Physica Scripta, Vol. 79, Nº 1, 015009 (2009).

Higher-order analytical approximate solutions to the nonlinear pendulum by He's homotopy method

A. Beléndez, C. Pascual, M. L. Álvarez, D. I. Méndez, M. S. Yebra and A. Hernández

Departamento de Física, Ingeniería de Sistemas y Teoría de la Señal.

Universidad de Alicante. Apartado 99. E-03080 Alicante. SPAIN

E-mail: a.belendez@ua.es

Corresponding author: A. Beléndez

Phone: +34-96-5903651

Fax: +34-96-5909750

ABSTRACT

A modified He's homotopy perturbation method is used to calculate the periodic solutions of a nonlinear pendulum. The method has been modified by truncating the infinite series corresponding to the first-order approximate solution and substituting a finite number of terms in the second order linear differential equation. As can be seen, the modified homotopy perturbation method works very well for high values of the initial amplitude. Excellent agreement of the analytical approximate period with the exact period has been demonstrated not only for small but also for large amplitudes A (the relative error is less than 1% for $A < 152º$). Comparison of the result obtained using this method with the exact ones reveals that this modified method is very effective and convenient.

Keywords: Nonlinear oscillator; Approximate solutions; Homotopy perturbation method.
PACS: 02.30.Jr, 02.30.Lt, 02.20.-f.

1. Introduction

In all areas of physics and engineering and, in general, in most of real world applications, there are some simple systems for which the equations governing their behaviour are easy to formulate but whose mathematical resolution is complicated [1, 2]. This is because, in most situations, these systems are governed by nonlinear equations forming a nonlinear system. Problems of nonlinear oscillations in conservative systems have a long history and of all such systems, perhaps the paradigm that is usually considered is the simple pendulum [2-4]. Application of Newton's second law to this physical system gives a differential equation with a non-linear term (the sine of an angle). It is possible to find the integral expression for the period of the pendulum and to express it in terms of elliptic functions and to find the exact solution in terms of the Jacobi elliptic function $sn(u;m)$ [4]. Although it is possible in many cases to replace the non-linear differential equation by a corresponding linear differential equation that approximates the original equation, such linearization is not always feasible. In such cases, the actual non-linear differential equation must be directly dealt with.

It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem [2]. There are several methods used to find approximate solutions to nonlinear problems, such as perturbation techniques [2, 5-10], variational approaches [11-20], decomposition [20-22], parameter expansion [23], exp-function [24-26] or harmonic balance based methods [2, 27-30]. Most of these techniques have been used to obtain analytical approximate solutions for the nonlinear pendulum [9, 27, 31-33]. Excellent reviews on some asymptotic methods for strongly nonlinear equations can be found in detail in Refs. [31] and [34]. In general, given the nature of a nonlinear phenomenon, the approximate methods can only be applied within certain ranges of the physical parameters and to certain classes of problems.

In the present paper we obtain an approximate expression for the periodic solutions of a nonlinear pendulum by means of a modified perturbation technique, the so-called He's homotopy perturbation method [31, 33-49]. As Lima [50] pointed out, the simple pendulum oscillatory motion is among the most investigated motion in physics and many nonlinear phenomena in many fields of science and technology are governed by pendulum-like differential equations.

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy concept as used in topology. In the homotopy perturbation method, which requires neither a small parameter nor a linear term in a differential equation, a homotopy with an imbedding parameter $p \in [0,1]$ is constructed. In Ref. [31] a basic idea of homotopy perturbation method for solving non-linear differential equations is presented while in Appendix A, we briefly described the basis of the method when it is applied to a nonlinear oscillator. The homotopy perturbation method has been applied to obtain analytical approximate solutions for different nonlinear oscillators. However, the first-order approximation is usually analyzed and only for nonlinear oscillators for which the restoring force has a polynomial form higher-order approximations have been derived [35, 51]. This is due to the fact that very complex equations appear when the method is applied to obtain higher-order approximations to nonlinear oscillators for which the restoring force has not a polynomial form. Beléndez et al [41] proposed a modification in the method that allows us to obtain easier this higher-order analytical approximation. This approach consists of the He's homotopy perturbation method with the extra simplification of considering a finite number of terms (N) in the first and second order approximate solution. For simplicity and convenience we choose $N = 1$ for the first order, $N = 2$ for the second order approximate solutions and so on. In this paper this modified method is applied to obtain approximate analytical solutions for nonlinear phenomena governed by pendulum-like differential equations. The results presented in this paper reveal that the method is very effective and convenient for conservative nonlinear oscillators for which the restoring force has not a polynomial form.

2. Solution procedure

The non-dimensional differential equation describing the free, undamped simple pendulum is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \sin x = 0 \tag{1}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \tag{2}$$

The periodic solution $x(t)$ of Eq. (1) and the period T depend on the amplitude A. For small x, the equation of motion approximates that of a linear oscillator

$$\frac{d^2x}{dt^2} + x = 0 \tag{3}$$

whose solution is $x(t) = A\sin t$ and then the dimensionless period is $T = 1$ for small A.

