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PROPORTIONAL REASONING: HOW TASK VARIABLES INFLUENCE THE DEVELOPMENT OF STUDENTS' STRATEGIES FROM PRIMARY TO SECONDARY SCHOOL

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Abstract. This study explores the development of students' strategies from primary to secondary school when solving proportional and additive problems. Its goal is to identify characteristics of the development of proportional reasoning and how the use of integer and non-integer ratios and the discrete or continuous nature of quantities influence this development. The findings indicate that primary school students use systematically the additive strategy in proportional and additive situations and that secondary school students present a wider variety of strategies, which are also used systematically. The type of ratio and the nature of the quantities influenced differently the development of these behaviors.

Résumé. Cette étude explore le développement de stratégies des élèves du primaire au secondaire pour résoudre des problèmes proportionnels et additifs. L'objectif est d'identifier les caractéristiques du développement du raisonnement proportionnel et comment l'utilisation de rapports entiers ou non et de la nature discrète ou continue des grandeurs influence cette évolution. Les résultats indiquent que les élèves du primaire utilisent systématiquement la stratégie additive dans tous les problèmes et que les élèves du secondaire présentent une plus grande variété de stratégies qui sont également utilisées systématiquement. Le type du rapport et la nature des quantités influencent différemment le développement de ces comportements.

Zusammenfassung. Diese Studie untersucht die Entwicklung der Schüler Strategien, die von der Grundschule zur weiterführenden Schule bei der Lösung proportional und additive Probleme. Das Ziel ist es, Charakteristika der Entwicklung proportionalen Denkens zu identifizieren und wie die Verwendung von Integer-und Fließkomma-Kennzahlen und die diskreten oder kontinuierlichen Charakter der Mengen beeinflussen diese Entwicklung. Die Ergebnisse zeigen, dass die Schüler der Grundschule systematisch die additive Strategie in allen Problemen und Schüler der Sekundarstufe eine breitere Palette von Strategien, die auch verwendet werden systematisch darzustellen. Die Art des Verhältnisses und der Art der Mengen beeinflusst unterschiedlich die Entwicklung dieser Verhaltensweisen.

Riassunto. Questo studio analizza lo sviluppo di strategie studenti 'dalla scuola primaria alla scuola secondaria quando la soluzione di problemi proporzionale e additivo. L'obiettivo è di identificare le caratteristiche dello sviluppo del ragionamento proporzionale e come l'uso di interi e non interi rapporti e la natura discreta o continua di quantità influenza questo sviluppo. I risultati indicano che gli studenti della scuola primaria utilizzata sistematicamente la strategia di additivo in tutti i problemi e che gli studenti della scuola secondaria presentano una più ampia varietà di strategie che vengono utilizzate anche in modo sistematico. Il tipo di rapporto e la natura delle quantità influenzato in modo diverso lo sviluppo di questi comportamenti.

Abstrakt. Táto štúdia sa zaoberá vývojom stratégií študentov zo základnej na strednú školu pri riešení proporcionálnych a doplnkových problémov. Cieľom je určiť charakteristiky vývoja proporcionálneho uvažovania a ako použitie celočíselného alebo neceločíselného pomeru a diskrétneho alebo kontinuálneho charakteru veličín ovplyvňuje tento vývoj. Zistenia ukazujú, že žiaci základných škôl systematicky použili doplnkové stratégie vo všetkých problémoch, a že študenti stredných škôl predložili širšiu škálu stratégií, ktoré sú taktiež používané systematicky. Typ pomeru a charakter veličín rôzne ovplyvnili rozvoj týchto správaní.

Key words: proportional reasoning, proportional strategies, additive strategies

1 INTRODUCTION

Proportionality is a fundamental concept in mathematics curriculum since it plays a critical role in students' mathematical development. It has been described as a watershed concept, a cornerstone of higher mathematics and the capstone of elementary concepts (Lesh, Post, & Behr, 1988). However, proportional reasoning is not only important in mathematics but also in our everyday life because many situations are organized around the idea of ratio and proportion.

