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A Novel Rational Harmonic Balance Approach for Periodic Solutions of Conservative Nonlinear Oscillators

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Abstract

An analytical approximate procedure for a class of conservative single degree-of-freedom nonlinear oscillators with odd non-linearity is proposed. This technique is based on the generalized harmonic balance method in which analytical approximate solutions have rational forms. Unlike the classical harmonic balance techniques, in this new procedure the approximate solution and the restoring force are expanded in Fourier series prior to substituting them in the nonlinear differential equation. This approach gives us not only a truly periodic solution but also the frequency of the motion as a function of the amplitude of oscillation. Four nonlinear oscillators are presented to illustrate the usefulness and effectiveness of the proposed technique. The most significant features of this method are its simplicity and its excellent accuracy for the whole range of oscillation amplitude values and the results reveal that this technique is very effective and convenient for solving a class of conservative nonlinear oscillatory systems.

Keywords: Nonlinear oscillator; Analytical approximations; Rational harmonic balance method; Fourier series.

1 Introduction

Physical and mechanical oscillatory systems are often governed by nonlinear differential equations. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. In particular the study of nonlinear oscillators is of great interest to many researchers. A large variety of approximate methods is commonly used for solving nonlinear oscillatory systems. The most commons and most widely studied methods of all approximation methods for nonlinear differential equations are perturbation methods [1,2]. Some of other techniques include variational and variational iteration methods [3-15], exp-function [16-19], homotopy perturbation [20-35], equivalent linearization [36,37], standard and modified Lindstedt-Poincaré [38-44], artificial parameter [45,46], parameter expanding [47,50], harmonic balance methods [1,51-56], etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [44] and [57].

The method of harmonic balance is a well-established procedure for determining analytical approximations to the solutions of differential equations, the time domain response of which can be expressed as a Fourier series. In the usual harmonic balance methods, the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series [1]. Being different from the other non-linear analytical methods, such as perturbation techniques, the harmonic balance method does not depend on small parameters, such that it can find wide application in nonlinear problems without linearization or small perturbations. Various modifications of the harmonic balance method have been made and one of them is the rational representation proposed by Mickens [59-61].

In this paper a new procedure to apply the generalized, rational harmonic balance method is proposed for constructing approximate analytical solutions to conservative nonlinear oscillations in which the nonlinear restoring force is an odd function of the displacement. In this method the approximate solution obtained approximates all of the harmonics in the exact solution [61], whereas the usual harmonic balance techniques provide an approximation to only the lowest harmonic components. Unlike the classical harmonic balance techniques, in this new procedure the approximate solution and the restoring force are expanded in Fourier series prior to substituting them in the nonlinear differential equation. The most interesting features of the proposed method are its simplicity and its excellent accuracy in a wide range of values of oscillation amplitude. We present four examples to illustrate the applicability and the effectiveness of the proposed approximate analytical solutions.

2 Solution procedure

Consider the following single degree-of-freedom conservative nonlinear oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+f(x)=0 \tag{1}$$

$$x(0)=A, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t}=0 \tag{2}$$

where the nonlinear function $f(x)$ is odd, i.e., $f(-x) = -f(x)$, and satisfies $xf(x)>0$ for $x\in[-A,A]$, $x\neq 0$. The equilibrium position is $x = 0$, the system oscillates between symmetric bounds $[+A,-A]$, and its frequency and the corresponding periodic solution are dependent on the amplitude A.

A new independent variable $\tau=\omega t$ is introduced, and then Eqs. (1) and (2) can be rewritten as follows

$$\omega^2\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2}+f(x)=0 \tag{3}$$

$$x(0)=A, \quad \frac{\mathrm{d}x}{\mathrm{d}\tau}(0)=0 \tag{4}$$

The new independent variable is chosen in such a way that the solution of Eq. (3) is a periodic function of τ of period 2π. The corresponding frequency of the nonlinear oscillator is ω and both periodic solution $x(\tau)$ and frequency ω depend on the initial amplitude A.

Following the lowest order harmonic balance approximation, we set

$$x_1(\tau)=A\cos\tau \tag{5}$$

which satisfies the initial conditions in Eq. (4). Substituting Eq. (5) into Eq. (3) and setting the coefficient of resulting $\cos\tau$ to zero yield the first approximation to the frequency in terms of the amplitude A

$$\omega_1(A)=\sqrt{\frac{c_1}{A}} \tag{6}$$

where

$$c_1=\frac{4}{\pi}\int_0^{\pi/2}f(x_1(\tau))\cos\tau\,\mathrm{d}\tau \tag{7}$$

is the first coefficient of the Fourier series expansion of function $f(x_1(\tau))$

$$f(x_1(\tau))=\sum_{n=0}^{\infty}c_{2n+1}\cos[(2n+1)\tau] \tag{8}$$

where only odd multiples of are presented because the nonlinear function $f(x)$ is odd.

In order to determine an improved approximation we use a generalized, rational form given by the following expression [61,62]

$$x_2(\tau)=\frac{A_1\cos\tau}{1+B_2\cos 2\tau} \tag{9}$$

In this equation A_1, B_2 and ω are to be determined as functions of the initial conditions expressed in Eq. (4) and $|B_2|<1$. From Eq. (4) we obtain $A_1=(1+B_2)A$ and Eq. (9) can be rewritten as follows

$$x_2(\tau)=\frac{(1+B_2)A\cos\tau}{1+B_2\cos 2\tau} \tag{10}$$

The Fourier representation of Eq. (17) is

$$x_2(\tau)=\sum_{n=0}^{\infty}a_{2n+1}\cos[(2n+1)\tau] \tag{11}$$

and the following result is obtained for the Fourier coefficients a_{2n+1} [62]

$$a_{2n+1} = (-1)^n 2^{n+1} A\sqrt{\frac{1+B_2}{1-B_2}}\left(\frac{B_2}{1-B_2}\right)^n\left(\frac{\sqrt{1-B_2}}{\sqrt{1-B_2}+\sqrt{1+B_2}}\right)^{2n+1} \qquad (12)$$

