k=1}^{K} p_k x_k = E\,,\ x_k \geq 0\,, \tag{1}$$

where $U(\boldsymbol{x})$ is the utility function corresponding to a consumption vector $\boldsymbol{x}$, p_k is the unit price of good *k*, and E is the total expenditure. Note that the formulation above is equally applicable to cases with complete or incomplete demand systems (that is, the modeling of demand for all commodities that enter preferences or the modeling of demand for a subset of commodities that enter preferences).[1]

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[1] A complete demand system involves the modeling of the demands of all consumption goods that exhaust the consumption space of consumers. However, complete demand systems require data on prices and consumptions of all commodity/service items, and can be impractical when studying consumptions in finely defined commodity/service categories. Thus, it is common to use an incomplete demand system, typically in the form of a two stage budgeting approach or in the form of the use of a Hicksian composite commodity assumption. In the former two stage budgeting approach, separabilility of preferences is invoked, and the allocation is pursued in two independent stages. The first stage entails allocation between a limited number of broad groups of consumption items, followed by the incomplete demand system allocation of the group expenditure to elementary commodities/services within the broad consumption group of primary interest to the analyst (the elementary commodities/services in the broad group of primary interest are commonly referred to as “inside” goods). The plausibility of such a two stage budgeting approach requires strong homothetic preferences within each broad group and strong separability of preferences, or the less restrictive conditions of weak separability of preferences and the price index for each broad group not being too sensitive to changes in the utility function (see Menezes *et al.,* 2005).

The functional form of the utility function $U(\boldsymbol{x})$ determines the characteristics of, and the solution for, the constrained utility maximization formulation of Equation (1). More importantly, the functional form determines whether the formulation corresponds to an SDC or an MDC model. For instance, Hanemann (1984) considers the "perfect substitutes" case when he writes the utility function $U(\boldsymbol{x})$ as follows (Hanemann considers an essential outside good, which we will assume to be good 1):

$$U(\boldsymbol{x}) = U^*\left(\sum_{k=2}^{K}\psi_k x_k, x_1\right), \tag{2}$$

where U^* is a bivariate utility function and ψ_k $(k = 2,3,...,K)$ represents the quality index specific to each inside good *k*. This functional form assures that, in addition to the outside good which is consumed, exactly one inside good ($k = 2,3,...,K$) is also consumed. Hanemann (1984) refers to this as the "extreme corner solution".[2] As is typical in SDC analysis, rather than deriving the consumption function based on solving the constrained maximization problem of the direct utility function in Equation (2), Hanemann assumes a functional form for the indirect utility function, introduces random stochasticity into the formulation, and then derives expressions for the probabilities of the discrete and continuous choices. Chiang (1991) and

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In the Hicksian composite commodity approach, the analyst assumes that the prices of elementary goods within each broad group of consumption items vary proportionally. Then, one can replace all the elementary alternatives within each broad group (that is not of primary interest) by a single composite alternative representing the broad group. The analysis proceeds then by considering the composite goods as "outside" goods and considering consumption in these outside goods as well as the "inside" goods representing the consumption group of main interest to the analyst. It is common in practice in this Hicksian approach to include a single outside good with the inside goods. If this composite outside good is not essential, then the consumption formulation is similar to that of a complete demand system. If this composite outside good is essential, then the formulation needs minor revision to accommodate the essential nature of the outside good. Please refer to von Haefen (2010) for a discussion of the Hicksian approach and other incomplete demand system approaches such as the one proposed by Epstein (1982) that we do not consider here. In this paper, we will consider incomplete demand systems in the form of the second stage of a two stage incomplete demand system with a finite, positive total budget as obtained from the first stage (for presentation ease, we will refer to this case as the "inside goods only" case in which at least one "inside" good has to be consumed and there are no essential outside goods) or in the form of a Hicksian composite approach with a single outside good that is essential and no requirement that at least one of the inside goods has to be consumed (for presentation ease, we will refer to this case simply as the "essential outside good" case; if the outside good is non-essential, the formulation becomes identical to the case of the "inside goods only" case, while if there are multiple outside goods, the situation is a very simple extension of the formulations presented here depending on whether the outside goods are all essential, all non-essential, or some combination of essential and non-essential). Finally, a complete demand system takes the same formulation as the "inside goods only" formulation.

[2] Of course, the utility function in Equation (2) is easily modified for the case when there is no outside good, and only one of the inside goods is consumed. In this case, the utility function becomes $U(\boldsymbol{x}) = \sum_{k=2}^{K}\psi_k x_k$. This corresponds to the case of the traditional discrete choice model, since all the expenditure is on the chosen good and so the continuous component drops out.

Chintagunta (1993) extend Hanemann's SDC formulation to include the possibility of no inside goods being selected for consumption by including a "reservation price". If the quality-adjusted prices of all the inside goods exceed the reservation price, no inside goods are selected, but if the quality adjusted prices of one or more of the inside goods are below the reservation price, an inside good is selected based on Hanemann's framework. The demand functions for the continuous components of choice are obtained using Roy's identity (Roy, 1947).

As indicated above, SDC analysis is usually undertaken using an indirect utility approach, based on the argument that it is usually difficult and, often intractable, to adopt a direct utility approach for estimating parameters and obtaining analytic expressions for demand functions. However, as clearly articulated by Bunch (2009), the direct utility approach has the advantage of being closely tied to an underlying behavioral theory, so that interpretation of parameters in the context of consumer preferences is clear and straightforward. Further, the direct utility approach provides insights into identification issues. Of course, when one moves to the MDC models, the indirect utility approach all but falls apart because multiple inside goods can be selected for consumption and non-negativity of the consumption vector must be guaranteed (see Wales and Woodland, 1983). Thus, in addition to conceptual and behavioral advantages, it has been the norm to examine MDC situations using the direct utility approach, especially because, through clever stochastic term distribution assumptions, one can obtain a closed form for the probability of the consumption patterns of goods.

Earlier direct utility-based MDC models have their origins in Hanemann's (1978) and Wales and Woodland's (1983) Karush-Kuhn-Tucker (KKT) first-order conditions approach for constrained random utility maximization. This approach assumes the utility function $U(\boldsymbol{x})$ to be random (from the analyst's perspective) over the population, and then derives the consumption vector for the random utility specification subject to the linear budget constraint by using the KKT conditions for constrained optimization. Several recent developments have sparked a renewed interest in applying the KKT-based approach to modeling MDC choices. A representative example is Bhat's (2008) multiple discrete-continuous extreme value (MDCEV) model formulation that provides a simple and parsimonious approach to model MDC choices.

To date, most MDC modeling frameworks, including the MDCEV model, have adopted an additively separable utility function, which assumes that the marginal utility of one good is independent of the consumption of another good. This assumption has at least two important implications. First, the marginal rate of substitution between any pair of goods is dependent only on the quantities of the two goods in the pair, and independent of the quantity of other goods. As indicated by Pollak and Wales (1992), this has consequences on the preferences directly. For example, let there be three food items: milk (x_1), cornflakes (x_2), and raisin bran (x_3). Consider an individual who tends to have milk and cornflakes, or milk and raisin bran, but not milk alone. Such an individual may prefer the triplet [20,1,20] over [10,10,20], but may also prefer [10,10,1] over [20,1,1]. This violates additive utility, because, if the individual prefers [20,1,20] over [10,10,20], she/he must prefer $[20,1,x_3]$ over $[10,10,x_3]$ according to additive utility. Then, the additively separable assumption substantially reduces the ability of the utility function to accommodate rich and flexible substitution patterns. Second, the specification of a quasi-concave and increasing utility function with respect to the consumption of goods, along with additive utility across goods, immediately implies that goods cannot be inferior and cannot be complements (*i.e.*, they must be strict substitutes; see Deaton and Muellbauer, 1980, page 139). Besides, additive utility structure makes it difficult to recognize that consumers might have a preference for certain specific combinations of alternatives. Overall, additively separable utility functions are substantially restricted in their ability to accommodate flexible dependencies (*e.g.*, complementarity and substitution) in the consumption of different goods.

The literature on MDC models that adopt a non-additively separable utility structure is relatively limited, and research in this area has arisen only in the last five years. Song and Chintagunta (2007) and Mehta (2007) accommodated complementarity and substitution effects in a MDC utility function to model purchase quantity decisions of house cleaning products. However, because of the model complexity, both studies use an indirect utility approach instead of a direct utility approach. Later, Lee and Allenby (2009) proposed a direct utility approach that incorporates a non-additively separable utility function. For this purpose, they grouped goods in categories assuming that goods in the same category are substitutes, while goods in different categories are complements. However, their modeling framework does not allow consumers to choose multiple goods within each category. Lee *et al.* (2010) proposed a direct utility model for measuring asymmetric complementarity. Their model formulation accommodates both inside and outside goods, but it was developed for the simple case of two goods. Vásquez-Lavín and Hanemann (2008) extended Bhat's (2008) additively separable linear form allowing the marginal utility of each good to be dependent on the level of consumption of other goods. However, as we discuss in Section 2, their utility function can become theoretically inconsistent for some combinations of the parameters.

1.1. Paper Objectives and Structure

The objective of this paper is to extend extant MDC formulations by relaxing the assumption of an additively separable utility function. In doing so, we propose a particular non-additively separable (NAS) utility functional form that remains within the class of flexible forms, while also retaining global theoretical consistency properties. The form also allows clarity in the interpretation of parameters and helps understand identification issues. In addition, we propose and discuss three different stochastic formulations to acknowledge two different sources of errors. The first source of errors arises when consumers make random "mistakes" in maximizing their utility function, and the second source of errors originates from the analyst's inability to observe all factors relevant to the consumer's utility formation. More specifically, we present the following three different non-additive utility formulations based on alternative specifications and interpretations of stochasticity: (1) The deterministic utility random maximization (DU-RM) formulation, which considers stochasticity due to the random "mistakes" consumers make during utility maximization, (2) The random utility deterministic maximization (RU-DM) formulation, which considers stochasticity due to the analyst's errors in characterizing the consumer's utility function, and (3) The random utility random maximization (RU-RM) formulation, which considers both analyst's errors and consumer's optimization "mistakes" within a unified framework. For each of these formulations, we are able to retain a relatively simple form for the model, and the structure of the Jacobian in the likelihood function is also relatively simple.

The rest of this paper is structured as follows. The next section formulates a functional form for the non-additive utility specification that enables the isolation of the role of different parameters in the specification. This section also identifies empirical identification considerations in estimating the parameters in the utility specification. Section 3 discusses alternative stochastic forms of the utility specification and the resulting general structures for the probability expressions. Section 4 provides an empirical demonstration of the model proposed in this paper for analyzing household expenditures in transportation-related categories. The final section concludes the paper.

2. FUNCTIONAL FORM OF UTILITY SPECIFICATION

The starting point for our utility functional form is Bhat (2008), who proposes a linear Box-Cox version of the constant elasticity of substitution (CES) direct utility function for MDC models:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \frac{\gamma_k}{\alpha_k} \psi_k \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\}, \tag{3}$$

where $U(\boldsymbol{x})$ is a strictly quasi-concave, strictly increasing, and continuously differentiable function with respect to the consumption quantity (K×1)-vector $\boldsymbol{x}$ ($x_k \geq 0$ for all k), and ψ_k, γ_k and α_k are parameters associated with good k. The function in Equation (3) is a valid utility function if $\psi_k > 0$, $\gamma_k > 0$, and $\alpha_k \leq 1$ for all k. For presentation ease, we assume temporarily that there is no essential outside good (that is, the case of "inside goods only"), so that corner solutions (*i.e.*, zero consumptions) are allowed for all the goods k (this assumption is being made only to streamline the presentation and should not be construed as limiting in any way; in fact, as we will show later, the econometrics become much easier when there is an essential outside good). We also assume for now that the utility function is deterministic to focus on functional form issues (important modeling issues arise when we introduce stochasticity, which we discuss in Section 3). The possibility of corner solutions implies that the term γ_k in Equation (3), which is a translation parameter, should be greater than zero for all k.[3] The reader will note that there is an assumption of additive separability of preferences in the utility form of Equation (3).

Bhat's utility form clarifies the role of the various parameters ψ_k, γ_k and α_k, and explicitly indicates the inter-relationships between these parameters that relate to theoretical and empirical identification issues (see Bhat, 2008 for an extensive discussion). In particular, ψ_k represents the baseline marginal utility, or the marginal utility at the point of zero consumption. γ_k, in addition to allowing corner solutions, controls satiation by translating consumption quantity, while α_k controls satiation by exponentiating consumption quantity. Clearly, both these effects operate in different ways, and different combinations of their values lead to different satiation profiles. However, empirically speaking, it is very difficult to disentangle the two effects separately, which leads to serious empirical identification problems and estimation

[3] As illustrated in Kim *et al.* (2002) and Bhat (2005), the presence of the translation parameters makes the indifference curves strike the consumption axes at an angle (rather than being asymptotic to the consumption axes), thus allowing corner solutions.

breakdowns when one attempts to estimate both γ_k and α_k parameters for each good. Thus, for identification purposes, earlier studies have either constrained α_k to zero for all goods (technically, assumed $\alpha_k \to 0 \;\forall k$) and estimated the γ_k parameters (*i.e.*, the γ-profile utility form), or constrained γ_k to 1 for all goods and estimated the α_k parameters (*i.e.*, the α-profile utility form).