A proportion is a second order relationship that implies an equivalent relationship between two ratios and can be expressed in the form of $a/b = c/d$ (Christou & Philippou, 2002). According to Freudenthal (1983), the ratio is a function of an ordered pair of numbers or quantities of magnitude. There are two kinds of relationships among quantities: "within" relationships, which are relationships between quantities of the same nature (internal ratio), and "between" relationships, which relate quantities of different nature (external ratio). Take, for example, the proportional problem: "Eight kilos of potatoes cost €4. If you want to buy 12 kilos of potatoes, how much will you pay?" If we relate the weight with the price, we obtain a "between" relationship (4 euros/8 kilos), whereas if we relate the first weight with the second weight we have a "within" relationship (12 kilos/8 kilos).

Vergnaud (1983) considered two measure spaces (in our example, euro and kilos) and the transformations that can be carried out within or between the variables-measures. These transformations reflect the different methods-strategies that students can use to solve proportional problems (Karplus, Pulos, & Stage, 1983; Modestou & Gagatsis, 2009). For instance, a response using the external ratio would be: "As the relationship between euro and kilos is 4 euros/8 kilos then 12 kilos cost 12 × (4/8) = 6 euros". A response using the internal ratio would be: "As the relationship between kilos and kilos is 12 kilos/8 kilos then 12 kilos cost 4 × (12/8) = 6 euros".

During many years, proportional reasoning has been considered synonymous with the ability to solve proportional missing-value problems (Cramer, Post, & Currier, 1993) and comparison problems (Noelting, 1980), although not as often. Missing-value proportional problems refer to tasks that include three quantities of a proportion and the fourth quantity is unknown and has to be computed. An example of a missing-value proportional task is the proportional problem mentioned above. In the proportional comparison problems, the four quantities are given and students have to compare the ratios. For instance: "Which vehicle has a faster average speed, a truck that covers 100 km in 90 minutes or a car that travels 120 km in 115 minutes?.

Recent research on the illusion of proportionality (De Bock, Verschaffel, & Janssens, 1998; Ebersbach, Van Dooren, Goudriaan, & Verschaffel, 2010; Fernández & Llinares, 2009; Fernández, Llinares, & Valls, 2008; Modestou & Gagatsis, 2007, 2010; Van Dooren, De Bock, Janssens, & Verschaffel, 2005) has suggested that proportional reasoning does not imply only the ability to solve missing-value proportional problems but also the ability to discriminate proportional from non-proportional situations. Taking into account these last studies, Modestou and Gagatsis (2010) suggested a new model of proportional reasoning where proportional reasoning does not only imply the success in solving a range of proportional problems, as routine missing-value and comparison problems, but it also involves handling verbal and arithmetical analogies, as well as the awareness of discerning non-proportional situations from other situations. They pointed out that the use of linearity in situations where it is not suitable could be explained considering the linearity as an epistemological obstacle (Modestou & Gagatsis, 2007).

On the other hand, there are studies (Alatorre & Figueras, 2005; Chistou & Philippou, 2002) that provide information about characteristics of the development of proportional reasoning and about the influence of some factors such as the type of ratio on the students' success (Cramer et al., 1993; Tourniaire & Pulos, 1985). Nevertheless we do not have information on possible changes of students' strategies according to their age as a result of the above mentioned factors.

The present study focuses on analyzing the evolution of strategies used by students when handling proportional and non-proportional situations from primary to secondary school. In particular, the main aim of this study is to explore the influence of the different types of ratios (integer or non-integer) and the nature of quantities (discrete or continuous) on this evolution.

2 THEORETICAL FRAMEWORK

The literature on proportional reasoning reveals a broad consensus that proportional reasoning develops from qualitative thinking, to build-up strategies, to multiplicative reasoning (Inhelder & Piaget, 1958; Kaput & West, 1994; Karplus et al., 1983; Noelting, 1980). This development represents different levels of sophistication in thinking about proportions. The qualitative thinking is characterized by the use of comparison words, such as bigger and smaller, more or less, to relate to the quantities of the questions. Build-up reasoning is an attempt to apply knowledge of addition or subtraction to proportion. In this strategy students note a pattern within a ratio and then iterate it to build up additively to the unknown quantity. For example, in the previously mentioned task, as 2 kilos cost 1 euro, then 2+2+2+2+2+2 kilos will cost 1+1+1+1+1+1 euros. Strategies influenced by multiplicative approaches are based on the properties of the linear function: $f(a+b) = f(a)+f(b)$ and $f(ma) = mf(a)$ taking into account the transformations between or within ratios. Inhelder and Piaget (1958) regarded students' mature use of proportional reasoning as indication of the formal operational thinking, in which students observed the consistency of a covariational relationship between and within variables (measure spaces).