Based on the odd function assumption, $f(-x)=-f(-x)$, $f(x_2(\tau))$ can be expanded in a Fourier series as

$$f(x_2(\tau))=\sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\tau] \qquad (13)$$

Substituting Eqs. (11) and (13) into Eq. (3) yields

$$-\omega^2\sum_{n=0}^{\infty}(2n+1)^2 a_{2n+1}\cos[(2n+1)\tau] + \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\tau]=0 \qquad (14)$$

and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zeros, respectively, yields

$$-\omega^2 a_1(A,B_2)+b_1(A,B_2)=0 \qquad (15)$$

$$-9\omega^2 a_3(A,B_2)+b_3(A,B_2)=0 \qquad (16)$$

From Eq. (12), we obtain

$$a_1(A,B_2)=\frac{2A\sqrt{1+B_2}}{\sqrt{1-B_2}+\sqrt{1+B_2}} \qquad (17)$$

$$a_3(A,B_2)=-\frac{4AB_2\sqrt{1+B_2}}{\left(\sqrt{1-B_2}+\sqrt{1+B_2}\right)^3} \qquad (18)$$

Solving Eqs. (15) and (16) we can obtain B_2 and the second order approximate frequency ω as a function of A. It should be clear how the procedure works for constructing the second-order analytical approximate solution. As you can see, the main problem is to obtain the Fourier series expansion for $f(x_2(\tau))$. To do this, we re-write Eq. (10) as follows

$$x_2(\tau)=\frac{(1+B_2)A\cos\tau}{1+2B_2\cos^2\tau-B_2} = \frac{(1+B_2)A\cos\tau}{(1-B_2)\left(1+\frac{2B_2}{1-B_2}\cos^2\tau\right)} = \frac{(1+B_2)A}{\sqrt{2B_2(1-B_2)}}\frac{u}{1+u^2} \qquad (19)$$

where

$$u=\sqrt{\frac{2B_2}{1-B_2}}\cos\tau \qquad (20)$$

As $|B_2|<<1$ we obtain $|u|<<1$. We will show in the following examples that this new technique provides excellent analytical approximations to frequency and corresponding periodic solutions for a class of conservative nonlinear oscillators.

3 Illustrative examples

In this section, we present four examples to illustrate the usefulness and effectiveness of the proposed technique.

Example 1. *Cubic truly nonlinear oscillator.*
This oscillator is governed by the following differential equation

$$\frac{d^2x}{dt^2}+x^3=0, \quad f(x)=x^3 \qquad (21)$$

with initial conditions given in Eq. (2).

From Eq. (19) we can be write

$$f(x_2(\tau))=x_2^3(\tau)=\frac{A^3(1+B_2)^3}{\sqrt{8B_2^3(1-B_2)^3}}\frac{u^3}{(1+u^2)^3} \qquad (22)$$

Now applying the Taylor series expansion, it follows that

$$\frac{u^3}{(1+u^2)^3}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!}{2\,n!}u^{2n+3} \qquad (23)$$

Substituting Eqs. (20) and (23) into Eq. (22), it can be derived that

$$f(x_2(\tau)) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! 2^{n-1} B_2^n (1+B_2)^3 A^3}{(1-B_2)^{n+3}\, n!} \cos^{2n+3}\tau$$
$$= \sum_{n=0}^{\infty} d_n \cos^{2n+3}\tau \qquad (24)$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2n+3}\tau = \frac{1}{2^{2n+2}} \sum_{j=0}^{n+1} \binom{2n+3}{n+1-j} \cos[(2j+1)\tau]$$
$$= \sum_{j=0}^{n+1} h_j^{(n)} \cos[(2j+1)\tau] \qquad (25)$$

Substituting Eq. (26) into Eq. (24) gives

$$f(x_2(\tau)) = \sum_{n=0}^{\infty}\sum_{j=0}^{n+1} d_n h_j^{(n)} \cos[(2j+1)\tau] =$$
$$= \sum_{n=0}^{\infty} d_n h_0^{(n)} \cos\tau + \sum_{n=0}^{\infty} d_n h_1^{(n)} \cos 3\tau +$$
$$+ \sum_{n=1}^{\infty} d_n h_2^{(n)} \cos 5\tau + \sum_{n=2}^{\infty} d_n h_3^{(n)} \cos 7\tau + \ldots \quad (26)$$

From Eqs. (13), (24), (25) and (26) we obtain b_1 and b_3 as follows

$$b_1 = \frac{3}{4} A^3 \Delta \qquad (27)$$

$$b_3 = \frac{A^3}{4B_2^3}[4 - 4\Delta + (12 - 8\Delta)B_2$$
$$- 2(-6+\Delta)B_2^2 + (4+5\Delta)B_2^3] \qquad (28)$$

where

$$\Delta = \sqrt{\frac{1+B_2}{1-B_2}} \qquad (29)$$

Substituting Eqs. (17), (18), (27), (28) and (29) into Eqs. (15) and (16), and solving the resulting equations for B_2 and ω, we obtain

$$B_2 = -0.0899952 \qquad (30)$$

$$\omega_2(A) = A\sqrt{\frac{3}{8}\left(1 + \sqrt{\frac{1+B_2}{1-B_2}}\right)} = 0.847138A \qquad (31)$$

As we can see, for $B_2 = 0$ the approximate frequency for the first-order harmonic balance approximation is obtained ($\omega_1 = \sqrt{3/8}A$).

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{x_2(t)}{A} = \frac{0.9100048\cos(0.847138At)}{1 - 0.0899952\cos(1.694276At)} \qquad (32)$$

From Eqs. (11) and (12) the Fourier series expansion for the second-order approximate solution given in Eq. (32) is

$$\frac{x_2(t)}{A} = 0.954911\cos\omega_2 t + 0.043056\cos 3\omega_2 t$$
$$+ 0.00194136\cos 5\omega_2 t + 0.0000875340\cos 7\omega_2 t + \ldots \qquad (33)$$

which has an infinite number of harmonics.

