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# ALPHA-DENSE CURVES IN INFINITE DIMENSIONAL SPACES

Gaspar Mora[1 §], Juan A. Mira[2]

Department of Mathematical Analysis
Faculty of Sciences
University of Alicante
Ap. Correus 99, E-03080 Alicante, SPAIN
[1]e-mail: gaspar.mora@ua.es
[2]e-mail: jamira@ua.es

**Abstract:** The theory of $\alpha-$dense curves in the euclidean space $R^n, n \geq 2$, was developed for finding algorithms for Global Optimization of multivariable functions ([1], [6]). The $\alpha$-dense curves, considered as a generalization of Peano curves or space-filling curves, densify the domain of definition $D$ of a multivariable function $f$ in the sense of the Hausdorff metric. Then, the restriction of $f$ on an $\alpha-$dense curve $\gamma$, contained in $D$, is a univariable function $f_\gamma$ for which will have less difficulty to locate its global minimum.

In this paper we shall study some properties of $\alpha-$dense curves that are Lipschitzian. Moreover, we shall point out that this theory of $\alpha-$dense curves is characteristic of the finite dimensional spaces. In fact, we shall prove that a Banach space has finite dimension iff its unit ball can be densified with arbitrary small density $\alpha$. From this, we shall deduce the classical Theorem of Riesz.

Finally, we shall construct a family of infinite dimensional $\alpha-$dense curves, whith controlled density $\alpha$, in the Hilbert parallelotope.

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§Correspondence author

## 1. Introduction and Notation

When we try to find the global minimum of a real continuous function $f$ defined on a compact set, say a parallelepiped $D = \sqcap_{i=1}^{n} [a_i, b_i]$ of $R^n$ with $n \geq 2$, many problems, closely connected with the location of that critical point, occur. These difficulties should be, certainly, less if the objetive function $f$ were univariable.

Assume that, for arbitrary small $\alpha > 0$, there exists a curve $\gamma_\alpha : I = [0,1] \to R^n$, satisfying the conditions:

i) $\gamma_\alpha^* = \gamma_\alpha(I) \subset D$.

ii) for any $x \in D$, the distance . (1)

Thus, the restriction $f_{\gamma_\alpha}$ of $f$ on $\gamma_\alpha^*$ is an approximant to $f$ in the sense that the Hausdorff distance $d_H(\gamma_\alpha^*, D)$ tends to 0 as $\alpha$ does (see [2, p. 61]). Now, by continuity the minimization of $f$ can be substituted by the minimization of $f_{\gamma_\alpha}$ and so the global minimum of $f_{\gamma_\alpha}$ is taken as an approximation to the global minimum of $f$ (see [1], [5], [6], [12]).

For a given $\alpha > 0$, a curve $\gamma_\alpha$, for which the above conditions (1) are fulfilled, is said to be $\alpha - dense$ in $D$ or that the curve $\gamma_\alpha$ densifies $D$ with density $\alpha$. In a general metric space $(E, d)$ a compact set $K$, containing $\alpha$-dense curves with arbitrary small $\alpha > 0$, is called *densifiable* (see [5] and [12], for more details). For instance, in $R^n$ all parallelepipeds are densifiable sets and, in fact, the $\alpha$-dense curves densifying them can be chosen of class $C^\infty$ (see [5], [12]).

As, in practice, many functions to minimize are Lipschitzian, it is convenient to take the $\alpha$-dense curves to be also Lipschitzian. We recall that a function $g : I \to E$ is said to be Lipschitzian of exponent $0 < \beta \leq 1$ iff there exists a constant $l > 0$ such that (for properties see, for instance, [13])

$$d(g(t), g(t')) \leq l \left|t - t'\right|^\beta . \qquad (2)$$

In [7] is proposed the problem of the existence of a minimal length curve densifying the unit square $I^2$ with fixed density $\alpha > 0$. Nevertheless, [11] deals with the same question but in a compact set $K$ belonging to a general metric space. The solution of this problem is given by considering the set $\Lambda_{\alpha,l}(K)$ of all Lipschitzian $\alpha$-dense curves in $K$, with constant $l$ and exponent $\beta = 1$. Since there is a connection between the degree of density $\alpha$ and the constant of Lipschitz $l$, in the present article we shall determine a lower bound of $l$ to guarantee that the set $\Lambda_{\alpha,l}(K)$ is non-void.