A characteristic in the development of proportional reasoning is the difficulty of students in distinguishing proportional from non-proportional situations revealed by the erroneous use of additive strategies in proportional situations and by using erroneous proportional strategies in non-proportional situations (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2009; Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, (in press); Modestou & Gagatsis, 2007; Van Dooren et al., 2005). The erroneous additive strategy used in proportional tasks consists in the use of the difference between the numbers in the ratio and then the application of this difference to the third number to find the unknown quantity (Hart, 1984; Tourniaire & Pulos, 1985).

For example, in the missing-value proportional task: "Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded?", the erroneous additive strategy would be "As the difference between the boxes loaded by Peter and Tom is 100-40 = 60 boxes, then Tom has loaded 60+60 = 120 boxes". Based on the literature, additive strategy is the most common incorrect strategy used on proportional problems (Hart, 1981; Karplus et al., 1983; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985).

On the other hand, the use of erroneous proportional strategies in non-proportional tasks can be exemplified by the following non-proportional problem that is modeled by $f(x) = x + b,\ b \neq 0$ (additive situation): "Victor and Ana are running around a track. They run equally fast but Ana started later.

When Ana has run 5 rounds, Victor has run 15 rounds. When Ana has run 30 rounds, how many has Victor run?" Cramer et al., (1993) showed that a large number of students and pre-service teachers solved this problem using a proportion: 30/5=x/15. In relation to this phenomenon, Van Dooren et al. (2005) observed that the primary students' performance on proportional word problems considerably improved from 3rd to 6th grade. However, during the same years, students' tendency to erroneously over-use proportional methods to non-proportional tasks also increased accordingly. Whereas in 3rd grade 30% of all non-proportional problems were answered proportionally, this percentage increased to 51% in the 6th grade (Van Dooren et al., 2005).

Literature reveals task variables that affect students' performance and the strategy used in proportional situations. Some of these variables that are studied in this paper are the type of ratio (integer or non-integer) and the nature of quantities (discrete or continuous). It is well known that the type of ratio influences students' use of the additive strategy and the proportional strategy. Students prefer the additive strategy when the ratios are non-integer and apply more often the proportional strategy when the ratios are integer, in both primary (Van Dooren, De Dock, Evers, & Verschaffel, 2009) and secondary education (Fernández et al., 2009).

In the present research we will try to answer the following research questions:

- How do the type of ratios influence the development of the primary and secondary school students' proportional and additive strategies when handling proportional and non-proportional situations?
- How the nature of quantities affects the evolution of students' strategies along primary and secondary school?

3 METHOD

3.1 PARTICIPANTS AND CONTEXT

The sample of the study consisted of 755 primary and secondary school students; 65 fourth graders, 68 fifth graders, 64 sixth graders, 124 seventh graders, 151 eighth graders, 154 ninth graders and 129 tenth graders. The participating schools were situated in different cities of Spain and pupils came from different socio-economic backgrounds.

Curriculum contents in Spain, relating to proportional reasoning in the involved age groups differ in relation to students' grade. In the first four years of primary school, there is not any content relating to proportionality, although in grades three and four the concept of fraction appears through equivalent fractions and fraction comparisons. In grades five and six, the computation of percentages is introduced, as well as the recognition of proportional and non-proportional situations and the application of the algorithm of the "rule of three" (cross product) in proportional situations.

According to the curriculum of secondary school, in the seventh grade students should identify proportional magnitudes from the analysis of data that are organized in tables and use the rule of three to calculate missing values in proportional situations. The eighth grade focuses on inverse ($f(x) = a/x$) proportionality with the study of tables and in the ninth and tenth grades the focus is on studying linear functions.

3.2 INSTRUMENT AND PROCEDURE

Students were given a test consisting of 12 missing-value word problems, four of which were proportional (P), four additive (A) and four buffer problems. Additive situations referred to situations reflecting $f(x) = ax + b$, $b \neq 0$. Buffer problems were included so as to create a variation of the given tasks and avoid stereotyped replies as well as the effects of learning.

Although there are many non-proportional situations, only situations with the structure $f(x) = x + b$, $b \neq 0$ were chosen in this study. This type of problems refers to additive situations because the strategy used to solve this kind of problems is based on identifying the additive relationships between quantities. These problems are very interesting as students not only over-use proportional methods in non-proportional situations but also over-use additive methods in proportional situations (Fernández et al., 2009; Fernández et al., in press; Van Dooren et al., 2009). These studies have shown that students' tendency to use proportional strategies on additive problems and additive strategies on proportional problems are complementary.