Vásquez-Lavín and Hanemann (2008) extended Bhat's additively separable linear Box-Cox form and presented a quadratic version of it, as below:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \left(\frac{\gamma_k}{\alpha_k} \left[\left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right] \left\{ \psi_k + \frac{1}{2} \sum_{m=1}^{K} \theta_{km} \frac{\gamma_m}{\alpha_m} \left[\left(\frac{x_m}{\gamma_m} + 1 \right)^{\alpha_m} - 1 \right] \right\} \right), \tag{4}$$

where $\psi_k > 0$, $\gamma_k > 0$, and $\alpha_k \leq 1$ for all *k*. The new interaction parameters θ_{km} allow quadratic effects (when $k = m$) as well as allow the marginal utility of good *k* to be dependent on the level of consumption of other goods (note that $\theta_{km} = \theta_{mk}$ for all *k* and *m*). Positive interaction parameters accommodate complementarity effects, while negative interaction parameters accommodate substitution effects. Of course, if $\theta_{km} = 0$ for all *k* and *m*, the utility function collapses to Bhat's linear Box-Cox form. If $\alpha_k \to 0 \;\forall k$, the function collapses to the well-known direct basic translog utility function (see Christensen *et al*., 1975), and if $\alpha_k = 1 \;\forall k$, we obtain the quadratic utility function used by Wales and Woodland (1983). The quadratic form of Equation (4) is a flexible functional form that has enough parameters to provide a second-order approximation to any true unknown direct twice-differentiable utility functional form. It also is a non-additive functional form. However, the flexibility is also a limitation, since the function can provide nonsensical results and be theoretically inconsistent for some combinations of the parameters and consumption bundles, an issue that has not received much attention in the literature (but see Sauer *et al*., 2006). For example, positive value of the θ_{kk} parameters can lead to situations with increasing (as opposed to diminishing) marginal utility with increasing consumption. Similarly, negative θ_{kk} parameters can lead to parabolic utility forms that do not comply with theory that utility is strictly increasing with consumption. In fact, due to the presence of the θ_{kk} parameters, it is not possible to achieve global consistency (over all consumption bundles) in terms of the strictly increasing and quasi-concave nature of the utility function using the translog form. In the next section, we extend Vásquez-Lavín and Hanemann's discussion to clarify the role of parameters, identify issues of theoretical consistency and restrictions that need to be maintained, present identification considerations, and recommend a flexible form similar to the translog but that is easier to estimate and reduces global inconsistency problems associated with the translog.

2.1. Role of Parameters in Non-Additively Separable Utility Specification

2.1.1. Role of ψ_k

The marginal utility of consumption with respect to good *k* can be written from Equation (5) as:

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k} = \left(\frac{x_k}{\gamma_k}+1\right)^{\alpha_k-1}\left\{\psi_k + \sum_{m=1}^{K}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\right\}. \tag{5}$$

The difference between the above expression and the corresponding one in Bhat's (2008) linear Box-Cox additively separable case is the presence of the second term in parenthesis, which includes the consumptions of other goods. Thus, the formulation is not additively separable, but one in which the marginal utility of a good is dependent on the consumption amounts of other goods. The marginal utility at zero consumption of good *k* collapses to:

$$\left.\frac{\partial U(\boldsymbol{x})}{\partial x_k}\right|_{x_k=0} = \tilde{\pi}_k = \psi_k + \sum_{m\neq k}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]. \tag{6}$$

From above, it is clear that ψ_k is no more the baseline (marginal) utility of good *k* at the point of zero consumption of good *k*. Rather, it should be viewed as the baseline (marginal) utility of good *k* at the point at which no good has yet been "consumed"; that is, when $x_m = 0\ \forall m$ (no consumption decision has yet been made). This also indicates that, if prices of all goods are the same, then the good with the highest value of ψ_k will definitely see some positive consumption.[4]

Another important point to note from Equation (5) is that for the utility function to be strictly increasing, the following condition should be satisfied for all possible values of the consumption vector $\boldsymbol{x}$:

[4] If there is price variation across goods the good with the highest price-normalized marginal utility ψ_k/p_k will definitely see some positive consumption (see Pinjari and Bhat, 2011).

$$\psi_k + \sum_{m=1}^{K} \theta_{km} \frac{\gamma_m}{\alpha_m} \left[\left(\frac{x_m}{\gamma_m} + 1 \right)^{\alpha_m} - 1 \right] > 0 \text{ for all } k. \tag{7}$$

This is in addition to the condition in the linear case where $\psi_k > 0 \; \forall \; k$. The condition above is needed because we are considering the case of economic goods. In addition, a sufficiency condition for maintaining the decreasing marginal utility (or strict quasi-concavity) of the utility function is that the left side of Equation (7) be a non-increasing function of x_k. We return to these conditions later in the paper.

2.1.2. Role of γ_k

As in the linear case, the γ_k parameter allows for corner solutions. In particular, the γ_k terms shift the position of the point at which the indifference curves are asymptotic to the axes from $(0,0,0,...,0)$ to $(-\gamma_1, -\gamma_2, -\gamma_3, ..., -\gamma_K)$, so that the indifference curves strike the positive orthant with a finite slope. This, combined with the consumption point corresponding to the location where the budget line is tangential to the indifference curve, results in the possibility of zero consumption of good *k*. In addition to allowing corner solutions, the γ_k terms also serve as satiation parameters. In general, the higher the value of γ_k, the less is the satiation effect in the consumption of x_k. However, unlike the linear case, γ_k affects satiation for good *k* in two ways. The first effect is through the first linear term on the left side of Equation (4), and the second is through the second term on the right side of Equation (4) that generates quadratic effects. The overall effect depends on the sign and magnitude of the parameter θ_{kk} in the second term. If this term is negative, and particularly for high values of γ_k, we can get an inappropriate parabolic shape for the contribution of alternative *k* to overall utility within the range of x_k. In particular, beyond a certain point of consumption of alternative *k*, there is negative marginal utility. This is because of the violation of the condition in Equation (7). An illustration is provided in Figure 1, which plots the utility contribution of alternative *k* for $\psi_k = 1$, $\alpha_k \to 0$, $\theta_{kk} = -0.02$, $\theta_{km} = 0 \;\; \forall m \neq k$, and different values of γ_k ($\gamma_k = 1$, 10, and 30). As can be observed, for the γ_k value of 30, we get a profile that peaks at about 110 units, and violates the requirement that the utility function be strictly increasing. On the other hand, if θ_{kk} is positive and quite high in magnitude, it is possible that, for high γ_k values, there is in fact an increase in the marginal utility effect at low values of x_k (essentially a violation of the strictly quasi-concave assumption of the utility function). This is because the left side of Equation (7) becomes an increasing function of x_k at low x_k values. Figure 2 illustrates such a case for $\psi_k = 1$, $\alpha_k \to 0$, $\theta_{kk} = +0.2$, $\theta_{km} = 0 \;\; \forall m \neq k$, and different values of γ_k ($\gamma_k = 1$, 10, and 30). For $\gamma_k = 10$, one can discern the increasing marginal utility until about 6.5 units after which the shape becomes one of decreasing marginal utility. The increasing marginal utility at low values is particularly pronounced for $\gamma_k = 30$, which continues until a value of 40 units before starting to decrease in marginal utility. We will return to these issues in Section 2.2.

The translation parameters γ_m of other goods also have an impact on the utility contribution of good *k*, through the influence on the baseline (marginal) utility of good *k* (see Equation (6)). Specifically, for a given value of x_m, the baseline (marginal) utility for good *k* increases as γ_m increases for positive θ_{km} values and decreases as γ_m increases for negative θ_{km} values.

2.1.3. Role of α_k

The express role of α_k is to reduce the marginal utility with increasing consumption of good *k*; that is, it represents a satiation parameter. However, as in the case of the γ_k effect on consumption of good *k*, there are two effects of the α_k parameter – one through the first linear term on the right side of Equation (4) and the second through the quadratic effect caused by the combination of the first and second terms on the right side of Equation (4). The overall α_k effect depends on the sign and magnitude of the parameter θ_{kk} in the second term. If this term is negative, and particularly for values of α_k close to 1, we can get a "nonsensical" parabolic shape for the utility contribution of alternative *k* within the usual possible range of x_k. An illustration is provided in Figure 3, which plots the utility contribution of alternative *k* for $\gamma_k = 1$, $\psi_k = 1$, $\theta_{kk} = -0.03$, $\theta_{km} = 0 \;\; \forall m \neq k$, and different values of α_k. As can be observed, at the α_k value of 0.6, we get a profile that peaks at about 150 units and decreases thereafter, violating the requirement that the utility function be strictly increasing. On the other hand, if θ_{kk} is positive and quite high in magnitude, it is possible that, for high α_k values, there is in fact an increase in the marginal utility effect at some low values of x_k. Figure 4 illustrates such a case for $\gamma_k = 1$, $\psi_k = 1$, $\theta_{kk} = +0.2$, $\theta_{km} = 0 \ \forall m \neq k$, and different values of α_k. The non-conforming utility profile is obvious for the α_k value of 0.8.

The α_m parameters for other goods also impact the baseline (marginal) utility of good k (see Equation (6)). For a given value of x_m, the baseline (marginal) utility for good k decreases as α_m falls down from 1 for positive θ_{km} values and increases as α_m falls down from 1 for negative θ_{km} values.

2.2. Empirical Identification Issues Associated with Utility Form

The total number of parameters in the flexible utility functional form of Equation (4) rises rapidly with the number of alternatives, especially in the θ_{km} terms ($k = 1,2,...,K;\ m = 1,2,...,K$). There are also empirical identification issues that arise with the utility form. As in the linear case, empirically speaking, it is next to impossible to disentangle the effects of the γ_k and α_k parameters for each good separately (see Bhat, 2008). Thus one has to impose some constraints on these parameters. While many combinations of constraints are possible, the easiest approaches are to constrain α_k to zero for all goods (technically, assume $\alpha_k \rightarrow 0 \ \forall k$) and estimate the γ_k parameters (the γ-profile), or constrain γ_k to 1 for all goods and estimate the α_k parameters (the α-profile).

In the case of the non-additively separable utility function, there is an additional empirical identification issue in both the γ-profile case and the α-profile case. This is because the θ_{kk} parameters in the quadratic utility functional form essentially also serve as "satiation" parameters by providing appropriate curvature to the utility function. However, empirically speaking, it is difficult to disentangle the θ_{kk} effects from the γ_k effects (for the γ-profile) and from the α_k effects (for the α-profile) as long as the θ_{kk} effects do not become that negative as to bring on a parabolic shape at even low to moderate consumption levels (this latter case would anyway be inappropriate to represent the utility function). In fact, a utility profile based on a combination of θ_{kk} and γ_k values for the γ-profile case can be closely approximated by a utility function based solely on γ_k values with $\theta_{kk}=0$. Similarly, a utility profile based on a combination of θ_{kk} and α_k values for the α-profile case can be closely approximated by a utility function based solely on α_k values with $\theta_{kk}=0$. This is illustrated in Figures 5 for the γ-profile, with $\psi_k=1$, $\alpha_k \to 0 \;\; \forall k$, and $\theta_{km}=0 \;\; \forall m \neq k$. The figure shows that alternative *k*'s contribution to utility based on a certain combination of γ_k and θ_{kk} values can be closely replicated by other combination values of γ_k and θ_{kk}. In particular, the utility profiles corresponding to combinations of γ_k and θ_{kk} values can be replicated very closely by a profile that corresponds to $\gamma_k=1$ and some specific θ_{kk} value, or by a profile that corresponds to $\theta_{kk}=0$ and some specific γ_k value. Thus, in Figure 5, the utility profiles corresponding to $\gamma_k=7.45$ and $\theta_{kk}=0$, and $\gamma_k=1$ and $\theta_{kk}=1.3$, are able to closely replicate all the other utility profiles. A similar situation may be observed from Figure 6 for the α-profile, where the utility profiles of different combinations of θ_{kk} and α_k values can be approximated closely by the profile corresponding to $\alpha_k=0.442$ and $\theta_{kk}=0$, and $\alpha_k=0$ and $\theta_{kk}=1.38$.