For this problem the exact frequency is [27]

$$\omega_e(A) = \frac{\pi A}{2K(1/2)} = 0.847213A \qquad (34)$$

where $K(m)$ is the complete elliptic integral of the first kind defined as follows

$$K(m) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-mz^2)}} \qquad (35)$$

The exact solution to Eq. (21) is [27]

$$\frac{x_e(t)}{A} = cn(At; 1/2) \qquad (36)$$

where cn is the Jacobi elliptic function which has the following Fourier expansion [1,27]

$$cn(u;m) = \frac{2\pi}{\sqrt{m}\,K(m)} \sum_{n=0}^{\infty} \frac{q^{n+1/2}(m)}{1+q^{2n+1}(m)} \cos\left[\frac{(2n+1)\pi u}{2K(m)}\right] \qquad (37)$$

where

$$q(m) = \exp\left[-\frac{\pi K(m')}{K(m)}\right] \qquad (38)$$

and $m' = 1 - m$. With these results, the Fourier expansion of Eq. (36) becomes

$$\frac{x_e(t)}{A} = cn(At;1/2) =$$
$$= \frac{2\pi\sqrt{2}}{K(1/2)}\sum_{n=0}^{\infty}\left(\frac{\exp[(n+1/2)\pi]}{1+\exp[(2n+1)\pi]}\right)\cos[(2n+1)\omega_e t] =$$
$$= 0.95501\cos\omega_e t + 0.043050\cos 3\omega_e t +$$
$$+ 0.0018605\cos 5\omega_e t + 0.0000804\cos 7\omega_e t + \ldots \quad (39)$$

If we compare the first four terms of the Fourier series expansions of $x_2(t)/A$ (Eq. (33) and $x_e(t)/A$ (Eq. (39)) we can see that the relative errors for these terms are 0.010%, 0.014%, 4.3% and 8.9%, respectively.

The percentage error (*PE*) of the approximate frequency ω_2 in relation to the exact one is

$$PE = \left|(\omega_e - \omega_2)/\omega_e\right| \cdot 100 = 0.0089\%$$

The 0.0089% accuracy is remarkably very good.

By applying the second approximation based on the standard method of harmonic balance method, Mickens achieved the following expression for the frequency [1]

$$\omega_{M2}(A) = 0.8507A \qquad (PE = 0.41\%) \qquad (40)$$

Wu et al. [53] approximately solved Eq. (21) using an improved harmonic balance method (LHBM) that incorporates salient features of both Newton's method and the harmonic balance method. They achieved the following results for the second and third approximation orders

$$\omega_{WSL2}(A) = \sqrt{\frac{23}{32}}A = 0.847791A (PE = 0.068\%) \qquad (41)$$

$$\omega_{WSL3}(A) = \sqrt{\frac{65856986475}{91739270448}}A = 0.847273A \quad (PE = 0.0070\%) \quad (42)$$

Beléndez et al. [27] approximately solved Eq. (21) using He's homotopy perturbation method (HPM). They achieved the following results for the second and third approximation orders

$$\omega_{B2}(A) = \omega_{B3}(A) = \frac{1}{4}\sqrt{6+\sqrt{30}}A = 0.84695136A \quad (PE = 0.031\%) \quad (43)$$

The normalized periodic exact solution, $x_e(t)/A$, achieved using Eq. (36), and the proposed second-order approximate solution, $x_2(t)/A$ (Eq. (32)), are plotted in Figure 1, whereas in Figure 2 we plotted the difference $(x_e - x_2)/A$. In these figures h is defined as $h = t/T_e$.

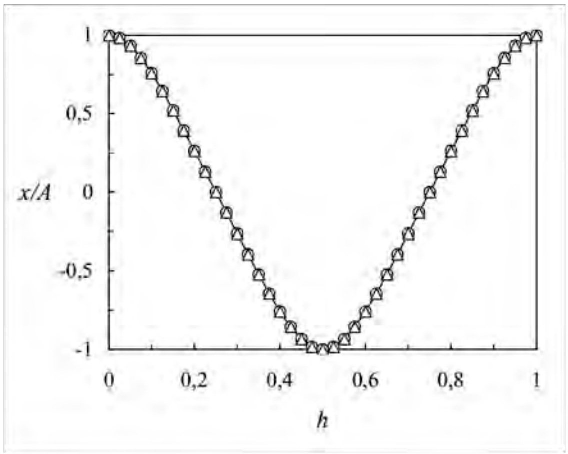

Fig. 1 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for the example 1.

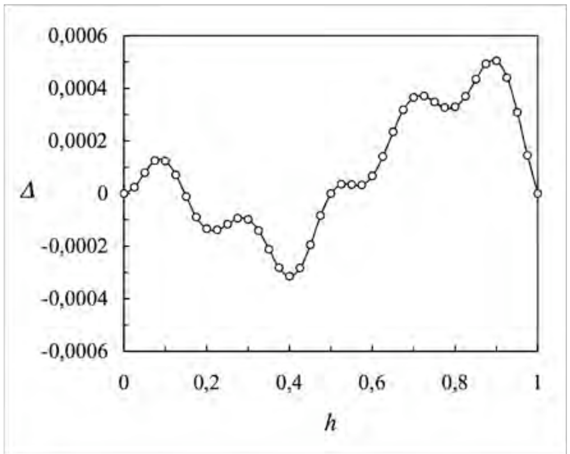

Fig. 2 Difference between normalized exact and second order approximate solutions (example 1).

Example 2. *Cubic Duffing nonlinear oscillator.* This oscillator is governed by the following differential equation

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = 0, \qquad f(x) = x + \varepsilon x^3 \quad (44)$$

with initial conditions given in Eq. (2).