We designed 8 discrete situations (D) – loading boxes – and 8 continuous situations (C) – skating a certain distance. Then, for each situation, proportional and additive tasks were created by manipulating only one sentence (see Table 1). For example, where in the proportional situation (P) the sentence "They started together but Tom loads faster" was used, in the respective additive situation (A) the sentence "They load equally fast but Peter started later" was used. In a second step, two different versions were considered, one with integer relationships between quantities (I, P1) and another with non-integer relationships between the quantities (N, P2). With the obtained set of problems (64 experimental problems and 4 buffer problems), 8 parallel tests were composed, in which the problems were presented in different order. Table 1 presents the design of the test, with the structural characteristics of each task, whereas Table 2 presents examples of the tasks used.

Table 1. *Design of the Test*

Proportional (P)	Integer Ratios (I)	Non-integer Ratios (N)
Discrete quantities (D)	P-D-I	P-D-N
Continuous quantities (C)	P-C-I	P-C-N
Additive (A)		
Discrete quantities	A-D-I	A-D-N
Continuous quantities	A-C-I	A-C-N

Table 2. *Examples of Word Problems Used in the Test*

1 – Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded? (P-D-I)	2 – Peter and Tom are loading boxes in a truck. They started together but Tom loads faster. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded? (P-D-N)
3 – Ann and Rachel are skating. They started together but Rachel skates faster. When Ann has skated 150 m, Rachel has skated 300 m. If Ann has skated 600 m, how many meters has Rachel skated? (P-C-I)	4 – Ann and Rachel are skating. They started together but Rachel skates faster. When Ann has skated 80 m, Rachel has skated 120 m. If Ann has skated 200 m, how many meters has Rachel skated? (P-C-N)
5 – Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 160 boxes. If Peter has loaded 80 boxes, how many boxes has Tom loaded? (A-D-I)	6 – Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded? (A-D-N)
7 – Ann and Rachel are skating. They skate equally fast but Rachel started earlier. When Ann has skated 150 m, Rachel has skated 300 m. If Ann has skated 600 m, how many meters has Rachel skated? (A-C-I)	8 – Ann and Rachel are skating. They skate equally fast but Rachel started earlier. When Ann has skated 80 m, Rachel has skated 120 m. If Ann has skated 200 m, how many meters has Rachel skated? (A-C-N)

Buffer problems were formulated similarly but with a different semantic structure. For instance consider the example: "Sam and John are playing a computer game. They play equally long, but Sam got fewer points than John. Sam got 320 points less than John in total. If John got 850 points, how many points did Sam get?". In this case students had to deal with a comparison problem which was presented in the same linguistic form as the other tasks used in the research.

Students had approximately 50 minutes to complete the test. They were allowed to use calculators but were asked to write down all the operations they had conducted, so as to be able to follow the problem solving path they used.

3.3 METHOD OF DATA ANALYSIS

Students' strategies were analyzed and categorized into five groups: additive strategies, build-up methods, use of ratios, rule of three and other strategies, by taking into account the way students handled the relationships between the given numbers in each situation. For additive problems, additive strategies were considered correct whereas proportional strategies (build-up method, use of ratios, the rule of three) and other strategies were not appropriate. For proportional problems, proportional strategies were considered correct and additive strategies and other strategies were incorrect. Table 3 and 4 show students' strategies when solving proportional and additive problems respectively:

Table 3. *Examples of Students' Strategies when Handling Proportional Problems (Problem 1, Table 2)*

	Proportional Strategies
Use of ratios (SR)	As 160 / 40 = 4 (external ratio) then Tom has loaded 80 × 4 = 320 boxes "As 80 / 40 = 2 (internal ratio) then Tom has loaded 160 × 2 = 320 boxes"
Build-up method (SBU)	As 40 + 40 + 40 + 40 = 160 boxes, then Tom has loaded 80 + 80 + 80 + 80 + 80 = 320 boxes
Rule of three (SRT)	80 × 160 = 12800; 12800 : 40 = 320 boxes
	Additive Strategies
Additive strategy (SAdd)	As the difference between the boxes loaded by Peter and Tom is 160 – 40 = 120, then Tom has loaded 120 + 80 = 200 boxes.

