Motzkin decomposition of closed convex sets

M A. Goberna

Dep. of Statistics and Operations Research, Universidad de Alicante, Spain.

E. González

Dep. of Statistics and Operations Research, Universidad de Alicante, Spain.

J. E. Martínez-Legaz

Dep. of Economics and Economic History, Universitat Autònoma de Barcelona, Spain.

M. I. Todorov

Dep. Actuary and Mathematics, Universidad de las Américas, Cholula, Puebla, Mexico. On leave IMI-BAS, Sofia, Bulgaria.

Abstract

Theodore Motzkin proved, in 1936, that any polyhedral convex set can be expressed as the (Minkowski) sum of a polytope and a polyhedral convex cone. This paper provides five characterizations of the larger class of closed convex sets in finite dimensional Euclidean spaces which are the sum of a compact convex set with a closed convex cone. These characterizations involve different types of representations of closed convex sets as the support functions, dual cones and linear systems whose relationships are also analyzed in the paper. The obtaining of information about a given closed convex set F and the parametric linear optimization problem with feasible set F from each of its different representations, including the Motzkin decomposition, is also discussed.

Key words: closed convex sets, linear inequality systems, semi-infinite optimization

1 Introduction

We say that a set $F \subset \mathbb{R}^n$ is decomposable in Motzkin's sense (*M-decomposable* in short) if there exist a compact convex set C and a closed convex cone D such that $F = C + D$. Then we say that $C + D$ is a *Motzkin representation* (or *decomposition*) of F with compact and conic components C and D, respectively. A Motzkin representation of F is *minimal* whenever its compact component is the smallest possible. Examples of M-decomposable sets are the compact convex sets, the closed convex cones, the polyhedral convex sets, and the sums of compact convex sets with linear subspaces. According to Klee representation theorem [13], a sufficient condition for a nonempty closed convex set to be M-decomposable is the boundedness of the set of extreme points of the intersection of F with the orthogonal subspace to the lineality space of F. The convex subsets of M-decomposable sets are related to inner aperture cones and barrier cones (see [2] and [3], where they are called *hyperbolic sets*). The M-decomposable sets with conic component $\mathbb{R}^n_-$ have been used in game theory under the name of *compactly generated and comprehensive* [14].

Motzkin decomposition can be seen as a new kind of representation for an important class of closed convex sets. The characterizations of M-decomposable sets provided in this paper involve other well-known types of representations, as indicator and support functions (see, e.g., [17] and [10]), and other less popular, as linear inequality systems and dual cones, we recall now briefly.

By the separation theorem, any closed convex set $F \subset \mathbb{R}^n$, $\emptyset \neq F \neq \mathbb{R}^n$, is the intersection of closed halfspaces. Thus F is the solution set of systems of the form

$$\sigma = \{a_t' x \geq b_t, t \in T\},$$

where T is a (possibly infinite) set, $a_t = (a_{t1}, ..., a_{tn}) \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T$. Then σ is said to be a *linear representation* of F. Any closed convex set admits infinitely many linear representations. One says that σ is an *ordinary linear system* if T is finite and it is a *linear semi-infinite system* (an LSIS in short) otherwise. LSISs have been studied from the point of view of existence of solutions, redundancy, and the geometry of F (see, e.g., [9], [11], [7], and references therein).

The *conic representation* of a nonempty closed convex set F is

$$K(F) := \left\{(a, b) \in \mathbb{R}^{n+1} : a'x \geq b \text{ for all } x \in F\right\}.$$

Since any linear representation of F is a subsystem of $\{a'x \geq b, (a, b) \in K(F)\}$, and this is also a linear representation of F by the separation theorem, this system is called the *maximal linear representation* of F. The *conic representations* (also called *reference cones* or *dual cones*) provide dual formulations for the inclusion of closed convex sets (see, e.g., the approach to the set containment problem in [12]). Different families of nonempty closed convex sets have been characterized ([9], [8]) in terms of the geometric properties of their conic representations: F is polyhedral if and only if $K(F)$ is also polyhedral, it is compact if and only if $(0_n, -1) \in \operatorname{int} K(F)$, and it is the sum of a compact convex set with a linear subspace if and only if $(0_n, -1) \in \operatorname{rint} K(F)$ (here 0_n denotes the zero vector in $\mathbb{R}^n$ whereas $\operatorname{int} K(F)$ and $\operatorname{rint} K(F)$ stand for the interior and the relative interior of $K(F)$, respectively).

The M-decomposable sets can be also characterized by means of the following parametric optimization problem:

$$P(c): \quad \operatorname{Min}_{x \in F} \ c'x,$$

with parameter $c \in \mathbb{R}^n$. If F is the solution set of a given LSIS, then $P(c)$ is a *linear semi-infinite programming* (LSIP) problem with *feasible set* F. We represent by $F^*(c)$ and $v(c)$ *the optimal set* and *the optimal value* of $P(c)$, respectively. LSIP problems arise frequently in economics, game theory, robust statistics, functional approximation, machine learning, etc. (see, e.g., the survey paper [5]). Observe that, if F is an M-decomposable set, then $P(c)$ is either solvable (i.e., $F^*(c) \neq \emptyset$) or unbounded ($v(c) = -\infty$) for all $c \in \mathbb{R}^n$.

Each of the next sections is devoted to a different type of representation, analyzing the way they can be obtained and how they can be exploited in order to get information on F and $P(c)$. More in detail, Section 2 provides linear representations of F from certain families of supporting halfspaces and extends to closed convex sets the Fourier's elimination theorem, whose classical version (see [18] and references therein) provides linear representations of the projections of a given polyhedral convex set onto the coordinate hyperplanes. Section 3 analyzes conic representations of closed convex sets, providing characterizations of closed convex cones and formulae for the conic representation of intersections and sums of closed convex sets. Finally, Section 4 characterizes the M-decomposable sets in five different ways and yields formulae for the effective computation of a compact component of a given M-decomposable set. The simplest formula derives from the constructive proof of a generalization of the classical decomposition theorem for polyhedral convex sets due to Motzkin [15] whereas another one, which involves a certain Pareto set, provides the minimal Motzkin representation.

Throughout the paper we use the following notation. The scalar product of $x, y \in \mathbb{R}^p$ is denoted by either $x'y$ or $\langle x, y \rangle$ whereas $\|x\|$ denotes the Euclidean norm of x. For any set $X \subset \mathbb{R}^p$, we denote by $\operatorname{cl} X$, $\operatorname{bd} X$, and $\operatorname{conv} X$, the *closure*, the *boundary*, and the *convex hull* of X, respectively. The *convex conical hull* of $X \cup \{0_p\}$ is denoted by $\operatorname{cone} X$. If $X \neq \emptyset$, we denote by $\operatorname{aff} X$ and $\operatorname{span} X$ the *affine hull* and the *linear hull* of X, respectively. The *orthogonal complement* of a linear subspace X is $X^{\perp} :=$

$\{y \in \mathbb{R}^p : x'y = 0 \text{ for all } x \in X\}$ and the *positive polar* of a convex cone X is $X^\circ := \{y \in \mathbb{R}^p : x'y \geq 0 \text{ for all } x \in X\}$. If X is a convex set, 0^+X and $\operatorname{lin} X := (0^+X) \cap (-0^+X)$ denote the *recession cone* and the *lineality space* of X, respectively, whereas $\mathbb{B}(X) := \{y \in \mathbb{R}^p : \exists \beta \in \mathbb{R} \text{ such that } x'y \leq \beta \; \forall x \in X\}$ is the *barrier cone* of X. A boundary point x of a closed convex set X is called *smooth* when there exists a unique supporting hyperplane to X at x.

Linear mappings and matrices are denoted in the same way. Given a linear mapping A, its adjoint mapping is denoted byA^*.

Given $x = (x_1, ..., x_p)$ we denote by $\widehat{x}$ the result of eliminating the last component, i.e., $\widehat{x} = (x_1, ..., x_{p-1})$. We identify $\widehat{x}$ with the orthogonal projection of $x \in \mathbb{R}^p$ onto the hyperplane $x_p = 0$, say $(\widehat{x}, 0)$. Coherently, we identify $\widehat{X} = \{\widehat{x} : x \in X\}$ with the orthogonal projection of X onto $x_p = 0$.

Given $h : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$, we denote by $\operatorname{dom} h$, $\operatorname{gph} h$, and $\operatorname{epi} h$ its *domain*, its *graph* and its *epigraph*, whereas $\nabla h(x)$ and $\partial h(x)$ denote the *gradient* and the convex *subdifferential of* h at $x \in \operatorname{dom} h$. The *conjugate* of h is the function $h^* : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that $h^*(u) := \sup\{\langle u, x\rangle - h(x) : x \in \operatorname{dom} h\}$.

$X \subset \mathbb{R}^p$ is represented in a unique way by its *indicator function*

$$\delta_X(x) := \begin{cases} 0, & \text{if } x \in X \\ +\infty, & \text{otherwise.} \end{cases}$$

The *support function* of X is $\delta_X^*(y) = \sup\{\langle y, x\rangle : x \in X\}$, whose domain is $\operatorname{dom} \delta_X^* = \mathbb{B}(X)$. If X is closed and convex, then $\mathbb{B}(X)^\circ = -(0^+X)$ (see, e.g., [17, Corol. 14.2.1]) and $\delta_X^{**} = \delta_X$. The latter equation implies the existence of a bijection between the nonempty closed convex sets and the lower semicontinuous (lsc in short) proper convex functions which are positively homogeneous.