From Eq. (19) we can be write

$$f(x_2(\tau)) = x_2(\tau) + \varepsilon x_2^3(\tau) = \frac{A(1+B_2)}{\sqrt{2B_2(1-B_2)}}\frac{u}{1+u^2} + \varepsilon\frac{A^3(1+B_2)^3}{\sqrt{8B_2^3(1-B_2)^3}}\frac{u^3}{(1+u^2)^3} \quad (45)$$

From Eqs. (11), (27), (28) and (45) we can easily obtain

$$b_1 = a_1 + \frac{3}{4}\varepsilon A^3\Delta \quad (46)$$

$$b_3 = a_3 + \frac{\varepsilon A^3}{4B_2^3}[4 - 4\Delta + (12 - 8\Delta)B_2 - 2(-6+\Delta)B_2^2 + (4+5\Delta)B_2^3] \quad (47)$$

where a_1, a_2 and Δ are given by Eqs. (17), (18) and (29), respectively. Substituting Eqs. (17), (18), (46), (47) and (29) into Eqs. (15) and (16), we can obtain ω for different values of ε and A

$$\omega_2(A) = \sqrt{1 + \frac{3}{8}\varepsilon A^2\left(1 + \sqrt{\frac{1+B_2}{1-B_2}}\right)} \quad (48)$$

where B_2 is the solution of Eq. (16) when ω is given by Eq. (48), i.e.

$$\frac{2B_2\sqrt{1+B_2}}{\left(\sqrt{1-B_2}+\sqrt{1+B_2}\right)^3}\left[64 + 27\varepsilon A^2\left(1+\sqrt{\frac{1+B_2}{1-B_2}}\right)\right] + \frac{\varepsilon A^2}{B_2^3}\left[4 - 4\sqrt{\frac{1+B_2}{1-B_2}} + B_2\left(12 - 8\sqrt{\frac{1+B_2}{1-B_2}}\right) + 2B_2^2\left(6 - \sqrt{\frac{1+B_2}{1-B_2}}\right) + B_2^3\left(4 + 5\sqrt{\frac{1+B_2}{1-B_2}}\right)\right] = 0 \quad (49)$$

which allows us to obtain B_2 as a function of εA^2. From Eqs. (49) and (29) we obtain

$$\lim_{\varepsilon A^2 \to \infty} B_2 = -0.0899952 \quad (50a)$$

$$\lim_{\varepsilon A^2 \to \infty} \Delta = 0.9137125 \quad (50b)$$

In Figure 3 we have plotted B_2 as a function of εA^2. As we can see B_2 is only defined for $\varepsilon A^2 > -1$. It should be noted that the maximum amplitude of the Duffing oscillator in Eq. (44) for $\varepsilon = -1$, is $A = 1$ which corresponds to the heteroclinic orbit with period $+\infty$ [53].

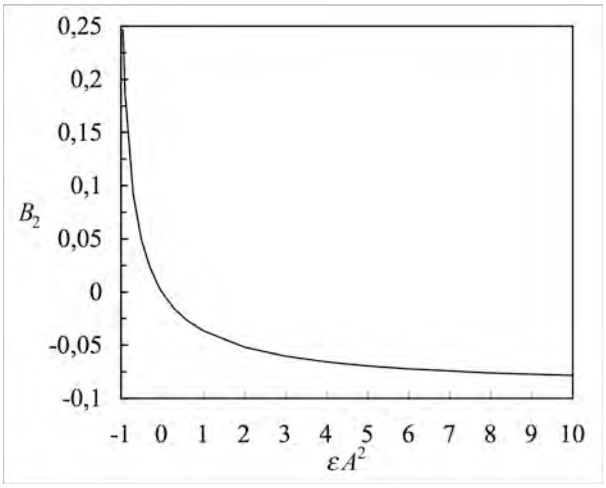

Fig. 3 B_2 as a function of εA^2 for the example 2.

As we can see in Eq. (48), for $B_2 = 0$, the approximate frequency for the first-order harmonic balance approximation is obtained [1]

$$\omega_1(A) = \sqrt{1 + \frac{3}{4}\varepsilon A^2} \quad (51)$$

The second approximation to the periodic solution of this nonlinear oscillator for the value $\varepsilon A^2 = 10$ is given by the following equation

$$\frac{x_2(t)}{A} = \frac{0.9216309\cos(2.866492t)}{1 - 0.0783691\cos(5.732984t)} \quad (52)$$

From Eqs. (11) and (12) the Fourier series expansion for the second-order approximate solution given in Eq. (52) for $\varepsilon A^2 = 10$ is

$$\frac{x_2(t)}{A} = 0.960755\cos\omega_2 t + 0.0377047\cos 3\omega_2 t + 0.00147972\cos 5\omega_2 t + 0.0000580714\cos 7\omega_2 t + \ldots \quad (53)$$

which has an infinite number of harmonics.

For this problem the exact frequency is [1]

$$\omega_e(A) = \frac{\pi\sqrt{1+\varepsilon A^2}}{2K(k)} \tag{54}$$

where

$$k = \frac{\varepsilon A^2}{2(1+\varepsilon A^2)} \tag{55}$$

and the exact solution to Eq. (44) is given as follows

$$\frac{x_e(t)}{A} = cn[t(1+\varepsilon A^2)^{1/2}; k] \tag{56}$$

where *cn* is the Jacobi elliptic function which has the following Fourier expansion [1]

$$\frac{x_e(t)}{A} = \frac{2\pi}{kK(k)} \sum_{n=0}^{\infty} \frac{q^{n+1/2}(k)}{1+q^{2n+1}(k)} \cos[(2n+1)\omega_e t] \tag{57}$$

where q is given by Eq. (38). The first four terms of the Fourier expansion of Eq. (57) for $\varepsilon A^2 = 10$ are

$$\frac{x_e(t)}{A} = 0.960817\cos\omega_e t + 0.0377014\cos 3\omega_e t + \\ +0.00142562\cos 5\omega_e t + 0.0000539044\cos 7\omega_e t + ... \tag{58}$$

If we compare the first four terms of the Fourier series expansions of $x_2(t)/A$ (Eq. (52) and $x_e(t)/A$ (Eq. (58)) for $\varepsilon A^2 = 10$, we can see that the relative errors for these terms are 0.0064%, 0.0089%, 3.8% and 7.7%, respectively.