Eq. (1) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. There exists no small parameter in Eq. (1), so the standard perturbation methods cannot be applied directly. Due to the fact that the homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation [31, 51], one possibility to approximately solve Eq. (1) is by means the homotopy perturbation method. Eq. (1) can be re-written in the form

$$\frac{d^2x}{dt^2} + x = x - \sin x \tag{4}$$

For Eq. (4) we can establish the following homotopy

$$\frac{d^2x}{dt^2} + x = p(x - \sin x) \tag{5}$$

where p is the homotopy parameter. When p = 0, equation (5) becomes the linearized equation and for the case p = 1, Eq. (5) becomes the original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + \ldots \tag{6}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - \ldots \tag{7}$$

where α_i (i = 1, 2, …) are to be determined.

Substituting Eqs. (6) and (7) into Eq. (5) gives

$$\begin{aligned}&(x_0'' + px_1'' + p^2x_2'' + \ldots) + (\omega^2 - p\alpha_1 - p^2\alpha_2 - \ldots)(x_0 + px_1 + p^2x_2 + \ldots)\\ &\quad = p\left[(x_0 + px_1 + p^2x_2 + \ldots) - \sin(x_0 + px_1 + p^2x_2 + \ldots)\right]\end{aligned} \tag{8}$$

and equating the terms with identical powers of p, we can obtain a series of linear equations, of which we write only the first three

$$x_0'' + \omega^2 x_0 = 0, \quad x_0(0) = A, \quad x_0'(0) = 0 \tag{9}$$

$$x_1'' + \omega^2 x_1 = (1+\alpha_1)x_0 - \sin x_0, \qquad x_1(0) = x_1'(0) = 0 \tag{10}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1+\alpha_1)x_1 - x_1\cos x_0, \quad x_2(0) = x_2'(0) = 0 \tag{11}$$

$$\vdots$$

In Eqs. (9)-(11) we have taken into account the following expression

$$\begin{aligned}&f(x) = f(x_0 + px_1 + p^2x_2 + \ldots) \approx\\ &\approx f(x_0) + px_1 f'(x_0) + p^2\left[x_2 f'(x_0) + \frac{1}{2}x_1^2 f''(x_0)\right] + O(p^3)\end{aligned} \tag{12}$$

where $f'(x) = \mathrm{d}f(x)/\mathrm{d}x$.

The solution of Eq. (9) is

$$x_0(t) = A\cos\omega t \tag{13}$$

Substitution of this result into the right side of Eq. (10), we obtain the following differential equation for x_1

$$x_1'' + \omega^2 x_1 = (1+\alpha_1) A\cos\omega t - \sin(A\cos\omega t) \tag{14}$$

It is possible to do the following Fourier series expansion

$$\sin(A\cos\omega t) = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] = a_1\cos\omega t + a_3\cos 3\omega t + \ldots \tag{15}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} \sin(A\cos\theta)\cos^{2n+1}\theta \mathrm{d}\theta = 2(-1)^n J_{2n+1}(A) \tag{16}$$

where $\theta = \omega t$ and J_{2n+1} is the $(2n+1)$-order Bessel function of the first kind. The first term of this expansion can be obtained by means of the following equation

$$a_1 = 2J_1(A) \tag{17}$$

where $J_1(A)$ is the first order Bessel function of the first kind. Substituting Eq. (17) into Eq. (14) gives

$$x_1'' + \omega^2 x_1 = (1+\alpha_1) A\cos\omega t - a_1\cos\omega t - \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] \tag{18}$$

No secular terms in $x_1(t)$ require eliminating contributions proportional to $\cos\omega t$ on the right-side of Eq. (18)

$$(1+\alpha_1)A - a_1 = 0 \tag{19}$$

Substituting Eq. (17) into Eq. (19) and reordering, we obtain

$$\alpha_1 = \frac{a_1}{A} - 1 = \frac{2J_1(A)}{A} - 1 \tag{20}$$

From Eqs. (7) and (20), writing $p = 1$, we can easily find that the first order approximate frequency is

$$\omega_1(A) = \sqrt{\frac{2J_1(A)}{A}} \tag{21}$$

Now in order to obtain the correction term x_1 for the periodic solution x_0 we consider the following procedure. Taking into account Eqs. (18) and (19), we re-write Eq. (18) in the form

$$x_1'' + \omega^2 x_1 = -\sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \tag{22}$$

with initial conditions $x_1(0) = A$ and $x_1'(0) = 0$. The periodic solution to Eq. (22) can be written as follows

$$x_1(t) = \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n+1)\omega t] \tag{23}$$