On the other hand, we shall prove that the degree of densification of the unit ball of a normed space determines its dimensionality (see the Theorem).

We shall use this idea to show that any curve in the unit ball of an infinite dimensional normed space densifies it with the worst of all densities, namely $\alpha \geq 1$. By virtue of this theorem we shall obtain, as corollary, the well-known Theorem of Riesz on the finite dimensionalty of locally compact normed spaces (see for instance [2], [4], [10]).

Also we shall construct a family of infinite dimensional curves, with arbitrary density $\alpha$, in the Hilbert parallelotope. The method that we shall exhibit here is different to that of Schöenberg curve that, recall, fills $K$, whereas our curves densify it with controlled density (see [ 8, p. 128]). The goal consists of using none Cantor set to define the curve, as that of [8] . Our construction is a natural generalization of the method for densifying finite dimensional domains exhibed in [5].

## 2. Lipschitzian $\alpha$-dense Curves

In this section we point out the relation between the degree of density $\alpha$ and the constant $l$ in such a way that $\Lambda_{\alpha,l}(K)$ to be a non-empty set. For this purpose, we begin to expose a simple example, where $\Lambda_{\alpha,l}(K) = \emptyset$ for certain values of $\alpha$ and $l$.

**Example 1.** In $R^2$ the segment line $K = \{(x, 0) : 0 \leq x \leq 1\}$ is a densifiable set for which $\Lambda_{\alpha,l}(K) = \emptyset$, for any $0 < \alpha < \dfrac{1}{4}$ and $0 < l \leq \dfrac{1}{2}$.

*Proof.* Suppose that there exists some $f \in \Lambda_{\alpha,l}(K)$. Thus,

i) $f : I \to K$ is continuous and has density $\alpha$ in $K$

and

ii)$\|f(t) - f(t')\| \leq l\,|t - t'|$ for any $t, t' \in I$.

From i), there exists a continuous function $\varphi : I \to I$ such that $f(t) = (\varphi(t), 0)$ for $t \in I$. Let $t_0, t_1$ be the values, where $\varphi$ attains the global extrema, minimum and maximum respectively. Because of the $\alpha-$density, $\varphi(t_0) \leq \alpha$ and $\varphi(t_1) \geq 1 - \alpha$.

Therefore

$$\varphi(t_1) - \varphi(t_0) \geq 1 - 2\alpha. \tag{1}$$

By the condition ii),

$$|\varphi(t_1) - \varphi(t_0)| \leq l\,|t_1 - t_0| \leq l$$

and from (1), $1-2\alpha \leq l \leq \frac{1}{2}$. Thus, $\frac{1}{4} \leq \alpha$, which is a contradiction since $\alpha < \frac{1}{4}$. □

As a first approximation to establish the dependance of $\alpha$ we include this easy result.

**Proposition 2.** *Let $K$ be a compact densifiable set in a metric space $(E,d)$. For $\alpha$ sufficiently small, if $\Lambda_{\alpha,l}(K) \neq \emptyset$, then, necessarily $l \geq \Delta - 2\alpha$, where $\Delta$ denotes the diameter of $K$.*

For the sake of completeness we give the next trivial lemma.

**Lemma 3.** *Let $(E,d)$ be a metric space. Any curve $\gamma : I \to E$, satisfying the Lipschitz condition of constant $l$ and exponent $\beta = 1$, is rectifiable and its length $L_\gamma \leq l$.*

Fixed the densifiable compact set, a lower bound, in function of the density, for the Lipschitz constant is stated in the following theorem.

**Theorem 4.** *For $0 < \alpha \leq \frac{1}{2}$, any Lipschitzian $\alpha$-dense curve in the unit square $I^2$, of constant $l$ and exponent $\beta = 1$, satisfies $l \geq \frac{1}{2\alpha} - 1$.*

*Proof.* Let $m = \left[\alpha^{-1}\right]$ be the entire part of $\alpha^{-1}$. The partition $P = \{0, \frac{2}{m}, \frac{4}{m}, \ldots, \frac{m}{m}\}$ produces, in the unit interval $I$, the $M = \left[\frac{m+1}{2}\right]$ subintervals

$$I_1 = \left[0, \frac{2}{m}\right], I_2 = \left[\frac{2}{m}, \frac{4}{m}\right], \ldots, I_M = \left[a_M, \frac{m}{m}\right],$$

where

$$a_M = \begin{cases} \dfrac{m-1}{m} & \text{if } m \text{ is odd,} \\ \dfrac{m-2}{m} & \text{if } m \text{ is even.} \end{cases}$$