2 Linear representations

Each nonempty closed convex set F admits a multiplicity of linear representations, all of them subsystems of the maximal linear representation of F, $\{a'x \geq b, (a, b) \in K(F)\}$, but in general there is no minimal representation of F, as the following example shows.

Example 1 $\{-(\cos t)x_1 - (\sin t)x_2 \geq -1, t \in T\}$, *with* $T = [0, 2\pi]$, *is a linear representation of* $F = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$. *It is easy to realize that the subsystem* $\{-(\cos t)x_1 - (\sin t)x_2 \geq -1, t \in S\}$, *with* $S \subset T$, *is a linear* *representation of F if and only if S is dense in T. Since there is no minimal dense subset of T, there is no minimal linear representation of F.*

Nevertheless we can consider the problem of obtaining linear representations of F which are small in some sense. In the case that $P(c)$ is either solvable or unbounded for all $c \in \mathbb{R}^n$ (e.g., when F is M-decomposable), we can proceed as follows: for each $(a, b) \in K(F)$ such that $\|a\| = 1$ and $P(a)$ is solvable, take a point $x^{(a,b)} \in \operatorname{bd} F$ such that $a'x \geq a'x^{(a,b)}$ for all $x \in F$, so that $a'x \geq b$ is consequence of $a'x \geq a'x^{(a,b)}$. Then

$$\left\{a'x \geq a'x^{(a,b)}, (a, b) \in T\right\}, \text{ with } T = \{(a, b) \in K(F) : \|a\| = 1\},$$

is a linear representation of F. A system larger than the previous one is

$$\{a'x \geq b, (a, b) \in S\}, \text{ with } S := \{(a, b) \in T : a'x = b \text{ for some } x \in \operatorname{bd} F\},$$

which is a linear representation of F by means of supporting halfspaces at the boundary points of F. In fact, according to [17, Theorem 18.8], the index set S in the latter linear representation can be replaced with the smaller set

$$Q := \{(a, b) \in T : a'x = b \text{ for some } x \text{ smooth point of } F\}.$$

Next we prove this statement in a different way, combining a class of the separation functional introduced in [1] to analyze the perturbation of convex sets with a well-known result of differential theory.

Proposition 2 *Let $F \subsetneqq \mathbb{R}^n$ be a closed convex set with nonempty interior and let $\mathcal{H}$ be the family of all the supporting hyperplanes at smooth points of F. Then every point in $\mathbb{R}^n \setminus F$ can be strictly separated from F by a member of $\mathcal{H}$.*

Proof. Let $x_0 \in \mathbb{R}^n \setminus F$ and pick $\widehat{x} \in \operatorname{int} F$. Without loss of generality, we assume that $x_0 = 0_n$ and $\widehat{x} = (0_{n-1}, 1)$. We define $f : \mathbb{R}^{n-1} \longrightarrow \mathbb{R} \cup \{+\infty\}$ by $f(y) := \min\{\lambda \in \mathbb{R} : (y, \lambda) \in F\}$ (with the convention $\min \emptyset := +\infty$). We can easily check that f is convex. Consider the convex open set $U := \{y \in \mathbb{R}^{n-1} : (y, 1) \in \operatorname{int} F\}$, which is a neighborhood of the origin because $(0_{n-1}, 1) \in \operatorname{int} F$; moreover, since f is bounded above by 1 on U, it is finite-valued on U and is therefore continuous on U. Thus, as $f(0_{n-1}) > 0$, there exists an open convex neighborhood $V \subseteq U$ of 0_{n-1} on which f is strictly positive. Let $y \in V$.

Since $(y, f(y)) \in \operatorname{bd} F$, there exists a supporting hyperplane to F at $(y, f(y))$, that is, there is a nonzero vector $(x^*, \lambda^*) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that

$$\langle z - y, x^* \rangle + (\lambda - f(y)) \lambda^* \leq 0 \text{ for all } (z, \lambda) \in F.$$

One can easily prove that $\lambda^* < 0$, so that, without loss of generality, we take $\lambda^* = -1$. With this choice, it turns out that $x^* \in \partial f(y)$; conversely, for every

$x^* \in \partial f(y)$ the hyperplane through $(y, f(y))$ orthogonal to $(x^*, -1)$ supports F. As f is finite-valued on V, we have $\partial f(0_{n-1}) \neq \emptyset$; hence, given that

$$\partial f(0_{n-1}) = \operatorname{conv}\{\lim_k \nabla f(y_k) : \{y_k\} \ \rightarrow\ 0_{n-1},\ f \text{ is differentiable at } y_k\ \forall k \\ \text{and } \{\nabla f(y_k)\} \text{ converges}\},$$

by [17, Theorem 25.6], there exists a sequence of points $y_k \rightarrow 0_{n-1}$ at which f is differentiable such that $\nabla f(y_k) \rightarrow x^* \in \partial f(0_{n-1})$. Since

$$0 < f(0_{n-1}) = \lim_k (f(y_k) - \langle y_k, \nabla f(y_k)\rangle),$$

for some k_0 we have $f(y_{k_0}) - \langle y_{k_0}, \nabla f(y_{k_0})\rangle > 0$. Consider the hyperplane through $(y_{k_0}, f(y_{k_0}))$ with normal $(\nabla f(y_{k_0}), -1)$. Since $\partial f(y_{k_0}) = \{\nabla f(y_{k_0})\}$, in view of the above arguments this is the unique supporting hyperplane to F at $(y_{k_0}, f(y_{k_0}))$; hence $(y_{k_0}, f(y_{k_0}))$ is a smooth point of F and therefore its supporting hyperplane belongs to $\mathcal{H}$. One has

$$\langle z - y_{k_0}, \nabla f(y_{k_0})\rangle - (\lambda - f(y_{k_0})) \leq 0 \text{ for all } (z, \lambda) \in F$$

and, on the contrary, this inequality does not hold for $(z, \lambda) = (0_{n-1}, 0)$. So we conclude that this hyperplane separates $x_0 = 0_n$ from F. □

Corollary 3 *Let $F \subseteq \mathbb{R}^n$ be a closed convex set with nonempty interior and $\mathcal{H}$ be a family of supporting hyperplanes of F such that at every point of* $\operatorname{bd} F$ *there is a supporting hyperplane to F belonging to $\mathcal{H}$. Then every point in $\mathbb{R}^n \diagdown F$ can be strictly separated from F by a member of $\mathcal{H}$.*

Proof. This is an immediate consequence of Proposition 2. □

The linear representation of F provided by Proposition 2 is minimal when F is polyhedral but not in general (recall Example 1). The examples in Section 4 illustrate the use of Proposition 2 in order to validate a given linear system as a representation of some closed convex set F.

Fundamental results in LSIS and LSIP theories provide information on F and $P(c)$ from the data, a given linear representation of F, say $\sigma = \{a_t' x \geq b_t, t \in T\}$, and c. For instance, it is well-known that the homogeneous system of σ, $\{a_t' x \geq 0, t \in T\}$, is a linear representation of 0^+F and that $F^*(c)$ is a nonempty compact set if and only if $c \in \operatorname{int}\operatorname{cone}\{a_t, t \in T\}$ (see, e.g., [9]).

To the authors' knowledge, the problem consisting of obtaining a linear representation of the image of F by a linear mapping is not considered in the existing literature. Observe that, if $A : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a linear mapping, then $\{a_t' Ay \geq b_t, t \in T\}$ is a linear representation of $A^{-1}(F)$. Nevertheless, finding a representation of $A(F)$ is a difficult task with some exceptions (e.g., if A

is an automorphism in $\mathbb{R}^n$, then $\{a_t' A^{-1}x \geq b_t, t \in T\}$ is a linear representation of $A(F)$). Observe that $A(F)$ is generally nonclosed even though F is a cone (recall that, in that case, the Farkas Lemma in [4] establishes that $(A^{-1}(F))^{\circ} = A^*(F^{\circ})$ if and only if $A^*(F^{\circ})$ is closed). The next result considers a type of mapping which arises frequently in practice (e.g., in the next two sections): the orthogonal projection onto a given hyperplane H, denoted by $\text{proj}_H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Obviously, $\text{proj}_H(x) = (x + \text{span}\{v\}) \cap H$, where $v \in \mathbb{R}^n \diagdown \{0_n\}$ is some vector orthogonal to H.

Proposition 4 *Let* $\sigma = \{a_t' x \geq b_t, t \in T\}$ *be a linear representation of* $F \neq \emptyset$ *and let* $v \in \mathbb{R}^n \diagdown \{0_n\}$ *be orthogonal to the hyperplane* H*. Then each of the following conditions guarantees that* $\text{proj}_H(F)$ *is closed:*
(i) $\{a_t' v : t \in T\}$ *contains positive and negative elements.*
(ii) $a_t' v = 0$ *for all* $t \in T$*.*
(iii) $P(\pm v)$ *is bounded.*

Proof. We can assume that $0_n \in H$. According to [17, Theorem 9.1], since the kernel of proj_H is $\text{span}\{v\}$, $\text{span}\{v\} \cap (0^+F) \subset \text{lin}\, F$ implies that $\text{proj}_H(F)$ is closed.

(i) If $\{a_t' v : t \in T\}$ contains positive and negative elements, then neither $a_t' v \geq 0$ for all $t \in T$ nor $a_t' v \geq 0$ for all $t \in T$, i.e, $\pm v \notin 0^+F$. Thus $\text{span}\{v\} \cap (0^+F) = \{0_n\} \subset \text{lin}\, F$.