Substituting Eq. (23) into Eq. (22) gives

$$-\omega^2 \sum_{n=0}^{\infty} 4n(n+1) b_{2n+1} \cos[(2n+1)\omega t] = -\sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \tag{24}$$

and then we can write the following expression for the coefficients b_{2n+1}

$$b_{2n+1} = \frac{a_{2n+1}}{4n(n+1)\omega^2} = \frac{2(-1)^n J_{2n+1}(A)}{4n(n+1)\omega^2} \tag{25}$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. (23) gives

$$b_1 = -\sum_{n=1}^{\infty} b_{2n+1} \tag{26}$$

To determine the second-order approximate solution it is necessary to substitute Eq. (23) into Eq. (11). Then secular terms are eliminated and parameter α_2 can be calculated. However, it is difficult to solve the new differential equation because, as x_1(t) has an infinite number of harmonics, it would be necessary to multiply this infinite series by $\cos x_0$. At this moment we consider a modification in He's homotopy perturbation method to simplify the solution procedure. $x_1(t)$ has an infinite number of harmonics, however we can truncate the series expansion in Eq. (23) and we can write an approximate equation $x_1^{(N)}(t)$ in the form [41]

$$x_1^{(N)} = \sum_{n=0}^{N} b_{2n+1} \cos[(2n+1)\omega t] \quad \text{and} \quad b_1^{(N)} = -\sum_{n=1}^{N} b_{2n+1} \tag{27}$$

which has only a finite number of harmonics. Comparing Eqs. (23) and (27), it follows that

$$\lim_{N\to\infty} x_1^{(N)}(t) = x_1(t) \tag{28}$$

and

$$\lim_{N\to\infty} b_1^{(N)} = b_1 \tag{29}$$

In the simplest case we can consider $N = 1$ in Eq. (27) and we obtain

$$x_1^{(1)}(t) = b_1^{(1)} \cos\omega t + b_3 \cos 3\omega t = b_3 (\cos 3\omega t - \cos\omega t) \tag{30}$$

where we have taken into account that from Eq. (27) we obtain $b_1^{(1)} = -b_3$. Eq. (30) has a similar form than the second order approximate solution considered in harmonic balance methods. It is possible to do this approximation because the absolute value of the coefficient b_{2n+1} decreases when n increase as we can easily verify from Eqs. (25).

From Eq. (25) the following expression for the coefficient b_3 is obtained

$$b_3 = \frac{a_3}{8\omega^2} \tag{31}$$

where from Eq. (16) we have

$$a_3 = -2J_3(A) \tag{32}$$

Substitution of Eq. (30) into Eq. (11) gives the following equation for $x_2(t)$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1+\alpha_1)x_1^{(1)} - x_1^{(1)} \cos x_0 \tag{33}$$

and taking into account Eqs. (13), (20), (30) and (31), Eq. (33) becomes

$$\begin{aligned} x_2'' + \omega^2 x_2 = \alpha_2 A\cos\omega t + \frac{a_1 a_3}{8A\omega^2}(\cos 3\omega t - \cos\omega t) \\ - \frac{a_3}{8\omega^2}\left[(\cos 3\omega t - \cos\omega t)\cos(A\cos\omega t)\right] \end{aligned} \tag{34}$$

It is possible to do the following Fourier series expansion

$$(\cos 3\omega t - \cos\omega t)\cos(A\cos\omega t) = \sum_{n=0}^{\infty} b_{2n+1}\cos[(2n+1)\omega t] = c_1\cos\omega t + c_3\cos 3\omega t + \ldots \tag{35}$$

where the first term of this expansion can be obtained by means of the following equation

$$c_1 = \frac{4}{\pi}\int_0^{\pi/2}(\cos 3\theta - \cos\theta)\cos(A\cos\theta)\cos\theta \mathrm{d}\theta = -\frac{8}{A^2}[J_2(A) - AJ_3(A)] \tag{36}$$

From Eqs. (34) and (36), the secular term in the solution for $x_2(t)$ can be eliminated if

$$\alpha_2 A - \frac{a_1 a_3}{8A\omega^2} - \frac{a_3 c_1}{8\omega^2} = 0 \tag{37}$$

Eq. (37) can be solved for α_2 and we obtain

$$\alpha_2 = \frac{3J_3(A)}{2A^3\omega^2}\left[AJ_1(A) - 4J_2(A)\right] \tag{38}$$

From Eqs. (7), (20) and (38), and taking $p = 1$, we can easily obtain the following expression for the second order approximate frequency

$$\omega_2(A) = \frac{1}{\sqrt{2}A}\sqrt{2AJ_1(A) + \sqrt{4A^2J_1^2(A) - 2A^2J_1(A)J_3(A) + 8AJ_2(A)J_3(A) - 8A^2J_3^2(A)}} \tag{39}$$

With the requirement of Eq. (37), we can re-write Eq. (34) in the form

$$x_2'' + \omega^2 x_2 = \frac{a_1 a_3 \cos 3\omega t}{8A\omega^2} - \frac{a_3}{8\omega^2}\sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] \tag{40}$$

with initial conditions $x_2(0) = 0$ and $x_2'(0) = 0$. The general solution of this equation is

$$x_2(t) = \sum_{n=0}^{\infty} d_{2n+1}\cos[(2n+1)\omega t] \tag{41}$$

Substituting Eq. (41) into Eq. (40) gives

$$-\omega^2 \sum_{n=0}^{\infty} 4n(n+1) d_{2n+1} \cos[(2n+1)\omega t] = \frac{a_1 a_3 \cos 3\omega t}{8A\omega^2} - \frac{a_3}{8\omega^2} \sum_{n=1}^{\infty} c_{2n+1} \cos[(2n+1)\omega t] \quad (42)$$

and then we can write the following expression for the coefficients d_{2n+1}

$$d_3 = \frac{a_3(c_3 A - a_1)}{64A\omega^2} \quad (43)$$

$$d_{2n+1} = \frac{a_3 c_{2n+1}}{32n(n+1)\omega^4} \quad \text{for} \quad n \geq 2 \quad (44)$$

where ω is given by Eq. (39) and c_3 can be calculated as follows

$$c_3 = \frac{4}{\pi} \int_0^{\pi/2} (\cos 3\theta - \cos\theta) \cos(A\cos\theta) \cos 3\theta \, \mathrm{d}\theta = \frac{8}{A^3} [11AJ_2(A) - (A^2 - 60) J_3(A)] \quad (45)$$