Table 4. *Examples of Students' Strategies when Handling Additive Problems (Problem 5, Table 2)*

	Additive Strategies
	As the difference between the boxes loaded by Peter and Tom is 160 – 40 = 120, then Tom has loaded 120 + 80 = 200 boxes
	Proportional Strategies
Using the external ratio	As 160 / 40 = 4 then Tom has loaded 80 × 4 = 320 boxes

The use of strategies in different problems was codified by adding the number of the task in the name of the strategy. For example the variable referring to the use of the additive strategy in problem 1 is SAdd1, whereas the variable referring to the use of the rule of three in problem 4 is SRT4.

The use of a strategy in a problem was codified as 1 and the absence as 0. In each problem the presence of the five categories of strategies (the four strategies listed in Table 3 plus SOth for other strategies) was encoded, and therefore 40 variables arose (eight problems × five categories of strategies). Next, the Implicative Statistical Analysis with the use of the computer software CHIC was conducted (Gras, Suzuki, Guillet, & Spagnolo, 2008*),* first with primary school students' data and then with secondary school students' data. The similarity diagrams that arose from this analysis allowed the arrangement of the strategies into groups according to the homogeneity, indicating the way these strategies were applied by students.

4 RESULTS

The results of this study are organized and presented into two parts. The first part focuses on the development of students' strategies used, from primary to secondary school, when solving proportional and additive problems. In the second part, the similarity diagrams of the strategies used by primary school students and secondary school students are presented. A comparison of these similarity diagrams follows, in order to identify possible changes in the strategies used in each problem taking into account the type of ratios (integer or non-integer) and the nature of the quantities (discrete or continuous).

4.1 DEVELOPMENT OF STUDENTS' STRATEGIES

Table 5 shows the percentages of students' use of strategies in proportional and additive problems in both primary and secondary education. In each grade, the percentages of using the strategies in each type of situation (proportional and additive) have been considered.

Table 5. *Percentages of Students' Use of Strategies in Additive and Proportional Problems in each Grade from Primary to Secondary School*

		Primary School				Secondary School				
		4	5	6	Total	1	2	3	4	Total
Build-up strategy (SBU)	P	0.8	1.5	2.0	1.4	4.4	6.3	5.4	4.5	5.1
	A	0.0	0.4	2.4	0.9	3.6	5.6	3.6	2.4	3.8
Internal and external ratios (SR)	P	4.6	9.2	12.1	8.6	9.1	12.3	16.1	12.8	12.6
	A	3.1	8.1	10.6	7.3	7.5	8.1	15.2	11.5	10.6
Rule of three (SRT)	P	0.0	0.0	0.0	0.0	0.0	7.9	27.3	54.1	22.4
	A	0.0	0.0	0.0	0.0	0.0	6.3	24.9	49.6	20.2
Additive strategy (SAdd)	P	50.8	53.3	57.1	53.7	52.0	47.0	34.3	21.1	38.6
	A	53.1	52.9	58.2	54.7	53.3	54.8	41.7	30.1	44.9
Other strategies (SOth)	P	43.8	36.0	28.8	36.3	34.5	26.5	16.9	7.5	21.3
	A	43.8	38.6	28.8	37.1	35.6	25.2	14.6	6.4	20.5

Taking into account the proportional strategies, the use of build-up methods (SBU) increased along primary school and in the first and second grade of secondary school in both type of problems. However, the use of this strategy decreased in the third and the fourth grade of secondary school, perhaps because of the systematic teaching of the use of the rule of three; rule that can be applied mechanically and can more easily give correct results to proportional tasks. The use of ratios (SR) increased during primary school (both in additive and proportional problems), but it decreased in the transition from primary to secondary school. This decrease was compensated by the increase of build-up methods (SBU) and the increase of other strategies (SOth). After that, it increased again during the secondary school and it was compensated by the decrease of other strategies. Primary school students did not use the rule of three, but there was a strong increase of this strategy after the second grade of secondary school; time when the rule of three is introduced. This could also explain the fact that there was a decrease of the use of build-up methods along secondary school.

The application of the additive strategy increased during primary school and decreased during secondary school, independently of the type of problem (additive or proportional). On the other hand, the use of other strategies (SOth) decreased from primary to secondary school but increased in the first and second grade of secondary school. This behavior was compensated by a decrease of the use of internal and external ratios (SR) from sixth grade of primary school to the first grade of secondary school.