(ii) $a_t' v = 0$ for all $t \in T$ if and only if $\text{span}\{v\} \subset \text{lin}\, F$.

(iii) If there exist scalars α and β such that $\alpha \leq v'x \leq \beta$ for all $x \in F$, then $\pm v \notin 0^+F$. Hence (iii)⇒(i). □

Obviously, if F is bounded, then condition (iii) holds. We consider now the problem of determining $\text{proj}_H(F)$ when H is some coordinate hyperplane. In the classical version of the elimination theorem, due to Fourier (1827), F is a polyhedral convex set (see [18]). We can take $H = \{x \in \mathbb{R}^n : x_n = 0\}$ without loss of generality (w.l.o.g. in brief), so that the problem consists of representing $\widehat{F} = \text{proj}_H(F)$. From now on e_n denotes the last vector of the canonical basis of $\mathbb{R}^n$.

We associate with $\sigma = \{a_t' x \geq b_t, t \in T\}$ the index sets

$$T_+ := \{t \in T : a_{tn} > 0\}, T_- := \{t \in T : a_{tn} < 0\}, T_0 := \{t \in T : a_{tn} = 0\} \tag{1}$$

(which form a partition of T), the vectors $c_t = \left(c_{t1}, ..., c_{t(n-1)}\right) \in \mathbb{R}^{n-1}$ such that

$$c_{tk} := \begin{cases} -\dfrac{a_{tk}}{a_{tn}} \ , \text{ if } t \in T_+ \cup T_- \\ -a_{tk}, \ , \text{ if } t \in T_0, \end{cases}$$

$k = 1, ..., n-1$, and the scalars $d_t \in \mathbb{R}$ such that

$$d_t := \begin{cases} \dfrac{b_t}{a_{tn}} \ , \text{ if } t \in T_+ \cup T_- \\ b_t, \ \ , \text{ if } t \in T_0. \end{cases}$$

Theorem 5 *(Generalized Fourier's Theorem) Let* $\sigma = \{a_t' x \geq b_t, t \in T\}$ *be a linear representation of* $F \neq \emptyset$*. Then* $\widehat{F}$ *is the solution set of the (reduced) system* $\widehat{\sigma}$*, defined as follows:*
(i) $\widehat{\sigma} := \left\{(c_t - c_s)' \, \widehat{x} \geq d_s - d_t, (t, s) \in T_- \times T_+; c_t' \widehat{x} + d_t \leq 0, t \in T_0\right\}$,
if $T_+ \neq \emptyset \neq T_-$*, i.e.,* $\pm e_n \notin 0^+ F$*.*
(ii) $\widehat{\sigma} := \left\{c_t' \widehat{x} + d_t \leq 0, t \in T_0; \sup_{t \in T_+} (c_t' \widehat{x} + d_t) < +\infty\right\}$,
if $T_+ \neq \emptyset = T_-$*, i.e.,* $e_n \in 0^+ F \diagdown \operatorname{lin} F$*.*
(iii) $\widehat{\sigma} := \left\{c_t' \widehat{x} + d_t \leq 0, t \in T_0; \inf_{t \in T_-} (c_t' \widehat{x} + d_t) > -\infty\right\}$,
if $T_+ = \emptyset \neq T_-$*, i.e.,* $-e_n \in 0^+ F \diagdown \operatorname{lin} F$*.*
(iv) $\widehat{\sigma} := \{c_t' \widehat{x} + d_t \leq 0, t \in T_0\}$,
if $T_+ = \emptyset = T_-$*, i.e.,* $e_n \in \operatorname{lin} F$*.*
Moreover, $\widehat{F}$ *is closed in cases (i) and (iv), and also in cases (ii) and (iii) provided the set* $\{(c_t, d_t), t \in T_+ \cup T_-\}$ *is bounded.*

Proof. We can write the inequality of index $t \in T$ in σ as

$$a_{tn} x_n \geq -\left(\widehat{a}_t\right)' \widehat{x} + b_t, \tag{2}$$

that can reformulate (2) as follows:

$$x_n \geq c_t' \widehat{x} + d_t, \quad \text{if } t \in T_+, \tag{3}$$
$$c_t' \widehat{x} + d_t \geq x_n, \quad \text{if } t \in T_-, \tag{4}$$
$$0 \geq c_t' \widehat{x} + d_t, \quad \text{if } t \in T_0. \tag{5}$$

By definition of $\widehat{\sigma}$, if $(\widehat{x}, x_n) \in F$ then $\widehat{x}$ is solution of $\widehat{\sigma}$. Now we discuss four possible cases for the emptiness or not of T_+ and T_- :

(i) Since $0^+ F$ is the solution set of the homogeneous system of σ, $T_+ \neq \emptyset \neq T_-$ if and only if $\pm e_n \notin 0^+ F$.

If $\widehat{x} = (x_1, ..., x_{n-1})$ is solution of $\widehat{\sigma}$, then there exist real numbers α and β such that

$$\beta = \inf_{t \in T_-} \{c_t' \widehat{x} + d_t\} \geq \sup_{t \in T_+} \{c_s' \widehat{x} + d_s\} = \alpha. \quad (6)$$

It is obvious that $(\widehat{x}, x_n)$ satisfies (3), (4) and (5) for any $x_n \in [\alpha, \beta]$, so that it is a solution of (2) for all $t \in T$, i.e., $(\widehat{x}, x_n) \in F$. Thus $\widehat{F}$ is the solution set of the reduced system $\widehat{\sigma}$. Since this is a linear system, $\widehat{F}$ is closed (also by Proposition 4 (i)).

(ii) $T_+ \neq \emptyset = T_-$ if and only if $e_n \in 0^+F$ and $-e_n \notin 0^+F$ if and only if $e_n \in 0^+F \diagdown \operatorname{lin} F$.

If $\widehat{x} = (x_1, ..., x_{n-1})$ is a solution of $\widehat{\sigma}$, we can take some

$$x_n \geq \sup_{t \in T_+} \{c_t' \widehat{x} + d_t\}.$$

Then $(\widehat{x}, x_n)$ satisfies (3) and (5), so that $(\widehat{x}, x_n) \in F$.

Now we assume that $\{(c_t, d_t), t \in T_+ \cup T_-\}$ is bounded. Then, by the Cauchy-Schwarz inequality, $\sup_{t \in T_+} \{c_t' \widehat{x} + d_t\} < +\infty$ for all $\widehat{x} \in \mathbb{R}^{n-1}$, and $\widehat{F}$ is closed because it is the solution set of a linear system.

(iii) The same argument as in (ii).

(iv) $T_+ = \emptyset = T_-$ if and only if $\pm e_n \in 0^+F$ if and only if $e_n \in \operatorname{lin} F$. In this case $\widehat{x} = (x_1, ..., x_{n-1})$ is solution of $\widehat{\sigma}$ if and only if $(\widehat{x}, x_n) \in F$ for all $x_n \in \mathbb{R}$. □

Let us discuss the latter statement in Theorem 5. First, observe that the boundedness of $\{(c_t, d_t), t \in T_+ \cup T_-\}$ depends on the given representation of F. In fact, if F is a polyhedral convex set, any linear representation $\{a_t' x \geq b_t, t \in T\}$ such that $|T| < \infty$ satisfies this boundedness condition, which gets lost by replacing a single inequality $a_t' x \geq b_t$, with $t \in T_+ \cup T_-$, by the equivalent system $\{a_t' x \geq b_t - s, s = 0, 1, ...\}$. In Example 1, since the partition of T is formed by $T_+ =]\pi, 2\pi[$, $T_- =]0, \pi[$, and $T_0 = \{0, \pi, 2\pi\}$, the reduced system of σ is

$$\hat{\sigma} = \left\{ \begin{array}{c} \left(\frac{\cos s}{\sin s} - \frac{\cos t}{\sin t}\right) x_1 \geq \frac{1}{\sin s} - \frac{1}{\sin t}, \ (t, s) \in T_- \times T_+ \\ (\cos t) x_1 \geq -1, \ t \in T_0 \end{array} \right\},$$

whose solution set $\widehat{F} = [-1, 1]$ is closed although $\{(c_t, d_t), t \in T_+ \cup T_-\}$ is unbounded. Thus, the boundedness condition is not necessary for the closedness of $\widehat{F}$. Even more, the next example shows that the closedness of $\widehat{F}$ does not imply the existence of some linear representation of F satisfying the boundedness condition.

Example 6 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be any convex function not being Lipschitz continuous. Since* f *is finite-valued, the projection of* $\operatorname{epi} f$ *onto the hyperplane* $y = 0$ *is the whole of* $\mathbb{R}^n$*, so it is closed. Let us suppose that there exists a linear representation*

$$\sigma = \{u_t x + v_t y \geq w_t, t \in T\}$$

of $\operatorname{epi} f$ *satisfying the boundedness condition. Obviously,* $T_- = \emptyset$ *and we can suppose w.l.o.g. that* $v_t = 1$ *for all* $t \in T_+$*. Then* σ *consists of some trivial constraints of the type* $0x + 0y \geq w_t$ *(with* $w_t \leq 0$*), if* $T_0 \neq \emptyset$*, and the linear representations of the epigraphs of a family of affine minorants of* f *whose pointwise supremum is* f*, that is,*

$$\sup_{t \in T_+} \{w_t - u_t' x\} = f(x) \ , \ \forall x \in \mathbb{R}^n.$$

By the boundedness of $\{u_t, t \in T\}$*, the family* $\{w_t - u_t' x, t \in T\}$ *would consist of affine functions with the same Lipschitz constant. Then* f *would be Lipschitz continuous, which is not the case.*

When F is bounded, as in Example 1, the variables can be eliminated in any order, the successive projections on intersections of coordinate hyperplanes being always obtained as in case (i). This provides an analytic method for the feasibility problem in LSISs with bounded solution set.

