On Motzkin decomposable sets and functions

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Abstract

A set is called Motzkin decomposable when it can be expressed as the Minkowski sum of a compact convex set with a closed convex cone. The main result in this paper establishes that a closed convex set is Motzkin decomposable if and only if the set of extreme points of its intersection with the linear subspace orthogonal to its lineality is bounded. The paper characterizes the class of the extended functions whose epigraphs are Motzkin decomposable sets showing, in particular, that these functions attain their global minima when they are bounded from below. The generation of functions of this class from other functions of the same type is also considered.

Key words: Motzkin decomposition, closed convex sets, convex functions

1 Introduction

We say that a nonempty set $F \subset \mathbb{R}^n$ is decomposable in Motzkin's sense (*M-decomposable* in short) if there exist a compact convex set C and a closed convex cone D such that $F = C + D$. Then we say that $C + D$ is a *Motzkin representation* (or *decomposition*) of F with compact and conic components C and D, respectively. Any M-decomposable set F has a unique conic component $D = 0^+F$ but multiple compact components when F is unbounded.

The classical Motzkin Theorem [6] asserts that any polyhedral convex set is M-decomposable. The convex subsets of M-decomposable sets were called *hyperbolic sets* in [1] and [2].

This class of closed convex sets was characterized in different ways in [3], two of them providing the smallest compact component when the checked set F turns out to be M-decomposable and contains no line. The mentioned characterizations involve a geometric object, the so-called Pareto-like set of the intersection of F with the linear subspace orthogonal to its lineality, and a certain linear representation of the so-called conic representation of F, i.e., the closed convex cone $\{(a,b) \in \mathbb{R}^{n+1} : a'x \geq b \ \forall x \in F\}$. The Pareto-like sets are characterized in different ways in Section 2.

In Section 3 we give two new characterizations of the M-decomposable sets, the main one showing that it is possible to replace the mentioned concept of Pareto-like set by the more intuitive one of the set of extreme points. We also show how to obtain new M-decomposable sets from a given finite family of sets of the same class by combining Minkowski sums and unions with convex hulls and closures.

Finally, Section 4 considers the so-called *M-decomposable functions*, i.e., those extended functions whose epigraphs are M-decomposable. These functions are characterized and its behavior in the optimization context is analyzed. In particular, it is shown that any M-decomposable function which is bounded from below attains its infimum on the whole space. It is also shown that the sum of an M-decomposable function with an affine function is M-decomposable, too, and we indicate how to build M-decomposable functions from other functions of the same class by combining pointwise minimum and infimal convolution with convex and lsc hulls. Concerning Section 4, the only antecedents are the properties of two particular classes of M-decomposable functions: the polyhedral convex functions and the support functions of nonempty closed convex sets, whose respective epigraphs (polyhedral convex sets and closed convex cones, respectively) are M-decomposable. Thus the common properties of both families of functions become conjectures on M-decomposable functions to be checked.

Throughout the paper we use the following notation. For any $X \subset \mathbb{R}^p$, we denote by $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{bd} X$, $\operatorname{rint} X$, $\operatorname{rbd} X$, $\operatorname{conv} X$, and $\operatorname{cone} X = \mathbb{R}_+ \operatorname{conv} X$, the *interior*, the *closure*, the *boundary*, the *relative interior*, the *relative boundary*, the *convex hull* of X, and the *convex conical hull* of X, respectively.

The scalar product of $x, y \in \mathbb{R}^p$ is denoted either by $x'y$ or by $\langle x, y \rangle$, the Euclidean norm of x by $\|x\|$, the Euclidean distance by ρ, the canonical basis by $\{e_1, ..., e_p\}$, the zero vector by 0_p, and the closed unit ball by B_p. The *orthogonal complement* of a linear subspace X is $X^\perp := \{y \in \mathbb{R}^p : \langle x, y \rangle = 0 \ \forall x \in X\}$.

If X is a convex set, $\operatorname{extr} X$, 0^+X and $\operatorname{lin} X := (0^+X) \cap (-0^+X)$ denote the *set of extreme points*, the *recession cone* and the *lineality space* of X, respectively. Given a convex set X and a point $a \in X$,

$$D(X; a) := \{u \in \mathbb{R}^p : \exists \lambda > 0 \text{ such that } a + \lambda u \in X\}$$

and

$$N_X(a) := \{u \in \mathbb{R}^p : \langle x - a, u \rangle \leq 0 \ \forall x \in X\}$$

are the *cone of feasible directions* and the *normal cone* at x, respectively. It is easy to prove that $N_X(a)$ is a linear subspace of $\mathbb{R}^p$ whenever $a \in \operatorname{rint} X$.

Given $x = (x_1, ..., x_p)$ we denote by $\widehat{x}$ the result of eliminating the last component of x, i.e., $\widehat{x} = (x_1, ..., x_{p-1})$. Coherently, we identify $\widehat{X} = \{\widehat{x} : x \in X\}$ with the (orthogonal) projection of $X \subset \mathbb{R}^p$ onto $\mathbb{R}^{p-1}$.

Given $f : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, we denote by $\operatorname{gph} f$, $\operatorname{epi} f$, and $\operatorname{dom} f = \widehat{\operatorname{epi} f}$ its *graph*, its *epigraph* and its *domain*, respectively, whereas $\partial f(x)$ denotes the *subdifferential of* f at $x \in \operatorname{dom} f$.

The *conjugate* of f is the function $f^* : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ such that

$$f^*(u) := \sup\{\langle x, u \rangle - f(x) : x \in \operatorname{dom} f\}.$$

Any set $X \subset \mathbb{R}^p$ is represented in a unique way by its *indicator function*

$$\delta_X(x) := \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{otherwise.} \end{cases}$$

The *support function* of X is $\delta_X^*(u) = \sup\{\langle x, u \rangle : x \in X\}$.

The *lower semicontinuous (lsc) envelope* of $f : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ is the function $\overline{f} : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ defined by

$$\overline{f}(x) := \inf\{t \in \mathbb{R} : \ (x, t) \in \operatorname{cl} \operatorname{epi} f\}.$$

Clearly we have $\operatorname{epi} \overline{f} = \operatorname{cl} \operatorname{epi} f$, which implies that $\overline{f}$ is the greatest lsc function minorizing f; so $\overline{f} \leq f$. If f is convex, then $\overline{f}$ is also convex, and then $\overline{f}$ does not take the value $-\infty$ if and only if f admits an affine minorant.

The *lsc convex hull* of f is the convex lsc function $\overline{\operatorname{conv}} f : \mathbb{R}^p \longrightarrow \overline{\mathbb{R}}$ such that

$$\operatorname{epi}(\overline{\operatorname{conv}} f) = \operatorname{cl} \operatorname{conv}(\operatorname{epi} f).$$

Obviously $\overline{\operatorname{conv}} f \leq \overline{f} \leq f$.

2 Pareto-like sets revisited

The *Pareto-like set* of a closed convex set F, $\emptyset \neq F \subset \mathbb{R}^n$, is

$$M(F) := \left\{ x \in F \cap (\operatorname{lin} F)^{\perp} : (x - K) \cap F = \{x\} \right\},$$

where

$$K := \left(0^{+} F\right) \cap (\operatorname{lin} F)^{\perp} \tag{1}$$

is a pointed convex cone. The next result characterizes $M(F)$ from any linear representation of F.