The discussion above suggests that, without loss of empirical generality, one can normalize $\gamma_k=1$ (and estimate θ_{kk}) or set $\theta_{kk}=0$ (and estimate γ_k) for each good *k* in the γ-profile case. In the α-profile case, one can normalize $\alpha_k=0$ (and estimate θ_{kk}) or set $\theta_{kk}=0$ (and estimate α_k) for each good *k* in the utility function. We propose to set $\theta_{kk}=0$ for each good, since this immediately removes the possibility of a parabolic shape for the utility contribution of good *k*. At the same time, we immediately ensure that the marginal utility is strictly decreasing over the entire range of consumption values of the good *k*. These are important theoretical considerations that we are able to maintain globally without any loss in functional form flexibility. In fact, the functional form proposed in this paper remains within the class of flexible forms, while also retaining global theoretical consistency properties (unlike the translog and related flexible quadratic functional forms). The result is also clarity in the interpretation of the γ_k and α_k parameters, which now have the same interpretation as satiation parameters corresponding to good *k* as in the linear utility function case of Bhat (2008). Besides, the baseline marginal utility of good k now remains unchanged with the consumption of good k, which is intuitive.

There is still, however, one remaining issue, which is that the baseline marginal utility of all goods should be positive for all consumption bundles ($\tilde{\pi}_k > 0$, $k = 1,2,...,K$). The only way this condition will hold globally is if $\theta_{km} \geq 0$ for all k and m (see Equation (6)). The condition $\theta_{km} > 0$ implies that the goods k and m are complements (since the consumption of good m would increase the baseline marginal utility of good k and therefore consumption of good k). However, we would also like to allow rich substitution patterns in the utilities of goods by allowing $\theta_{km} < 0$ for some pairs of goods. As we discuss later, our methodology accommodates this, while also recognizing the constraint $\tilde{\pi}_k > 0$ ($k = 1,2,...,K$) during estimation and ensuring that it holds in the range of consumptions observed in the data.

To summarize, we propose the following general formulation for the non-additively separable utility specification:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \left(\frac{\gamma_k}{\alpha_k} \left[\left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right] \left\{ \psi_k + \frac{1}{2} \sum_{m \neq k} \theta_{km} \frac{\gamma_m}{\alpha_m} \left[\left(\frac{x_m}{\gamma_m} + 1 \right)^{\alpha_m} - 1 \right] \right\} \right). \tag{8}$$

Note that the above function is obtained by simply setting θ_{kk} parameters to zero in the Vásquez-Lavín and Hanemann (2008) function in Equation (3). Further, as discussed earlier, the analyst will need to estimate the γ-profile or the α-profile. The γ-profile takes the following form:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \left(\gamma_k \ln\left(\frac{x_k}{\gamma_k} + 1 \right) \left\{ \psi_k + \frac{1}{2} \sum_{m \neq k} \theta_{km} \, \gamma_m \ln\left(\frac{x_m}{\gamma_m} + 1 \right) \right\} \right), \tag{9}$$

and the α-profile takes the following form:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K} \left(\frac{1}{\alpha_k} \left[(x_k + 1)^{\alpha_k} - 1 \right] \left\{ \psi_k + \frac{1}{2} \sum_{m \neq k} \frac{\theta_{km}}{\alpha_m} \left[(x_m + 1)^{\alpha_m} - 1 \right] \right\} \right). \tag{10}$$

In the case that a γ-profile is estimated, the γ_k values need to be greater than zero, which can be maintained by reparameterizing γ_k as $\exp(\kappa_k)$. Additionally, the translation parameters can be allowed to vary across individuals by writing $\kappa_k = \tilde{\boldsymbol{\kappa}}_k' \boldsymbol{w}_k$, where $\boldsymbol{w}_k$ is a vector of individual characteristics for the k^{th} alternative, and $\tilde{\boldsymbol{\kappa}}_k'$ is a corresponding vector of parameters. In the case when a α - profile is estimated, the α_k values need to be bounded from above at the value of 1. To enforce these conditions, α_k can be parameterized as $[1-\exp(-\delta_k)]$, with δ_k being the parameter that is estimated. Further, to allow the satiation parameters (*i.e.*, the α_k values) to vary across individuals, Bhat (2005) writes $\delta_k = \tilde{\boldsymbol{\delta}}_k' \boldsymbol{y}_k$, where $\boldsymbol{y}_k$ is a vector of individual characteristics impacting satiation for the k^{th} alternative, and $\tilde{\boldsymbol{\delta}}_k$ is a corresponding vector of parameters. In actual application, it would behoove the analyst to estimate models based on both the estimable profiles above, and choose the one that provides a better statistical fit. In the rest of this paper, we will use the general form in Equation (8) for the "no-outside good" case for ease in presentation.

Thus far, the discussion has assumed that there is no essential outside good. If an essential outside good is present, label the outside good as the first good which now has a unit price of one (*i.e.*, $p_1 = 1$). This good, being an outside good, has no interaction term effects with the inside goods; *i.e.*, $\theta_{1m} = 0 \ \forall m \ (m \neq 1)$. The utility functional form of Equation (8) now needs to be modified as follows:

$$U(\boldsymbol{x}) = \frac{1}{\alpha_1}(x_1+\gamma_1)^{\alpha_1}\psi_1 + \sum_{k=2}^{K}\left(\frac{\gamma_k}{\alpha_k}\left[\left(\frac{x_k}{\gamma_k}+1\right)^{\alpha_k}-1\right]\left\{\psi_k + \frac{1}{2}\sum_{\substack{m\neq k\\ m\neq 1}}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\right\}\right). \qquad (11)$$

In the above formula, we need $\gamma_1 \leq 0$, while $\gamma_k > 0$ for $k > 1$. Also, we need $x_1 + \gamma_1 > 0$. The magnitude of γ_1 may be interpreted as the required lower bound (or a "subsistence value") for consumption of the essential outside good. As in the "inside goods only" case, the analyst will generally not be able to estimate both α_k and γ_k for the outside and inside goods. The analyst will have to use either an α - profile or a γ - profile, though we will use the general form above for ease in presentation. For identification, we impose the condition that $\psi_1 = 1$.

3. THE ECONOMETRIC MODEL

We first consider the "no-outside" good setting, because the econometrics is more involved in this case. When an essential outside good is also present, the econometrics simplify considerably, as we will show after discussing the more involved case.

3.1. Optimal Consumption Allocations

The consumer maximizes utility $U(\boldsymbol{x})$ as provided by Equation (8) subject to the budget constraint that $\sum_{k=1}^{K} p_k x_k = E$, where p_k is the unit price of good *k* and *E* is total expenditure across all goods. The analyst can solve for the optimal consumption allocations by forming the Lagrangian and applying the KKT conditions. The Lagrangian function for the problem is:

$$L = U(\boldsymbol{x}) - \lambda\left[\sum_{k=1}^{K} x_k p_k - E\right], \tag{12}$$

where λ is the Lagrangian multiplier associated with the budget constraint (that is, it can be viewed as the marginal utility of total expenditure or income). The KKT first-order conditions for the optimal consumption allocations (the x_k^* values) are given by:

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k^*} - \lambda p_k = 0, \text{ if } x_k^* > 0, \; k = 1,2,...,K \tag{13}$$

$$\frac{\partial U(\boldsymbol{x})}{\partial x_k^*} - \lambda p_k < 0, \text{ if } x_k^* = 0, \; k = 1,2,...,K .$$

The precise form of the KKT conditions depends on how stochasticity is introduced in the model, and determines the model structure (note that the discussions in Section 2 were based on the assumption of a deterministic utility function).

3.2. Introducing Stochasticity in the Additively Separable (AS) Case

To complete the econometric model, the analyst needs to introduce stochasticity. This is an important component of the model formulation. As in Bhat (2008), we maintain that a stochastic component must be included in the context of each alternative *k*, rather than ignoring the stochastic component for one of the alternatives. This is because the probability expressions and the probability values of consumption of the different alternatives completely change based on which alternative's stochastic term is suppressed.[5] In Bhat's additively separable (AS) form of

[5] Studies that adopt a non-stochastic approach for one of the alternatives do so for an outside good that is always consumed. However, there is little reason to expect that the outside good is any different than the inside goods in terms of utility perceptions, and so the authors consider it conceptually and structurally inconsistent to consider the outside good's utility to be non-stochastic and the inside good's utilities to be stochastic. Besides, if an analysis is being conducted without an explicit outside good, it is essential to consider stochasticity in each alternative's utility contribution.

the utility function in Equation (3) (and in other restricted versions of this formulation), stochasticity is introduced using the following random specification:

$$U(\boldsymbol{x}) = \sum_k \frac{\gamma_k}{\alpha_k} \left[\psi(\boldsymbol{z}_k)\exp(\varepsilon_k)\right] \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\}, \tag{14}$$

where $\boldsymbol{z}_k$ is a set of attributes characterizing alternative k and the decision maker, and the ε_k terms are independent and identically distributed (IID) across alternatives with an extreme value distribution. ε_k captures idiosyncratic (unobserved) characteristics that impact the baseline utility for good k (the above stochastic utility form is equivalent to assuming a stochastic baseline (marginal) utility function given by $\psi(\boldsymbol{z}_k)\exp(\varepsilon_k)$). The exponential form for the introduction of the random term guarantees the positivity of the baseline marginal utility as long as $\psi(\boldsymbol{z}_k) > 0$. To ensure this latter condition, $\psi(\boldsymbol{z}_k)$ is further parameterized as $\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)$, where $\boldsymbol{\beta}$ is a vector of parameters. The KKT conditions corresponding to the random utility functional form of Equation (14) are thus stochastic and take the following form:

$$\psi(\boldsymbol{z}_k)\exp(\varepsilon_k) \left(\frac{x_k^*}{\gamma_k} + 1 \right)^{\alpha_k - 1} - \lambda p_k = 0, \text{ if } x_k^* > 0, \; k = 1,2,...,K \tag{15}$$

$$\psi(\boldsymbol{z}_k)\exp(\varepsilon_k) \left(\frac{x_k^*}{\gamma_k} + 1 \right)^{\alpha_k - 1} - \lambda p_k < 0, \text{ if } x_k^* = 0, \; k = 1,2,...,K \,.$$

According to this approach, any stochasticity in the KKT conditions originates from the analyst's inability to observe all factors relevant to the consumer's utility formation. Individuals are assumed to know all relevant factors impacting choice, and make an error-free maximization of overall utility (subject to the budget constraint) to determine their consumption patterns (this is the random utility-deterministic maximization or RU-DM decision postulate).

Note, however, that the stochastic KKT conditions above of the AS model could as well have been obtained using a deterministic utility specification (rather than a random utility specification) as follows:

$$U(\boldsymbol{x}) = \sum_k \frac{\gamma_k}{\alpha_k} \psi_k \left\{ \left(\frac{x_k}{\gamma_k} + 1 \right)^{\alpha_k} - 1 \right\}. \tag{16}$$

The KKT conditions corresponding to the above form are also deterministic (the conditions are identical to Equation (15), without the presence of the term $\exp(\varepsilon_k)$). But stochasticity can then be introduced explicitly in the KKT conditions in a multiplicative exponential form to once again obtain Equation (15). According to this view, not only is the consumer aware of all factors relevant to utility formation, but the analyst observes all of these factors too. However, consumers are assumed to make random mistakes ("errors") in maximizing utility (subject to the budget constraint), which gets manifested in the form of stochasticity in the KKT conditions (this is the deterministic utility-random maximization or DU-RM decision postulate; though they do not characterize this perspective as the DU-RM postulate, Wales and Woodland explicitly identify this alternative perspective for KKT models – see footnote 5 in their paper, page 268). While the DU-RM postulate is seldom used for KKT models in the econometric literature, it certainly is a plausible one that should not be summarily dismissed. It also allows the usual computations of compensating variation for welfare analysis (a common reason for modeling consumer preferences) as does the RU-DM postulate.

In the AS case, both the DU-RM and RU-DM decision postulates lead to exactly the same model (further, when the error terms ε_k are assumed to be IID across alternatives, the resulting model collapses to the surprisingly simple MDCEV model after using a logarithm transformation on the KKT conditions of Equation (15), as illustrated by Bhat, 2008). Since the two postulates are empirically indistinguishable, one can use either postulate to motivate the model. However, this ceases to be the case when moving from the AS utility form to the non-additively separable (NAS) utility functional form of Equation (8). In the next two sections, we discuss the DU-RM and RU-DM formulations, and show how a formulation that combines these two formulations in a random utility-random maximization (RU-RM) decision postulate is particularly convenient and general for the NAS case.