Thus, the square is divided in $M$ strips $I_1 \times I$, $I_2 \times I$, $\ldots$, $I_M \times I$ having each $2m^{-1}$ in width. Since for densifying each strip is needed at least a segment line (actually the mean line) of length 1, any curve of density less or equal than $m^{-1}$ in the square will have, at least, $M$ of length. Now, noticing the definitions of $m$ and $M$ we have

$$M > \frac{m}{2} - \frac{1}{2} > \frac{1}{2\alpha} - 1.$$

Therefore the length $L_\gamma$ of any $\alpha-$dense curve in the unit square verifies

$$L_\gamma \geq M > \frac{1}{2\alpha} - 1.$$

Now, by applying the above lemma, the proof is completed. □

**Remark 5.** We have omitted, in our study, the case $0 < \beta < 1$. The reason consists of that it could lead to obtain curves, whose graphs have nonzero Lebesgue measure. Indeed, a lemma of [9] proves that the image of a Lipschitzian function $f : R \to R^n$ of exponent $\beta$, with $\beta > \dfrac{1}{n}$, has Lebesgue measure zero and, of course, these curves are our subject to study.

## 3. $\alpha$-dense Curves and Dimensionality

From a geometrical intuitive point of view, it appears that is always possible to put a curve, with arbitrary small density $\alpha$, into the unit ball. As soon we shall see, this is a peculiar property of the finite dimensional spaces.

We proceed by proving this elementary result.

**Lemma 6.** *In a metric space $(E, d)$, any densifiable set $K$ is precompact.*

*Proof.* Given $\varepsilon > 0$, let $\gamma_{\frac{\varepsilon}{2}} : I \to E$ be a curve of density $\dfrac{\varepsilon}{2}$ in $K$. Then, the image $\gamma_{\frac{\varepsilon}{2}}(I)$, denoted by $\gamma^*_{\frac{\varepsilon}{2}}$, satisfies

$$d(x, \gamma^*_{\frac{\varepsilon}{2}}) \leq \frac{\varepsilon}{2} \quad \text{for all } x \in K. \tag{1}$$

On the other hand, $\gamma^*_{\frac{\varepsilon}{2}}$ is a compact set of $E$, so precompact. Thus, there exists, a finite set $F \subset \gamma^*_{\frac{\varepsilon}{2}}$ such that

$$d(y, F) \leq \frac{\varepsilon}{2} \quad \text{for all } y \in \gamma^*_{\frac{\varepsilon}{2}}. \tag{2}$$

From (1) and (2)

$$d(x, F) \leq \varepsilon \text{ for all } x \in K,$$

implying that $K$ is precompact, as claimed. □

With the help of the precedent lemma, we show a characterization of the finite dimension spaces by means of the densification of its unit ball.

**Theorem 7.** *A Banach space $(E, \|\|)$ is finite dimensional if and only if the closed unit ball $B$ is densifiable.*

*Proof.* Consider the unit closed ball $B$ in the finite dimensional space $R^n$ . Then, there exists, for each $\varepsilon > 0$, a finite quantity of parallelepipeds $H_i$, with $1 \leq i \leq p$, such that:

i) If $H_i, H_j$ are adjacent, then, they have a common part of boundary but disjoint interiors.

ii) $\cup_{i=1}^{p} H_i \subset B$.

iii) For any $x \in B$, $d(x,\, H_i) \leq \varepsilon$ for some $i = 1, 2, \ldots, p$.

Since each $H_i$ can be densified by an $\alpha$-dense curve $\gamma_i$ (see [5]), i), ii) and iii) imply that there exists an $\alpha$-dense curve $\gamma$ into $B$ by considering the union of all $\gamma_i$ (it could be added or cutted a finite quantity of pieces of each curve to define the $\alpha$-dense curve in $B$). Hence, as $\varepsilon$ and $\alpha$ are arbitrary small, the unit ball is densifiable.