Proposition 1 *Let $\{a_t' x \geq b_t, t \in T\}$ and $\{c_s' x = 0, s \in S\}$ be linear representations of $F \neq \emptyset$ and $(\operatorname{lin} F)^{\perp}$ ($S = \emptyset$ if F does not contain lines), respectively. Then $x \in M(F)$ if and only if $a_t' x \geq b_t$ $\forall t \in T$, $c_s' x = 0$ $\forall s \in S$, and*

$$\pm \begin{pmatrix} e_i \\ e_i' x \end{pmatrix} \in \operatorname{cl} \operatorname{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; - \begin{pmatrix} a_t \\ a_t' x \end{pmatrix}, t \in T; \pm \begin{pmatrix} c_s \\ c_s' x \end{pmatrix}, s \in S; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\},$$

$i = 1, .., n.$

Proof. Let K be as in (1). Since $0^{+} F = \{x \in \mathbb{R}^n : a_t' x \geq 0, t \in T\}$, we have

$$K = \{x \in \mathbb{R}^n : a_t' x \geq 0, t \in T; c_s' x = 0, s \in S\}.$$

Then $\overline{x} \in M(F)$ if and only if $\overline{x} \in F \cap (\operatorname{lin} F)^{\perp}$ and $x = \overline{x}$ is a consequence of $x \in (\overline{x} - K) \cap F$, i.e., the equations $e_i' x = e_i' \overline{x}$, $i = 1, .., n$, are consequences of the linear system

$$\{a_t' x \geq b_t, t \in T; a_t' (\overline{x} - x) \geq 0, t \in T; c_s' (\overline{x} - x) = 0, s \in S\}.$$

The result follows from the nonhomogeneous Farkas Lemma for semi-infinite linear systems (see, e.g., [4, Theorem 3.1]). □

In the next two statements, we shall consider M-decomposable sets with pointed recession cones. It is not too restrictive condition because, if it is not the case for a certain M-decomposable set F, we refer to $F \cap (\operatorname{lin} F)^{\perp}$, whose recession cone $0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ is always pointed.

Proposition 2 *Let F be an M-decomposable set with a pointed recession cone and $x \in F$. Then $x \in M(F)$ iff $(-0^{+} F) \cap D(F; x) = \{0_n\}$. Hence, if F is unbounded then $M(F) \subset \operatorname{rbd} F$.*

Proof. The proof is evident, because $M(F)$ is the set of efficient points of F with respect to the cone $0^{+} F$. □

Proposition 3 *Let F be an M-decomposable set with a pointed recession cone and $x \in \operatorname{rbd} F$. If there exists a supporting hyperplane H of F at x such that $H \cap F$ is a bounded set then $x \in M(F)$. If $x \in M(F)$ and $D(F;x)$ is closed, then there exists a supporting hyperplane H of F at x such that $H \cap F$ is a bounded set.*

Proof. Let there exist a supporting hyperplane H of F at x such that $H \cap F$ is a bounded set, H^+ and H^- be the closed halfspaces determined by H, and assume that $F \subset H^+$. Since $H \cap F$ is a bounded set, then $(x + 0^+F) \cap H = \{x\} = (x - 0^+F) \cap H$. Therefore $x - 0^+F \subset H^-$. This implies that $x \in M(F)$.

Now, let $x \in M(F)$ and $D(F;x)$ be a closed cone. We point out that $F \subset x + D(F;x)$. From the previous proposition we have $(-0^+F) \cap D(F;x) = \{0_n\}$. Let us consider the set $\operatorname{conv}((-0^+F) \cap B_n)$. Since 0^+F is a pointed closed convex cone, $\operatorname{conv}((-0^+F) \cap B_n)$ is a compact base of -0^+F, $0_n \notin \operatorname{conv}((-0^+F) \cap B_n)$ and $\operatorname{conv}((-0^+F) \cap B_n) \cap D(F;x) = \emptyset$. There exists an $\varepsilon > 0$ sufficiently small such that still $0_n \notin \operatorname{conv}((-0^+F) \cap B_n) + \varepsilon B_n$ and $(\operatorname{conv}((-0^+F) \cap B_n) + \varepsilon B_n) \cap D(F;x) = \emptyset$. Now, we shall consider the closed convex pointed cone K generated by the compact base $\operatorname{conv}((-0^+F) \cap B_n) + \varepsilon B_n$. First, we have that $(-0^+F) \setminus \{0_n\} \subset \operatorname{int} K$. Second, if we suppose that $K \cap D(F;x) \neq \{0_n\}$ we shall get an element $y \in (\operatorname{conv}((-0^+F) \cap B_n) + \varepsilon B_n) \cap D(F;x)$, which is a contradiction. So, $K \cap D(F;x) = \{0_n\}$ and we can separate both closed convex cones by means a hyperplane $\overline{H}$. Let us translate these cones and hyperplane at the point x. The translated hyperplane H separates the closed sets $x + D(F;x)$ and $x + K$. Moreover $x + 0^+F \subset x + D(F;x)$. Let $d \neq 0_n$, $d \in 0^+F$ and $x + d \in H$. Then $x - d \in H$, which contradicts the inclusion $(-0^+F) \setminus \{0_n\} \subset \operatorname{int} K$. Hence $F \cap H$ is a bounded set. The proof is complete. $\square$

If the M-decomposable set is a polyhedral convex set, the above closedness assumption on $D(F;x)$ is automatically satisfied. The first part of Example 9 below shows that this assumption is not superfluous in the nonpolyhedral case (consider the points $\pm(1,0,0)$).

The next characterization of the M-decomposable sets is Theorem 19 in [3]. Here $F^*(c)$ represents the set of global minima of the linear form $\langle c, . \rangle$ on F.

Theorem 4 *Let F be a closed convex set, $\emptyset \neq F \subset \mathbb{R}^n$. Then the following statements hold:*
(i) F is M-decomposable if and only if $M(F)$ is bounded. In that case,

$$F = \operatorname{cl}\operatorname{conv} M(F) + 0^+F \tag{2}$$

is a Motzkin representation of F.
(ii) If F is an M-decomposable set containing no lines, then $\operatorname{cl}\operatorname{conv} M(F)$ *is*

the smallest compact component of F, *with*

$$M(F) = \left\{ x \in F : \left(x - 0^+F \right) \cap F = \ \{x\} \right\} \tag{3}$$

satisfying

$$\begin{array}{c} \emptyset \neq \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\} \subset M(F) \\ \subset \bigcup \left\{ F^*(c) : 0_n \neq c \in \operatorname{cl} \widehat{K(F)} \right\}. \end{array} \tag{4}$$

3 Identifying and generating M-decomposable sets

The first characterization of M-decomposable sets requires the next simple lemma.

Lemma 5 *Let* $F \subset \mathbb{R}^n$ *be a closed convex set. Then*

$$0^+F = 0^+ \left(F \cap (\operatorname{lin} F)^\perp \right) + \operatorname{lin} F. \tag{5}$$

Proof. It is consequence of the well-known decomposition of a convex set F as the sum of a closed convex set containing no lines with a linear subspace:

$$F = F \cap (\operatorname{lin} F)^\perp + \operatorname{lin} F \tag{6}$$

(see, e.g., [7, p. 65]). □

According to Klee representation theorem [5], a sufficient condition for a nonempty closed convex set F to be M-decomposable is the boundedness of $F \cap (\operatorname{lin} F)^\perp$. The next result shows that this condition is also necessary.

Theorem 6 *A closed convex set* $F \subset \mathbb{R}^n$ *is M-decomposable if and only if* $F \cap (\operatorname{lin} F)^\perp$ *is M-decomposable. In this event, any compact component of* $F \cap (\operatorname{lin} F)^\perp$ *is a compact component of* F *too. Consequently,* F *is M-decomposable whenever* 0^+F *is a linear subspace.*

Proof. *"If". Let* C *be a* compact convex set such that $F \cap (\operatorname{lin} F)^\perp = C + 0^+ \left(F \cap (\operatorname{lin} F)^\perp \right)$. Then, by (6),

$$F = C + 0^+ \left(F \cap (\operatorname{lin} F)^\perp \right) + \operatorname{lin} F. \tag{7}$$

Since $0^+ \left(F \cap (\operatorname{lin} F)^\perp \right) \subset (\operatorname{lin} F)^\perp$, the convex cone $0^+ \left(F \cap (\operatorname{lin} F)^\perp \right) + \operatorname{lin} F$ is closed, and hence (7) is an M-decomposition of F with compact component C.