3.2.1 The DU-RM non-additively separable (NAS) utility formulation and model

Consider the utility form of Equation (8). For this deterministic NAS utility form, the corresponding deterministic KKT conditions are:

$$\tilde{\pi}_k \left(\frac{x_k^*}{\gamma_k} + 1 \right)^{\alpha_k - 1} - \lambda p_k = 0, \text{ if } x_k^* > 0, \; k = 1,2,...,K \tag{17}$$

$$\tilde{\pi}_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}-\lambda p_k<0,\ \text{if}\ x_k^*=0,\ k=1,2,...,K,$$

where $\tilde{\pi}_k$ is the baseline marginal utility as provided in Equation (6). Stochasticity may be introduced explicitly in the KKT conditions in the usual multiplicative exponential form as follows:

$$\tilde{\pi}_k\exp(\varepsilon_k)\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}-\lambda p_k=0,\ \text{if}\ x_k^*>0,\ k=1,2,...,K \tag{18}$$

$$\tilde{\pi}_k\exp(\varepsilon_k)\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}-\lambda p<0,\ \text{if}\ x_k^*=0,\ k=1,2,...,K.$$

Note that, unlike in the AS case, one cannot develop a random utility specification that corresponds to the KKT stochastic conditions in the equation above.[6]

The optimal demand satisfies the conditions in Equation (18) plus the budget constraint. The structure is now exactly the same as the MDCEV model of Bhat (2005, 2008). Specifically, consider an extreme value distribution for ε_k and assume that ε_k is independent of ψ_k, γ_k, and α_k ($k=1,2,...,K$). The ε_k terms are also assumed to be independently distributed across alternatives with a scale parameter of σ (σ can be normalized to one if there is no variation in unit prices across goods; see Bhat, 2008 for a detailed discussion of identification issues). In this case, the probability expression collapses to the following MDCEV closed-form:

[6] A random utility function of the form: $U(\boldsymbol{x})=\sum_{k=1}^{K}\left\{\left(\frac{\gamma_k}{\alpha_k}\left[\left(\frac{x_k}{\gamma_k}+1\right)^{\alpha_k}-1\right]\left\{\psi_k+\frac{1}{2}\sum_{m\neq k}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\right\}\right)\exp(\varepsilon_k)\right\}$ leads to stochastic KKT conditions that have the following leading term on the left side:

$$\left\{\psi_k+\frac{1}{2}\sum_{m\neq k}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\right\}\exp(\varepsilon_k)+\frac{1}{2}\sum_{m\neq k}\left(\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\exp(\varepsilon_m)\right)\left(\frac{1}{p_k}\right)\left(\frac{p_k x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}.$$

The expression above, which is effectively the baseline (marginal) utility of good *k*, is a function of all the error terms, and does not collapse to $\tilde{\pi}_k\exp(\varepsilon_k)$ as in the DU-RM NAS model. In any case, the random utility function form above is also not theoretically and conceptually intuitive, as we discuss later (see footnote 8).

$$P\left(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0\right) = |J_M| \frac{1}{\sigma^{M-1}} \left[\frac{\prod_{i=1}^{M} e^{V_i/\sigma}}{\left(\sum_{k=1}^{K} e^{V_k/\sigma} \right)^M} \right] (M-1)!, \tag{19}$$

where $V_k = \ln(\tilde{\pi}_k) - \ln p_k + (\alpha_k - 1)\ln\left(\frac{x_k^*}{\gamma_k} + 1\right)$ ($k = 1,2,...,K$), and the elements of the Jacobian J_M are given by:

$$J_{ih} = \frac{\partial[V_1 - V_{i+1} + \varepsilon_1]}{\partial x_{h+1}^*} = \frac{\partial[V_1 - V_{i+1}]}{\partial x_{h+1}^*}, \; i = 1,2,...,M-1; \; h = 1,2,...,M-1, \tag{20}$$

where the first alternative is an alternative to which the consumer allocates some non-zero budget amount (note that the consumer should allocate budget to at least one alternative, given that the total expenditure across all alternatives is a positive quantity). To write these Jacobian elements, define $z_{ih} = 1$ if $i = h$, and $z_{ih} = 0$ if $i \neq h$. Also, define the following:

$$\omega_k = \frac{1}{p_k}\left(\frac{x_k^*}{\gamma_k} + 1\right)^{\alpha_k - 1} \text{ for } k = 1, 2, ..., K. \tag{21}$$

Then, the elements of the Jacobian can be derived to be:[7]

$$J_{ih} = \omega_{h+1} p_{h+1} \left[\frac{\theta_{1,h+1}}{\tilde{\pi}_1} - (1 - z_{ih}) \frac{\theta_{i+1,h+1}}{\tilde{\pi}_{i+1}} \right] + \omega_1 p_1^2 \frac{\theta_{1,i+1}}{\tilde{\pi}_{i+1}} + p_{h+1}\left[L_1 p_1 + z_{ih} L_{i+1}\right], \tag{22}$$

where $L_k = \frac{1-\alpha_k}{p_k(x_k^* + \gamma_k)}$. Unfortunately, there is no concise form for the determinant of the Jacobian for $M > 1$ (unlike the case of the additively separable case, where Bhat derived a simple form for any value of *M*). When $M = 1$ (*i.e.*, only one alternative is chosen) for all individuals, there are no satiation effects ($\alpha_k = 1$ for all *k*), $\theta_{km} = 0 \; \forall k, m \; (k \neq m)$ and the Jacobian term drops out (that is, the continuous component drops out, because all expenditure is allocated to good 1). Then, the model in Equation (19) collapses to the standard MNL model.

In estimating the DU-RM model, as discussed in Section 2.1.1, we should ensure $\tilde{\pi}_k > 0$ for each good *k*. This is recognized in the logarithmic transformation of $\tilde{\pi}_k$ appearing in V_k. At

[7] The derivation is rather straightforward, but requires some cumbersome differentiation. Interested readers may obtain the derivation from the authors.

the same time, we also require that $\psi_k > 0$, which is ensured (as in the AS case) by writing $\psi_k = \exp(\boldsymbol{\beta}'\boldsymbol{z}_k)$. Also, since only differences in the V_k from V_1 matters in the KKT conditions, a constant cannot be identified in the term for one of the K alternatives. Similarly, individual-specific variables are introduced in the V_k's for (K-1) alternatives, with the remaining alternative serving as the base. The parameters in the DU-RM NAS-based model may be estimated in a straightforward way using the maximum likelihood inference approach. However, it is difficult to motivate generalized extreme value error structures and variable-specific random coefficients in the context of the DU-RM formulation. These extensions, however, are quite natural in the context of the RU-DM decision postulate, which we discuss in the next section.

For the DU-RM formulation with an essential outside good, the econometrics simplify considerably. One can go through the same procedure as earlier by writing the KKT conditions and introducing stochasticity corresponding to the deterministic utility expression in Equation (11) instead of Equation (8). For the outside good (say, the first alternative), we have the following: $\boldsymbol{\beta}'\boldsymbol{x}_1 = 0, \psi_1 = 1,$ and $p_1 = 1.$ The final expression for probability in this outside good case is the same as in Equation (19) with the following modifications to the V_k terms:

$$V_k = \ln(\tilde{\pi}_k) - \ln p_k + (\alpha_k - 1)\ln\left(\frac{x_k^*}{\gamma_k} + 1\right) \; (k > 2)\,;\; V_1 = (\alpha_1 - 1)\ln(x_1^* + \gamma_1)\,. \tag{23}$$

The Jacobian elements in this case simplify relative to Equation (22), with $\theta_{1m} = 0 \; \forall m \, (k \neq 1)$. The elements now are given as follows:

$$J_{ih} = \omega_{h+1}\left[-(1 - z_{ih})\frac{\theta_{i+1,h+1}}{\tilde{\pi}_{i+1}}\right] + p_{h+1}\left[L_1 + z_{ih}L_{i+1}\right]. \tag{24}$$

3.2.2. The RU-DM non-additively separable (NAS) utility formulation and model

Consider the following random utility form originating from the NAS utility function form of Equation (8) for the no-outside good case:

$$U(\boldsymbol{x}) = \sum_{k=1}^{K}\left(\frac{\gamma_k}{\alpha_k}\left[\left(\frac{x_k}{\gamma_k} + 1\right)^{\alpha_k} - 1\right]\left\{\psi_k \exp(\xi_k) + \frac{1}{2}\sum_{m \neq k}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m} + 1\right)^{\alpha_m} - 1\right]\right\}\right), \tag{25}$$

where ξ_k is an IID (across alternatives) random error term with a scale parameter of σ (σ can be normalized to one if there is no variation in the unit prices across alternatives). ξ_k captures idiosyncratic (unobserved) characteristics that impact the baseline (marginal) utility of good *k* at the point at which no expenditure outlays have yet been made on any alternative.[8,9] The KKT conditions then are:

$$\eta_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}-\lambda p_k=0\text{, if } x_k^*>0\text{, } k=1,2,...,K \tag{26}$$

$$\eta_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}-\lambda p_k<0\text{, if } x_k^*=0\text{, } k=1,2,...,K\text{ ,}$$

where $\eta_k=\psi_k\exp(\xi_k)+W_k$ and $W_k=\sum_{m\neq k}\theta_{km}\dfrac{\gamma_m}{\alpha_m}\left[\left(\dfrac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]$.

Define ω_k as in Equation (21). Let $R_k=\eta_1\cdot\dfrac{\omega_1}{\omega_k}-W_k$ and $\psi_k=\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)$, and let the first alternative be the one to which the consumer allocates some non-zero budget amount. Then, the KKT conditions may be simplified as follows:

$$\exp(\xi_k)=\frac{R_k\mid\xi_1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)}\text{, if } x_k^*>0\text{, } k=2,3,...,K \tag{27}$$

$$\exp(\xi_k)<\frac{R_k\mid\xi_1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)}\text{, if } x_k^*=0\text{, } k=2,3,...,K\text{ .}$$

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[8] Vásquez-Lavín and Hanemann indicate that introducing stochasticity in the multiplicative exponential form as in Equation (25) does not help in any way simplify the KKT first-order conditions. They proceed by writing the utility function as:
$U(\boldsymbol{x})=\sum_{k=1}^{K}\left(\dfrac{\gamma_k}{\alpha_k}\left[\left(\dfrac{x_k}{\gamma_k}+1\right)^{\alpha_k}-1\right]\left\{\boldsymbol{\beta}'\boldsymbol{z}_k+\varepsilon_k+\dfrac{1}{2}\sum_{m\neq k}\theta_{km}\dfrac{\gamma_m}{\alpha_m}\left[\left(\dfrac{x_m}{\gamma_m}+1\right)^{\alpha_m}-1\right]\right\}\right)$. The problem with this, however, is that it allows negative values for $\boldsymbol{\beta}'\boldsymbol{z}_k+\varepsilon_k$, which is theoretically inappropriate since this term has to be positive for $U(\boldsymbol{x})$ to be a valid utility function. Besides, as we will show below, there is really no computational advantage whatsoever in assuming a linear form relative to a multiplicative exponential form.

[9] Stochasticity in utility is assumed in a specific form in Equation (25), where the supposition is that the analyst does not observe all factors that influence the baseline (marginal) utility for each alternative at the point when no consumption decisions have yet been made (we will refer to this baseline utility as the "no-consumption" baseline (marginal) utility). This stochastic specification is quite intuitive, since it indicates an intrinsic (unobserved) individual preference for each alternative whose magnitude remains stable as the consumer navigates to reach her/his optimal expenditure point. The alternative way of including utility stochasticity as in the equation in footnote 5 is very difficult to justify and interpret, since it postulates that the magnitude of unobserved individual factors influencing the baseline (marginal) utility for any specific alternative varies continuously during the navigation process and is a complex parametric function of the extents of the "no-consumption" unobserved individual preferences of all alternatives. Besides, the econometrics involved with such a utility specification is extremely difficult. Thus, from both an intuitive and econometric perspective, we suggest that Equation (25) is the appropriate one to use in the context of a random utility specification for the NAS case.

Next, let $\zeta_k = \exp(\xi_k)$, and assume that $g(.)$ and $G(.)$ are the standardized versions of the probability density function and standard cumulative distribution function characterizing ζ_k. Then, the probability that the individual allocates expenditure to the first *M* of the *K* goods may be derived to be:

$$P\left(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0\right)$$
$$= \int_{\zeta_1=0}^{\zeta_1=+\infty} |J_M \mid \zeta_1| \left\{\left(\prod_{i=2}^{M} \frac{1}{\sigma} g\left[\frac{1}{\sigma} \frac{R_k \mid \zeta_1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)}\right]\right)\right\} \times \left\{\prod_{s=M+1}^{K} G\left[\frac{1}{\sigma} \frac{R_k \mid \zeta_1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_k)}\right]\right\} f(\zeta_1) d\zeta_1, \tag{28}$$

where $f(.)$ refers to the density function characterizing ζ_1, and $J_M \mid \xi_1$ is the Jacobian whose elements are given by ($i, h = 1,2,..., M-1$):

$$J_{ih} = \frac{\partial}{\partial x_{h+1}^*}\left[\frac{R_{i+1} \mid \zeta_1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_{i+1})}\right] = \frac{1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_{i+1})}\left\{\frac{\omega_1}{\omega_{i+1}}\left[(\eta_1 \mid \zeta_1)(p_1^2 L_1 + p_{h+1} L_{i+1} z_{ih}) + p_{h+1}\theta_{1,h+1}\omega_{h+1}\right]\right.$$
$$\left. + p_{h+1}\left[p_1 \theta_{1,h+1}\omega_1 - \theta_{i+1,h+1}\omega_{h+1}(1 - z_{ih})\right]\right\}. \tag{29}$$

In the above expression, $z_{ih} = 1$ if $i = h$, and $z_{ih} = 0$ if $i \neq h$ and $L_k = \dfrac{1-\alpha_k}{p_k(x_k^* + \gamma_k)}$ ($k = 1,2,..., K$).