Taking into account that $x_2(0) = 0$, Eq. (41) gives

$$d_1 = -\sum_{n=1}^{\infty} d_{2n+1} \quad (46)$$

and truncating the infinite series in Eq. (41), it is possible to obtain the following second-order approximate solution for x_2

$$x_2^{(N)}(t) = \sum_{n=0}^{N} d_{2n+1} \cos[(2n+1)\omega t] \quad \text{and} \quad d_1^{(N)} = -\sum_{n=1}^{N} d_{2n+1} \quad (47)$$

which has only a finite number of harmonics. Comparing Eqs. (41) and (47), it follows that

$$\lim_{N\to\infty} x_2^{(N)}(t) = x_2(t) \tag{48}$$

and

$$\lim_{N\to\infty} d_1^{(N)} = d_1 \tag{49}$$

As we are analyzing the second-order approximation we consider N = 2 in Eq. (47), in other words, only three harmonics (n = 0, 1, 2). In this situation, it is easy to verify that

$$d_5 = \frac{a_3 c_5}{192\omega^4} \tag{50}$$

where ω is given by Eq. (41) and c_5 can be calculated as follows

$$\begin{aligned} c_5 &= \frac{4}{\pi}\int_0^{\pi/2}(\cos 3\theta - \cos\theta)\cos(A\cos\theta)\cos 3\theta \mathrm{d}\theta = \\ &= -\frac{8}{A^5}[35(A^2-48)J_2(A) - (10080 - 420A^2 + A^4)J_3(A)] \end{aligned} \tag{51}$$

From Eq. (47) we obtain the following value for d_1

$$d_1^{(2)} = -\sum_{n=1}^{2} d_{2n+1} = -d_3 - d_5 \tag{52}$$

Taking this into account, $x_2^{(2)}$ can be written as follows

$$x_2^{(2)}(t) = d_1^{(2)}\cos\omega t + d_3\cos 3\omega t + d_5\cos 5\omega t =$$
$$= d_3(\cos 3\omega t - \cos\omega t) + d_5(\cos 5\omega t - \cos\omega t) \tag{53}$$

From Eqs. (6), (13), (30) and (52), and taking $p = 1$, one can easily obtain the following expression for the second-order approximate solution

$$x_a(t) = x_0(t) + x_1^{(1)}(t) + x_2^{(2)}(t) = (A - b_3 - d_3 - d_5)\cos\omega_2 t + (b_3 + d_3)\cos 3\omega_2 t + d_5\cos 5\omega_2 t =$$
$$= \left(A - \frac{a_3}{8\omega_2^2} - \frac{a_1 a_3}{64 A\omega_2^4} + \frac{a_3 c_3}{64\omega_2^4} - \frac{a_3 c_5}{192\omega_2^4}\right)\cos\omega_2 t + \left(\frac{a_3}{8\omega_2^2} + \frac{a_1 a_3}{64 A\omega_2^4} - \frac{a_3 c_3}{64\omega_2^4}\right)\cos 3\omega_2 t + \frac{a_3 c_5}{192\omega_2^4}\cos 5\omega_2 t \tag{54}$$

which has a similar form to the third-order approximate solution considered in harmonic balance methods.

3. Comparison with the exact solution

We illustrate the accuracy of the modified approach by comparing the approximate solutions previously obtained with the exact period $T_{ex} = 2\pi/\omega_{ex}$ for the nonlinear pendulum. There are some nonlinear oscillatory systems whose nonlinear differential equations have exact solutions [2] such as a particle-in-a-box, the antisymmetric, constant force oscillator, the Duffing oscillator or the nonlinear pendulum. For the nonlinear pendulum it can be obtained that the exact solution of Eq. (1) can be expressed in terms of the Jacobi elliptic function $sn(u;m)$ [2, 4], while the exact period can be expressed in terms of the complete elliptic integral of the first kind [2]. The exact value of the period of oscillations can be obtained integrating Eq. (1). In Ref. [2] we can see as the exact period can be derived and its value is

$$T_{\mathrm{ex}} = T_0\frac{2}{\pi}K(k) \tag{55}$$

where $k = \sin^2(A/2)$ and $K(k)$ is the complete elliptic integral of the first kind defined as follows

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k\sin^2\theta}} \tag{56}$$

For the second order approximation, the relative error between the approximate and the exact values of the period is less than 1% for $A < 152^{\circ}$.