Reciprocally, assume the unit ball $B$ of a Banach space $(E, \|\|)$ is densifiable. Then, by the above lemma, $B$ is precompact and, by completeness, compact. Therefore, by applying the Theorem of Riesz (see, for instance [2], [4], [10]) the space has finite dimension, and so the proof is completed. □

Let $(E, \|\|)$ be an infinite dimensional Banach space. By the above theorem, the unit ball $B$ is not densifiable but it will have a certain degree of densification that we are going to determine. To precise this idea we introduce the following definitions.

**Definition 8.** Let $f : I \to B$ be an arbitrary continuous function. By compactness, for each $x \in B$, the distance

$$\delta_x = d(x, f(I) = \inf\{\|x - f(t)\| : t \in I\}$$

is attained.

Since $\delta_x \leq 2$ for all $x \in B$, the number $m_f = \sup\{\delta_x : x \in B\}$ exists and satisfies $m_f \leq 2$. Thus $m_f$ is called the degree of density of $f$ in $B$.

**Definition 9.** Denote $C(I, B)$ the set of all continuous functions $f : I \to B$. Then, the number $M_B = \inf\{m_f : f \in C(I, B)\ \}$ is defined as the degree of densification of $B$.

From the above definition, one has that $M_K\ = 0$ iff $K$ is a densifiable set.

Then, in an infinite dimensional normed space, the unit ball $B$ satisfies that $M_B > 0$. But, can we determine a strictely positive bound for $M_B$? Before to give the answer to this question, we begin with an example of curve, with inifinite length, which densifies the unit ball with a rough degree of density, namely greater than, or equal to 1.

**Example 10.** Let $L^2(I)$ be the Hilbert space of all real functions of integrable square defined on $I = [0, 1]$. The curve $f : I \to L^2(I)$, defined by $f(t) = \aleph_{[0,t]}$ (the characteristic function of the set $[0, t]$), is contained in the unit ball $B$, has infinite length and a degree of density $m_f \geq 1$.

*Proof.* Suppose $t' > t$, then, $f(t') - f(t) = \aleph_{[t,t']}$ and so

$$\left\|f(t') - f(t)\right\|^2 = t' - t = \left|t' - t\right|. \tag{1}$$

Hence $f$ is a Lipschitzian of constant 1 and exponent $\beta = \frac{1}{2}$, so it is continuous on $I$.

The length of $f$ is

$$L_f = \sup\{\sum_{i=1}^{n} \|f(t_i) - f(t_{i-1})\| : P \in \Pi\}, \tag{2}$$

where $\Pi$ denotes the set of all partitions of $I$.

Taking the partition $P_n = \{\frac{i}{n} : i = 0, 1, \ldots, n\}$, by (2) and (1) $L_f \geq \sum_{i=1}^{n} (\frac{1}{n})^{\frac{1}{2}} = n^{\frac{1}{2}}$ for all $n$, so $f$ has infinite length.

Let $g(t) = -1$ be for all $t \in I$. Denote $\delta_g = \inf\{\|g - f(t)\| : t \in I\}$, i. e. the distance from $g$ to the graph of the curve $f$. Noticing the inner product in $L^2(I)$,

$$\|g - f(t)\|^2 = \langle g - f(t), g - f(t)\rangle = \|g\|^2 + \|f(t)\|^2 - 2\langle g, f(t)\rangle \geq 1. \tag{3}$$

Since $g \in B$, from (3)

$$m_f = \sup\{\delta_h : h \in B\} \geq \delta_g \geq 1\,,$$

and the example is proved. □

**Remark 11.** Observe that relation (1), in the above example, implies the pythagorean property

$$\left\|f(t') - f(t)\right\|^2 + \left\|f(t'') - f(t')\right\|^2 = \left\|f(t'') - f(t)\right\|^2 \text{ for all } t < t' < t''.$$

Therefore the vectors $f(t') - f(t)$ and $f(t'') - f(t')$ are orthogonal for all $t < t' < t''$. This function $f(t)$ is called a Brownian curve (see [3]).

The above example is not an anomalous property of the space $L^2(I)$. In fact, we shall prove below that the density $\alpha$ of any curve in the unit ball of any infinite dimensional normed space is the worst of all, namely $\alpha \geq 1$.