From Eqs. (48), (50) and (54) we have

$$\lim_{\varepsilon A^2 \to \infty} \frac{\omega_2}{\omega_e} = \frac{2}{\pi}\sqrt{\frac{3}{8}} 1.9137125 K(1/2) = 0.999911 \tag{59}$$

Wu et al. [53] approximately solved Eq. (44) using an improved harmonic balance method (LHBM). They achieved the following results for the second approximation order

$$\omega_{WSL2}(A) = \sqrt{\frac{128+192\varepsilon A^2 + 69\varepsilon^2 A^4}{128+96\varepsilon A^2}} \tag{60}$$

and

$$\lim_{\varepsilon A^2 \to \infty} \frac{\omega_{WSL2}}{\omega_e} = \frac{1}{\pi}\sqrt{\frac{69}{24}} K(1/2) = 1.000682 \tag{61}$$

In Figure 4 we plotted the relative errors for the approximate frequencies ω_2 and ω_{WSL2}. This figure indicates that ω_2 is more accurate than ω_{WSL2} and provides excellent approximations to the exact frequency for the whole range of parameter $\varepsilon A^2 > -1$.

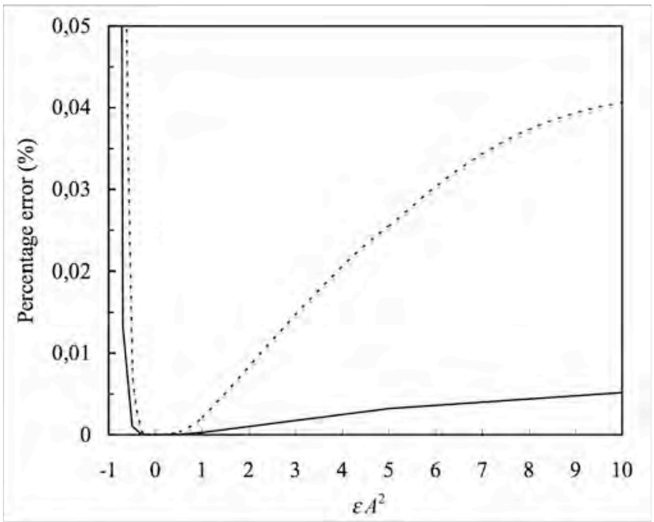

Fig. 4 Relative error for the approximate frequencies ω_{WSL2} (dashed line) and ω_2 (continuous line).

For $\varepsilon A^2 = -0.9$, 10 and 100, the exact periodic solution x_e/A obtained using Eq. (56) and the approximate periodic solution x_2/A computed by Eq. (52), are plotted as a function of $h = t/T_e$ in Figures 5, 6 and 7, respectively, whereas in Figures 8, 9 and 10 we plotted the difference $(x_e - x_2)/A$. These figures show that the proposed formulas (48) and (52) provide excellent approximations to exact periodic solutions for both soft and hard nonlinear oscillators.

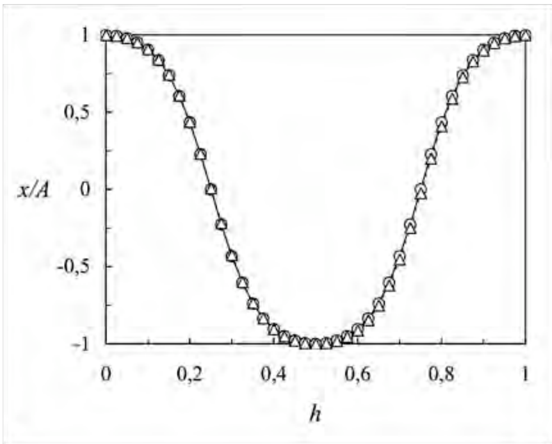

Fig. 5 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for $\varepsilon A^2 = -0.9$.

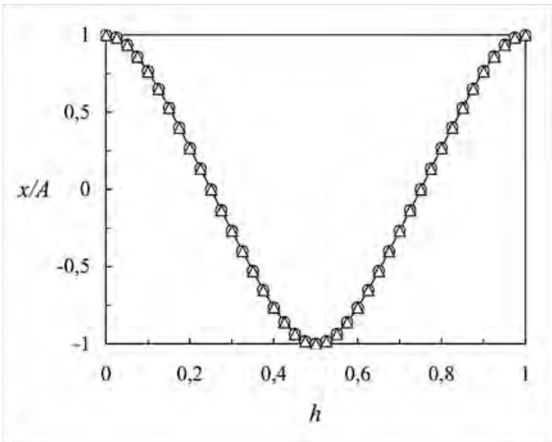

Fig. 6 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for $\varepsilon A^2 = 10$.

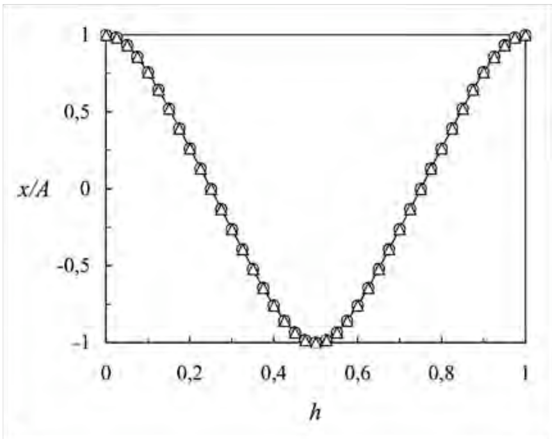

Fig. 7 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for $\varepsilon A^2 = 100$.