These data give two important indications. Firstly, students use a strategy without taking into account the additive or proportional character of the task. That is, students apply the same strategy in additive and proportional problems without examining first if the strategy is appropriate or inappropriate for the situation at hand. Secondly, there are two tendencies that exist in parallel: There is a decrease of the use of additive strategies on proportional and additive problems along secondary school but, at the same time, there is an increase of the use of proportional strategies in both types of problems, mainly due to the systematic teaching of proportions and the rule of three in secondary education. However, the results presented so far do not provide information about the influence of the type of ratio and the nature of quantities on students' strategies. Do these factors affect the use of these strategies? In which way? Are students' strategies different in additive and proportional problems because of these factors?

4.2 THE INFLUENCE OF THE TYPE OF RATIO AND THE NATURE OF QUANTITIES ON STUDENTS' STRATEGIES

4.2.1 Primary education

Figure 1 shows the similarity diagram of the strategies used by primary school students in the different problems of the test. The similarity diagram was generated only with 23 variables clustered in three similarity clusters (i.e., groups of variables). Cluster A consisted of variables referring to the use of additive strategies in all additive and proportional problems. This indicates that primary school students used this strategy systematically, without being able to discriminate proportional or additive problems. Furthermore, the way in which the use of the additive strategy in the different problems has been grouped suggests that the nature of quantities has an influence on its use. Two sub-groups appear in the diagram: one with the problems including discrete quantities (SAdd1, SAdd2, SAdd5, SAdd6) and another with the tasks with continuous quantities (SAdd3, SAdd4, SAdd8). This implies that there are students who used additive strategies only in problems with discrete quantities and that there are students who used additive strategies only in problems with continuous quantities.

Cluster B referred to the use of other strategies in discrete situations (SOth1, SOth2, SOth5 and SOth6), and in the additive problem with continuous quantities and integer ratios (SOth7). This way of grouping suggests that the discrete nature of quantities had also an influence on the use of other strategies

Finally, Cluster C grouped three strategies, the category of "Other strategies" in the problems with continuous quantities (SOth3, SOth4 and SOth8); the use of ratios (internal or external) in problems with integer multiplicative relationships between quantities (SR3, SR5 and SR7) and the use of build-up methods in problems with non-integer ratios (SBU2, SBU4 and SBU6). This grouping indicates, firstly that the integer multiplicative relationship between the given numbers drove primary school students' use of the ratios regardless of the additive or proportional character of the problems, and the presence of non-integer multiplicative relationships drove the primary school students' to use of build-up strategies. This fact can be easily explained as for primary school students the relation 160 : 40 = 4 is easier than the relation 100 : 40 = 20,5. In this sense, when the multiplicative relationship was easy (integer multiplicative relationships), primary school students were more prone to use it (ratios), but when the multiplicative relation included non-integer ratio, students were derived to use build-up methods.

Finally, the way in which these clusters have been formed, indicates that the similarity groups are based on the strategies used and not on the proportional or additive character of the tasks. In addition, the formation of the sub-groups (defined by the use of strategies) is mainly based on the nature of quantities (discrete or continuous) or on the number structure (integer or non-integer ratios) and not on the proportional or additive character of problems.