The probability expression in Equation (28) is a simple one-dimensional integral, which can be computed using quadrature techniques. Note that the distribution for ξ_k can be any univariate distribution, though the normal distribution may be convenient if there are also random normal coefficients in the $\boldsymbol{\beta}$ vector to capture unobserved individual heterogeneity (then the one-dimensional normal integral becomes simply a part of a multi-dimensional normal integration that can be evaluated using familiar simulation techniques). Such a random-coefficients specification allows a flexible covariance structure between the elements of the $\boldsymbol{\beta}$ vector, and can also include covariances among the baseline utilities of alternatives (as in a mixed multinomial logit structure). The model may be estimated using traditional maximum likelihood techniques, as for the DU-RM formulation. Note, however, that the marginal utility of any good at any point of consumption should be positive (for strictly increasing utility functions). This condition is met by setting $\eta_k > 0$ (see Equation (26)) for each good *k*.

When an essential outside good is present, the econometrics again simplify considerably. For the outside good (say, the first alternative), we have the following: $W_1 = 0$, $\boldsymbol{\beta}'\boldsymbol{z}_1 = 0$, $\psi_1 = 1$, $p_1 = 1$, and $\eta_1 = \zeta_1$. The random utility function originates from Equation (11) and takes the following form:

$$U(\boldsymbol{x}) = \frac{1}{\alpha_1}(x_1 + \gamma_1)^{\alpha_1}\ \psi_1 \exp(\xi_1) \\ + \sum_{k=2}^{K}\left(\frac{\gamma_k}{\alpha_k}\left[\left(\frac{x_k}{\gamma_k}+1\right)^{\alpha_k} - 1\right]\left\{\psi_k \exp(\xi_k) + \frac{1}{2}\sum_{\substack{m\neq k\\ m\neq 1}}\theta_{km}\frac{\gamma_m}{\alpha_m}\left[\left(\frac{x_m}{\gamma_m}+1\right)^{\alpha_m} - 1\right]\right\}\right). \tag{30}$$

The probability expression takes the same form as in Equation (28) with the following modifications to the ω_k terms:

$$\omega_k = \frac{1}{p_k}\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k - 1} \text{ for } k = 2, ..., K;\ \ \omega_1 = (x_1^* + \gamma_1)^{\alpha_1 - 1}. \tag{31}$$

The Jacobian elements are as follows ($i, h = 1,2,..., M-1$):

$$J_{ih} = \frac{1}{\exp(\boldsymbol{\beta}'\boldsymbol{z}_{i+1})}\left\{\frac{\omega_1}{\omega_{i+1}}\left[(\eta_1 \mid \zeta_1)(L_1 + p_{h+1}L_{i+1}z_{ih})\right] - p_{h+1}\theta_{i+1,h+1}\omega_{h+1}(1 - z_{ih})\right\}. \tag{32}$$

3.2.3. The RU-RM non-additively separable (NAS) utility formulation and model

Consider the random utility function of Equation (25) for the case with no essential outside good. The KKT conditions are given by Equation (26), but we now add stochasticity originating from consumer mistakes in the optimizing process.[10] The KKT conditions take the form shown below:

$$\eta_k \exp(\varepsilon_k)\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k - 1} - \lambda p_k = 0, \text{ if } x_k^* > 0,\ k = 1,2,...,K \tag{33}$$

$$\eta_k \exp(\varepsilon_k)\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k - 1} - \lambda p_k < 0, \text{ if } x_k^* = 0,\ k = 1,2,...,K\ ,$$

[10] Intuitively, we are able to distinguish between random preferences and random maximization errors in the NAS case because the former is associated with the "no-consumption" baseline (marginal) utilities that then remains fixed during the consumer's navigation through the optimization process, while the latter is essentially associated with overall mistakes represented by random errors in the baseline (marginal) utilities after including substitution/complementarity effects. In the AS case, both these sources of stochasticity become one and the same because there are no substitution/complementarity effects, leading to an identification problem.

where η_k is as defined earlier in Equation (26) (and has the error term ξ_k embedded within), and the ε_k terms are independent and identically (across alternatives) extreme value distributed. Recall that the ξ_k terms represent stochasticity due to the analyst's inability to capture consumer preferences, while the ε_k terms represent stochasticity due to consumer errors in utility maximization. Let $Var(\varepsilon_k) + Var(\xi_k) = (\pi^2\sigma^2)/6 \ (k = 1,2,...K)$.[11] In the RU-RM formulation, we assume that the ξ_k terms are normally distributed. This is particularly convenient when one wants to accommodate a flexible error covariance structure through a multivariate normal-distributed coefficient vector $\boldsymbol{\beta}$ and/or account for covariance in utilities across alternatives through the appropriate random multivariate specification for the ξ_k terms. To develop the probability function for consumptions, let $Var(\varepsilon_k) = \mu^2(\pi^2\sigma^2)/6$ and $Var(\xi_k) = (1-\mu^2)(\pi^2\sigma^2)/6 \ (k = 1,2,...,K)$, where μ is a parameter to be estimated $(0 \le \mu \le 1)$. Then, if $\mu \to 0$, and when there is no covariance among the ξ_k terms across alternatives, the RU-RM formulation approaches the RU-DM formulation of Section 3.2.2 in which the scale parameter σ is innocuously rescaled to $(\pi/\sqrt{6})\,\sigma$, so that the variance of the error terms ξ_k in the RU-DM formulation is comparable to the variance of the corresponding terms in the RU-RM formulation. However, as $\mu \to 1$, the RU-RM formulation approaches the DU-RM formulation. Thus, the parameter μ determines the extent of the mix of the RU-DM and DU-RM decision postulates leading up to the observed behavior of consumers. One can impose the constraint that $0 \le \mu \le 1$ through the use of a logistic transform $\mu = 1/(1+\exp(-\mu^*))$ and estimate the parameter μ^*.

The probability expression for consumptions in the RU-RM model formulation takes the following mixed MDCEV form:

$$P\left(x_1^*, x_2^*, x_3^*, ..., x_M^*, 0, 0, ..., 0\right) = \int_{\xi=-\infty}^{\infty} \left[\left|J_M \mid \xi\right| \frac{1}{(\mu\sigma)^{M-1}} \left[\frac{\prod_{i=1}^{M} e^{[V_i/(\mu\sigma)] \mid \xi_i}}{\left(\sum_{k=1}^{K} e^{[V_k/(\mu\sigma)] \mid \xi_k}\right)^M} \right] (M-1)! \right] dF(\xi), \quad (34)$$

[11] As earlier, we will impose the normalization that $\sigma^2 = 1$ if there is no price variation across the alternatives.

where $V_k = \ln(\eta_k) - \ln p_k + (\alpha_k - 1)\ln\left(\frac{x_k^*}{\gamma_k} + 1\right)$, $\eta_k = \psi_k \exp(\xi_k) + W_k$, W_k is defined as earlier, and F is the multivariate normal distribution of the random element vector $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_K)$ (each of whose elements has a variance of $(1-\mu^2)(\pi^2\sigma^2)/6$). The elements of the Jacobian are given by:

$$J_{ih} = \omega_{h+1} p_{h+1} \left[\frac{\theta_{1,h+1}}{(\eta_1 \mid \xi_1)} - (1 - z_{ih}) \frac{\theta_{i+1,h+1}}{(\eta_{i+1} \mid \xi_{i+1})} \right] + \omega_1 p_1^2 \frac{\theta_{1,i+1}}{(\eta_{i+1} \mid \xi_{i+1})} + p_{h+1}\left[L_1 + z_{ih} L_{i+1}\right]. \tag{35}$$

When there is an essential outside good, the probability expression remains the same as in Equation (34), but with $V_k = \ln(\eta_k) - \ln p_k + (\alpha_k - 1)\ln\left(\frac{x_k^*}{\gamma_k} + 1\right)$ $(k > 2)$, $V_1 = (\alpha_1 - 1)\ln(x_1^* + \gamma_1)$, $\theta_{1m} = 0 \ \forall m \ (m \neq 1)$, $W_1 = 0$, $\boldsymbol{\beta}'\boldsymbol{z}_1 = 0$, $\psi_1 = 1$, $p_1 = 1$, and $\eta_1 = \exp(\xi_1)$. The Jacobian elements in this case are given as follows:

$$J_{ih} = \omega_{h+1} \left[-(1 - z_{ih}) \frac{\theta_{i+1,h+1}}{(\eta_{i+1} \mid \xi_{i+1})} \right] + p_{h+1}\left[L_1 + z_{ih} L_{i+1}\right]. \tag{36}$$

Similar to the earlier two formulations, the theoretical condition that the marginal utility of consumption for any alternative should always be positive must be ensured during model estimation. Thus, we should ensure $\eta_k > 0$ for each good *k*.

4. EMPIRICAL DEMONSTRATION

4.1. The Context

In 2010, transportation expenses accounted for nearly 20% of total household expenses and 12-15% of total household income (U.S. Bureau of Labor Statistics, 2012). In fact, this is the second largest family expense category after housing, with an average expenditure of $7,677 per year (or, equivalently, about $650 per month). It is little surprise, therefore, that the study of transportation expenditures has been of much interest in recent years (Gicheva *et al.*, 2007, Cooper, 2005, Hughes *et al.*, 2006, Thakuriah and Liao, 2006, Choo *et al.*, 2007a,b, Sanchez *et al.*, 2006). Several of these studies examine the factors that influence total household transportation expenditures and/or examine transportation expenditures in relation to expenditures on other commodities and services (such as in relation to housing, telecommunications, groceries, and eating out). But there has been relatively little research on identifying the many disaggregate-level components of transportation expenditures, with all transportation expenditures usually lumped into a single category. Besides, many of these earlier efforts use the almost ideal demand system (AIDS) proposed by Deaton and Muelbauer (1980), which assumes that all families expend their budgets in all possible expenditure categories (that is, the AIDS model does not allow corner solutions, as does our proposed model).

In the current paper, we demonstrate the use of the proposed model for an empirical case of household transportation expenditures in six disaggregate categories: (1) Vehicle purchase, (2) Gasoline and motor oil (termed as gasoline in the rest of the document), (3) Vehicle insurance, (4) Vehicle operation and maintenance (labeled as vehicle maintenance from hereon), (5) Air travel, and (6) Public transportation. In addition, we consider all other household expenditures in a single "outside good" category that lumps all non-transportation expenditures, so that total transportation expenditure is endogenously determined. Households expend some positive amount on the "outside good" category, while expenditures can be zero for one or more transportation categories for some households. A non-additively separable utility form is adopted to accommodate rich substitution patterns as well as to allow complementarity among the transportation expenditure categories.

Data for the analysis is drawn from the 2002 Consumer Expenditure (CEX) Survey, which is a national level survey conducted by the U.S. Census Bureau for the Bureau of Labor Statistics (U.S. Bureau of Labor Statistics, 2003). This survey has been administered regularly since 1980 and is designed to collect information on incomes and expenditures/buying habits of households in the United States. In addition, information on individual and household socio-economic, demographic, employment and vehicle characteristics is also collected. Details of the data and sample extraction process for the current analysis are available in Ferdous *et al*. (2010). Essentially, the 109 categories of expenditure and income defined by the CEX were consolidated, defining 17 broad categories of annual expenditures (including the six categories of transportation expenditures identified in the previous paragraph). Next, the 11 non-transportation categories were all grouped into a single "outside good" category, and the proportion of total expenditures (across the six transportation categories and the "outside good" category) spent in each of the six transportation categories and the "outside" non-transportation category were constructed as the dependent variables in the analysis.

The final sample for analysis includes 4100 households. About one-quarter of the sample reports expenditures on vehicle purchase. 94% of the sample incurs expenditures on gasoline, and 90% of the sample indicates vehicle maintenance expenses. About 80% of the sample has vehicle-insurance related expenses, suggesting that a sizeable number of households operate motor vehicles with no insurance or have insurance costs paid for them (possibly by an employer or self-employed business). About one-third of the sample reports spending money on public transportation and air travel. Only 2.6% of the households expend no money in transportation-related expenses. These households may undertake trips using non-motorized modes, or rely on someone else to travel. Altogether, expenditures on transportation-related items account for about 15% of household income, a figure that is quite consistent with reported national figures. Of the 4100 households, a random sample of 3600 households was used for model estimation and the remaining sample of 500 households was held for out-of-sample validation.

4.2 Model Specification and Estimation

The additively separable (AS) and non-additively separable (NAS) models were estimated using the GAUSS matrix programming language.[12] We first estimated the best empirical specification for the MDCEV model (assuming additive separability) based on intuitive and statistical significance considerations, and then explored alternative specifications for the interaction parameters in the NAS model for the three model formulations proposed. The γ-profile of Equation (9) was used in all specifications, since it consistently provided a better model fit than the α-profile. Also, the γ_1 value for the outside good was set to zero for estimation stability.