"Only if". *Let* C be a compact convex set such that

$$F = C + 0^{+}F. \tag{8}$$

Without loss of generality, we can assume that $C \subseteq (\operatorname{lin} F)^{\perp}$ (see the first paragraph of the proof of [3, Theorem 19]). Since $C \subseteq F \cap (\operatorname{lin} F)^{\perp}$, we have $C + 0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right) \subseteq F \cap (\operatorname{lin} F)^{\perp}$. To prove the opposite inclusion, let $x \in F \cap (\operatorname{lin} F)^{\perp}$. In view of (8) and (5), there exist $y \in C + 0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ and $d \in \operatorname{lin} F$ such that $x = y + d$. Since $x, y \in (\operatorname{lin} F)^{\perp}$, we have $0 = \langle x, d\rangle = \langle y, d\rangle + \|d\|^{2} = \|d\|^{2}$, so that $d = 0$ and therefore $x = y \in C + 0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$. We have thus proved the inclusion $F \cap (\operatorname{lin} F)^{\perp} \subseteq C + 0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ and hence the equality between these two sets, which shows that $F \cap (\operatorname{lin} F)^{\perp}$ is M-decomposable.

>From (7) it is clear that any compact component of $F \cap (\operatorname{lin} F)^{\perp}$ is a compact component of F too.

Now we assume that $0^{+}F$ is a linear subspace. Given $y \in 0^{+}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$, $y \in 0^{+}F = \operatorname{lin} F$ (by assumption) and $y \in 0^{+}(\operatorname{lin} F)^{\perp} = (\operatorname{lin} F)^{\perp}$, so that $y = 0_{n}$. Thus $F \cap (\operatorname{lin} F)^{\perp}$ is M-decomposable because it is a compact convex set. $\square$

Observe that an M-decomposable set F has a smallest compact component if and only if $\operatorname{lin} F = \{0_{n}\}$ (i.e., $\operatorname{ext} F \neq \emptyset$).

Corollary 7 *Let* $f : \mathbb{R}^{n} \longrightarrow \overline{\mathbb{R}}$ *be a proper convex function. Then* $\partial f(x)$ *is M-decomposable for any* $x \in \operatorname{rint} \operatorname{dom} f$.

Proof. Let $x \in \operatorname{rint} \operatorname{dom} f$. Then $\partial f(x) \neq \emptyset$ and this implies that $0^{+}\partial f(x) = N_{\operatorname{dom} f}(x)$ (see, e.g. [7, p. 218, l. 9-15]), this cone being actually a linear subspace because $x \in \operatorname{rint} \operatorname{dom} f$. The conclusion follows from Theorem 6. $\square$

The preceding result does not hold if the assumption $x \in \operatorname{rint} \operatorname{dom} f$ is removed, since every nonempty closed convex set is the subdifferential of its support function at the origin.

To get a counterpart of Theorem 4 in terms of $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ instead of $M(F)$ we need a lemma.

Lemma 8 *Let* F *be a closed convex set,* $\emptyset \neq F \subset \mathbb{R}^{n}$ *and let* $L := \operatorname{lin} F$. *Then* $M(F) = F \cap L^{\perp}$ *if* $F \cap L^{\perp}$ *is bounded and*

$$\operatorname{extr}\left(F \cap L^{\perp}\right) \subset M(F) \subset \operatorname{conv} \operatorname{extr}\left(F \cap L^{\perp}\right) \cap \operatorname{rbd}\left(F \cap L^{\perp}\right), \tag{9}$$

otherwise. Hence, $M(F)$ *and* $\operatorname{extr}\left(F\cap L^{\perp}\right)$ *have the same convex hull and, so, both sets are simultaneously bounded or unbounded.*

Proof. Let K be as in (1).

Assume that $F\cap L^{\perp}$ is bounded. Then $K=(0^{+}F)\cap L^{\perp}=0^{+}\left(F\cap L^{\perp}\right)=\{0_n\}$, so that

$$M(F)=\left\{x\in F\cap L^{\perp}:\{x\}\cap F=\ \{x\}\right\}=F\cap L^{\perp}=\operatorname{conv}\operatorname{extr}\left(F\cap L^{\perp}\right).$$

Assume now that $F\cap L^{\perp}$ is unbounded. Let $x\in\operatorname{extr}\left(F\cap L^{\perp}\right)$ and $y\in(x-K)\cap F$. We have $y\in F\cap L^{\perp}$ because $x-K\subset L^{\perp}-L^{\perp}=L^{\perp}$. Since $x-y\in K\subset 0^{+}F$, we also have $2x-y\in F\cap L^{\perp}$; therefore, as $x\in\operatorname{extr}\left(F\cap L^{\perp}\right)$ is the midpoint of the segment with endpoints $y,2x-y\in F\cap L^{\perp}$, it follows that $y=x$. We have thus proved that $(x-K)\cap F=\{x\}$, that is, $x\in M(F)$. So, $\operatorname{extr}\left(F\cap L^{\perp}\right)\subset M(F)$.

Next we prove that $M(F)\subset\operatorname{rbd}\left(F\cap L^{\perp}\right)$.

Let us take any point $x\in M(F)$, i.e., $x\in F\cap L^{\perp}$ such that $(x-K)\cap F=\{x\}$. Obviously,

$$\operatorname{rint}(x-K)\cap\operatorname{rint}\left(F\cap L^{\perp}\right)\subset\{x\}. \tag{10}$$

On the other hand, $0_n\in\operatorname{rbd}K$ because K is pointed and does not reduce to $\{0_n\}$, so that $x\in\operatorname{rbd}(x-K)$ which together with (10) gives $\operatorname{rint}(x-K)\cap\operatorname{rint}\left(F\cap L^{\perp}\right)=\emptyset$. Let H be a hyperplane separating $x-K$ and $F\cap L^{\perp}$ properly. We have $F\cap L^{\perp}\not\subset H$ (otherwise, since $K=0^{+}\left(F\cap L^{\perp}\right)\subset H-H$, $x-K\subset H$ and H does not separate both sets properly). Hence, $x\in\operatorname{rbd}\left(F\cap L^{\perp}\right)$ (otherwise, $x\in\operatorname{rint}\left(F\cap L^{\perp}\right)$ entails $F\cap L^{\perp}\subset H$).

Observe that H supports $F\cap L^{\perp}$ properly at x, and the same is true for the hyperplane $H\cap L^{\perp}+\operatorname{lin}F$ which supports F properly at x too.

We will prove that $M(F)\subset\operatorname{conv}\operatorname{extr}\left(F\cap L^{\perp}\right)$ by induction on $k:=\dim F$.

Let $k=1$. Then F is a closed halfline (if F is a whole line, then $\operatorname{lin}F=F$, $L^{\perp}$ is a hyperplane orthogonal to F and, so, $F\cap L^{\perp}$ is singleton, contradicting the unboundedness of $F\cap L^{\perp}$). If $F=\{x+\lambda y:\lambda\geq 0\}$, where $x,y\in\mathbb{R}^n$ and $y\neq 0_n$, then $\operatorname{lin}F=\{0_n\}$, $L^{\perp}=\mathbb{R}^n$, and $M(F)=\{x\}=\operatorname{extr}\left(F\cap L^{\perp}\right)$.

Let $k>1$. Let H be a hyperplane which supports properly F at x (we have already shown the existence of such a hyperplane). Obviously, $L=\operatorname{lin}F\subset H-x$ (the linear subspace parallel to H). Let $\widetilde{F}:=F\cap H$, with $\dim\widetilde{F}<k$,

and let $\widetilde{L} := \operatorname{lin} \widetilde{F} = \operatorname{lin} F \cap (H - x) = \operatorname{lin} F$. We have $\widetilde{L}^{\perp} := \left(\operatorname{lin} \widetilde{F}\right)^{\perp} = L^{\perp}$ and

$$\widetilde{K} := \left(0^+ \widetilde{F}\right) \cap \widetilde{L}^{\perp} = 0^+ (F \cap H) \cap L^{\perp} = \left(0^+ F\right) \cap L^{\perp} \cap (H - x) = K \cap (H - x).$$

Let $y \in \left(x - \widetilde{K}\right) \cap \widetilde{F}$. Since $x - y \in \widetilde{K} \subset K$ and $y \in \widetilde{F} \subset F$, and $x \in M(F)$, we have $y = x$. Hence, $x \in M\left(\widetilde{F}\right) := \left\{x \in \widetilde{F} \cap \widetilde{L} : \left(x - \widetilde{K}\right) \cap \widetilde{F} = \{x\}\right\}$ and $x \in \operatorname{conv} \operatorname{extr}\left(\widetilde{F} \cap L^{\perp}\right)$ by the induction hypothesis. Because H supports $F \cap L^{\perp}$ at x, $\operatorname{extr}\left(\widetilde{F} \cap L^{\perp}\right) = \operatorname{extr}\left(F \cap H \cap L^{\perp}\right) \subset \operatorname{extr}\left(F \cap L^{\perp}\right)$ and we get $x \in \operatorname{conv} \operatorname{extr}\left(F \cap L^{\perp}\right)$.