3 Conic representations

Let F be the solution set of $\sigma = \{a_t' x \geq b_t, t \in T\}$. The *characteristic cone* and the *first moment cone* of σ are $\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}$ and $\operatorname{cone}\{a_t, t \in T\}$, respectively. For instance, the characteristic cone and the first moment cone of the maximal linear representation of F are $K(F)$ and its vertical projection $\widehat{K(F)}$, respectively.

Two basic results on LSISs involve the closure of the characteristic cone: first, σ is consistent (i.e., $F \neq \emptyset$) if and only if

$$(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}$$

(existence theorem); and second, if $F \neq \emptyset$, a linear inequality $a' x \geq b$ is consequence of σ (i.e., $a' x \geq b$ for all $x \in F$) if and only if

$$(a, b) \in \operatorname{cl}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}$$

(nonhomogeneous Farkas Lemma). From the latter result and the separation theorem we get that the conic representation of $F \neq \emptyset$ is

$$K(F) = \operatorname{cl}\operatorname{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}.$$

This means that associating to each nonempty closed convex set its reference cone, we have established a bijection between nonempty closed convex sets in $\mathbb{R}^n$ and closed convex cones in $\mathbb{R}^{n+1}$ containing $(0_n, -1)$ but not containing $(0_n, 1)$. Observe that $(a, b) \in K(F)$ if and only if $\langle (a, b), (x, -1) \rangle \geq 0 \ \forall x \in F$. Thus,

$$K(F) = (\operatorname{cone}(F \times \{-1\}))^{\circ}.$$

On the other hand, by definition of conic representation, given $\lambda > 0$, $K(\lambda F) = \left\{(a, b) : \left(a, \frac{b}{\lambda}\right) \in K(F)\right\}$. Moreover, given two nonempty closed convex sets F and G, $F \subset G$ if and only if $K(G) \subset K(F)$ and $F = G$ if and only if $K(G) = K(F)$. These statements are also consequence of the next result together with well known properties of the support functions.

Proposition 7 *Let $F \subset \mathbb{R}^n$ be a nonempty closed convex set. Then $K(F) = -\operatorname{epi}\delta_F^*$.*

Proof. Let $F \neq \emptyset$ be closed and convex. Since $K(F)$ provides the maximal linear representation of F and $\{a'x \geq b, (a, b) \in -\operatorname{epi}\delta_F^*\}$ is another linear representation of F because

$$\begin{aligned} x \in F &\Leftrightarrow \delta_F(x) \leq 0 \\ &\Leftrightarrow \langle u, x \rangle \leq \delta_F^*(u), \ \forall u \in \operatorname{dom}\delta_F^* \\ &\Leftrightarrow \langle u, x \rangle \leq \delta_F^*(u) + \beta, \ \forall u \in \operatorname{dom}\delta_F^*, \ \forall \beta \in \mathbb{R}_+, \end{aligned}$$

we have $-\operatorname{epi}\delta_F^* \subset K(F)$. Conversely, if $(a, b) \in K(F)$, then $-a'x \leq -b$ for all $x \in F$, i.e., $\delta_F^*(-a) \leq -b$. Thus $-(a, b) \in \operatorname{epi}\delta_F^*$. □

We have incidentally shown that $\{a'x \geq b, (a, b) \in -\operatorname{gph}\delta_F^*\}$ is another linear representation of F. The next result provides information on F and $P(c)$ from $K(F)$.

Proposition 8 *Let $F \subset \mathbb{R}^n$ be a nonempty closed convex set. Then the following statements hold:*
(i) Given $x \in \mathbb{R}^n$, $x \in F$ if and only if $(x, -1) \in K(F)^{\circ}$. Moreover, $x_{n+1} \leq 0$ for all $(x, x_{n+1}) \in K(F)^{\circ}$.
(ii) F contains an extreme point if and only if $\operatorname{int}\widehat{K(F)} \neq \emptyset$.
(iii) $\mathbb{B}(F) = -\widehat{K(F)}$ *and* $0^+F = \left[\widehat{K(F)}\right]^{\circ}$.
(iv) $\operatorname{aff} F = \{x \in \mathbb{R}^n : a'x = b$ *for all* $(a, b) \in \operatorname{lin} K(F)\}$.
(v) $\operatorname{bd} F = \bigcup\left\{F^*(c) : 0_n \neq c \in \operatorname{cl}\widehat{K(F)}\right\}$.

(vi) $K(\mathrm{cl}\,\mathrm{conv}(F \cup G)) = K(F) \cap K(G)$ *for any closed convex set* G.
(vii) If $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is a linear mapping such that* AF *is closed, then* $K(AF) = \{(a,b) : (A^*a, b) \in K(F)\}$.
(viii) Given $c \in \mathbb{R}^n$ *and* $x^* \in F$, $x^* \in F^*(c)$ *if and only if* $(c, c'x^*) \in K(F)$.
(ix) If $c \in \mathrm{int}\,\widehat{K(F)}$, *then* $F^*(c) \neq \emptyset$.

Proof. (i) $x \in F$ if and only $a'x \geq b$ (i.e., $\langle (a,b), (x,-1) \rangle \geq 0$), for all $(a,b) \in K(F)$ if and only $(x,-1) \in K(F)^{\circ}$. On the other hand, since $(0_n,-1) \in K(F)$, $\langle (0_n,-1), (x, x_{n+1}) \rangle = -x_{n+1} \geq 0$ for all $(x, x_{n+1}) \in K(F)^{\circ}$.

(ii) F contains an extreme point if and only if $\mathrm{lin}\,F = \{0_n\}$ if and only if

$$\mathrm{span}\,\{a : (a,b) \in K(F)\} = \mathrm{span}\,\widehat{K(F)} = \mathbb{R}^n,$$

i.e., $\mathrm{int}\,\widehat{K(F)} \neq \emptyset$.

(iii) The orthogonal projection of both members of $\mathrm{epi}\,\delta_F^* = -K(F)$ on the hyperplane $x_{n+1} = 0$ yields $\mathbb{B}(F) = -\widehat{K(F)}$ and, taking positive polars, $0^+F = \left[\widehat{K(F)}\right]^{\circ}$.

(iv) Given $(a,b) \in \mathbb{R}^{n+1}$, $a'x = b$ for all $x \in F$ if and only $\pm(a,b) \in K(F)$ if and only $(a,b) \in \mathrm{lin}\,K(F)$.

(v) Let $\overline{x} \in \mathrm{bd}\,F$ and $c \in \mathbb{R}^n \setminus \{0_n\}$ be such that $\{x \in \mathbb{R}^n : c'x \geq c'\overline{x}\}$ is a supporting halfspace to F at $\overline{x}$, in which case $\overline{x} \in F^*(c)$. Given $d \in 0^+F$, since $\overline{x} + d \in F$, $c'(\overline{x} + d) \geq c'\overline{x}$, i.e., $c'd \geq 0$. By (iii), $c \in [0^+F]^{\circ} = \left[\widehat{K(F)}\right]^{\circ\circ} = \mathrm{cl}\,\widehat{K(F)}$. Thus

$$\mathrm{bd}\,F \subset \bigcup \left\{ F^*(c) : 0_n \neq c \in \mathrm{cl}\,\widehat{K(F)} \right\}.$$

The reverse inclusion is trivial.

(vi) and (vii) are the result of combining Proposition 7 with Corollaries 16.3.1 and 16.5.1 in [17], respectively.

(viii) Let $c \in \mathbb{R}^n$ and $x^* \in F$. Then $x^* \in F^*(c)$ if and only if $c'x \geq c'x^*$ for all $x \in F$, i.e., $(c, c'x^*) \in K(F)$.

(ix) If $c \in \mathrm{int}\,\widehat{K(F)} = \mathrm{int}\,\mathrm{cone}\,\{a : (a,b) \in K(F)\}$, then $F^*(c) \neq \emptyset$. □

From statement (i) in Proposition 8 we conclude that, if $F \neq \emptyset$, then $K(F)^{\circ}$ is contained in the halfspace $x_{n+1} \leq 0$ but not in its boundary $x_{n+1} = 0$.

We consider now the characterization of closed convex cones in terms of its conic representation. This characterization allows us to give conditions guaranteeing the dual equations $K(F\cap G)=K(F)+K(G)$ and $K(F+G)=K(F)\cap K(G)$.

Proposition 9 *Let $F\subset\mathbb{R}^n$ be a nonempty closed convex set. Then the following statements are equivalent to each other:*
(i) F is a cone.
(ii) $K(F)=F^\circ\times\mathbb{R}_-$.
(iii) $K(F)=\widehat{K(F)}\times\mathbb{R}_-$.
(iv) There exists a set $D\subset\mathbb{R}^n$ such that $K(F)=D\times\mathbb{R}_-$.

Proof. (i)⇒(ii) Since F is a closed convex cone, Farkas Lemma for closed convex cones yields

$$F=F^{\circ\circ}=\{x\in\mathbb{R}^n : y'x\geq 0 \text{ for all } y\in F^\circ\},$$

so that $\{y'x\geq 0,\ y\in F^\circ\}$ is a linear representation of F and $K(F)=\operatorname{cl}(F^\circ\times\mathbb{R}_-)=F^\circ\times\mathbb{R}_-$.