For small values of A it is possible to do the power series expansion of the exact and approximate angular periods $T_{\mathrm{ex}}(A)$ (Eq. (55)) and $T_2(A) = 2\pi/\omega_2(A)$ (Eq. (39)). Doing these expansions, the following equations can be obtained

$$T_{\mathrm{ex}} = 2\pi\left(1 + \frac{1}{16}A^2 + \frac{11}{3072}A^4 + \frac{173}{737280}A^6 + \frac{22931}{1321205760}A^8 + \ldots\right) \tag{57}$$

$$T_2 = 2\pi\left(1 + \frac{1}{16}A^2 + \frac{11}{3072}A^4 + \frac{173}{737280}A^6 + \frac{23898}{1321205760}A^8 + \ldots\right) \tag{58}$$

These series expansions were carried out using MATHEMATICA. As can be seen, in Eq. (58), the first four terms are the same as the first four terms obtained from the expansion of the exact period T_{ex} (Eq. (57)), whereas the fifth term of the expansion of the exact period is $\frac{22931}{1321205760}$ compared with $\frac{23898}{1321205760}$ obtained in our study, that is, the relative error in this term is 4.2%. As we can see the second-order approximate period $T_2(A) = 2\pi/\omega_2(A)$ obtained in this paper provides excellent approximations to the exact period $T_{\mathrm{ex}}(A)$ for high values of the oscillation amplitude A.

The normalized exact periodic solution x_{ex}/A (see Appendix, Eq. (B22))

$$\frac{x_{ex}(t)}{A} = \frac{2}{A}\arcsin\left\{\sin\left(\frac{A}{2}\right)\mathrm{sn}\left[K\left(\sin^2\left(\frac{A}{2}\right)\right) - t\,;\sin^2\left(\frac{A}{2}\right)\right]\right\} \tag{59}$$

and the second-order approximate periodic solution, x_a / A in Eq. (54), are plotted in Figure 1 for A = 140º. In this figure, parameter h is defined as follows

$$h = \frac{t}{T_{ex}(A)} \tag{60}$$

It can be observed that the second-order approximate solution provides excellent approximation to the exact periodic solution.

4. Conclusions

The homotopy perturbation method has been used to obtain the second-order approximate frequency for the nonlinear oscillations of a simple pendulum. Although the lowest order homotopy perturbation method approximation to the nonlinear pendulum [33] is good for high values of oscillation amplitude (Eq. (21)), the second analytical approximation derived here is even better. Excellent agreement between the second-order approximate period, $T_2(A)$, and the exact one has been demonstrated and discussed, and the discrepancy of this second-order approximate period with respect to the exact one is less than 1% for A < 152º. We think that the method has great potential and can be applied to other strongly nonlinear oscillators with non-polynomial terms.

Appendix A [31, 51]

The homotopy perturbation method provides an approach to introduce an expanding parameter in the nonlinear equation governing the behaviour of the oscillatory system. To illustrate the basic ideas of this method, we consider the following nonlinear differential equation for a conservative nonlinear oscillator

$$\frac{d^2x(t)}{dt^2} + f(x(t)) = 0 \quad \text{(A1)}$$

with initial conditions in Eq. (2). In this equation $F(x) = -f(x)$ is the dimensionless restoring force and $f(x)$ is a nonlinear function. Eq. (A1) corresponds to a nonlinear oscillatory system for which there is no linear term and no perturbation parameter exists. In Eq. (A1) the nonlinear function $f(x)$ cannot be treated as a perturbation and thus it is evident that standard perturbation techniques cannot be applied to Eq. (A1). Following the homotopy perturbation method, firstly a linear terms is introduced in Eq. (A1) by rewriting this equation as follows

$$\frac{d^2x}{dt^2} + x = x - f(x) \quad \text{(A2)}$$

and secondly, a "perturbation parameter" p can be introduced in the following manner

$$\frac{d^2x}{dt^2} + x = p\left[x - f(x)\right] \quad \text{(A3)}$$

The embedding parameter p monotonically increases from zero to unit as the trivial problem. For $p = 0$ we obtain

$$\frac{d^2x}{dt^2} + x = 0 \quad \text{(A4)}$$

which is continuously deformed to the original problem, Eq. (A1). So if we can construct an iteration formula for Eq. (A3), the series approximations comes along the solution path, by incrementing the imbedding parameter p from zero to one; this continuously maps the initial solution into the solution of the original Eq. (A1). This is a basic idea of homotopy method, which is to continuously deform a simple problem easy to solve into the difficult problem under study.

The basic assumption is that the solution of Eq. (A3) can be written as a power series in p

$$x = x_0 + x_1 p + x_2 p^2 + \ldots \tag{A5}$$

Setting $p = 1$ results in the approximate solution of Eq. (A1).

Appendix B [2, 4]

In order to obtain the exact solution of Eq. (1), this equation is multiplied by $\mathrm{d}x/\mathrm{d}t$, so that it becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \sin x \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \tag{B1}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \cos x\right] = 0 \tag{B2}$$

From Eqs. (B2) and (2) we can obtain

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 2(\cos x - \cos A) \tag{B3}$$

which can be written as follows

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = 4\left[\sin^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right] \tag{B4}$$

Now let

$$y = \sin^2\left(\frac{x}{2}\right) \tag{B5}$$

and

$$k = \sin^2\left(\frac{A}{2}\right) \tag{B6}$$

From Eqs. (2), (B5) and (B6) we have $y(0) = \sqrt{k}$.