Recall that the DU-RM formulation assumes extreme value random error terms for the random mistakes made by the consumer during his/her optimization process, while the RU-DM specification assumes normally distributed random terms for the analyst's errors in characterizing the consumer's utility functions. In the absence of interactions between the sub-utility functions of different alternatives, the DU-RM formulation collapses to the AS MDCEV model, while the RU-DM formulation collapses to an AS MDC model with IID normal (or probit) error terms (label this as the MDCP for MDC probit model). Thus, for model evaluation purposes, the analyst can compare the performance of the DU-RM model to its special case

[12] Gauss^{TM}, Aptech Systems Inc., Maple Valley, WA, USA, http://www.aptech.com.

MDCEV and that of the RU-DM model to its special case MDCP. The RU-RM formulation utilizes a combination of extreme value error terms and normally distributed error terms for the consumer's mistakes and the analyst's errors, respectively. Thus, for this last formulation there is no direct AS model to be compared with. However, as discussed in Section 3.2.3, the RU-DM and DU-RM formulations are limiting case of the RU-RM formulation.

The estimation of the three model formulations was undertaken to explicitly consider the constraint that the marginal utility of any good at any consumption point for each good *k* should always be positive. In the current empirical application, our attempts to use the constrained maximum likelihood module of GAUSS to estimate the models encountered estimation instability and convergence problems. Therefore, the models were estimated using the traditional maximum likelihood module of GAUSS, while checking for the positivity of the marginal utility at each iteration and heuristically updating parameters to cause the least departure from the iteration-search parameters and still ensuring positivity if positivity was not maintained (in most iterations, positivity was maintained automatically). The DU-RM NAS model was estimated imposing $\tilde{\pi}_k > 0$ for each good *k* (see Equation (6)), since the term $\tilde{\pi}_k$ is inside a logarithmic function. For the RU-DM and RU-RM NAS models, the baseline marginal utility is given by $\eta_k\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}$. Because the term $\left(\frac{x_k^*}{\gamma_k}+1\right)^{\alpha_k-1}$ is always positive, we have to constrain $\eta_k > 0$. In the estimation of the RU-DM and RU-RM NAS models formulations, we imposed the more restrictive condition $W_k > 0$ to ensure that the condition is fulfilled for all values of ξ_k (ξ_k is embedded in η_k; see Equation (26)). Quadrature techniques for log-normally distributed variables were used to evaluate the integral in Equation (28) for the RU-DM NAS model formulation (details are available from the authors). To evaluate the multivariate integral of Equation (34) for the RU-RM NAS model, we used the Halton sequence to draw realizations for $\boldsymbol{\xi} = (\xi_1, \xi_2, ..., \xi_K)$ from a normal distribution, assuming in the empirical analysis that these error terms are independent and identically distributed across alternatives. Details of the Halton sequence and the procedure to generate this sequence are available in Bhat (2003). We tested the sensitivity of parameter estimates with different numbers of Halton draws per observation, and found the results to be very stable with as few as 75 draws. In this analysis we used 100 draws per household in the estimation.

4.3 Model Results

The estimation results are provided in Table 1. At the outset, we should note that the intent of this empirical analysis is not to contribute in a substantive way to an analysis of household expenditures. Rather, the emphasis is on demonstrating the applicability of the three different NAS formulations proposed in this paper, and showing the advantage of the NAS formulations relative to the traditional AS formulations. To that extent, the focus is on in-sample and out-of-sample data fits of the NAS and AS formulations, as well as on demonstrating the significant presence of NAS interaction parameters in our NAS utility formulations.

Table 1 is organized in three main columns. The first main column provides the parameters estimates of the DU-RM NAS model and its restrictive MDCEV formulation, while the second main column presents the results of the RU-DM NAS model and its restrictive MDCP formulation. The third column provides the parameters estimates of the RU-RM non-AS model. As discussed in Section 3, one of the alternatives forms the base category for the introduction of the family-specific variables in the baseline utility in Table 1. This base alternative is the essential outside good, which is the non-transportation good category in the current analysis. If, in addition, some transportation categories do not appear for a variable in Table 1, it implies that these transportation categories also constitute the base expenditure category along with the non-transportation category. For example, for the effect of "Number of workers in the household", the base categories include the non-transportation category as well as the air travel and public transportation categories. A positive (negative) coefficient for a certain variable-category combination implies that an increase in the explanatory variable increases (decreases) the likelihood of budget being allocated to that expenditure category relative to the base expenditure categories.

Overall, the empirical results are intuitive. Also, while there are differences in the estimated coefficients between the AS and NAS models, the general pattern and direction of variable effects are similar. Regarding the baseline parameters ($\boldsymbol{\beta}$), the alternative specific constants in the baseline utility for all the transportation categories are negative, indicating the generally higher baseline utility of the "outside" non-transportation good category relative to each transportation category (this is a reflection of the higher expenditure on the outside good than on the transportation categories). Similar to the results found by Thakuriah and Liao (2005), as the number of workers in the household increases, so does the proportion of income allocated to all vehicle-related transportation expenses, presumably to support the transportation needs of multi-worker households (an exception is in the RU-DM model, in which the coefficient associated with vehicle insurance is negative but statistically insignificant). The effect of income was considered in a continuous linear form, in a piecewise linear form to introduce non-linearities, as well as in the form of dummy variables for specific income categories. At the end, a dummy variable specification with low income (less than 30K), mid-range income (30-70K), and high income (>70K) provided the best results. The effect of this discrete representation of income is incorporated with the low income category constituting the base category (and so the low income category does not appear in Table 1). The results indicate that, relative to families in the low income group, families in the middle and high income groups expend a higher proportion of their income on vehicle purchases and air travel. These families also spend a lower proportion of their income on gasoline relative to the low income group, suggesting that gasoline expenditures constitute a particularly high proportion of the income budgets of low income families. A detailed discussion of this result from a social and environmental justice perspective can be found in Deka (2004). Households with more vehicles tend to allocate a larger proportion of their income to all the transportation categories, except on public transportation. Finally, non-Caucasians, those residing in urban areas, and those living in the Northeast and West regions of the U.S. spend a higher proportion on public transportation than Caucasians, those residing in non-urban areas, and those living in the South and Midwest regions of the U.S, respectively.

The satiation parameters (γ_k) in Table 1 capture the variation in the extent of non-linearity across different expenditure categories. The satiation parameter is highest for the vehicle purchase category, indicating that households are likely to allocate a large proportion of their budget to acquiring a vehicle, if they expend any money in this category. The satiation parameter is lowest for gasoline, indicating that households allocate a relatively small proportion of their overall budget in gasoline consumption.

Several interaction parameters (θ_{km}) are statistically significant in the final model specification presented in Table 1. The interaction parameters of the DU-RM NAS model indicate a significant complementary effect in vehicle purchase and gasoline expenditures, and in vehicle purchase and vehicle maintenance expenditures. Also, as expected, there are complementary effects in the expenditures on gasoline, vehicle insurance, and vehicle maintenance, as well as between air travel and public transportation expenditures. This last complementary effect perhaps reflects the use of public transportation to get to/from the airport and the use of public transportation at the non-home end. On the other hand, there are particularly sensitive substitution effects in gasoline and air transportation expenditures, presumably a reflection of the choice between auto travel and air transportation mode travel for long-distance trips. For the RU-DM NAS model formulation, only complementarity effects were statistically significant, which align with the results of the DU-RM NAS model. The RU-RM model interaction parameters show significant complementarity effects similar to those from the DU-RM and RU-DM models, along with a strong substitution effect between vehicle purchase and public transportation expenditures. This latter substitution effect is more intuitive than the complementary effect between vehicle purchase and public transportation expenditures, as implied by the RU-DM model.

As mentioned in Section 3.2.3, the RU-RM NAS formulation combines the RU-DM and DU-RM postulates of consumer behavior via the parameter μ. In the current empirical analysis, we obtained $\mu = 0.379$. The parameter is statistically different from zero (with a t-stat of 58.51 as shown in Table 1) and statistically different from one (with a t-stat of 95.60). The μ parameter is closer to zero than it is to one, indicating that the predominant source of stochasticity (62%) is due to the analyst's errors in characterizing the consumer's utility function. To a lesser extent (38%), stochasticity arises also from the random "mistakes" consumers make during utility maximization.

4.4. Model Evaluation

In this section, we compare the model performance of the AS and NAS models, both in the estimation sample of 3600 households as well as a validation sample of 500 households.

In terms of model fit in the estimation data, the log-likelihood value at convergence of the DU-RM NAS model is -36,645, while that of the MDCEV model is -37,045. A likelihood ratio test between these two models returns a value of 799, which is larger than the chi-squared statistic value with 7 degrees of freedom at any reasonable level of significance, indicating the substantially superior fit of the DU-RM NAS model compared to the MDCEV model. Similarly, the log-likelihood value at convergence of the RU-DM NAS model is -35,086, while the same figure for the MDCP model is -35,269. The likelihood ratio test between the RU-DM and MCDP models is 366, which again indicates a statistically significant difference in data fit between the models. The log-likelihood value at convergence of the RU-RM NAS model is -34,168, which is considerably higher than the corresponding value for the MDCEV and MDCP models. The RU-RM NAS model log-likelihood is also far superior to the log-likelihood values of the DU-RM amd RU-DM models, underscoring the presence of stochasticity on the part of both the analyst and the consumer.[13]

To further compare the performance of the MDCEV and NAS models, we computed an out-of-sample log-likelihood function (OSLLF) using the validation sample of 500 observations. The OSLLF is computed by plugging in the out-of-sample (*i.e*., validation) observations into the log-likelihood function, while retaining the estimated parameters from the estimation sample. As indicated by Norwood *et al.* (2001), the model with the highest value of OSLLF is the preferred one, since it is most likely to generate the set of out-of-sample observations. Table 2 reports the OSLLF values for the entire validation sample (of 500 households) as well as for different socio-demographic segments within the sample. As can be observed from the first row, the OSLLF value for the DU-RM model is better than for the MDCEV model, and the OSLLF value for the RU-DM model is better than for the MDCP model. This result is also maintained, in general, for all socio-demographic segments. Also, in general, the RU-RM formulation outperforms all other formulations, except in a few isolated segments with few observations.

In summary, the data fits of the NAS models are superior to that of the AS models in both the estimation and validation samples.

5. CONCLUSIONS

Classical discrete and discrete-continuous models deal with situations where only one alternative is chosen from a set of mutually exclusive alternatives. Such models assume that the alternatives are perfectly substitutable for each other. On the other hand, many consumer choice situations

[13] An interesting result from Table 1 that is not directly relevant to the current paper, but of general interest, is that the MDCP model provides a much better data fit relative to the MDCEV model. Note that these differences are simply an artifact of using an IID extreme value distribution for the errors in the MDCEV as opposed to an IID normal distribution for the errors in the MDCP (since, in the AS case, the DU-RM and RU-DM formulations become identical up to the distribution chosen for the error terms). Thus, the choice of the error term distribution in MDC models does not seem as innocuous as that in traditional discrete choice models (where, for example, the results from a binary probit model and a binary logit model tend to be almost identical after parameter scaling). This result is not entirely surprising, since the MDC model uses both the probability density function as well as the cumulative distribution function of the error terms (or, more precisely, of the error term differences) to accommodate the combined discrete-continuous nature of the formulation, while traditional discrete choice models use only the cumulative distribution function of error differences (and cumulative distribution functions can be relatively similar even for random distributions with quite different continuous probability density functions).

are characterized by the simultaneous demand for multiple alternatives that are imperfect substitutes or even complements for one another. Traditional MDC models developed in the literature adopt an additively-separable utility form that assumes that the marginal utility of a good is independent of the consumption amounts of other goods. It also is not able to allow complementarity among goods. This paper develops model formulations that allow a non-additive utility structure and complementarity effects. As importantly, the utility functional form proposed here remains within the class of flexible forms, while also retaining global theoretical consistency properties (unlike the Translog and related flexible quadratic functional forms). The result is also clarity in the interpretation of the model parameters. Stochasticity is introduced in the formulation in three different ways to develop three possible models for non-additive utility structures. In the first stochastic formulation, labeled as the deterministic utility–random maximization or DU-RM decision postulate, consumers are assumed to make random mistakes in maximizing utility. In the second stochastic formulation, labeled as the random utility-deterministic maximization or RU-DM decision postulate, consumers are assumed to know all relevant factors impacting their choices and make an error-free maximization of overall utility, but the analyst is not aware of all the factors influencing consumer's choice. The third stochastic formulation combines the two previous postulates into a random utility-random maximization (RU-RM) decision postulate.