We have thus proved the required inclusion. $\square$

The next two examples show that the three sets in (9) may coincide (even simultaneously) or not.

Example 9 *Consider the closed convex set*

$$F := \left\{x \in \mathbb{R}^3 : \ -1 \leq x_1 \leq 1, \ x_2 \geq -\sqrt{1 - x_1^2}\right\}.$$

We have $0^+ F = \operatorname{cone}\{(0,1,0), \pm(0,0,1)\}$, $L^{\perp} = (\operatorname{lin} F)^{\perp} = \mathbb{R}^2 \times \{0\}$, $K := (0^+ F) \cap L^{\perp} = \operatorname{cone}\{(0,1,0)\}$, *and*

$$\begin{aligned} \operatorname{extr}\left(F \cap L^{\perp}\right) &= M(F) = \operatorname{conv} \operatorname{extr}\left(F \cap L^{\perp}\right) \cap \operatorname{rbd}\left(F \cap L^{\perp}\right) \\ &= \left\{(x_1, x_2, 0) \in \mathbb{R}^3 : \ -1 \leq x_1 \leq 1, \ x_2 = -\sqrt{1 - x_1^2}\right\}. \end{aligned}$$

Notice that $M(F)$ *is the smallest compact component of* $F \cap L^{\perp}$. *Observe also that the unique plane supporting properly* F *at* $x = (1,0,0) \in M(F)$ *is* $H = \{x \in \mathbb{R}^3 \ : x_1 = 1\}$ *whereas any plane containing the line* $\{x \in \mathbb{R}^3 \ : x_1 = 1, x_3 = 0\}$, *except* $\{x \in \mathbb{R}^3 \ : x_3 = 0\}$, *separates properly* $x - K$ *and* $F \cap L^{\perp}$. *Any of the latter planes contains the translated cone* $x - K$ *whereas its intersection with* F *yields the facet* $\{x \in \mathbb{R}^3 \ : x_1 = 1, x_2 \geq 0, x_3 = 0\}$.

Example 10 *Consider the polyhedral closed convex set*

$$F := \left\{x \in \mathbb{R}^3 \ : x_1 + x_2 \leq 1, x_3 \geq \ x_i \geq 0, \ i = 1,2\right\}.$$

Obviously, $L^{\perp} = \mathbb{R}^3$, $0^+ F = \operatorname{conv}\{(0,0,1)\}$, $\operatorname{extr} F = \{c_1, c_2, c_3, c_4\}$, *where* $c_1 = (0,0,0)$, $c_2 = (1,0,1)$, $c_3 = (0,1,1)$, *and* $c_4 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$,

$$M(F) = \operatorname{conv}\{c_1, c_2, c_4\} \cup \operatorname{conv}\{c_1, c_3, c_4\},$$

and

$$\operatorname{conv} \operatorname{extr} F \cap \operatorname{bd} F = M(F) \cup \operatorname{conv}\{c_2, c_3, c_4\}.$$

Thus,

$$\operatorname{extr} F \subsetneqq M(F) \subsetneqq \operatorname{conv} \operatorname{extr} F \cap \operatorname{bd} F.$$

Here the smallest compact component of F *is*

$$F = \operatorname{conv}\{c_1, c_2, c_3, c_4\}.$$

Theorem 11 *Let* F *be a closed convex set,* $\emptyset \neq F \subset \mathbb{R}^n$*. Then the following statements hold:*
(i) F *is M-decomposable if and only if* $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ *is bounded. In that case,*

$$F = \operatorname{cl} \operatorname{conv} \operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right) + 0^{+}F \tag{11}$$

is a Motzkin representation of F*.*
(ii) If F *is an M-decomposable set containing no lines, then the compact component of* F *in (11) is the smallest one, with* $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right) = \operatorname{extr} F$ *satisfying*

$$\begin{aligned} \emptyset \neq \bigcup \{(F)^{*}(c) : |(F)^{*}(c)| = 1, c \in \mathbb{R}^n\} \subset \operatorname{extr} F \\ \subset \operatorname{cl}\left(\bigcup \{(F)^{*}(c) : |(F)^{*}(c)| = 1, c \in \mathbb{R}^n\}\right). \end{aligned} \tag{12}$$

Proof. Statement (i) and the first part of (ii) are straightforward consequences of Theorem 4 and Lemma 8 whereas (12) follows from Straszewicz's Theorem (see, e.g., [7, Theorem 18.6]). □

If F is a polyhedral convex set, $F\cap(\operatorname{lin} F)^{\perp}$ is polyhedral too, so that $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ is finite. Thus, any polyhedral convex set is M-decomposable. Even more, from (11), the smallest compact component of F is a polytope (this proves the classical Motzkin's Theorem in [6]). Thus, for polyhedral convex sets, the first inclusion in (9) is generally strict except in particular cases (as lines and hyperplanes) because $\operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right)$ is finite whereas $M(F)$ is commonly infinite.

On the other hand, because $\left\{\operatorname{lin}\left[F \cap (\operatorname{lin} F)^{\perp}\right]\right\}^{\perp} = \{0_n\}^{\perp} = \mathbb{R}^n$,

$$\operatorname{extr}\left(\left(F \cap (\operatorname{lin} F)^{\perp}\right) \cap \left\{\operatorname{lin}\left[F \cap (\operatorname{lin} F)^{\perp}\right]\right\}^{\perp}\right) = \operatorname{extr}\left(F \cap (\operatorname{lin} F)^{\perp}\right).$$

Thus, by Theorem 11, $F \cap (\operatorname{lin} F)^{\perp}$ is M-decomposable if and only if F is decomposable (this is an alternative proof of Theorem 6).

Obviously, the decomposability property is preserved by the product by scalars. Moreover, if F is M-decomposable and $\varepsilon > 0$, the set $\{x \in \mathbb{R}^n : \rho(x, F) \leq \varepsilon\}$ is M-decomposable too. Concerning the ordinary binary operations with sets (Cartesian product, sum, union, intersection), only the Cartesian product is closed in the class of the M-decomposable sets (this has been shown in [3] for the intersection whereas it is obvious for the sum and union). The next result shows that applying convex hulls and/or closures to sums and unions (but not to intersections, because they are already closed and convex) we get M-decomposable sets.

Theorem 12 *If $F_1, ..., F_m$ are M-decomposable sets, then* $\operatorname{cl}\left(\sum_{i=1}^m F_i\right)$ *and* $\operatorname{cl}\operatorname{conv}\left(\bigcup_{i=1}^m F_i\right)$ *are M-decomposable.*

Proof. It is sufficient to prove the statement for $m = 2$. Let $F_i = C_i + D_i$, with C_i compact convex and D_i closed convex cone, $i = 1, 2$.

First of all, recall that if C is a compact and convex subset of $\mathbb{R}^n$ and $D \subset \mathbb{R}^n$ is a convex cone, then $\operatorname{cl}(C + D) = C + \operatorname{cl} D$. Thus,

$$\operatorname{cl}(F_1 + F_2) = \operatorname{cl}(C_1 + C_2 + D_1 + D_2) = C_1 + C_2 + \operatorname{cl}(D_1 + D_2),$$

which is obviously an M-decomposable set.

Let $x \in \operatorname{conv}(F_1 \cup F_2)$. Then there exist $\alpha \in [0, 1]$ and $(c_i, d_i) \in (C_i, D_i)$, $i = 1, 2$ such that

$$x = \alpha (c_1 + d_1) + (1 - \alpha)(c_2 + d_2) =$$

$$= \alpha c_1 + (1 - \alpha) c_2 + \alpha d_1 + (1 - \alpha) d_2 \in \operatorname{conv}(C_1 \cup C_2) + D_1 + D_2.$$

Hence,

$$\operatorname{conv}(F_1 \cup F_2) \subset \operatorname{conv}(C_1 \cup C_2) + D_1 + D_2, \tag{13}$$

where $\operatorname{conv}(C_1 \cup C_2)$ is a compact set by Mazur's Theorem.