It is easy to obtain the value of $\mathrm{d}x/\mathrm{d}t$ as a function of $\mathrm{d}y/\mathrm{d}t$ as follows. Firstly, from Eq. (B6) we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}\cos\left(\frac{x}{2}\right) \tag{B7}$$

and secondly

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{1}{4}\cos^2\left(\frac{x}{2}\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{1}{4}\left[1 - \sin^2\left(\frac{x}{2}\right)\right]\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{1}{4}\left(1 - y^2\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 \tag{B8}$$

Then, we have

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{4}{1-y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \tag{B9}$$

Substituting Eqs. (B5), (B6) and (B9) into Eq. (B4) gives

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = k(1-y^2)\left(1 - \frac{y^2}{k}\right) \tag{B10}$$

We define a new variable z as follows

$$z = \frac{y}{\sqrt{k}} \tag{B11}$$

then Eq. (B10) becomes

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = (1-z^2)(1-kz^2) \tag{B11}$$

where $0 < k < 1$, and

$$z(0)=1 \qquad \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)_{t=0}=0 \tag{B12}$$

Solving Eq. (B11) for dt gives

$$\mathrm{d}t = \pm\frac{\mathrm{d}z}{\sqrt{(1-\mathrm{z}^2)(1-kz^2)}} \tag{B13}$$

The time τ to go from point (1,0) to the point (z,dz/dt) in the lower half –plane of the graph of dz/dt as a function of z, is

$$t=-\int_1^z \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-ku^2)}} \tag{B14}$$

Eq. (B14) can be re-written as follows

$$t=\int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-ku^2)}}-\int_0^z \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-ku^2)}} \tag{B15}$$

which allows us to obtain t as a function of z and k as

$$t(z)=K(k)-F(\arcsin z;k) \tag{B16}$$

where $K(m)$ and $F(\varphi;m)$ are the complete and the incomplete elliptical integral of the first kind, defined as follows

$$K(m)=\int_0^1 \frac{\mathrm{d}u}{\sqrt{(1-u^2)(1-mu^2)}} \tag{B17}$$

$$F(\varphi;m)=\int_0^\varphi \frac{\mathrm{d}z}{\sqrt{(1-u^2)(1-mu^2)}} \tag{B18}$$

and $z = \sin\varphi$.

Eq. (B16) can be written as follows

$$F(\arcsin z;k) = K(k) - t \tag{B19}$$

which can be written in terms of the Jacobi elliptic function sn(v;m)

$$z = \mathrm{sn}(K(k) - t;k) \tag{B20}$$

In terms of the original variables Eq. (B20) becomes

$$\sin\left(\frac{x}{2}\right) = \sin\left(\frac{A}{2}\right)\mathrm{sn}\left[K\left(\sin^2\left(\frac{A}{2}\right)\right) - t\,;\sin^2\left(\frac{A}{2}\right)\right] \tag{B21}$$

which allows us to express x as a function of t as follows

$$x(t) = 2\arcsin\left\{\sin\left(\frac{A}{2}\right)\mathrm{sn}\left[K\left(\sin^2\left(\frac{A}{2}\right)\right) - t\,;\sin^2\left(\frac{A}{2}\right)\right]\right\} \tag{B22}$$

Acknowledgements

This work was supported by the “Generalitat Valenciana”, Spain, under project GVPRE/ 2008/182.

References

[1] He J H 2006, *Non-perturbative methods for strongly nonlinear problems* (dissertation.de-Verlag im Internet GmbH, Berlin).

[2] Mickens R E 1996, *Oscillations in Planar Dynamics Systems* (World Scientific: Singapore).

[3] Baker G L and Blackburn J A 2005 *The Pendulum: A Case Study in Physics* (Oxford University Press: Oxford)

[4] Beléndez A, Pascual C, Méndez D I, Beléndez T and Neipp C 2007 Exact solution for the nonlinear pendulum *Rev. Brasileira Ens. Fisica* **29**, 645-8

[5] Amore P, Raya A and Fernández F M 2005 Comparison of alternative improved perturbative methods for nonlinear oscillations *Phys. Lett. A* **340**, 201-8

[6] He J H 2000 A new perturbation technique which is also valid for large parameters *J. Sound Vib.* **229**, 1257-63

[7] He J H 2001 Modified Lindstedt-Poincare methods for some non-linear oscillations. Part III: double series expansion", *Int. J. Non-linear Sci. Numer. Simul.* **2**, 317-20

[8] He J H 2002 Modified Lindstedt-Poincare methods for some non-linear oscillations. Part I: expansion of a constant *Int. J. Non-linear Mech.* **37**, 309-14

[9] Amore P and Aranda A 2005 Improved Lindstedt-Poincaré method for the solution of nonlinear problems *J. Sound. Vib.* **283**, 1115-36

[10] Demiray H 2007 The modified reductive perturbation method as applied to the Boussinesq equation *Z. Naturforsch.* **632**, 347-52

[11] Assas L M B 2007 Approximate solutions for the generalized KdV–Burgers' equation by He's variational iteration method *Phys. Scr.* **76**, 161-4

[12] Shou D H 2008 Variational approach to the nonlinear oscillator of a mass attached to a stretched wire *Phys. Scr.* **77**, 045006.