(ii)⇒(iii)⇒(iv) These implications are trivial.

(iv)⇒(i) Let $K(F)=D\times\mathbb{R}_-$, with $D\subset\mathbb{R}^n$. Let $y\in F$ and $\lambda>0$. Let $(a,b)\in D\times\mathbb{R}_-$. Since $\frac{b}{\lambda}\in\mathbb{R}_-$ and F is the solution set of its maximal linear representation $\{a'x\geq b,(a,b)\in D\times\mathbb{R}_-\}$, we have $a'y\geq\frac{b}{\lambda}$, i.e., $a'(\lambda y)\geq b$. Thus $\lambda y\in F$. □

Theorem 10 *Let F and G be nonempty closed convex sets in $\mathbb{R}^n$. Then the following statements are true:*
(i) $K(F)+K(G)\subset K(F\cap G)$. The equality holds if $K(F)\cap(-K(G))$ is a linear subspace of $\mathbb{R}^{n+1}$.
(ii) If G is a cone, then

$$K(F+G)=K(F)\cap(G^\circ\times\mathbb{R}). \tag{7}$$

Moreover, $K(F+G)=K(F)\cap K(G)$ if, additionally, $F\cap(-G)\neq\emptyset$.

Proof. (i) The aggregation of the maximal linear representations of F and G gives a linear representation of $F\cap G$, with characteristic cone

$$\operatorname{cone}(K(F)\cup K(G))=K(F)+K(G),$$

so that

$$K(F\cap G)=\operatorname{cl}[K(F)+K(G)] \tag{8}$$

and (i) holds.

If $K(F)\cap(-K(G))$ is a linear subspace of $\mathbb{R}^{n+1}$, then $K(F)+K(G)$ is closed convex cone [17, Corollary 9.1.3] and (8) becomes $K(F\cap G)=K(F)+K(G)$.

(ii) Assume that G is a cone. Given $(a,b)\in\mathbb{R}^{n+1}$,

$$\begin{aligned}(a,b)\in K(F+G) &\Leftrightarrow a'(x+y)\geq b\ \forall x\in F \text{ and } \forall y\in G\\ &\Leftrightarrow a'x\geq b\ \forall x\in F \text{ and } a'y\geq 0\ \forall y\in G\\ &\Leftrightarrow (a,b)\in K(F) \text{ and } a\in G^{\circ}.\end{aligned}$$

Thus, (16) holds.

By Proposition 9 and (7),

$$K(F)\cap K(G)=K(F)\cap\left(G^{\circ}\times\mathbb{R}_{-}\right)\subset K(F)\cap\left(G^{\circ}\times\mathbb{R}\right)=K(F+G). \quad (9)$$

Let us prove the reverse inclusion when, additionally, $F\cap(-G)\neq\emptyset$. Let $(a,b)\in K(F+G)$. Obviously, $(a,b)\in K(F)$ and $a\in G^{\circ}$. Since $F\cap(-G)\neq\emptyset$ means that $0_n\in F+G$, $b\leq a'0_n=0$, so that $(a,b)\in G^{\circ}\times\mathbb{R}_{-}=K(G)$, again by Proposition 9. Therefore, $(a,b)\in K(F)\cap K(G)$. □

Statement (i) in Theorem 10 is also a consequence of [6, Proposition 3.2]. The next examples show that the additional assumptions in both statements of Theorem 10 are not superfluous.

Example 11 *Since the compact convex set*

$$\operatorname{conv}\left\{\left(t,1,-t^{2}\right),t\in[-1,1]\,;(0,0,-1)\right\}$$

does not contain the origin, its conic hull is a closed convex cone that does not contain $(0,0,1)$. *So, it is the reference cone of some nonempty closed convex set* F. *Consider the closed convex cone* $G=\mathbb{R}\times\mathbb{R}_{-}$, *with* $K(G)=\{0\}\times\mathbb{R}_{-}^{2}$. *Since*

$$K(F)+K(G)=\operatorname{cone}\left\{\left(t,1,-t^{2}\right),t\in[-1,1]\,;(0,-1,0)\,;(0,0,-1)\right\}$$

is not closed, we cannot have $K(F)+K(G)=K(F\cap G)$. *Here* $K(F)\cap(-K(G))=\{0\}\times\mathbb{R}_{+}\times\{0\}$ *is not a linear subspace of* $\mathbb{R}^{3}$.

Example 12 *Let* $F=\{(1,1)\}$ *and* $G=\mathbb{R}_{+}^{2}$. *We have*

$$K(F)=\operatorname{cone}\{(1,0,1),(-1,0,-1),(0,1,1),(0,-1,-1),(0,0,-1)\},$$

$$K(G)=\operatorname{cone}\{(1,0,0),(0,1,0),(0,0,-1)\},$$

and

$$K(F+G)=\operatorname{cone}\{(1,0,1),(0,1,1),(0,0,-1)\},$$

with $K(G)=K(F)\cap K(G)\subsetneq K(F+G)$. *Obviously,* $F\cap(-G)=\emptyset$.

4 Motzkin representations

First we establish some simple properties of the M-decomposable sets.

Proposition 13 *Let $F = C + D$ be a Motzkin representation of F. Then:*
(i) $K(F) = K(C) \cap (D^{\circ} \times \mathbb{R})$.
(ii) $\operatorname{aff} F = \operatorname{aff} C + \operatorname{span} D$.
(iii) If $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear mapping and D is polyhedral, then $A(F) = A(C) + A(D)$ is a Motzkin representation of $A(F)$.
(iv) $F^*(c) \cap C \neq \emptyset$ for each $c \in \mathbb{R}^n$ such that $v(c) > -\infty$.
(v) $v(c) = \begin{cases} \min\{c'x : x \in C\}, & \text{if } c \in D^{\circ} \\ -\infty, & \text{otherwise.} \end{cases}$
(vi) $\mathbb{B}(F) = -D^{\circ}$ *and* $0^+F = D$.

Proof. (i) It is Theorem 10 (ii).

(ii) By assumption, any $x \in \operatorname{aff} F$ can be written as $x = \sum_{i=1}^{m} \lambda_i c^i + \sum_{i=1}^{m} \lambda_i d^i$, with $\sum_{i=1}^{m} \lambda_i = 1$, $c^i \in C$, $d^i \in D$, $i = 1, ..., m$. Since $\sum_{i=1}^{m} \lambda_i c^i \in \operatorname{aff} C$ and $\sum_{i=1}^{m} \lambda_i d^i \in \operatorname{span} D$, we get $\operatorname{aff} F \subset \operatorname{aff} C + \operatorname{span} D$. The reverse inclusion is trivial.

(iii) Under the assumptions, $A(C)$ is a compact convex set and $A(D)$ a polyhedral (and so closed) convex cone.

(iv) and (v) are immediate, whereas (vi) is a straightforward consequence of (v). □

From (vi) it follows that the barrier cone of any M-decomposable set is closed, but the converse statement is not true (consider the convex hull of a branch of hyperbola). On the other hand, the M-decomposable sets in $\mathbb{R}^2$ have no asymptotes (a halfline L is an *asymptote* of F if $F \cap L = \emptyset$ and $d(F, L) = 0$), but there are also sets in $\mathbb{R}^2$ which are not M-decomposable but have no asymptotes, like, e.g., the set $F = \{(x, y) : x^2 \leq y\}$. In higher dimensions one can even find hyperbolic sets with these properties, as the next example shows.

Example 14 *Let $F = \operatorname{cl} \operatorname{conv} X$, with $X = \left\{\left(\cos s, \sin s, \frac{s}{2\pi - s}\right) : s \in [0, 2\pi[\right\}$. Since F is closed and convex, its recession cone 0^+F coincides with its asymptotic cone*

$$F_{\infty} := \left\{d \ : \exists \lambda_k \rightarrow +\infty, \ x_k \in F \ \text{such that } d = \lim_{k \rightarrow \infty} \frac{x_k}{\lambda_k}\right\}.$$

Then, given three sequences $\{s_k\} \subset [0, 2\pi[$, $\{x_k\} \subset X$, *and* $\{\lambda_k\} \subset \mathbb{R}_+$ *such that* $s_k \to 2\pi$, $x_k = \left(\cos s_k, \sin s_k, \frac{s_k}{2\pi - s_k}\right)$, *and* $\lambda_k = \|x_k\|$ *for all* $k \in \mathbb{N}$, *we have* $\lim_{k\to\infty} \frac{x_k}{\lambda_k} = (0,0,1) \in 0^+F$. *Even more,* $0^+F = \mathbb{R}_+(0,0,1)$. *If* $L = \{x_0 + \lambda y : \lambda \geq 0\}$ *is an asymptote of* F, *necessarily* $y \in 0^+F = \mathbb{R}_+(0,0,1)$. *But then* $d(F, L) = 0$ *implies that* $\|\widehat{x}_0\| = 1$. *If* $\widehat{x}_0 = (\cos s_0, \sin s_0)$, *with* $s_0 \in [0, 2\pi[$, *then* $(\cos s_0, \sin s_0, \mu) \in F \cap L$ *for* μ *large enough (contradiction). Thus* F *is a closed convex set with no asymptote, but it is not M-decomposable. Moreover* F *is hyperbolic, since it is contained in the M-decomposable set* $\{(x, y, z) : x^2 + y^2 \leq 1\} = \{(x, y, z) : x^2 + y^2 \leq 1, z = 0\} + \{(0, 0, z) : z \in \mathbb{R}\}$.