Now, let $x \in \operatorname{conv}(C_1 \cup C_2) + D_1 + D_2$. Therefore, there exist $\alpha \in [0, 1]$ and $(c_i, d_i) \in C_i \times D_i$, $i = 1, 2$ such that

$$x = \alpha c_1 + (1 - \alpha) c_2 + d_1 + d_2.$$

If $\alpha \in]0, 1[$, then

$$x = \alpha \left(c_1 + \frac{d_1}{\alpha}\right) + (1 - \alpha)\left(c_2 + \frac{d_2}{1 - \alpha}\right) \in \operatorname{conv}(F_1 \cup F_2).$$

If $\alpha = 1$, then

$$x = \lim_{\alpha \nearrow 1} \left[\alpha \left(c_1 + \frac{d_1}{\alpha}\right) + (1 - \alpha)\left(c_2 + \frac{d_2}{1 - \alpha}\right)\right] \in \operatorname{cl}\operatorname{conv}(F_1 \cup F_2),$$

where $c_2 \in C_2$ is an arbitrary point.

The same is true when $\alpha = 0$. Hence, we have

$$\operatorname{conv}(C_1 \cup C_2) + D_1 + D_2 \subset \operatorname{cl}\operatorname{conv}(F_1 \cup F_2). \tag{14}$$

>From (13) and (14) we get

$$\begin{aligned}\operatorname{cl}\operatorname{conv}(F_1 \cup F_2) &= \operatorname{cl}\left[\operatorname{conv}(C_1 \cup C_2) + D_1 + D_2\right] \\ &= \operatorname{conv}(C_1 \cup C_2) + \operatorname{cl}(D_1 + D_2),\end{aligned}$$

where the latter set is the sum of a compact convex set and a closed convex cone. □

4 M-decomposable functions

A function $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is *decomposable in Motzkin's sense* (*M-decomposable* in short) if $\operatorname{epi} f$ is M-decomposable. In this event, f is convex, lower semicontinuous (also abbreviated as lsc) and non identically $+\infty$. Moreover, the conic component of $\operatorname{epi} f$ is $0^+(\operatorname{epi} f) = \operatorname{epi} f0^+$, where $f0^+$ denotes the *recession function* of f (obviously, any recession function is M-decomposable). The next two propositions characterize the proper and the improper M-decomposable functions, respectively. We will need the following Lemma:

Lemma 13 *Let f be a proper convex function and M be an affine manifold parallel to* $(\operatorname{lin}\operatorname{epi} f)^{\perp}$. *Then*

$$\operatorname{extr}(\operatorname{epi} f \cap M) \subset \operatorname{gph} f.$$

Proof. Let $(x, \alpha) \in \operatorname{extr}(\operatorname{epi} f \cap M)$, and denote by π the orthogonal projection mapping from $\mathbb{R}^n$ onto M. Since $(x, \alpha) \in M$, we have

$$\begin{aligned}(x, \alpha) = \pi(x, \alpha) &= \pi\left(\tfrac{1}{2}\left[(x, f(x)) + (x, 2\alpha - f(x))\right]\right) \\ &= \tfrac{1}{2}\left[\pi(x, f(x)) + \pi(x, 2\alpha - f(x))\right].\end{aligned}$$

>From $(x, \alpha) \in \operatorname{epi} f$ it follows that $(x, 2\alpha - f(x)) \in \operatorname{epi} f$. Consequently, $\pi(x, f(x)), \pi(x, 2\alpha - f(x)) \in (\operatorname{epi} f + \operatorname{lin}\operatorname{epi} f) \cap M = \operatorname{epi} f \cap M$ and hence, by $(x, \alpha) \in \operatorname{extr}(\operatorname{epi} f \cap M)$, we must have $(x, \alpha) = \pi(x, f(x))$. This equality implies that $(x, \alpha) - (x, f(x)) \in \operatorname{lin}\operatorname{epi} f$ and therefore, since $\operatorname{epi} f$ contains no vertical lines (as f is proper), we conclude that $(x, \alpha) = (x, f(x)) \in \operatorname{gph} f$. □

Theorem 14 *Let f be an lsc proper convex function. Then the following statements hold:*

(i) f is M-decomposable if and only if $\operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$ *is bounded.*
(ii) If $\operatorname{gph} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}$ *is bounded, then f is M-decomposable.*
(iii) If $\operatorname{dom} f$ *is bounded and f is bounded on* $\operatorname{dom} f$, *then f is M-decomposable.*
(iv) If f is M-decomposable and finite-valued, then f cannot be strictly convex.

Proof. (i) It is straightforward consequence of Theorem 11 applied to the nonempty closed convex set $\operatorname{epi} f$.

(ii) By Lemma 13,

$$\operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right] \subset \operatorname{gph} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp},$$

and the conclusion follows from (i).

(iii) Since $\operatorname{gph} f \subset \operatorname{dom} f \times f(\operatorname{dom} f)$ and this set is bounded, the conclusion follows from (ii).

(iv) If f is finite-valued and strictly convex, then $\operatorname{epi} f$ does not contain lines and $\operatorname{extr}\operatorname{epi} f = \operatorname{gph} f$, so that

$$\operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right] = \operatorname{gph} f$$

is unbounded and the conclusion follows again from (i). □

Corollary 15 *Let $\emptyset \neq F \subset \mathbb{R}^n$ be a closed set. Then the following statements are equivalent:*
(i) F is M-decomposable.
(ii) The indicator function δ_F is M-decomposable.
(iii) The distance function $\rho(\cdot, F)$ is M-decomposable.

Proof. The three statements (i)-(iii) imply the convexity of F because

$$\{x \in \operatorname{epi}\delta_F : x_{n+1} = 0\} = \{x \in \operatorname{epi}\rho(\cdot, F) : x_{n+1} = 0\} = F \times \{0\}.$$

Let $L = \operatorname{lin} F$.

(i)$\Leftrightarrow$ (ii) It follows from theorems 14 and 11 applied to δ_F and F, respectively. Indeed, since

$$(\operatorname{lin}\operatorname{epi}\delta_F)^{\perp} = [\operatorname{lin}(F \times \mathbb{R}_+)]^{\perp} = L^{\perp} \times \mathbb{R},$$

δ_F is M-decomposable if and only if

$$\begin{aligned}\operatorname{extr}\left[(F \times \mathbb{R}_+) \cap \left(L^{\perp} \times \mathbb{R}\right)\right] &= \operatorname{extr}\left[\left(F \cap L^{\perp}\right) \times \mathbb{R}_+\right] \\ &= \operatorname{extr}\left(F \cap L^{\perp}\right) \times \operatorname{extr}\mathbb{R}_+ \\ &= \operatorname{extr}\left(F \cap L^{\perp}\right) \times \{0\}\end{aligned}$$

is bounded if and only if F is M-decomposable.

(i)$\Leftrightarrow$ (iii) The argument is similar to the previous one, replacing $\delta_F(x)$ with $f(x) := \rho(x, F)$. In fact, since

$$(\operatorname{lin} \operatorname{epi} f)^{\perp} = [\operatorname{lin}(F \times \{0\})]^{\perp} = L^{\perp} \times \mathbb{R},$$

f is M-decomposable if and only if

$$\operatorname{extr}\left\{(x, \gamma) \in L^{\perp} \times \mathbb{R} : \rho(x, F) \leq \gamma\right\} = \left[\operatorname{extr}\left(F \cap L^{\perp}\right)\right] \times \{0\}$$

is bounded if and only if F is M-decomposable. $\square$

Proposition 16 *Let f be an improper lsc convex function non identically $+\infty$. Then, f is M-decomposable if and only if $f(x) = -\infty$ for all $x \in \operatorname{dom} f$ and $\operatorname{dom} f$ is an M-decomposable set.*

Proof. The lower semicontinuity assumption on f entails that $f(x) = -\infty$ for all $x \in \operatorname{rint} \operatorname{dom} f$. Let $x \in \operatorname{rbd} \operatorname{dom} f$ such that $f(x) \in \mathbb{R}$. Because $\operatorname{cl} \operatorname{dom} f = \operatorname{cl} \operatorname{rint} \operatorname{dom} f$, there exists a sequence $\{x_k\} \subset \operatorname{rint} \operatorname{dom} f$ such that $x_k \to x$. Then, $\{(x_k, f(x) - 1)\} \subset \operatorname{epi} f$ for all $k \in \mathbb{N}$ and $(x_k, f(x) - 1) \to (x, f(x) - 1) \notin \operatorname{epi} f$ (contradiction). Thus $f(x) = -\infty$.