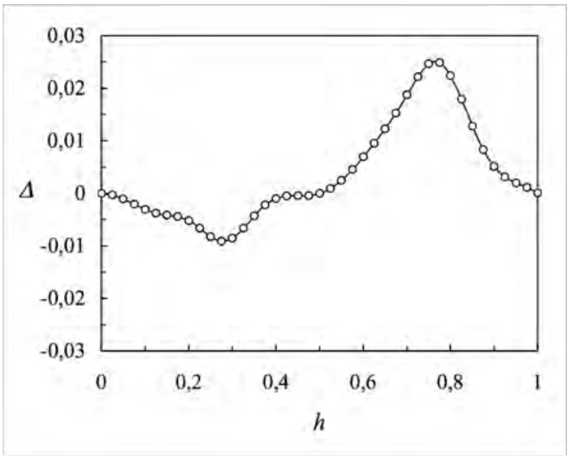

Fig. 8 Difference between normalized exact and approximate solutions for $\varepsilon A^2 = -0.9$.

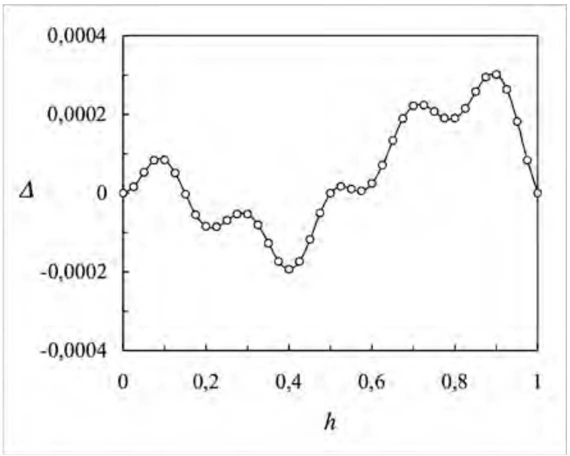

Fig. 9 Difference between normalized exact and approximate solutions for $\varepsilon A^2 = 10$.

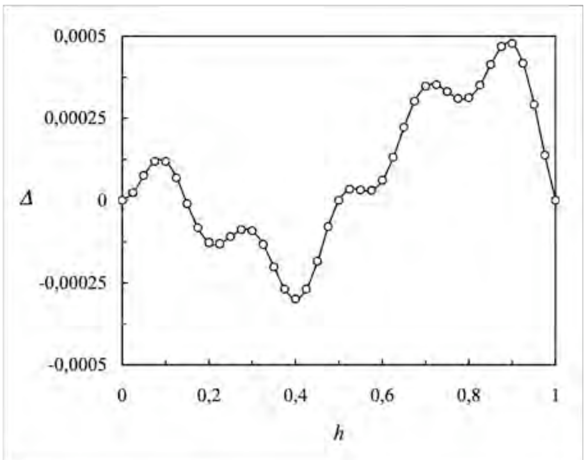

Fig. 10 Difference between normalized exact and approximate solutions for $\varepsilon A^2 = 100$.

Example 3. *Antisymmetric, constant force oscillator.*

This example corresponds to

$$\frac{d^2x}{dt^2}+\operatorname{sgn}(x)=0, \qquad f(x)=\operatorname{sgn}(x) \tag{62}$$

with initial conditions given in Eq. (2). sgn(x) is +1 and −1 for $x > 0$ and $x < 0$, respectively. As $|B_2|<1$, Eq. (19) we can be written as follows

$$f(x_2(\tau))=\operatorname{sgn}(x_2(\tau))=\operatorname{sgn}\left(\frac{A(1+B_2)\cos\tau}{1+B_2\cos 2\tau}\right) \tag{63}$$

The Fourier series expansion for $\operatorname{sgn}(\cos\tau)$ is

$$\operatorname{sgn}(\cos\tau)=\sum_{n=0}^{\infty}\frac{(-1)^n 4}{(2n+1)\pi}\cos[(2n+1)\tau] \tag{64}$$

From Eqs. (13), (63) and (64) we obtain b_1 and b_3 as follows

$$b_1=\frac{4}{\pi} \tag{65}$$

$$b_3=-\frac{4}{3\pi} \tag{66}$$

Substituting Eqs. (17), (18), (65) and (66) into Eqs. (15) and (16), and solving the resulting equations for B_2 and ω, yield

$$B_2=\frac{27}{365}=0.0739726 \tag{67}$$

$$\omega_2(A)=\sqrt{\frac{2}{\pi A}\left(1+\sqrt{\frac{1-B_2}{1+B_2}}\right)}=\sqrt{\frac{27}{7\pi A}}=\frac{1.108046}{\sqrt{A}} \tag{68}$$

As we can see, for $B_2 = 0$, the approximate frequency for the first-order harmonic balance approximation is obtained ($\omega_1=2/\sqrt{\pi A}$) [1].

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{x_2(t)}{A}=\frac{392\cos[\sqrt{27/(7\pi A)}\,t]}{365+27\cos[2\sqrt{27/(7\pi A)}\,t]} \tag{69}$$

From Eqs. (11) and (12) the Fourier series expansion for the second-order approximate solution given in Eq. (69) is

$$\frac{x_2(t)}{A}=\frac{28}{27}\cos\omega_2 t-\frac{28}{729}\cos 3\omega_2 t+\frac{28}{19683}\cos 5\omega_2 t-\frac{28}{531441}\cos 7\omega_2 t+\ldots$$
$$=1.03704\cos\omega_2 t-0.0384088\cos 3\omega_2 t+ +0.0014226\cos 5\omega_2 t-0.000052687\cos 7\omega_2 t+\ldots \tag{70}$$

which has an infinite number of harmonics.