[13] He J H 2007 The variational iteration method for eight-order initial-boundary value problems *Phys. Scr.* **76**, 680-2

[14] He J H 2007 Variational approach for nonlinear oscillators *Chaos, Solitons & Fractals* **34**, 1430-9

[15] Tao Z L 2008 The frequency-amplitude relationship for some nonlinear oscillators with discontinuity by He's variational method *Phys. Scr.* **78**, 015004

[16] Dehghan M and Tatari M 2007 The use of He's variational iteration method for solving multipoint boundary value problems *Phys. Scr.* **72**, 672-6

[17] Abbasbandy S and Shivanian E 2008 Application of the variational iteration method to nonlinear Volterra's integro-differential equations *Z. Naturforsch.* **63a**, 538-42

[18] Bathia B, Noorani M S M, Hashim I and Ismail E S 2007 The multistage variational iteration method for a class of nonlinear system of ODEs *Phys. Scr.* **76** 388-92

[19] Dehghan M and Tatari M 2006 The use of He's variational iteration method for solving a Fokker-Planck equation *Phys. Scr.* **74**, 310-6

[20] Inc M and Cavlak E 2008 On numerical solutions of a new coupled MKdV system by using the Adomian decomposition method and He's variational iteration method *Phys. Scr.* **78**, 045008

[21] Inc M 2007 A reliable treatment for solving nonlinear two-point boundary value problems *Z. Naturforsch.* **62a**, 483-9

[22] Ray S S 2007 Exact solutions for time-fractional diffusion-wave equations by decomposition method *Phys. Scr.* **75**, 53-61

[23] Zhang L-N and Xu L 2007 Determination of the limit cycle by He's parameter-expansion for oscillators in a $u^3/(1+u^2)$ potential *Z. Naturforsch.* **62a**, 396-8

[24] Dai S Q and Wang Y Y 2008 Exact travelling wave solutions of the discrete nonlinear Schrödinger equation and the hybrid lattice equation obtained via the exp-function method *Phys. Scr.* **78** 01013

[25] Xu F 2007 A generalized soliton solution of the Konopelchenko-Dubrovsky equation using He's Exp-function method *Z. Naturforsch.* **62a**, 685-8

[26] Biazar J and Ayati Z *2008* Application of the Exp-function method to the equal-width wave equation *Phys Scr.* **78**, 045005

[27] Beléndez A, Hernández A, Márquez A, Beléndez T and Neipp C 2006 Analytical approximations for the period of a simple pendulum *Eur. J. Phys.* **27**, 539-51

[28] Hu H 2006 Solution of a quadratic nonlinear oscillator by the method of harmonic balance *J. Sound Vib.* **293**, 462-8

[29] Beléndez A, Hernández A, Beléndez T, Alvarez M L, Gallego S, Ortuño M and Neipp C 2007 Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire *J. Sound Vib.* **302**, 1018-29

[30] Beléndez A and Pascual C 2007 Harmonic balance approach to the periodic solutions of the (an)harmonic relativistic oscillator *Phys. Lett. A* **371**, 291-9

[31] He J H 2006 Some asymptotic methods for strongly nonlinear equations *Int. J. Mod. Phys. B*, **20**, 1141-99

[32] Beléndez A, Hernández A, Beléndez T, Márquez A and Neipp C 2007 An improved 'heuristic' approximation for the period of a nonlinear pendulum: linear analysis of a classical nonlinear problem *Int. J. Non-linear Sci. Numer. Simul.* **8** 329-34

[33] Beléndez A, Hernández A, Beléndez T, Neipp C and Márquez A 2007 Application of the homotopy perturbation method to the nonlinear pendulum *Eur. J. Phys.* **28**, 93-104

[34] He J H 2008 An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering *Int. J. Mod. Phys. B* **22**, 3487-578

[35] He J H 2006 New interpretation of homotopy perturbation method *Int. J. Mod. Phys. B* **20**, 2561-8

[36] He J H 2004 The homotopy perturbation method for nonlinear oscillators with discontinuities *Appl. Math. Comp.* **151**, 287-92

[37] Mousa M M and Ragab S F 2008 Application of the homotopy perturbation method to linear and nonlinear Schrödinger equation *Z. Naturforsch.* **63a** 140-4

[38] Beléndez A, Hernández A, Beléndez T, Fernández E, Alvarez M L and Neipp C 2007 Application of He's homotopy perturbation method to the Duffing-harmonic oscillator *Int. J. Non-linear Sci. Numer. Simul.* **8**, 79-88

[39] Roohi E, Marzabadi F R and Farjami Y 2008 Application of the homotopy perturbation method to linear and nonlinear fourth-order boundary value problems *Phys. Scr.* **77**, 055004

[40] He J H 2006 Homotopy perturbation method for solving boundary value problems *Phys. Lett. A* **350**, 87-8

[41] Beléndez A, Pascual C, Fernández E, Neipp C and Beléndez T 2008 Higher-order approximate solutions to the relativistic and Duffing-harmonic oscillators by modified He's homotopy methods *Phys. Scr.* **77**, 025004