We have shown that $f(x) = -\infty$ for all $x \in \operatorname{dom} f$. Since $\operatorname{dom} f \times \mathbb{R} = \operatorname{epi} f$ is a closed convex set, $\operatorname{dom} f$ is also closed and convex. Moreover, we have

$$\operatorname{lin} \operatorname{epi} f = (\operatorname{lin} \operatorname{dom} f) \times \{0\} + \{0_n\} \times \mathbb{R}.$$

Thus, applying Theorem 6 to $\operatorname{epi} f$ and $\operatorname{dom} f$, we conclude that $\operatorname{epi} f$ is M-decomposable if and only if

$$\operatorname{epi} f \cap (\operatorname{lin} \operatorname{epi} f)^{\perp} = \left[\operatorname{dom} f \cap (\operatorname{lin} \operatorname{dom} f)^{\perp}\right] \times \{0\}$$

is M-decomposable if and only if $\operatorname{dom} f$ is M-decomposable. $\square$

The next result gives an interesting property of the M-decomposable functions in the optimization framework.

Proposition 17 *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be an M-decomposable function bounded from below on $\mathbb{R}^n$. Then f achieves a global minimum on $\mathbb{R}^n$.*

Proof. Let $\alpha \in \mathbb{R}$ be such that $f(x) \geq \alpha$ for all $x \in \mathbb{R}^n$. Then $\alpha \leq x_{n+1}$ for all $(x_1, ..., x_{n+1}) \in \operatorname{epi} f$. Since the linear mapping $(x_1, ..., x_{n+1}) \mapsto x_{n+1}$ is bounded from below on the M-decomposable set $\operatorname{epi} f$, there exists $(\overline{x}_1, ..., \overline{x}_{n+1}) \in \operatorname{epi} f$ such that $\overline{x}_{n+1} \leq x_{n+1}$ for all $(x_1, ..., x_{n+1}) \in \operatorname{epi} f$. Obviously, we must have $\overline{x}_{n+1} = f(\overline{x}_1, ..., \overline{x}_n)$ (otherwise $(\overline{x}_1, ..., \overline{x}_n, f(\overline{x}_1, ..., \overline{x}_n)) \in \operatorname{epi} f$ is preferable to $(\overline{x}_1, ..., \overline{x}_{n+1})$).

Since $f(\overline{x}_1, ..., \overline{x}_n) \leq x_{n+1}$ for all $(x_1, ..., x_{n+1}) \in \operatorname{epi} f$ and $(x_1, ..., x_n, f(x_1, ..., x_n)) \in \operatorname{epi} f$, we get $f(\overline{x}_1, ..., \overline{x}_n) \leq f(x_1, ..., x_n)$ for all $(x_1, ..., x_n) \in \mathbb{R}^n$. Then $(\overline{x}_1, ..., \overline{x}_n)$ is a global minimizer of f on $\mathbb{R}^n$. $\square$

Given that support functions are sublinear and hence M-decomposable (as their epigraphs are closed convex cones), from the observation we have made in Section 3 that every nonempty closed convex set is the subdifferential of its support function at the origin, it follows that the subdifferential of an M-decomposable function at a relative boundary point of its domain is not necessarily M-decomposable.

It is well known that, if f is a polyhedral convex function bounded from below on a polyhedral convex set F, then f attains its minimum on F ([7, Corollary 27.3.2]). The next example shows that we cannot replace in this statement "polyhedral convex" by "M-decomposable".

Example 18 *Consider the closed convex cone*

$$K := \operatorname{cone}\left\{\left(1, t, \frac{1-t}{t}\right), t > 0; (0, 1, 0); (0, 0, 1)\right\}.$$

Let $f : \mathbb{R}^2 \longrightarrow \overline{\mathbb{R}}$ *be the M-decomposable function whose epigraph is* $\operatorname{epi} f = K$ *and let* $F = [0, 1] \times \mathbb{R}$. *We have* $\inf \{f(x) : x \in F\} = -1$ *but* $f(x) \neq -1$ *for all* $x \in F$.

It is worth observing that any lsc proper convex function is the pointwise limit of a sequence of M-decomposable functions as an immediate consequence of the next result.

Proposition 19 *Every lsc proper convex function is the pointwise limit of a sequence of polyhedral convex functions.*

Proof. Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be an lsc proper convex function. Let $\varphi : \mathbb{N} \longrightarrow \mathbb{Q}^n$ be an arbitrary bijection. Then, for every $x \in \mathbb{R}^n$ one has

$$\begin{aligned} f(x) &= f^{**}(x) = \sup\nolimits_{u \in \mathbb{R}^n} \{\langle x, u\rangle - f^*(x)\} = \sup\nolimits_{u \in \mathbb{Q}^n} \{\langle x, u\rangle - f^*(x)\} \\ &= \sup\nolimits_{k \in \mathbb{N}} \sup\nolimits_{u \in \{\varphi(0), \dots, \varphi(k)\}} \{\langle x, u\rangle - f^*(x)\} \\ &= \lim\nolimits_{k \longrightarrow \infty} \sup\nolimits_{u \in \{\varphi(0), \dots, \varphi(k)\}} \{\langle x, u\rangle - f^*(x)\}, \end{aligned}$$

where $\sup_{u \in \{\varphi(0), \dots, \varphi(k)\}} \{\langle x, u\rangle - f^*(x)\}$ is a polyhedral convex function for all k. $\square$

Finally, we analyze the usual operations which provide convex functions from other convex functions from the point of view of the preservation of the M-decomposability.

Proposition 20 *Let* $f:\mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ *be M-decomposable and* $\lambda > 0$. *Then* λf *is M-decomposable.*

Proof. Let A be the $(n+1)\times(n+1)$ matrix obtained by replacing the last element of the diagonal of the identity matrix by λ and let $\operatorname{epi} f = C + D$, where C is a compact convex set and D is a closed convex cone. Then, $\operatorname{epi}(\lambda f) = A \operatorname{epi} f = AC + AD$ is the sum of a compact convex set with a closed convex cone. $\square$

Lemma 21 *Let* $f:\mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ *and let* $g:\mathbb{R}^n \longrightarrow \mathbb{R}$ *be linear. Then, given* $u \in \mathbb{R}^n$ *and* $\gamma \in \mathbb{R}$, $(u,\gamma) \in \operatorname{lin}\operatorname{epi} f$ *if and only if* $(u, \gamma + g(u)) \in \operatorname{lin}\operatorname{epi}(f+g)$.