For this nonlinear problem, the exact frequency and periodic solution are [53,63]

$$\omega_e(A)=\frac{\pi}{2\sqrt{2A}}=\frac{1.110721}{\sqrt{A}} \tag{71}$$

$$x_e(t)=\begin{cases}-\dfrac{t^2}{2}+A, & 0\le t\le \dfrac{T_e}{4}\\ \dfrac{t^2}{2}-2\sqrt{2A}\,t+3A, & \dfrac{T_e}{4}<t\le\dfrac{3T_e}{4}\\ -\dfrac{t^2}{2}+4\sqrt{2A}\,t-15A, & \dfrac{3T_e}{4}<t\le T_e\end{cases} \tag{72}$$

An easy and direct calculation gives the following series representation for the exact solution $x_e(t)$

$$x_e(t)=\frac{32A}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos[(2n+1)\omega_e t] \tag{73}$$

The first terms of the Fourier expansion in Eq. (73) are

$$\frac{x_e(t)}{A}=1.03205\cos\omega_e t-0.03822\cos 3\omega_e t +0.008256\cos 5\omega_e t-0.003009\cos 7\omega_e t+\ldots \tag{74}$$

If we compare the first two terms of the Fourier series expansions of $x_2(t)/A$ (Eq. (70) and $x_e(t)/A$ (Eq. (74)) we can see that the relative errors for these terms are 0.48% and 0.49%, respectively. The percentage error (PE) of the approximate frequency in relation to the exact one is

$$PE=\left|(\omega_e-\omega_2)/\omega_e\right|\cdot 100=0.24\%$$

The 0.24% accuracy is remarkably good.

Wu et al. [53] approximately solved Eq. (62) using an improved harmonic balance method (LHBM). They achieved the following results for the second approximation order

$$\omega_{WSL2}=\sqrt{\frac{104}{27\pi A}}=\frac{1.107286}{\sqrt{A}}\quad (PE=0.31\%) \tag{75}$$

Beléndez et al. [63] approximately solved Eq. (62) using He's homotopy perturbation method (HPM). They achieved the following results for the second approximation order

$$\omega_{B2} = \sqrt{\frac{2+2\sqrt{4-\pi}}{\pi A}} = \frac{1.107452}{\sqrt{A}} \qquad (PE = 0.30\%) \qquad (76)$$

The normalized periodic exact solution, x_e / A, achieved using Eq. (72) and the proposed second-order approximate solution, x_2 / A (Eq. (69)) are plotted in Figure 11. In Figure 12 we plotted the difference $(x_e - x_2)/A$.

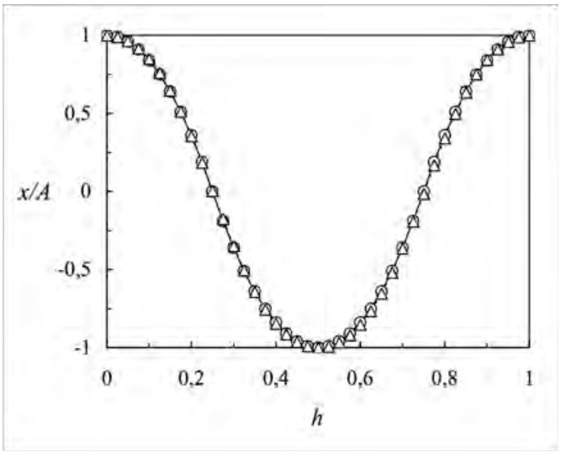

Fig. 11 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for the example 3.

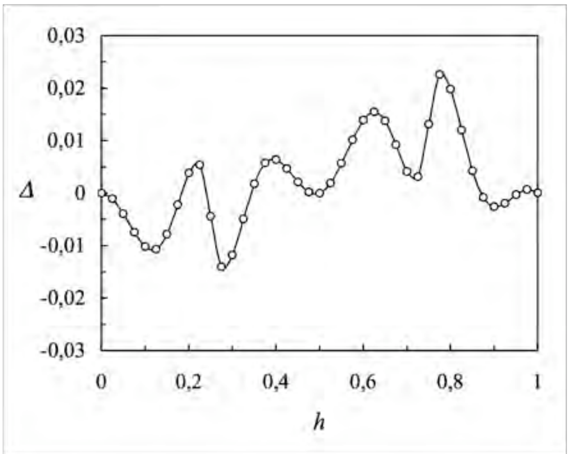

Fig. 12 Difference between normalized exact and second order approximate solutions (example 3).

Figures 11 and 12 show that Eqs. (68) and (69) provide excellent approximations to the exact periodic solution.

Example 4. *Nonlinear oscillator in which the restoring force is inversely proportional to the dependent variable.*

This oscillator is governed by the following second order nonlinear differential equation

$$\frac{d^2x}{dt^2} + \frac{1}{x} = 0, \qquad f(x) = \frac{1}{x} \qquad (77)$$

with initial conditions given in Eq. (2).

Eq. (19) we can be written as follows

$$f(x_2(\tau)) = \frac{1}{x_2(\tau)} = \\ = \frac{1-B_2}{A(1+B_2)}\frac{1}{\cos\tau} + \frac{2B_2}{A(1+B_2)}\cos\tau \qquad (78)$$

The Fourier series expansion for $\cos^{-1}\tau$ is

$$\cos^{-1}\tau = \sum_{n=0}^{\infty}(-1)^n 2\cos[(2n+1)\tau] \qquad (79)$$

From Eqs. (13), (78) and (79) we obtain b_1 and b_3 as follows

$$b_1 = \frac{2}{A(1+B_2)} \qquad (80)$$

$$b_3 = -\frac{2(1-B_2)}{A(1+B_2)} \qquad (81)$$

Substituting Eqs. (17), (18), (80) and (81) into Eqs. (15) and (16), and solving the resulting equations for B_2 and ω, we obtain

$$B_2 = \frac{1}{3}\left[2 - \frac{296}{(81\sqrt{4017}-649)^{1/3}} + \right. \\ \left. +(81\sqrt{4017}-649)^{1/3}\right] = 0.180593 \qquad (82)$$

$$\omega_2(A) = \sqrt{\frac{\sqrt{1+B_2}+\sqrt{1-B_2}}{A^2(1+B_2)^{3/2}}} = \frac{1.246073}{A} \qquad (83)$$

As we can see, for $B_2 = 0$, the approximate frequency for the first-order harmonic balance approximation is obtained ($\omega_1 = \sqrt{2}/A$) [64,65].