**Theorem 12.** *Let $(E, \|\|)$ be an infinite dimensional normed space. Thus, the degree of densification of the unit ball, $M_B \geq 1$.*

*Proof.* Let $\gamma : I \to E$ an arbitrary curve of density $\alpha$ in $B$. Then, for any $x \in B$ one has

$$\inf\{\|x - \gamma(t)\| : t \in I\} \leq \alpha. \tag{1}$$

Assume $\alpha < 1$. Thus, choose $0 < 2\varepsilon < 1 - \alpha$. By compactness, given $\varepsilon$ there exists a finite set $F = \{\gamma(t_i) : i = 1, \ldots, n\}$ such that

$$\inf\{\|y - \gamma(t_i)\| : i = 1, \ldots, n\} \leq \varepsilon \text{ for any } y \in \gamma(I). \tag{2}$$

From (1) and (2),

$$\inf\{\|x - \gamma(t_i)\| : i = 1, \ldots, n\} \leq \alpha + \varepsilon \text{ for any } x \in B. \tag{3}$$

Define $V$ as the subspace generated by $F$. Since $V$ has finite dimension, it is closed. Therefore there exists a vector $x_0 \in E$ such that

$$\inf\{\|x_0 - v\| : v \in V\} = \lambda \geq 1, \tag{4}$$

and so we can determine a vector $v_0 \in V$ such that

$$\lambda \leq \|x_0 - v_0\| \leq \lambda + \varepsilon. \tag{5}$$

Define $z_0 = \dfrac{x_0 - v_0}{\|x_0 - v_0\|}$ and so $z_0 \in B$. By applying (3), there exists $1 \leq k \leq n$ such that

$$\|z_0 - \gamma(t_k)\| \leq \alpha + \varepsilon. \tag{6}$$

On the other hand, we can write

$$x_0 = v_0 + \|x_0 - v_0\| .\gamma(t_k) + \|x_0 - v_0\| .(z_0 - \gamma(t_k).$$

Since $v_0 + \|x_0 - v_0\| .\gamma(t_k) \in V$, from (4) we deduce

$$\| \|x_0 - v_0\| .(z_0 - \gamma(t_k).\| = \|x_0 - v_0\| . \|z_0 - \gamma(t_k)\| \geq \lambda \geq 1.$$

Consequently,

$$\|x_0 - v_0\| \geq \frac{\lambda}{\|z_0 - \gamma(t_k)\|},$$

and by using (6)

$$\|x_0 - v_0\| \geq \frac{\lambda}{\alpha + \varepsilon}.$$

Now, from (5) we obtain $\lambda + \varepsilon \geq \dfrac{\lambda}{\alpha + \varepsilon}$ or equivalently $\dfrac{\lambda}{\lambda + \varepsilon} \leq \alpha + \varepsilon$. From the choise of $\varepsilon$, we are led to $\lambda \leq 1 - \varepsilon$, which contradits (4). Thus, $\alpha \geq 1$ for any arbitrary curve $\gamma$ and, noticing Definition 8 and Definition 9, the degree of densification of the unit ball $M_B \geq 1$, and so the theorem follows. □

**Corollary 13.** (The Theorem of Riesz) *Any normed space* $(E, \|\|)$ *locally compact is finite dimensional.*

*Proof.* Without loss of generality, we can suppose that the unit ball $B$ is compact. Assume $E$ to be infinite dimensional, then by the above theorem

$$M_B \geq 1. \tag{1}$$

Choose $0 < \varepsilon < 1$, by compactness there exists a finite set

$$F = \{x_i : i = 1, \ldots, p\} \subset B \text{ such that } B \subset \cup_{i=1}^{p} B(x_i, \varepsilon),$$

where $B(x_i, \varepsilon)$ denotes, as usual, the closed ball of center $x_i$ and radius $\varepsilon$. Consider a polygonal line $L$ joining all centers of the balls, then, there exists a continuous function $\gamma : I \to E$, whose graph is $L$. By construction, $\gamma$ is a curve in $B$ of density $\alpha \leq \varepsilon$. Noticing Definition 8 and Definition 9 we have $m_\gamma \leq \varepsilon$, $M_B \leq \varepsilon < 1$, which contradits (1) and so the result is showed. □

**Remark 14.** The Riesz Theorem establishes the finite dimensionality of the space as consequence of the compactness of the unit ball. Nevertheless, it does not say what is the size of the compacts contained in the unit ball, which it is exactly the heart of the Theorem 12. Therefore, obviously, the Riesz Theorem does not imply Theorem 12.