On the other hand, M-decomposable sets in $\mathbb{R}^n$ may have asymptotes if $n \geq 3$.

Example 15 *Consider the set* $F = \left\{x \in \mathbb{R}^n : x_n^2 \geq \sum_{i=1}^{n-1} x_i^2, x_n \geq 0\right\}$ *(the ice-cream cone in* $\mathbb{R}^n$*). It is a closed convex cone, hence it is M-decomposable. However, every bidimensional vertical section of* F *not containing the origin is a hyperbolic set with asymptotes (for instance, the intersection of* F *with the plane* $x_2 = \cdots = x_{n-1} = 1$ *has two asymptotes, namely, the intersections of the hyperplanes* $x_n = x_1$ *and* $x_n = -x_1$ *with that plane), and such asymptotes are obviously asymptotes of* F, *too.*

Although no topological or geometric characterization of the M-decomposable sets is available, the next five results characterize these sets in terms of the corresponding parametric problems, the support functions, the conic representations, the Pareto efficient sets (to be defined later), and the linear representations of their dual cones, respectively. Moreover, the two latter results provide the minimal Motzkin representation of F and a simple formula for obtaining a Motzkin representation of F, respectively.

Proposition 16 *A set* $F \subset \mathbb{R}^n$ *is M-decomposable if and only if there exists a compact set* $C \subset F$ *such that* $F^*(c) \cap C \neq \emptyset$ *for each* $c \in \mathbb{R}^n$ *such that* $v(c) > -\infty$.

Proof. The "only if" part is consequence of statement (v) in Proposition 13. To prove the "if" statement, let us consider the support functions of F and C. Our assumption clearly implies that $\delta_F^*(c) = \delta_C^*(c)$ for every $c \in \mathbb{R}^n$ such that $v(-c) > -\infty$. In other words, $\delta_F^* = \delta_C^* + \delta_{\{c\in\mathbb{R}^n : v(-c) > -\infty\}}$. Since δ_F^* is lsc, δ_C^* is continuous (as it is finite-valued), and $\mathbb{B}(F)^\circ = (\operatorname{dom} \delta_F^*)^\circ = -0^+F$, it turns out that $\delta_{\{c\in\mathbb{R}^n : v(-c) > -\infty\}} = \delta_F^* - \delta_C^*$ is lsc, which amounts to saying that the set $\{c \in \mathbb{R}^n : v(-c) > -\infty\}$ is closed, so that it coincides with $-(0^+F)^\circ$. We thus have $\delta_{\{c\in\mathbb{R}^n : v(-c) > -\infty\}} = \delta_{-(0^+F)^\circ} = \delta_{0^+F}^*$ and therefore

$$\delta_F^* = \delta_C^* + \delta_{0^+F}^* = \delta_{\operatorname{cl}\operatorname{conv} C}^* + \delta_{0^+F}^* = \delta_{\operatorname{cl}\operatorname{conv}(C+0^+F)}^*. \tag{10}$$

Given that $\operatorname{cl}\operatorname{conv} C$ is compact and 0^+F is convex and closed, $\operatorname{cl}\operatorname{conv} C + 0^+F$ is a closed convex set, too. Hence, from (10) we deduce that $F = \operatorname{cl}\operatorname{conv} C + 0^+F$, which shows that F is M-decomposable. $\square$

Proposition 17 *A closed convex set $F \subset \mathbb{R}^n$ is M-decomposable if and only if* $\operatorname{dom} \delta_F^*$ *is closed and the restriction of* δ_F^* *to* $\operatorname{dom} \delta_F^*$ *has a finite sublinear extension to the whole of* $\mathbb{R}^n$.

Proof. Assume first that $F = C + D$ for some compact convex set C and some closed convex cone D. Then $\operatorname{dom} \delta_F^* = \mathbb{B}(F) = -D^\circ$, hence it is closed. Moreover, $\delta_F^* = \delta_C^* + \delta_D^* = \delta_C^* + \delta_{\mathbb{B}(F)}$, which shows that δ_F^* coincides with the finite valued sublinear function δ_C^* on $\mathbb{B}(F)$.

Conversely, suppose that $\mathbb{B}(F)$ is closed and the restriction of δ_F^* to $\mathbb{B}(F)$ has a finite sublinear extension to the whole of $\mathbb{R}^n$. This finite sublinear extension is the support function δ_C^* of some compact convex set C. On the other hand, since $\mathbb{B}(F)$ is a closed convex cone it coincides with its second negative polar, so that $\mathbb{B}(F)$ is the negative polar of some closed convex cone D. We thus have $\delta_F^* = \delta_C^* + \delta_{\mathbb{B}(F)} = \delta_C^* + \delta_D^* = \delta_{C+D}^*$. Given that both F and $C + D$ are closed convex sets, from these equalities we conclude that $F = C + D$, which shows that F is M-decomposable. $\square$

Proposition 18 *Let F be a nonempty closed convex set in $\mathbb{R}^n$. Then F is M-decomposable if and only if there exist two closed convex cones $K \subset \mathbb{R}^{n+1}$ and $L \subset \mathbb{R}^n$ such that $K(F) = K \cap (L \times \mathbb{R})$, $(0_n, 1) \notin K$ and $(0_n, -1) \in \operatorname{int} K$.*

Proof. Let $F = C + D$ be such that C is a compact convex set and D a closed convex cone. The direct statement follows from Proposition 13 (i), taking $K = K(C)$ and $L = D^\circ$.

Conversely, assume that there exist K and L as in the statement and let $C = \{x \in \mathbb{R}^n : a'x \geq b \quad \forall (a, b) \in K\}$. The set C is convex and compact (as $(0_n, -1) \in \operatorname{int} K$). Moreover, one has $C + L^\circ \subseteq F$; indeed, if $x \in C$, $d \in L^\circ$ and $(a, b) \in K(F)$ then, by $(a, b) \in K$ and $a \in L$, we have $a'x \geq b$ and $a'd \geq 0$, so that $a'(x + d) \geq b$. Thus we only need to prove the opposite inclusion. To this aim, let $x_0 \in F$ and assume, towards a contradiction, that $x_0 \notin C + L^\circ$. Since $C + L^\circ$ is a closed convex set, by the separation theorem there exists $(a, b) \in \mathbb{R}^{n+1}$ such that $a'(x + d) \geq b > a'x_0$ for every $x \in C$ and $d \in L^\circ$. We thus have $a'x \geq b$ for every $x \in C$, and hence $(a, b) \in K(C) = K$ (this equality following from the fact that K is a closed convex cone that contains $(0_n, -1)$ and does not contain $(0_n, 1)$) and $a \in L^{\circ\circ} = L$. Consequently, $(a, b) \in K \cap (L \times \mathbb{R}) = K(F)$, which contradicts the assumption $x_0 \in F$. $\square$

Theorem 19 *Let F be a closed convex set, $\emptyset \neq F \subset \mathbb{R}^n$. Let $L := \operatorname{lin} F$, $K := 0^+F \cap L^\perp$, and*

$$M(F) := \left\{ x \in F \cap L^\perp : (x - K) \cap F = \{x\} \right\}.$$

Then the following statements hold:

(i) F is M-decomposable if and only if $M(F)$ is bounded. In that case,

$$F = \operatorname{cl}\operatorname{conv} M(F) + 0^+F \tag{11}$$

is a Motzkin representation of F.
(ii) If F is M-decomposable and contains an extreme point, then (11) is the minimal Motzkin representation of F, with

$$M(F) = \left\{x \in F : \left(x - 0^+F\right) \cap F = \{x\}\right\} \tag{12}$$

satisfying

$$\begin{aligned} \emptyset \neq \bigcup \left\{F^*(c) : c \in \operatorname{int} \widehat{K(F)}\right\} \subset M(F) \\ \subset \bigcup \left\{F^*(c) : 0_n \neq c \in \operatorname{cl} \widehat{K(F)}\right\}. \end{aligned} \tag{13}$$

Proof. (i) Assume that F is M-decomposable. Let $F = C + 0^+F$, with C compact convex. We can assume that $C \subset L^\perp$ (otherwise we replace it with its orthogonal projection on $L^\perp$, i.e., the compact convex set $(C + L) \cap L^\perp$, which is another compact component of F).

First we show that $M(F)$ is bounded. Let $x \in M(F)$. Since $x \in F = L + K + C$, we can write $x = a + b + c$, $a \in L$, $b \in K$, $c \in C$. Since $x - a - c = b \in K$, $a + c \in L + C \subset F$, and $x \in M(F)$, we have $x = a + c$. Then $x = c$ because $x, c \in L^\perp$. We get $M(F) \subset C$ and so $M(F)$ is bounded.

Now we assume that $M(F)$ is bounded. Obviously,

$$\operatorname{cl}\operatorname{conv} M(F) + 0^+F \subset F + 0^+F = F. \tag{14}$$

Next we prove the reverse inclusion of (14).

Let $v \in F$. We can write in a unique way $v = u + y$, $u \in L$, $y \in L^\perp$. Since $y = v - u \in F + 0^+F = F$, we have $y \in F \cap L^\perp \cap (y - 0^+F)$, with $0^+\left[F \cap L^\perp \cap (y - 0^+F)\right] = L \cap L^\perp = \{0_n\}$, so that $F \cap L^\perp \cap (y - 0^+F) = F \cap (y - K)$ is a nonempty compact convex set.