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{x_2(t)}{A} = \frac{1.180593\cos(1.246073A^{-1}t)}{1+0.180593\cos(2.492146A^{-1}t)} \quad (84)$$

From Eqs. (11) and (12) the Fourier series expansion for the second-order approximate solution given in Eq. (84) is

$$\frac{x_2(t)}{A} = 1.09104\cos\omega_2 t - 0.0993342\cos 3\omega_2 t + $$
$$+0.00904388\cos 5\omega_2 t - 0.0008234\cos 7\omega_2 t + ... \quad (85)$$

As we can see, Eq. (85) gives an expression that approximates all of the harmonics in the exact solution whereas the usual harmonic balancing techniques provide and approximation to only the lowest harmonic components.

For this problem the exact frequency is [64]

$$\omega_e(A) = \frac{\sqrt{2\pi}}{2A} = \frac{1.253314}{A} \quad (86)$$

and the percentage error (*PE*) of the approximate frequency ω_2 in relation to the exact one is

$$PE = \left|(\omega_e - \omega_2)/\omega_e\right| \cdot 100 = 0.58\%$$

The 0.58% accuracy is remarkably good.

In order to approximately solve Eq. (77), Mickens rewrote this equation in a form that does not contain the x^{-1} expression [64]

$$x\frac{d^2x}{dt^2} + 1 = 0 \quad (87)$$

Using the second-order standard harmonic balance approximation to the periodic solution of Eq. (87), he obtained the second-order approximate frequency [64]

$$\omega_{M2}(A) = \frac{\sqrt{162}}{10A} = \frac{1.272792}{A} \quad (PE = 1.6\%) \quad (88)$$

Beléndez et al [65] solved Eq. (77) applying the standard harmonic balance method. Using the second-order approximation, they obtained the approximate frequency

$$\omega_{B2}(A) = \frac{1.23733}{A} \quad (PE = 1.3\%) \quad (89)$$

The (numerical) normalized exact periodic solution, x_e / A, obtained by numerically integrating Eq. (77) and the proposed second-order approximate solution x_2 / A (Eq. (84)) are plotted in Figure 13 as a function of h.

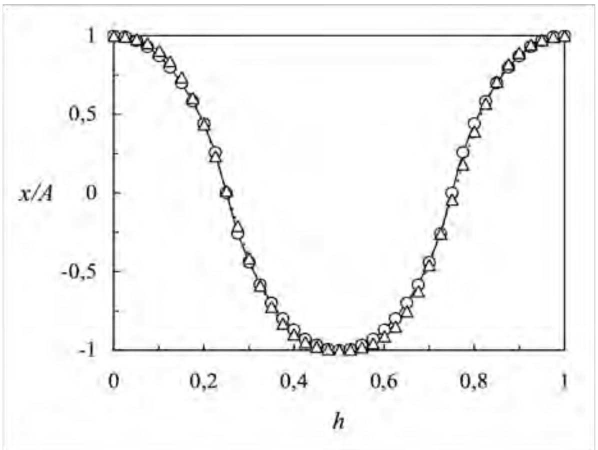

Fig. 13 Normalized approximate solution (triangles and dashed line) and exact solution (circles and continuous line) for the example 4.

In Figure 14 we plotted the difference $(x_e - x_2)/A$. These figures show that Eqs. (83) and (84) can provide high accurate approximations to the exact frequency and the exact periodic solution. These results are an indication of the accuracy of the proposed modified generalized harmonic balance method as applied to this particular problem and show that it provides an excellent approximation to the solution of Eq. (77).

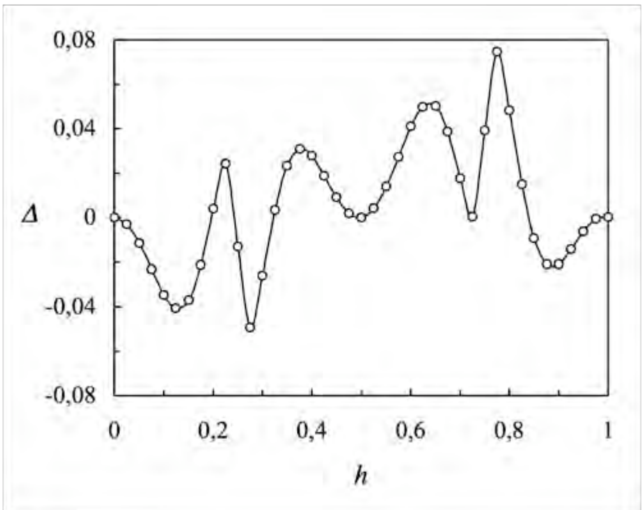

Fig. 14 Difference between normalized exact and approximate solutions (Example 4).

5 Conclusions

Based on the rational harmonic balance method a new procedure has been developed and used to determine analytical approximate solutions of conservative nonlinear oscillators with odd nonlinearities. Unlike the classical harmonic balance techniques, in this new procedure the approximate solution and the restoring force are expanded in Fourier series prior to substituting them in the nonlinear differential equation. Four examples have been presented to illustrate excellent accuracy of the analytical approximate frequencies. The analytical representations obtained using this technique give excellent approximations to the exact solutions for the whole range of values of oscillation amplitudes. For a simple cubic oscillator, it has been shown that the relative error of the second-order analytical approximate frequency obtained using the approach considered in this paper is as low as 0.0089%. In summary, this new procedure to apply the rational harmonic balance method is very simple in its principle, and it can be used to solve other nonlinear oscillators.

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