## 4. Infinite Dimensional $\alpha$-dense Curves

In [8] we can find the construction of the Schöenberg curve as example of space-filling curve in an infinite dimensional space. Of course any space-filling curve is, in particular, an $\alpha$-dense curve, but our interest is to give a curve with controlled and strictly positive density $\alpha$. Furthermore, we would like to exhibe a technique other than [8], where we recall, was used a Cantor set.

Let us denote by $R^\omega$ the product of countably many copies of $R$ endowed with the metric defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} . \frac{|x_n - y_n|}{1 + |x_n - y_n|},$$

with $x = (x_n)_{n=1,2,...}$ and $y = (y_n)_{n=1,2,...}$.

From elementary considerations over the above distance, the next lemma trivially follows

**Lemma 15.** *A sequence of continuous functions $\Phi_i : I \to R$, with $i = 1, 2, \dots$, determines a continuous function $\Phi : I \to R^\omega$, defined by $\Phi(t) = (\Phi_i(t))_{i=1,2,...}$.*

The simplicity of the following technical result is clear.

**Lemma 16.** *Let $\varphi : I \to I$ be a continuous surjective function of period $0 < \tau < 1$. For any interval $A \subset I$ with length $|A| > \tau$, one has $A \cap \varphi^{-1}(B) \neq \emptyset$ for all nonvoid open set $B \subset I$.*

For our aim, the following lemma is crucial.

**Lemma 17.** *Let $n \geq 2$ be an integer and $\varphi : I \to I$ a continuous and surjective function of period $0 < \tau < 1$. Consider the function $\Phi_n : I \to R^n$, defined by $\Phi_n(t) = (\Phi_{n,i}(t))_{i=1,...,n}$, where $\Phi_{n,i}(t) = \varphi^{i-1}(t)$ (the exponent $i-1$ denotes the composition of $\varphi$ with itself $i-1$ times, and $\varphi^0$ the identity function). Then, $\Phi_n$ densifies the parallelepiped $I^n$ with density $\alpha = \tau\sqrt{n}$.*

*Proof.* Let $x = (x_i)_{i=1,...,n}$ be a point of $I^n$. Let us take $\varepsilon > \tau$ and define the intervals $B_i = (x_i - \varepsilon, x_i + \varepsilon) \cap I$ for $i = 1, \dots, n$. Since the length of $B_1$ is greater than $\tau$, by applying the above lemma, the set

$$\Phi_{n,1}^{-1}(B_1) \cap \Phi_{n,2}^{-1}(B_2) \cap \cdots \cap \Phi_{n,n}^{-1}(B_n) \\ = B_1 \cap \varphi^{-1}(B_2) \cap \varphi^{-2}(B_3) \cap \cdots \cap \varphi^{-n+1}(B_n), \quad (1)$$

where ($\varphi^{-2}, \varphi^{-3}, \dots, \varphi^{-n+1}$ denotes the composition of $\varphi^{-1}$, with itself, 2, 3, $\dots$, $n-1$ times, will be nonvoid if

$$\varphi^{-1}(B_2) \cap \varphi^{-2}(B_3) \cap \cdots \cap \varphi^{-n+1}(B_n) \quad (2)$$

is a nonvoid open set in $I$.

The expression (2) can be written as

$$\varphi^{-1}(B_2 \cap \varphi^{-1}(B_3) \cap \cdots \cap \varphi^{-n+2}(B_n)), \quad (3)$$

so, as the length of $B_2$ is greater than $\tau$. By using the above lemma, the set (3) will be nonvoid if

$$\varphi^{-1}(B_3) \cap \cdots \cap \varphi^{-n+2}(B_n)$$

is a nonvoid open set in $I$.

Therefore, repeating this process we are led, finally, to consider the set

$$B_{n-1} \cap \varphi^{-1}(B_n),$$

which is nonvoid by the Lemma 16. Thus, the set defined by the expression (1) is nonvoid and consequently there exists $t \in I$ with $\Phi_{n,i}(t) \in B_i$ for all $i = 1, 2, \ldots, n$. This implies that

$$|\Phi_{n,i}(t) - x_i| < \varepsilon \text{ for all } i = 1, 2, \ldots, n.$$

Hence, the euclidean distance

$$d(\Phi_n(t), x) < \varepsilon\sqrt{n}, \text{ for any } \varepsilon > \tau$$

and so $d(\Phi_n(t), x) \leq \tau\sqrt{n}$, which proves the lemma. □

The construction of $\alpha$-dense curves, with arbitrary $\alpha > 0$, in infinite dimensional spaces is given by means of the following result.