Proof. We have

$$\begin{aligned}
&(u,\gamma) \in \operatorname{lin}\operatorname{epi} f \\
&\Leftrightarrow (\overline{x},\overline{y}) + \lambda(u,\gamma) \in \operatorname{epi} f \ \forall (\overline{x},\overline{y}) \in \operatorname{epi} f \ \forall \lambda \in \mathbb{R} \\
&\Leftrightarrow f(\overline{x} + \lambda u) \leq \overline{y} + \lambda\gamma \ \forall (\overline{x},\overline{y}) \in \operatorname{epi} f \ \forall \lambda \in \mathbb{R} \\
&\Leftrightarrow (f+g)(\overline{x} + \lambda u) \leq \alpha + \lambda(\gamma + g(u)) \ \ \forall (\overline{x},\alpha) \in \operatorname{epi}(f+g) \ \ \forall \lambda \in \mathbb{R} \\
&\Leftrightarrow (u, \gamma + g(u)) \in \operatorname{lin}\operatorname{epi}(f+g). \qquad \square
\end{aligned}$$

Lemma 22 *Let* f *be a proper convex function and let* $g:\mathbb{R}^n \longrightarrow \mathbb{R}$ *be linear. Then,* $(x,y) \in \operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$ *if and only if*

$$(x, y + g(x)) \in \operatorname{extr}\left\{\operatorname{epi}(f+g) \cap \left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp} + (x, y + g(x))\right]\right\}. \quad (15)$$

Proof. First, we prove the direct statement. Let $(x,y) \in \operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$. By Lemma 13, $(x,y) = (x, f(x))$. Let

$$(x, (f+g)(x)) = (1-\lambda)(x_1,y_1) + \lambda(x_2,y_2), \quad (16)$$

with $\lambda \in]0,1[$, $(x_1,y_1) \neq (x_2,y_2)$, and

$$(x_i,y_i) \in \operatorname{epi}(f+g) \cap \left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp} + (x, y + g(x))\right], \ i = 1,2. \quad (17)$$

Then,

$$0_{n+1} \neq (x_1 - x_2, y_1 - y_2) \in (\operatorname{lin}\operatorname{epi}(f+g))^{\perp} \quad (18)$$

because $(x_i,y_i) \in (\operatorname{lin}\operatorname{epi}(f+g))^{\perp} + (x, y+g(x))$, $i = 1,2$.

If $(x_1 - x_2, y_1 - y_2 - g(x_1) + g(x_2)) \in \operatorname{lin}\operatorname{epi} f$, Lemma 21 yields $(x_1 - x_2, y_1 - y_2) \in \operatorname{lin}\operatorname{epi}(f+g)$, and this contradicts (18). Thus we have

$$(x_1 - x_2, y_1 - g(x_1) - (y_2 - g(x_2))) \notin \operatorname{lin}\operatorname{epi} f. \quad (19)$$

>From (16), $x=(1-\lambda)x_1+\lambda x_2$, so that $g(x)=(1-\lambda)g(x_1)+\lambda g(x_2)$. Summing up $(0_n,-g(x))$ to both members of (16), we get

$$(x,f(x))=(1-\lambda)(x_1,y_1-g(x_1))+\lambda(x_2,y_2-g(x_2)), \tag{20}$$

with $(x_i,y_i-g(x_i))\in \operatorname{epi} f,\ i=1,2$, by (17). According to (19), these two points have different orthogonal projections on $(\operatorname{lin}\operatorname{epi} f)^{\perp}$, say $(\widetilde{x}_i,\widetilde{y}_i)$, $i=1,2$. Because the projection is along lines contained in $\operatorname{epi} f$, $(\widetilde{x}_i,\widetilde{y}_i)\in \operatorname{epi} f$, $i=1,2$. Applying the orthogonal projection on the linear subspace $(\operatorname{lin}\operatorname{epi} f)^{\perp}$ to both members of (20) we get $(x,f(x))=(1-\lambda)(\widetilde{x}_1,\widetilde{y}_1)+\lambda(\widetilde{x}_2,\widetilde{y}_2)$, with $(\widetilde{x}_i,\widetilde{y}_i)\in \operatorname{epi} f\cap(\operatorname{lin}\operatorname{epi} f)^{\perp}$, so that $(x,f(x))\notin \operatorname{extr}\left[\operatorname{epi} f\cap(\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$.

Now, we shall prove the converse statement. Let

$$(x,y+g(x))\in \operatorname{extr}\left\{\operatorname{epi}(f+g)\cap\left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp}+(x,y+g(x))\right]\right\}.$$

Since $f+g$ is a proper convex function, by Lemma 13 we have $(x,y+g(x))\in \operatorname{gph}(f+g)$, that is, $y=f(x)$. Suppose there exist $\lambda\in\,]0,1[$ and $(x_1,y_1)\neq(x_2,y_2)$, such that $(x_i,y_i)\in \operatorname{epi} f\cap(\operatorname{lin}\operatorname{epi} f)^{\perp}$, $i=1,2$, and

$$(x,f(x))=(1-\lambda)(x_1,y_1)+\lambda(x_2,y_2).$$

Obviously, $0_{n+1}\neq(x_1-x_2,y_1-y_2)\in(\operatorname{lin}\operatorname{epi} f)^{\perp}$, $x=(1-\lambda)x_1+\lambda x_2$, and $g(x)=(1-\lambda)g(x_1)+\lambda g(x_2)$. So,

$$(x,y+g(x))=(x,f(x)+g(x))=(1-\lambda)(x_1,y_1+g(x_1))+\lambda(x_2,y_2+g(x_2)).$$

We have that $(x_i,y_i+g(x_i))\in \operatorname{epi}(f+g)$, $i=1,2$. If

$$(x_1-x_2,y_1+g(x_1)-(y_2+g(x_2)))\in \operatorname{lin}\operatorname{epi}(f+g),$$

Lemma 21 yields $(x_1-x_2,y_1-y_2)\in \operatorname{lin}\operatorname{epi} f$, which is not true. Thus, we have

$$(x_1-x_2,y_1+g(x_1)-(y_2+g(x_2)))\notin \operatorname{lin}\operatorname{epi}(f+g).$$

The points $(x_i,y_i+g(x_i))\in \operatorname{epi}(f+g)$, $i=1,2$, have different orthogonal projections on the linear manifold $(\operatorname{lin}\operatorname{epi}(f+g))^{\perp}+(x,y+g(x))$, say $(\widetilde{x}_i,\widetilde{y}_i)$, $i=1,2$. Because the projection is along lines contained in $\operatorname{epi}(f+g)$, $(\widetilde{x}_i,\widetilde{y}_i)\in \operatorname{epi}(f+g)$, $i=1,2$. So, finally we get

$$(x,f(x)+g(x))=(1-\lambda)(\widetilde{x}_1,\widetilde{y}_1)+\lambda(\widetilde{x}_2,\widetilde{y}_2),$$

with

$$(\widetilde{x}_i,\widetilde{y}_i)\in \operatorname{epi}(f+g)\cap\left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp}+(x,f(x)+g(x))\right],$$

whereby

$$(x,f(x)+g(x))\notin \operatorname{extr}\left\{\operatorname{epi}(f+g)\cap\left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp}+(x,f(x)+g(x))\right]\right\},$$

which is a contradiction. □

Theorem 23 *Let $f : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be an M-decomposable function and let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be an affine function. Then, $f + g$ is M-decomposable.*

Proof. If f is improper, the conclusion follows from Proposition 16. Let us consider the case when f is proper. We can assume w.l.o.g. that g is linear. By Theorem 14, $\operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$ is a bounded set, so that its orthogonal projection onto $\mathbb{R}^n \times \{0\}$ is bounded too. Since g is linear, it is bounded on the latter set. Let k_1 and k_2 be scalars such that $\|(x, f(x))\| \leq k_1$ and $|g(x)| \leq k_2$ for all $(x, f(x)) \in \operatorname{extr}\left[\operatorname{epi} f \cap (\operatorname{lin}\operatorname{epi} f)^{\perp}\right]$. Then, by Lemma 22,

$$\|(x, (f+g)(x))\| \leq \|(x, f(x))\| + \|(0_n, g(x))\| \leq k := k_1 + k_2$$

for all

$$\begin{aligned}(x, (f+g)(x)) \in \operatorname{extr}&\left\{\operatorname{epi}(f+g) \cap \left[(\operatorname{lin}\operatorname{epi}(f+g))^{\perp} + (x, (f+g)(x))\right]\right\} \\ = \operatorname{extr}&\left\{[\operatorname{epi}(f+g) - (x, (f+g)(x))] \cap (\operatorname{lin}\operatorname{epi}(f+g))^{\perp}\right\} \\ &+ (x, (f+g)(x)). \end{aligned} \tag{21}$$

Since $\operatorname{epi}(f+g) - (x, (f+g)(x)) = \operatorname{epi} h$, $h : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ being the function defined by $h(y) = (f+g)(y+x) - (f+g)(x)$, and $\operatorname{lin}\operatorname{epi}(f+g) = \operatorname{lin}\operatorname{epi} h$ (because $\operatorname{epi} h$ is a translate of $\operatorname{epi}(f+g)$), it follows that the set $\operatorname{extr}\left\{\operatorname{epi} h \cap (\operatorname{lin}\operatorname{epi} h)^{\perp}\right\}$ is bounded. Hence, by Theorem 11, the set $\operatorname{epi} h$ is M-decomposable and therefore the set $\operatorname{epi}(f+g) = \operatorname{epi} h + (x, (f+g)(x))$ is M-decomposable, too. This proves that the function $f+g$ is M-decomposable. □

When f and g are proper polyhedral convex functions, $f + g$ is a polyhedral convex function. Analogously, when f and g are support functions of two nonempty convex sets C_1 and C_2, their sum is the support function of $C_1 + C_2$ and, so, it is an M-decomposable function. Nevertheless, neither the sums of support functions with proper polyhedral convex functions nor the sums of support functions with translated support functions are necessarily M-decomposable, as the next two examples show.