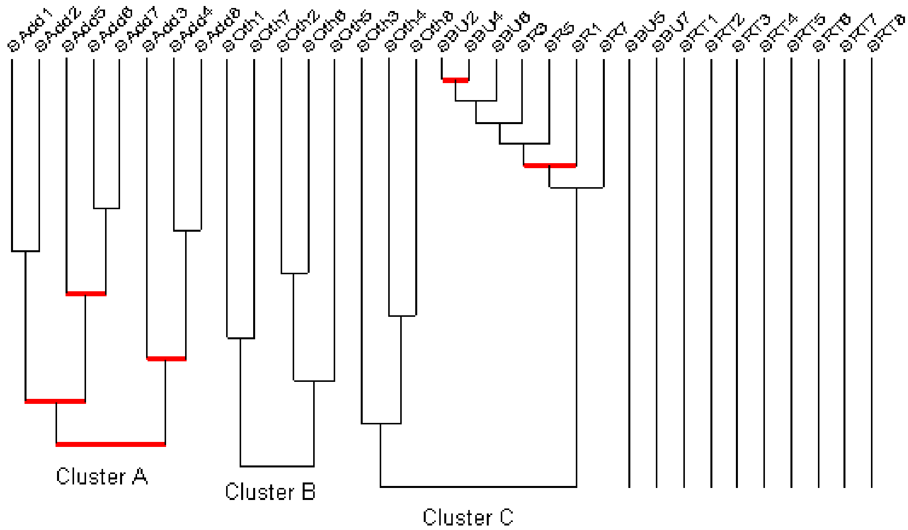

Figure 1. Similarity diagram of the strategies used by primary school students

4.2.2 Secondary education

Figure 2 presents the similarity diagram of the strategies used by secondary school students in the different problems of the test. In this case, six similarity clusters grouped all 40 variables. Cluster A grouped the use of additive strategies in all problems, where Cluster B grouped the use of the other strategies. The use of build-up strategies in problems with integer multiplicative relationships were grouped in Cluster C, and the use of build-up strategies in problems with non-integer multiplicative relationships were grouped in Cluster D. Finally, the use of ratios (Cluster E) and the use of the rule of three (Cluster F) were grouped in two other distinct clusters.

These similarity groups are formed in the same way as in the case of the primary school; that is based on the strategies used and not on the additive or proportional character of the tasks. These data suggest that secondary school students were systematic in the use of strategies in each type of problem. Therefore, the secondary school students that used additive strategies (Cluster A), they used them in all problems. The same pattern appeared for all the other different strategies. These data indicate essentially that secondary school students do not differentiate between additive and proportional situations, and handle them in the same way.

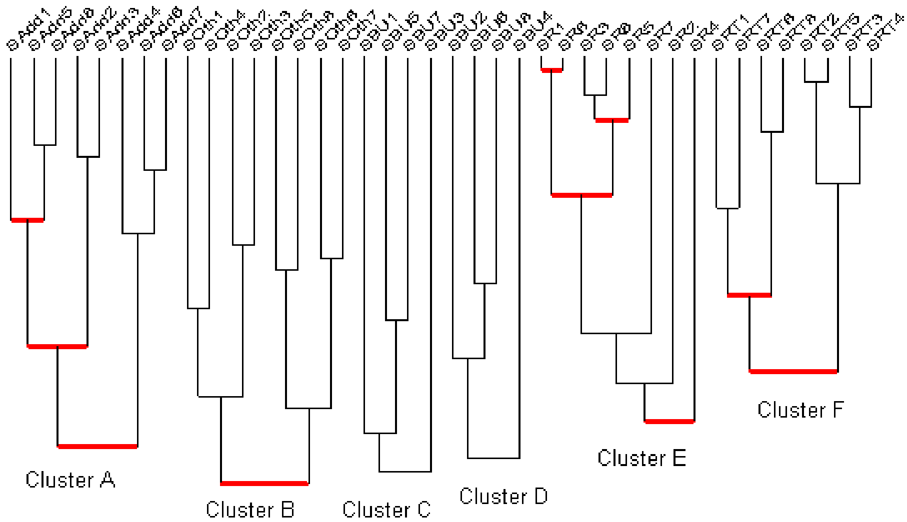

Figure 2. Similarity diagram of the strategies used by secondary school students

Only in Cluster B, the type of problem (additive/proportional) appeared to have a minor affect on the use of other strategies. Cluster B is formed by two groups. Firstly, the group formed by the use of other strategies in proportional problems (SOth1, SOth2, SOth3, SOth4), and secondly the group formed by the use of other strategies in the additive problems (SOth5, SOth6, SOth7, SOth8). This grouping suggests that the type of the problem (additive or proportional) influenced the approach adopted by the students when they did not use additive or proportional strategies adopting other different erroneous approaches. Regarding Cluster C and D jointly, it shows that the use of build-up strategies were grouped with the reference to the integer and non-integer ratios that were included in the problem. Cluster C grouped the use of build-up strategies in problems with integer ratios (SBU1, SBU3, SBU5 and SBU7) and Cluster D grouped the use of build-up strategies in problems with non-integer ratios (SBU2, SBU4, SBU6 and SBU8). These data suggest that the type of ratio affects the use of the build-up method.

Cluster E grouped the application of the ratio approach in proportional and additive problems. However, this grouping does not provide a clear indication about the influence of the nature of quantities, the type of ratio or the additive or proportional character of the problems.