**Theorem 18.** *The Hilbert parallelotope $I^w$ is a densifiable set in the metric space $(R^w, d)$.*

*Proof.* Let $\alpha$ be a positive arbitrary real number. Given $\varepsilon > 0$, determine a positive integer $n$ such that

$$\sum_{i=n+1}^{\infty} \frac{1}{2^i} < \varepsilon.$$

Noticing the above lemma, for $\tau = \dfrac{\alpha}{\sqrt{n}}$ there exists a curve $\Phi_n = (\Phi_{n,i})_{i=1,2,\ldots,n}$ densifying $I^n$ with density $\alpha$. Define the function $\Omega : I \to R^w$ by $\Omega(t) = (\Omega_i(t))_{i=1,2,\ldots}$ with $\Omega_i(t) = \Phi_{n,i}(t)$ for $i = 1, 2, \ldots, n$ and $\Omega_i(t) = \Psi_i(t)$ for $i > n$, where $\Psi_i$ are arbitrary continuous functions defined and valued on $I$. From Lemma 15, $\Omega$ is continuous, so it defines a curve in the Hilbert parallelotope.

Let $x = (x_i)_{i=1,2,\ldots}$ be a point of $I^w$. Let us denote $X_n = (x_i)_{i=1,2,\ldots,n}$. Then, by applying Lemma 17, there exists $t \in I$ such that the euclidean norm

$$\|X_n - \Phi_n(t)\| \leq \alpha.$$

According to the metric defined in $R^\omega$,

$$d(x,\Omega(t)) = \sum_{i=1}^{\infty} \frac{1}{2^i} . \frac{|x_i - \Omega_i(t)|}{1 + |x_i - \Omega_i(t)|}$$

$$\leq \sum_{i=1}^{n} \frac{1}{2^i} \|X_n - \Phi_n(t)\| + \sum_{i=n+1}^{\infty} \frac{1}{2^i} < \alpha + \varepsilon \,,$$

for arbitrary $\varepsilon$. Therefore the theorem follows. □

**Remark 19.** As we have just seen, Theorem 18 gives us a very large class of $\alpha$-dense curves densifying the Hilbert parallelotope. This family of curves can be easily constructed, according Lemma 16. By choosing a surjective and periodical continuous function $\varphi : I \to I$. After, the process is merely completed by taking arbitrary continuous functions $\Psi_i : I \to I$ from a sufficiently large positive integer $n$.

## References


[1] H. Ammar, Y. Cherruault, Approximation of a several variables function by a one variable function and application to global optimization, *Math. Comput. Modelling*, **18**, No. 2 (1993), 17-21.

[2] J. Dieudonné, *Fondements de L'Analyse Moderne*, Tome **I**, Gauthier-Villars, Paris (1969).

[3] J.P. Kahane et al, *Leçons de Mathématiques D'aujourd'hui*, Cassini, Paris (2000).

[4] G. Köthe, *Topological Vector Spaces I*, Springer-Verlag, Berlin (1969).

[5] G. Mora, Y. Cherruault, Characterization and generation of $\alpha$-dense curves, *Computers Math. Applic.*, **33**, No. 9 (1997), 83-91.

[6] G. Mora, Optimization by space-densifying curves as a natural generalization of the Alienor method, *Kybernetes*, **29**, No.5/6 (2000), 746-754.

[7] G. Mora, Y. Cherruault, On the minimal length curve that densifies the square, *Kybernetes*, **28**, No. 9 (1999), 1054-1064.

[8] H. Sagan, *Space-Filling Curves*, Springer-Verlag, New-York (1994).

[9] J.D. Stegeman, On a theorem of N.Th. Varopoulos, *Math. Scand.*, **27** (1970), 50-52.

[10] A.E. Taylor, *Introduction to Functional Analysis*, John Wiley & Sons, New York (1958).

[11] A. Ziadi, Y. Cherruault, G. Mora, The existence of $\alpha$-dense curves with minimal length in a metric space, *Kybernetes*, **29**, No. 2 (2000), 219-230.

[12] A. Ziadi, Y. Cherruault, Generation of $\alpha$-dense curves and application to global optimization, *Kybernetes*, **29**, No. 1 (2000), 71-82.

[13] A. Zigmund, *Trigonometric Series*, Cambridge University Press, Cambridge (1959).