The proposed non-additively separable model formulations should have several applications. In the current paper, we demonstrate the application of the formulations to the empirical case of household transportation expenditures in six disaggregate categories: (1) Vehicle purchase, (2) Gasoline and motor oil, (3) Vehicle insurance, (4) Vehicle operation and maintenance, (5) Air travel, and (6) Public transportation. In addition, we consider other household expenditures in a single "outside good" category that lumps all non-transportation expenditures, so that total transportation expenditure is endogenously determined. Households expend some positive amount on the "outside good" category, while expenditures can be zero for one or more transportation categories for some households. Data for the analysis is drawn from the 2002 Consumer Expenditure (CEX) Survey, which is a national level survey conducted by the US Census Bureau for the Bureau of Labor Statistics. The results of the DU-RM, RU-DM and RU-RM non-additively separable formulations suggest statistically significant complementary and substitution effects in the utilities of selected pairs of transportation categories, and show the substantially superior data fit of the proposed formulations relative to ones that assume an additively separable utility structure. The proposed non-additive separable models performed better in a validation sample as well.

In summary, the paper has successfully formulated and applied different forms of MDC models with non-additively separable utility functional forms. One area for further research is to develop more formal and rigorous methods to ensure the positivity of the marginal utility for each observation at each estimation iteration. Currently, we aided the estimation procedure by heuristically (and somewhat in an *ad hoc* manner) updating parameters to cause the least departure from the iteration-search parameters and still ensuring positivity (if positivity was not maintained automatically).

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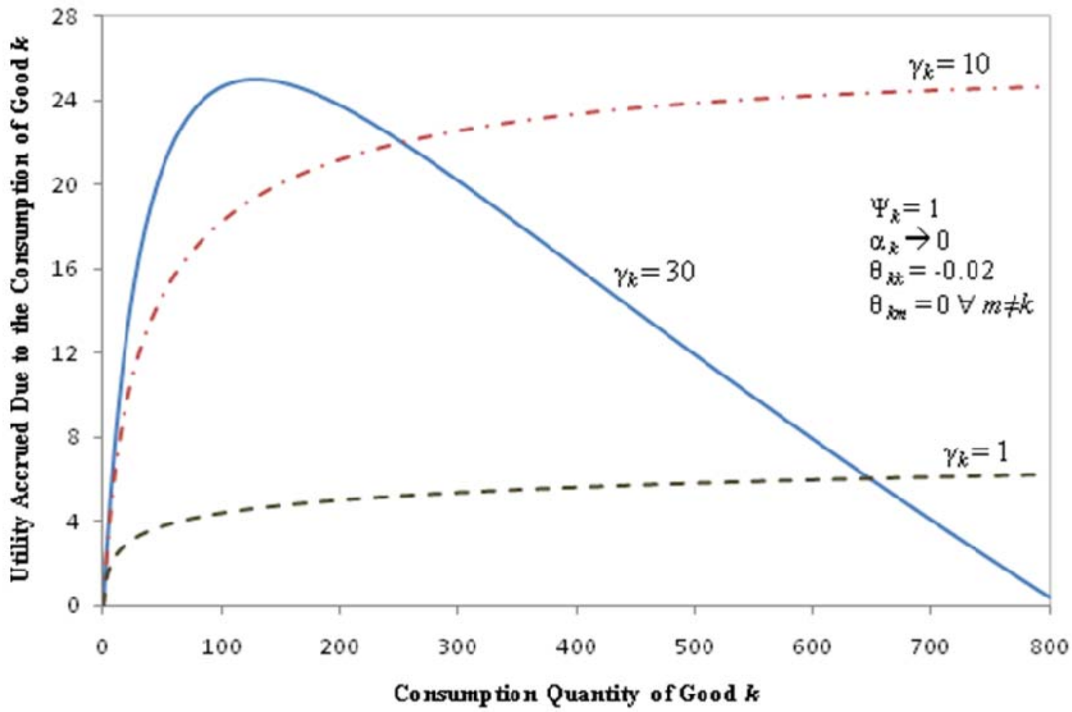

Figure 1. Effect of γ_k , due to Negative θ_{kk} , on Good *k*'s Subutility Function Profile

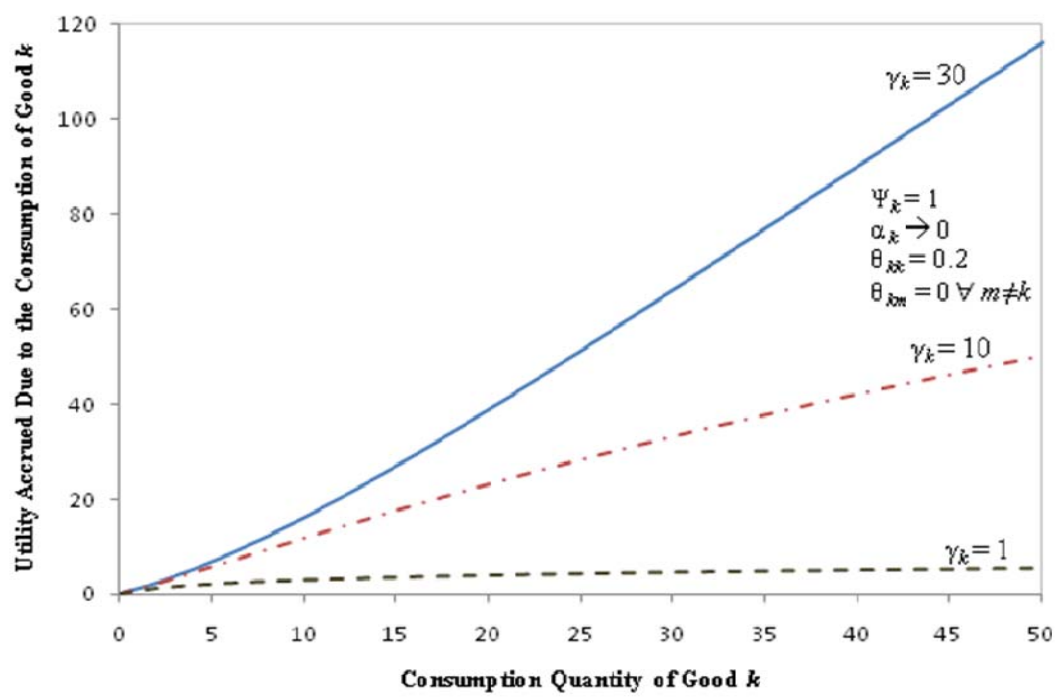

Figure 2. Effect of γ_k , due to Positive θ_{kk} , on Good *k*'s Subutility Function Profile

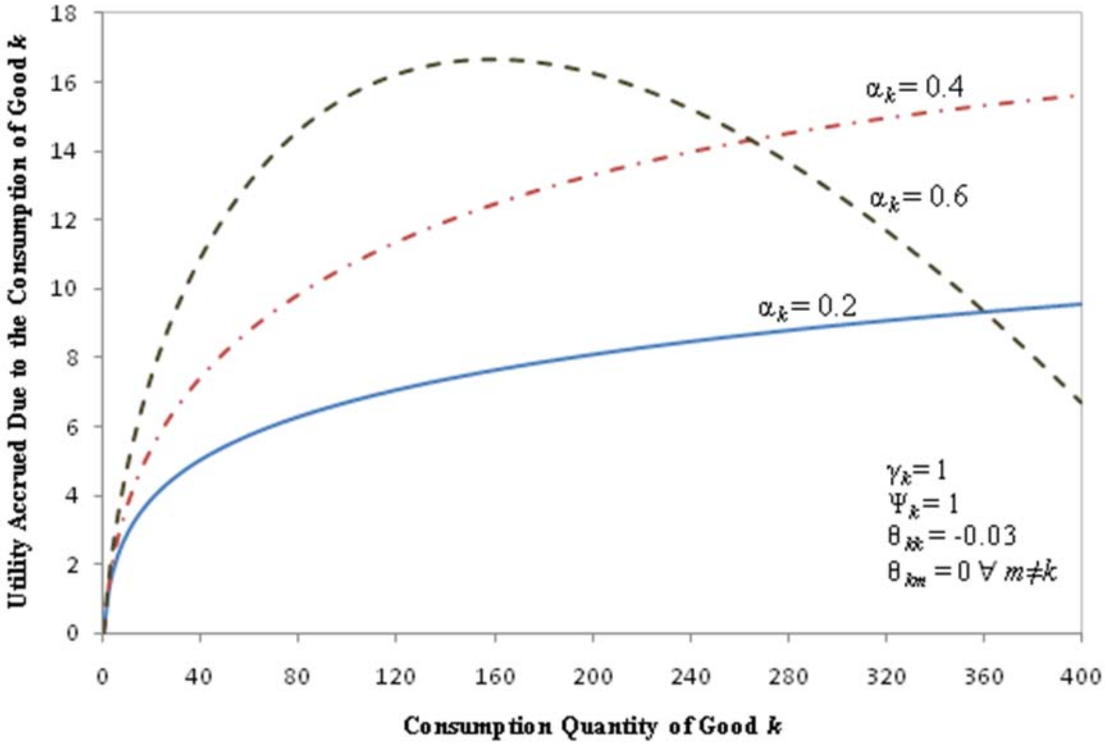

Figure 3. Effect of α_k, due to Negative θ_{kk}, on Good *k*'s Subutility Function Profile

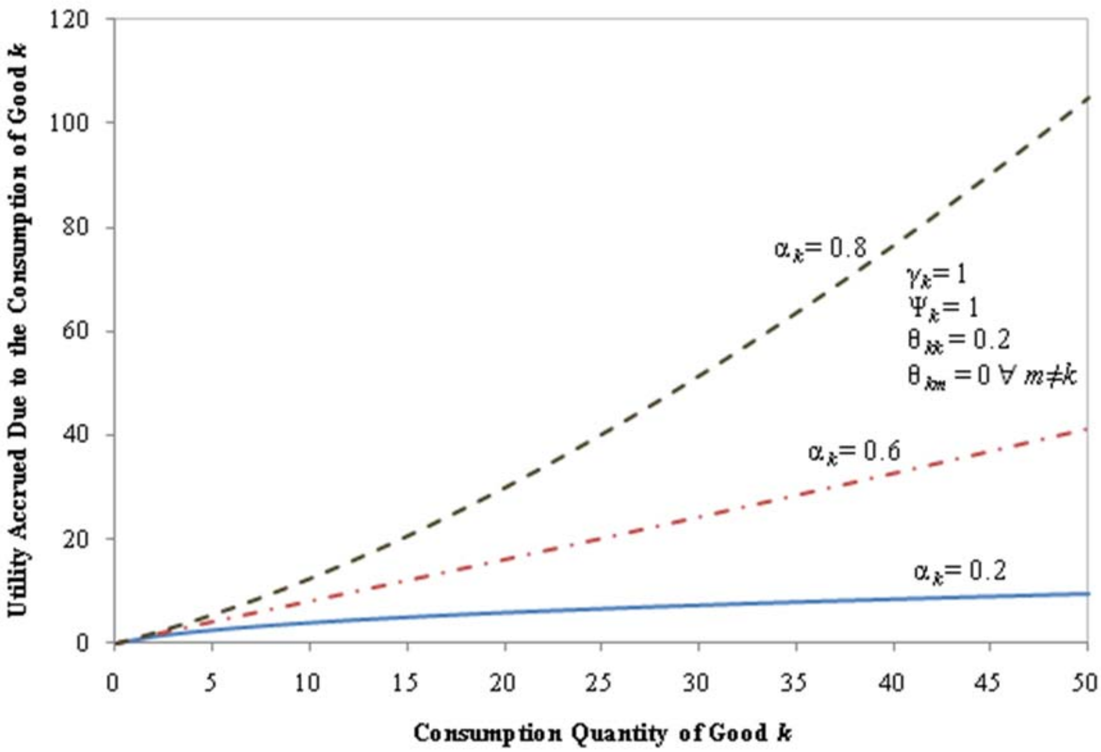

Figure 4. Effect of α_k, due to Positive θ_{kk}, on Good *k*'s Subutility Function Profile