Finally, Cluster F was formed by two groups. The one grouped the use of the algorithm “rule of three” in the proportional problem with discrete quantities and integer ratios (SRT1), in the additive problems with continuous quantities (SRT7 and SRT8) and in the additive problem with discrete quantities and non-integer relationships between quantities (SRT6). The other group was formed by the use of the same algorithm in the proportional problems with continuous quantities (SRT3 and SRT4), in the proportional problem with discrete quantities and non-integer ratios (SRT2) and in the additive problem with discrete quantities and integer relationships between quantities (SRT5). The way in which the groups are formed in Cluster F indicates that the type of ratio (integer or non-integer) determined the first sub-group (SRT1 and SRT7 form a group, and SRT6 and SRT8 the other group), but the nature of quantity (discrete or continuous) determined the second sub-group (SRT2 and SRT5 form one group, and SRT3 and SRT4 the other group).

4.2.3 Comparison of the primary and secondary school students’ similarity diagrams

When the two similarity diagrams (Figure 1 and 2) are compared, differences appear. Firstly, primary school students’ similarity diagram grouped only 23 variables, while the secondary school students’ similarity diagram grouped all the 40 variables. Clusters were always formed in a way that they grouped the use of one strategy in all the tasks. This indicates that primary and secondary school students were systematic in the use of strategies independently of the additive or proportional character of the problem.

On the other hand, the variables nature of quantities and presence or absence of integer or non-integer ratios affected more primary school students' use of a particular strategy than secondary school students' decisions. In fact, the nature of quantities does not seem to affect secondary school students, whereas the number structure only affects the use of proportional strategies.

5 DISCUSSION

The main goal of the present study was to explore the development of students' strategies when solving proportional and additive problems along primary and secondary education and identify how number structure and nature of quantities affect this development. This was put forward through the identification of similarities or disparities along different age groups (primary and secondary), when solving additive and proportional problems with different characteristics (integer and non-integer multiplicative relationships and discrete or continuous quantities).

Primary school students used additive strategies and build-up methods to solve proportional and additive problems, whereas secondary school students applied other strategies as the use of ratios and the rule of three. Based on the similarity clusters, the compartmentalization of the strategies becomes evident. In fact the additive and proportional character of tasks did not seem to have an influence on primary and secondary school students' strategies because they used systematically the same strategy to solve all types of problems. The exclusive use of one strategy without a meaningful understanding of multiplicative reasoning became a procedurally oriented operation that obstructed students' initial sense making of proportional reasoning. Therefore, this study supports the idea that the development of proportional reasoning seems to rely on the ability to discriminate additive and multiplicative relationships between numbers (De Bock et al., 1998; Fernández et al., 2008; Modestou & Gagatsis, 2010; Van Dooren, De Bock, Janssens, & Verschaffel, 2008). This claim is supported by the evidence of the compartmentalization of the strategies. There are students who use proportional strategies in proportional problems but also use these strategies in additive problems, and inversely, there are students who use additive strategies in additive problems but also use these strategies in proportional problems.

A characteristic of students' development of strategies is that the increase of the use of proportional strategies from primary to secondary school was accompanied by a decrease of the use of additive strategies and other incorrect strategies. However, it does not mean that secondary school students perform better than primary school students because secondary school students applied proportional strategies not only on proportional problems but also on additive problems. So the transition between primary to secondary school could be defined as: "the tendency of primary school students to apply additive strategies in all problems shift to the tendency of secondary school students to apply proportional strategies in all problems".

With regard to the influence of variables, students use more ratio strategies when the relationships between quantities are integer and they use more build-up methods when the relationships between quantities are non-integer. Furthermore the decrease of the use of ratios in non-integer ratios was accompanied by an increase of the use of additive strategies in this type of problems. This evidence indicates that the type of ratio influences the development of the use of strategies along primary and secondary school. On the other hand, the nature of quantities does not seem to have an influence on the characteristics of the development of students' strategies from primary to secondary school.

Consequently, as these results show, the instruction from primary to secondary school should pay more explicit attention to the different mathematical relationships between quantities in order to prevent that students base their strategies on superficial associations and that they use them systematically in all situations. Students apparently learn to apply procedures that are, at least partially, based on superficial associations such as the type of ratios between the given numbers, the nature of quantities mentioned in the problem or the missing-value formulation, rather than by reflecting on the underlying mathematical structure. In other words a successful learning of the proportional situations must be based on the de-compartmentalization of the strategies.

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