Let $\widetilde{y}$ be an optimal solution of the optimization problem

$$P: \quad \operatorname{Min}_{x \in F \cap (y-K)} \quad d'x,$$

where d is an element of K such that $d'x > 0$ for all $x \in K \setminus \{0_n\}$ (the existence of such a vector is consequence of [16, Theorem 3.13], taking into account that K is a pointed closed convex cone).

Now we prove that $\widetilde{y} \in M(F)$. Let $\widehat{y} \in F$ be such that $\widetilde{y} - \widehat{y} \in K$. We must show that $\widetilde{y} = \widehat{y}$. In fact, since

$$\widehat{y} - y = (\widehat{y} - \widetilde{y}) + (\widetilde{y} - y) \in (-K) + (-K) = -K,$$

because $\widetilde{y} \in y - K$, we have $\widehat{y} \in F \cap (y-K)$, and so $d'\widetilde{y} \leq d'\widehat{y}$, i.e., $d'(\widetilde{y} - \widehat{y}) \leq 0$, with $\widetilde{y} - \widehat{y} \in K$. This implies $\widetilde{y} = \widehat{y}$ by the assumption on d.

Since $\widetilde{y} \in y - K \subset y - 0^+F$ and $\widetilde{y} \in M(F)$, we get

$$v = y + u \in \left(\widetilde{y} + 0^+F\right) + L = y + 0^+F \subset M(F) + 0^+F.$$

Hence $F = M(F) + 0^+F$, and we conclude that

$$\begin{aligned} F &= \operatorname{cl}\operatorname{conv} F = \operatorname{cl}\operatorname{conv}\left(M(F) + 0^+F\right) = \operatorname{cl}\left(\operatorname{conv} M(F) + \operatorname{conv} 0^+F\right) \\ &= \operatorname{cl}\left(\operatorname{conv} M(F) + 0^+F\right) = \operatorname{cl}\operatorname{conv} M(F) + \operatorname{cl} 0^+F \\ &= \operatorname{cl}\operatorname{conv} M(F) + 0^+F \end{aligned}$$

and F is M-decomposable.

(ii) Since 0^+F is pointed, $M(F) \subset C$, which implies $\operatorname{cl}\operatorname{conv} M(F) \subset C$, with $M(F)$ as in (12) because $L^\perp = \mathbb{R}^n$. Now we shall prove (13).

$\bigcup\left\{F^*(c) : c \in \operatorname{int}\widehat{K(F)}\right\} \neq \emptyset$ by Proposition 8, statements (ii) and (ix).

Now let $x^* \in F^*(c)$, with $c \in \operatorname{int}\widehat{K(F)}$. By (iii) in Proposition 8,

$$c \in \operatorname{int}\widehat{K(F)} = \operatorname{int}\operatorname{cl}\widehat{K(F)} = \operatorname{int}\left[\widehat{K(F)}\right]^{\circ\circ} = \operatorname{int}\left(0^+F\right)^\circ,$$

so that $c'd > 0$ for all $d \in 0^+F \setminus \{0_n\}$ (this is part of the argument of [16, Theorem 3.13 (iii)]). Let $y \in F$ be such that $y \neq x^*$ and $y \in x^* - 0^+F$. Since $x^* - y \in 0^+F \setminus \{0_n\}$, $c'(x^* - y) > 0$, in contradiction with $x^* \in F^*(c)$. Thus, $x^* \in M(F)$.

Finally, Given $x \in M(F)$ and $d \in 0^+F \setminus \{0_n\}$, since $\left\{x - \frac{d}{k}\right\}_{k=1}^\infty$ is contained in $\mathbb{R}^n \setminus F$ and $x - \frac{d}{k} \to x$, we have

$$x \in \operatorname{bd} F = \bigcup\left\{F^*(c) : 0_n \neq c \in \operatorname{cl}\widehat{K(F)}\right\}$$

by (v) in Proposition 8. This completes the proof. □

We can interpret $M(F)$ as the *Pareto efficient set* of $F \cap L^\perp$ relative to its recession cone. Actually, in part (i), we have shown that $M(F)$ bounded entails $F = M(F) + 0^+F$, but this is not a Motzkin representation of F because $M(F)$ is generally nonconvex. Observe also that, if F contains an extreme point and every element of

$$M(F) \diagdown \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\}$$

is an extreme point of F, then

$$\operatorname{cl} M(F) = \operatorname{cl} \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\}.$$

In fact, given $\overline{x} \in M(F) \diagdown \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\}$, by [17, Theorem 18.6] there exists a sequence $\left\{ x^k \right\}_{k=1}^{\infty}$ of exposed points of F such that $x^k \to \overline{x}$. Let $c^k \in \mathbb{R}^n$ be such that $F^*\left(c^k\right) = \left\{x^k\right\}$, $k = 1, 2., , , .$ Let $d \in 0^+F \diagdown \{0_n\}$. Since $x^k + d \in F \diagdown \left\{x^k\right\}$, $\left(c^k\right)'\left(x^k + d\right) > \left(c^k\right)' x^k$, i.e., $\left(c^k\right)' d > 0$. Then $c^k \in \operatorname{int} 0^+F$ by [16, Theorem 3.13]. Hence $\overline{x} \in \operatorname{cl} \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\}$. This proves that $\operatorname{cl} M(F) \subset \operatorname{cl} \bigcup \left\{ F^*(c) : c \in \operatorname{int} \widehat{K(F)} \right\}$ whereas the converse inclusion is a consequence of (13).

Along the next proof, given an arbitrary set S, we denote by $\mathbb{R}^{(S)}$ the set of mappings from S to $\mathbb{R}$ with a finite support set and by $\mathbb{R}_+^{(S)}$ the positive cone in the linear space $\mathbb{R}^{(S)}$. We also maintain the notation of (1) for the partition of the index set of a linear system based on the sign of the coefficient of the last variable. Thus, for an homogeneous system $\{\langle (c_s, d_s), (x, x_{n+1}) \rangle \geq 0, s \in S\}$, $s \in S_- \Leftrightarrow d_s < 0$ and $s \in S_0 \Leftrightarrow d_s = 0$.

Theorem 20 *(Generalized Motzkin Theorem) Let $F \subset \mathbb{R}^n$ be a nonempty closed convex set. Then F is M-decomposable if and only if there exists a linear representation of $K(F)$, $\{\langle (c_s, d_s), (x, x_{n+1}) \rangle \geq 0, s \in S\}$, such that $\left\{ \frac{c_s}{d_s} : s \in S_- \right\}$ is bounded. In such a case,*

$$F = \operatorname{cl} \operatorname{conv} \left\{ -\frac{c_s}{d_s} : s \in S_- \right\} + \operatorname{cl} \operatorname{cone} \{c_s : s \in S_0\} \tag{15}$$

is a Motzkin representation of F.

Proof. First we assume that F is M-decomposable such that $F = C + D$, where C is a compact convex set and D is a closed convex cone. Then, by Proposition 13 (i),

$$\{\langle (c, -1), (x, x_{n+1}) \rangle \geq 0, c \in C; \langle (d, 0), (x, x_{n+1}) \rangle \geq 0, d \in D\} \tag{16}$$

is a linear representation of $K(F)$ satisfying the boundedness condition.

Conversely, assume that $\{\langle (c_s, d_s), (x, x_{n+1}) \rangle \geq 0, s \in S\}$ is a linear representation of $K(F)$ such that $\left\{ \frac{c_s}{d_s} : s \in S_- \right\}$ is bounded. Given $s \in S$, since

$(c_s, d_s) \in K(F)^\circ$, $d_s \leq 0$ according to Proposition 8 (i).

If $d_s = 0$ for all $s \in S$, then

$$K(F)^\circ = \operatorname{cl}\operatorname{cone}\{(c_s, 0), s \in S\} \subset \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\right\},$$

in contradiction with statement (i) of Proposition 8 (we are assuming $F \neq \emptyset$). Hence there exists some $s \in S$ such that $d_s < 0$. Dividing by $|d_s|$ if it is necessary, we can assume w.l.o.g. that $d_s = -1$ for all $s \in S_-$. Obviously,

$$K(F)^\circ = \operatorname{cl}\operatorname{cone}\{(c_s, -1), s \in S_-; (c_s, 0), s \in S_0\}. \tag{17}$$

Let $C := \operatorname{cl}\operatorname{conv}\{c_s, s \in S_-\} \neq \emptyset$ and $D := \operatorname{cl}\operatorname{cone}\{c_s, s \in S_0\}$. We must prove that $F = C + D$.

First we show that $C + D \subset F$. Given $x \in C + D$, there exist sequences $\{\delta^k\} \subset \mathbb{R}_+^{(S_-)}$ and $\{\xi_k\} \subset \mathbb{R}_+^{(S_0)}$ such that

$$x = \lim_k \sum_{s \in S_-} \delta_s^k c_s + \lim_k \sum_{s \in S_0} \xi_s^k c_s \text{ and } \sum_{s \in S_-} \delta_s^k = 1 \ \forall k.$$

Then, according to (17),

$$(x, -1) = \lim_k \left[\sum_{s \in S_-} \delta_s (c_s, -1) + \sum_{s \in S_0} \xi_s (c_s, 0)\right] \in K(F)^\circ,$$

so that $x \in F$. Thus, $C + D \subset F$.