Example 24 $f(x, y) = \|(x, y)\|$ *is the support function of the closed unit ball and* $g(x, y) = |y - 1|$ *is a finite polyhedral convex function, but their sum is not M-decomposable because* $\operatorname{extr}\operatorname{epi}(f+g) = \operatorname{gph}(f+g)$.

Example 25 *Let f be the same function as in Example 24 and* $g(x, y) = \|(x, y-1)\|$. *Even though g is the composition of f with a translation, its sum with f is not M-decomposable because the projection of* $\operatorname{extr}\operatorname{epi}(f+g) \subset \mathbb{R}^3$ *on* $\mathbb{R}^2$, $[\mathbb{R}^2 \diagdown \mathbb{R}(0,1)] \cup \{(0,0), (0,1)\}$, *is unbounded.*

Proposition 26 *Let $f, g : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be M-decomposable functions. Then $\overline{\operatorname{conv}} \min\{f, g\}$ is M-decomposable.*

Proof. Let $h := \min\{f, g\}$. Then $\operatorname{epi} h = \operatorname{epi} f \cup \operatorname{epi} g$, where $\operatorname{epi} f$ and $\operatorname{epi} g$ are M-decomposable sets. By Theorem 12, $\operatorname{cl} \operatorname{conv} \operatorname{epi} h$ is M-decomposable, i.e., the function $\overline{\operatorname{conv}} h$ is M-decomposable. $\square$

The next three examples show that the Fenchel conjugate, the maximum and the infimal convolution of M-decomposable functions are not necessarily M-decomposable neither (although the three operations are closed in the class of polyhedral convex functions).

Example 27 *If F is a nonempty closed convex set not M-decomposable, its support function δ_F^* is an M-decomposable function whose Fenchel conjugate $\delta_F^{**} = \delta_F$ is not M-decomposable by Theorem 14.*

Example 28 *Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be such that $f = \|\cdot\|$ and $g = \delta_H$, where $H \subset \mathbb{R}^2$ is an arbitrary line such that $0_2 \notin H$. Both functions are M-decomposable but $\operatorname{epi} \max\{f, g\}$ is the convex hull of a branch of hyperbola, so that $\max\{f, g\}$ is not M-decomposable.*

Example 29 *Let F_1 and F_2 be two M-decomposable sets in $\mathbb{R}^3$ whose intersection is closed but not M-decomposable (see [3, Example 24]) for the existence of such sets). According to Corollary 15, the indicator functions δ_{F_1} and δ_{F_2} are M-decomposable but the lsc envelope of their maximum $\overline{\max\{\delta_{F_1}, \delta_{F_2}\}} = \overline{\delta_{F_1 \cap F_2}} = \delta_{\operatorname{cl} F_1 \cap F_2} = \delta_{F_1 \cap F_2}$ is not.*

Example 30 *Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be*

$$f(x, y) = \begin{cases} \frac{x^2+y^2}{2}, & y > 0, \\ 0, & (x, y) = (0, 0), \\ -\infty, & \text{otherwise}, \end{cases} \quad \text{and } g(x, y) = \begin{cases} -\infty, & (x, y) = (0, 0), \\ +\infty, & \text{otherwise.} \end{cases}$$

Since $\operatorname{epi} f = \operatorname{cone}\{(\cos t, 1 + \sin t), t \in [0, 2\pi]\}$ and $\operatorname{epi} g = \mathbb{R}(0, 0, 1)$ are a closed convex cone and a line, respectively, both functions are M-decomposable. Moreover, $\operatorname{epi} f + \operatorname{epi} g = [(\mathbb{R} \times \mathbb{R}_{++}) \cup \{(0, 0)\}] \times \mathbb{R}$, so that the infimal convolution of f and g is

$$\begin{aligned} (f \Box g)(x, y) &= \inf\{\mu : (x, y, \mu) \in \operatorname{epi} f + \operatorname{epi} g\} \\ &= \begin{cases} -\infty, & (x, y) \in (\mathbb{R} \times \mathbb{R}_{++}) \cup \{(0, 0)\}, \\ +\infty, & , \text{ otherwise}, \end{cases} \end{aligned}$$

whose epigraph, $\operatorname{epi}(f \Box g) = \operatorname{epi} f + \operatorname{epi} g$, is not even closed.

However, the next proposition states that the lsc hull of the infimal convolution of two M-decomposable functions is M-decomposable.

Proposition 31 *Let $f, g : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ be M-decomposable functions. Then $\overline{f \Box g}$ is M-decomposable.*

Proof. Since $\operatorname{epi} \overline{f \Box g} = \operatorname{cl} \operatorname{epi} (f \Box g) = \operatorname{cl} (\operatorname{epi} f + \operatorname{epi} g)$, the statement follows from Theorem 12. $\square$

Proposition 19 shows that the pointwise limit of M-decomposable functions is not necessarily M-decomposable. The next example shows that this statement still holds for the uniform limit.

Example 32 *The convex non M-decomposable function $f : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ defined by $f(x) = \sqrt{x^2 + 1}$. is the uniform limit of a sequence of polyhedral convex because the second order derivative of f is bounded and the graph of f has the asymptotes $\{(x, y) \in \mathbb{R}^2 : y = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y = -x\}$.*

Finally, we show that the Motzkin decomposability of a function is independent of the corresponding property of its sublevel sets. This is obvious in one sense (the non M-decomposable function $f(x) = \|x\|^2$ has M-decomposable sublevel sets). In the particular case that f is a polyhedral convex function, given $\alpha \in \mathbb{R}$, the sublevel set $\{x \in \mathbb{R}^n : f(x) \leq \alpha\}$ is the projection of the polyhedral convex set $\operatorname{epi} f \cap \{x \in \mathbb{R}^{n+1} : x_{n+1} \leq \alpha\}$ on $\mathbb{R}^n \times \{0\}$, so that it is a polyhedral convex set too. The last example in this section shows that we cannot replace "polyhedral convex" with "M-decomposable" in the latter statement.

Example 33 *Consider the function $f : \mathbb{R}^2 \longrightarrow \overline{\mathbb{R}}$ such that*

$$f(x, y) = \begin{cases} -\sqrt{y^2 - x^2}, & \text{if } y \geq |x|, \\ +\infty, & \text{otherwise,} \end{cases}$$

whose epigraph is the closed convex cone

$$\operatorname{cone} \{(\cos t, 1, \sin t), t \in [\pi, 2\pi], (0, 0, 1)\}.$$

Obviously, f is an M-decomposable function but

$$\left\{(x, y) \in \mathbb{R}^2 : f(x, y) \leq \alpha\right\} = \left\{(x, y) \in \mathbb{R}^2 : y \geq |x|\right\}$$

(a closed convex cone) when $\alpha \geq 0$ and

$$\left\{(x, y) \in \mathbb{R}^2 : f(x, y) \leq \alpha\right\} = \left\{(x, y, \alpha) \in \mathbb{R}^3 : y^2 \geq x^2 + \alpha^2, y \geq 0\right\}$$

(the convex hull of a branch of hyperbola and, so, a non M-decomposable set), when $\alpha < 0$.

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