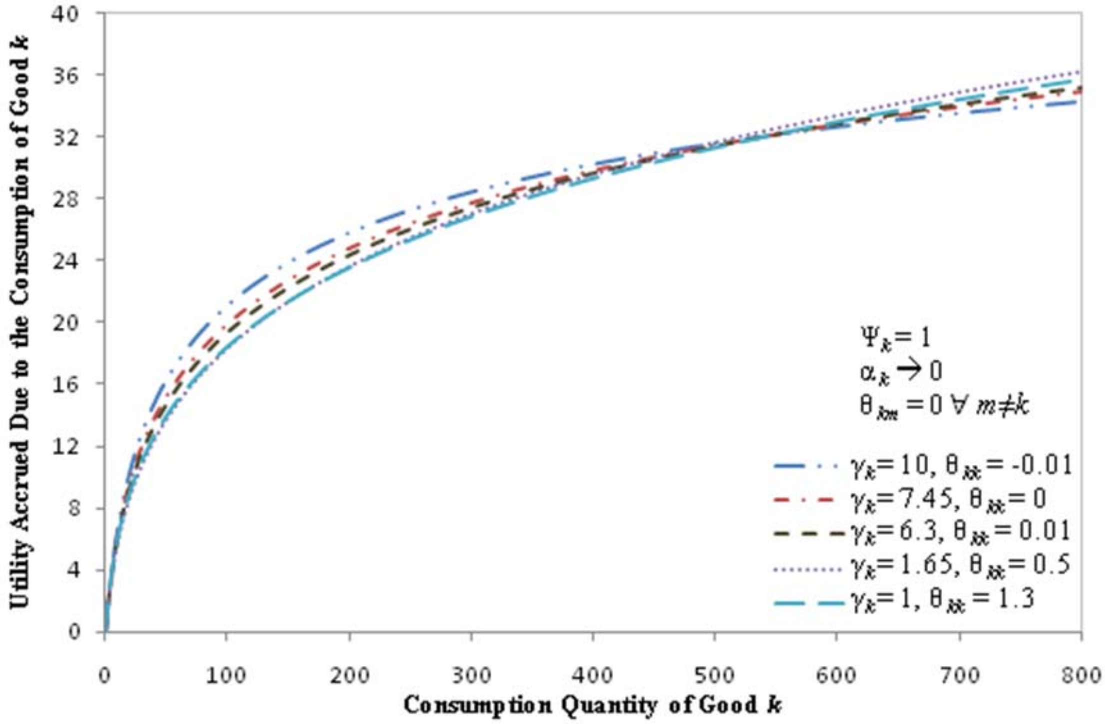

Figure 5. Alternative Subutility Profiles (for Good *k*) with Different θ_{kk} and γ_k Values

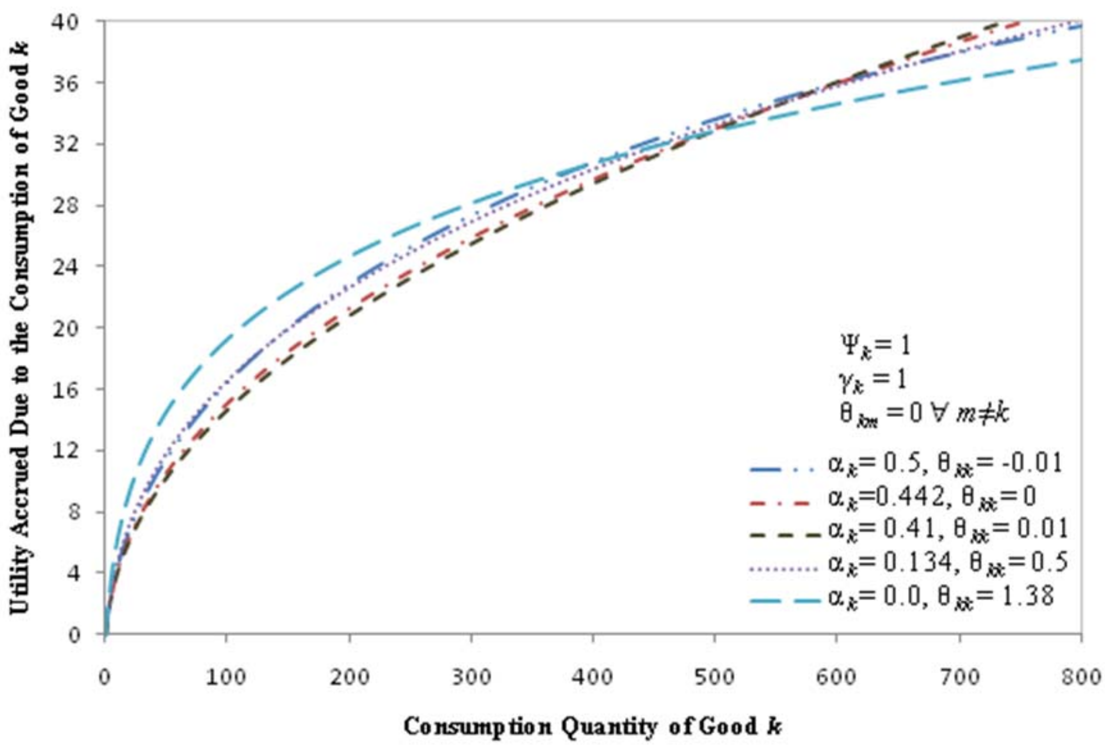

Figure 6. Alternative Subutility Profiles (for Good *k*) with Different θ_{kk} and α_k Values

Table 1. Model Estimation Results

Variables	MDCEV and DU-RM Models				MDCP and RU-DM Models				RU-RM NAS	
	MDCEV		DU-RM NAS		MDCP		RU-DM NAS			
	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat
Baseline Utility Parameters (β)										
Baseline Constants										
Veh. purchase	-7.126	-70.59	-8.059	-19.59	-5.865	-117.32	-5.279	-88.92	-5.926	-106.70
Gasoline/oil	-2.523	-37.62	-2.955	-38.11	-3.280	-79.61	-2.366	-56.75	-3.453	-100.43
Veh. insurance	-3.975	-72.08	-4.565	-28.01	-4.116	-106.60	-3.829	-84.00	-4.329	-120.54
Veh. maintenance	-3.446	-60.82	-4.247	-30.02	-3.893	-90.77	-3.486	-78.87	-4.169	-135.08
Air travel	-6.144	-72.87	-5.487	-50.12	-5.334	-125.56	-4.646	-101.04	-5.931	-76.82
Public transp.	-5.819	-42.16	-5.596	-52.95	-5.171	-78.27	-4.489	-52.26	-5.893	-38.35
Number of workers in household										
Veh. purchase	0.182	4.41	0.194	3.59	0.085	3.70	0.060	1.94	0.079	3.62
Gasoline	0.209	7.74	0.264	5.78	0.175	8.40	0.184	6.64	0.165	10.64
Veh. Insurance	0.081	2.89	0.111	2.52	0.058	3.40	-0.003	-0.14	0.039	2.30
Veh. Maintenance	0.192	7.36	0.288	6.02	0.139	8.74	0.116	5.32	0.098	7.71
Annual HH income 30-70K										
Veh. purchase	0.808	7.97	1.368	4.24	0.446	9.69	0.580	9.57	0.513	10.37
Gasoline	-0.284	-5.60	-0.337	-3.32	-0.198	-4.27	-0.346	-6.23	-0.219	-7.51
Air travel	0.756	8.80	0.414	7.26	0.400	9.10	0.511	9.97	0.330	4.43
Annual HH income >70K										
Veh. purchase	0.805	6.34	1.395	4.07	0.430	6.28	0.525	6.10	0.509	7.88
Gasoline	-0.793	-10.89	-0.964	-5.51	-0.656	-8.26	-1.006	-11.18	-0.636	-13.91
Veh. insurance	-0.337	-5.26	-0.379	-2.94	-0.308	-5.18	-0.356	-4.66	-0.251	-5.34
Air travel	1.189	11.31	0.695	7.16	0.587	8.13	0.670	8.70	0.290	2.80
Number of vehicles in household										
Veh. purchase	0.304	11.75	0.340	10.59	0.171	10.77	0.126	6.82	0.149	11.68
Gasoline	0.305	15.70	0.350	12.65	0.263	15.83	0.247	14.26	0.177	20.44
Veh. insurance	0.275	14.04	0.317	10.50	0.220	15.14	0.166	9.46	0.151	12.76
Veh. maintenance	0.269	13.62	0.326	11.86	0.198	14.87	0.141	9.25	0.105	11.96
Air travel	0.073	2.56	0.100	7.30	0.056	3.29	0.007	0.38	-0.030	-1.20
Public transp.	-0.122	-3.82	-0.555	-15.84	-0.051	-3.71	-0.131	-8.74	-0.698	-25.25

Table 1. Model Estimation Results (cont.)

Variables	MDCEV and DU-RM Models				MDCP and RU-DM Models				RU-RM NAS	
	MDCEV		DU-RM NAS		MDCP		RU-DM NAS			
	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat	Parameter	t-stat
Baseline Utility Parameters (β)										
Non-Caucasian HH – Public transp.	0.417	5.29	0.559	7.28	0.340	10.22	0.347	6.67	0.712	9.75
Urban location – Public transp.	0.490	3.96	0.580	6.09	0.261	4.88	0.287	3.78	0.487	3.36
North East Region – Public transp.	0.722	9.04	0.944	11.10	0.510	14.73	0.585	10.79	0.873	11.32
Western Region – Public transp.	0.590	8.28	0.709	8.73	0.292	8.60	0.370	7.49	0.398	5.59
Translation Parameters (γ_k)										
Veh. purchase	20.888	15.31	21.429	10.95	70.739	12.20	66.645	10.86	80.185	9.82
Gasoline	0.196	17.49	0.179	9.57	0.510	17.73	0.348	33.50	0.744	18.24
Veh. insurance	0.613	27.13	0.607	17.58	1.176	26.99	1.791	29.67	1.568	26.30
Veh. maintenance	0.284	21.08	0.270	17.55	0.879	23.94	1.153	28.82	1.809	23.95
Air travel	0.677	19.58	0.500	14.43	1.879	22.90	1.280	20.05	8.314	16.48
Public transp.	0.237	19.64	0.160	17.47	0.918	30.57	0.577	26.02	1.330	18.33
Interaction Parameters (θ_{km})										
Veh. purchase and gasoline	-	-	1.278×10^{-3}	3.23	-	-	1.126×10^{-3}	37.29	-	-
Veh. purchase and veh. insurance	-	-	-	-	-	-	0.406×10^{-3}	22.78	0.300×10^{-4}	4.36
Veh. purchase and veh. maintenance	-	-	0.338×10^{-3}	2.26	-	-	0.467×10^{-3}	19.41	0.131×10^{-3}	9.98
Veh. purchase and air travel	-	-	-	-	-	-	-	-	-	-
Veh. purchase and public transp.	-	-	-	-	-	-	0.212×10^{-3}	5.70	-0.890×10^{-4}	-4.85
Gasoline and veh. insurance	-	-	2.023×10^{-2}	4.53	-	-	1.954×10^{-2}	32.58	2.436×10^{-3}	10.17
Gasoline and veh. maintenance	-	-	5.095×10^{-2}	7.00	-	-	2.151×10^{-2}	37.93	0.909×10^{-3}	4.95
Gasoline and air travel	-	-	-5.023×10^{-3}	-5.81	-	-	-	-	-	-
Gasoline and public transp.	-	-	-	-	-	-	-	-	-	-
Veh. insurance and veh. maintenance	-	-	4.103×10^{-3}	2.90	-	-	8.879×10^{-3}	25.34	0.366×10^{-3}	4.19
Veh. insurance and air travel	-	-	-	-	-	-	-	-	-	-
Veh. insurance and public transp.	-	-	-	-	-	-	-	-	-	-
Veh. maintenance and air travel	-	-	-	-	-	-	-	-	-	-
Veh. maintenance and public transp.	-	-	-	-	-	-	-	-	-	-
Air travel and public transp.	-	-	8.623×10^{-3}	14.45	-	-	1.199×10^{-3}	7.48	9.204×10^{-3}	35.87
μ parameter	-		-		-		-		0.379	58.51
Number of parameters	33		40		33		41		40	
Log-likelihood at convergence	-37,045		-36,645		-35,269		-35,086		-34,168	

Table 2. Out-of-sample log-likelihood function (OSLLF) in the Validation Sample

Sample details	Number of observations	MDCEV and DU-RM Models		MDCP and RU-DM Models		RU-RM NAS
		MDCEV	DU-RM NAS	MDCP	RU-DM NAS	
Full validation sample	500	-5575.23	-5518.59	-5271.59	-5263.30	-5179.57
Number of workers in HH						
0	14	-147.99	-148.82	-139.89	-142.17	-136.35
1	109	-1139.69	-1126.78	-1075.22	-1149.70	-1059.11
2	240	-2667.62	-2623.82	-2527.78	-2521.63	-2433.19
>2	137	-1619.94	-1619.16	-1528.63	-1520.63	-1515.07
Household income ($/year)						
< 30K	10	-100.62	-101.93	-95.85	-95.28	-100.27
30K-70K	168	-1862.08	-1845.04	-1742.09	-1743.00	-1702.33
>70K	322	-3612.53	-3571.61	-3433.48	-3425.01	-3362.76
Number of vehicles						
0	9	-98.68	-98.00	-95.69	-96.03	-100.06
1	81	-854.90	-846.73	-805.70	-808.38	-783.27
2	173	-1763.61	-1746.95	-1671.01	-1689.54	-1690.87
More than 2	237	-2858.05	-2826.90	-2698.78	-2666.61	-2571.36
Race						
Non-Caucasian	47	-527.42	-520.27	-491.55	-483.70	-494.47
Caucasian	453	-5047.80	-4998.31	-4779.76	-4779.63	-4630.92
Residential location						
Urban	469	-5217.53	-5167.21	-4933.27	-4929.99	-4855.93
Rural	31	-357.72	-351.37	-337.88	-333.33	-321.51