Now we assume that $x \in F$. Then $(x, -1) \in K^\circ$ and, again by (17), there exists $\left\{\lambda^k\right\} \subset \mathbb{R}_+^{(S)}$ such that

$$(x, -1) = \lim_k \left[\sum_{s \in S_-} \lambda_s^k (c_s, -1) + \sum_{s \in S_0} \lambda_s^k (c_s, 0)\right],$$

i.e.,

$$x = \lim_k \left[\sum_{s \in S_-} \lambda_s^k c_s + \sum_{s \in S_0} \lambda_s^k c_s\right] \text{ and } \lim_k \sum_{s \in S_-} \lambda_s^k = 1. \tag{18}$$

Let $\rho_k := \sum_{s \in S_-} \lambda_s^k, k = 1, 2, \ldots$. Since $\lim_k \rho_k = 1$ we can assume w.l.o.g. $\rho_k > 0$ for all k. From (18) we get

$$x = \lim_k \left[\sum_{s \in S_-} \frac{\lambda_s^k}{\rho_k} c_s + \sum_{s \in S_0} \frac{\lambda_s^k}{\rho_k} c_s\right]. \tag{19}$$

Let

$$x^k := \sum_{s\in S_-} \frac{\lambda_s^k}{\rho_k} c_s \in \text{conv}\{c_s, s \in S_-\} \subset C.$$

Since C is compact, we can assume by considering a suitable subsequence that there exists $\overline{x} \in C$ such that $\lim_k x^k = \overline{x} \in C$. Defining

$$y^k := \sum_{s\in S_0} \frac{\lambda_s^k}{\rho_k} c_s \in \text{cone}\{c_s, s \in S_0\} \subset D,$$

(19) implies that $\lim_k y^k = x - \overline{x} \in D$, so that $x = \overline{x} + (x - \overline{x}) \in C + D$. We conclude that $F \subset C + D$. $\square$

For the linear representation of $K(F)$ given by (16), $\left\|\frac{c_s}{d_s}\right\| \leq \max_{c\in C} \|c\|$ for all $s \in S_-$. An alternative bound can be obtained observing that, by the compactness of C, $K(C)$ contains a ball centered at $(0_n, -1)$ and radius $\rho > 0$. Since the distance from $(0_n, -1)$ to any hyperplane $c'x - x_{n+1} = 0$ is at least ρ, $\left\|\frac{c_s}{d_s}\right\| \leq \frac{1}{\rho}$ for all $s \in S_-$. Once again, the boundedness assumption depends on the available linear representation of $K(F)$ and not on $K(F)$ itself. In fact, taking two arbitrary indexes $u \in S_-$ and $v \in S_0$, and aggregating the redundant constraints $\langle (c_u, d_u) + r(c_v, d_v), (x, x_{n+1})\rangle \geq 0, r \in \mathbb{N}$, to the given linear representation of $K(F)$, we obtain a new linear representation of $K(F)$ that violates the boundedness condition.

Notice that the boundedness assumption of Theorem 20 holds if $|S_-| < \infty$ (e.g., when F is a polyhedral convex set S can be taken finite). Under the mentioned boundedness assumption, by Theorem 5 and Proposition 8 (iii), we have

$$\widehat{K(F)} = \{x \in \mathbb{R}^n : c_s' x \geq 0, s \in S_0\} = \left(0^+F\right)^\circ.$$

In relation with equation (v) in Proposition 13, observe that $v(c)$ can be expressed in terms of the data (a linear representation of $K(F)$) under the assumption of Theorem 20 because

$$\min\{c'x : x \in C\} = \inf\left\{\frac{c'c_s}{|d_s|} : s \in S_-\right\}.$$

Example 21 *Let*

$$\left\{\left(s^2 - 2s + 1\right)x_1 + s^2 x_2 - x_3 \geq 0, s \in [0,1]; x_1 \geq 0, x_2 \geq 0\right\}$$

be a linear representation of the reference cone of certain set F to be described. Since the arch of parabola (or the astroid $\sqrt{|x_1|} + \sqrt{|x_2|} = 1$)

$$\left\{\left(s^2 - 2s + 1, s^2\right), s \in [0,1]\right\}$$

is bounded, F is the sum of the convex hull of this arch of parabola and the convex conical hull of $\{(1,0),(0,1)\}$, i.e., $\mathbb{R}^2_+$. The mentioned arch coincides also with $M(F)$. All the boundary points of such an M-decomposable set F are smooth, so that we get, from Proposition 2, the linear representation of F,

$$\left\{tx_1 + (1-t)\,x_2 \geq t - t^2, t \in [0,1]\right\},$$

from which we obtain another description of the conic representation of F :

$$K(F) = \operatorname{cone}\left\{\left(t, 1-t, t-t^2\right), t \in [0,1]; (0,0,-1)\right\}.$$

Example 22 *Let $\{s^2x_1 + x_2 - sx_3 \geq 0, s \in \mathbb{R}_{++}\}$ be a linear representation of $K(F)$. Here $\left\{\frac{c_s}{|d_s|} : d_s < 0, s \in S\right\}$ is unbounded. In fact, we have $F = C + D$ with $C = \operatorname{cl}\operatorname{conv}\{(s, s^{-1}), s \in \mathbb{R}_{++}\}$ (the unbounded convex hull of a branch of the hyperbola $x_1x_2 = 1$) and $D = \operatorname{cl}\operatorname{cone}\emptyset = \{0_2\}$. Observe that, since any boundary point of F is smooth, $\{x_1 + t^2x_2 \geq 2t, t \in \mathbb{R}_{++}\}$ is a linear representation of F and the conic representation of F turns out to be the closure of its characteristic cone, i.e.,*

$$K(F) = \operatorname{cone}\left\{\left(1, t^2, 2t\right), t \in \mathbb{R}_{++}; (0,1,0), (0,0,-1)\right\}.$$

Obviously, the decomposability property is preserved by the cartesian product and by the product by scalars, but not by the Minkowski sum (recall that the sum of closed convex cones is not necessarily closed) unless an additional condition holds.

Proposition 23 *Let $\{F_i, i \in I\}$ be a finite family of M-decomposable sets in $\mathbb{R}^n$ satisfying the following condition: if $z_i \in 0^+F_i$ for all $i \in I$ and $\sum_{i \in I} z_i = 0_n$ then $z_i \in \operatorname{lin} F_i$ for all $i \in I$. Then $\sum_{i \in I} F_i$ is M-decomposable.*

Proof. Let $\{C_i, i \in I\}$ be a family of compact convex sets such that $F_i = C_i + 0^+F_i$ for all $i \in I$. Then $\sum_{i \in I} F_i = \sum_{i \in I} C_i + \sum_{i \in I} 0^+F$, where $\sum_{i \in I} C_i$ is a compact convex set whereas $\sum_{i \in I} 0^+F$ is a closed convex cone by [17, Corollary 9.1.3]. □

Finally, the intersection of M-decomposable sets is not necessarily M-decomposable, as the next example shows.

Example 24 *Let $F = \{x \in \mathbb{R}^3 : x_3^2 \geq x_1^2 + x_2^2, x_3 \geq 0\}$ (the ice-cream cone in $\mathbb{R}^3$) and $F_i = F + \left(0, (-1)^{i+1}, 0\right)$, i.e.,*

$$F_i = \left\{x \in \mathbb{R}^3 : x_3^2 \geq x_1^2 + \left(x_2 + (-1)^i\right)^2, x_3 \geq 0\right\},$$

with $0^+F_i = F$, $i = 1, 2$. The intersection of both M-decomposable sets, $F_1 \cap F_2$, has extreme points (e.g., $(0, 0, 1)$) and its recession cone is $0^+(F_1 \cap F_2) = F$. In fact, $F_1 \cap F_2$ is the epigraph of the convex function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$f(x_1, x_2) = \begin{cases} \sqrt{x_1^2 + (x_2 - 1)^2}, & \text{if } x_2 < 0 \\ \sqrt{x_1^2 + (x_2 + 1)^2}, & \text{if } x_2 \geq 0. \end{cases}$$

On the other hand, given $z \in \operatorname{bd}(F_1 \cap F_2)$ (i.e., the graph of f), $z - F$ is the hypograph of the concave function $g_z : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

$$g_z(x_1, x_2) = f(z_1, z_2) - \sqrt{(x_1 - z_1)^2 + (x_2 - z_2)^2}.$$

Observe that $g_z(x_1, x_2) \leq f(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$ and the equality holds at the points (z_1, z_2) and

$$(\widetilde{z}_1, \widetilde{z}_2) := \begin{cases} \left(\frac{z_1}{1 - z_2}, 0\right), & \text{if } z_2 < 0 \\ \left(\frac{z_1}{1 + z_2}, 0\right), & \text{if } z_2 \geq 0. \end{cases}$$

Defining $\widetilde{z} := (\widetilde{z}_1, \widetilde{z}_2, f(\widetilde{z}_1, \widetilde{z}_2))$, we have

$$(z - F) \cap (F_1 \cap F_2) = \begin{cases} [\widetilde{z}, z], & \text{if } z_2 \neq 0 \\ \{z\}, & \text{if } z_2 = 0, \end{cases}$$

and, by Theorem 19, we get

$$M(F_1 \cap F_2) = \left\{x \in \mathbb{R}^3 : x_3^2 = x_1^2 + 1, x_2 = 0, x_3 \geq 0\right\},$$

which is unbounded. Hence $F_1 \cap F_2$ is not M-decomposable.

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