[42] Özis T and Yildirim A 2007 A comparative Study of He's homotopy perturbation method for determining frequency-amplitude relation of a nonlinear oscillator with discontinuities *Int. J. Non-linear Sci. Numer. Simul.* **8**, 243-8

[43] Gorji M, Ganji D D and Soleimani S 2007 New application of He's homotopy perturbation method *Int. J. Non-linear Sci. Numer. Simul.* **8**, 319-28

[44] Beléndez A, Pascual C, Méndez D I and Márquez A 2007 Application of He's homotopy perturbation method to the relativistic (an)harmonic oscillator. I: Comparison between approximate and exact frequencies *Int. J. Nonlin. Sci. Num. Simul.* **8**, 483-91

[45] Shakeri F and Dehghan M 2007 Inverse problem of diffusion by He's homotopy perturbation method *Phys. Scr.* **75**, 551-6

[46] Dehghan M and Shakeri F 2007 Solution of a partial differential equation subject to temperature overspecification by He's homotopy perturbation method *Phys. Scr.* **75**, 778-87

[47] Biazar J, Eslami M and Ghazvini H 2007 Homotopy perturbation method for systems of partial differential equations *Int. J. Non-linear Sci. Numer. Simul.* **8**, 413-8

[48] Sadighi A and Ganji D D 2007 Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation method *Int. J. Non-linear Sci. Numer. Simul.* **8**, 435-44

[49] Rafiq A, Ahmed M and Hussain S 2008 A general approach to specific second order ordinary differential equations using homotopy perturbation method *Phys. Lett. A* **372**, 4973-6

[50] Lima F M S 2008 Simple 'log formulae' for pendulum motion valid for any amplitude *Eur. J. Phys.* **29**, 1091-8

[51] Beléndez A, Beléndez T, Márquez A and Neipp C 2008 Application of He's homotopy perturbation method to conservative truly nonlinear oscillators *Chaos, Solitions & Fractals* **37**, 770-80

FIGURE CAPTIONS

Figure 1.- Comparison of normalized approximate periodic solution x_a/A (circles and dashed line) with the normalized exact solution x_{ex}/A (continuous line) for an oscillation amplitude A = 140º.

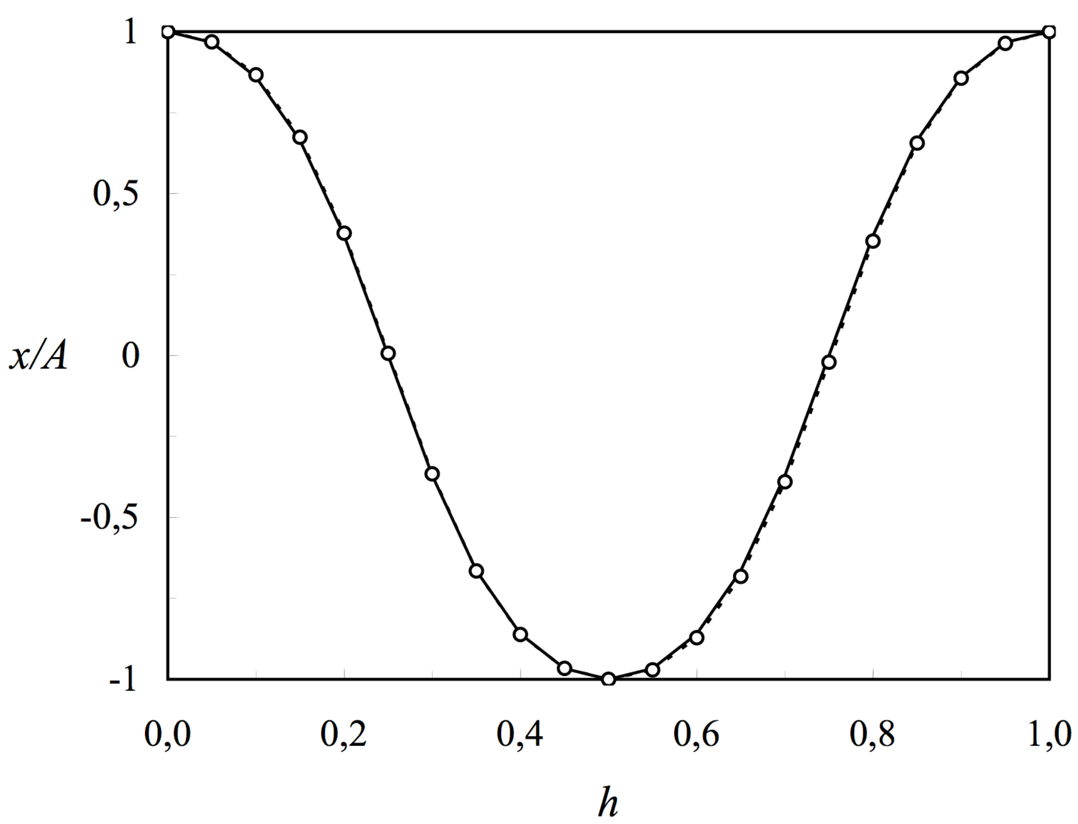

FIGURE 1