

ILL-POSEDNESS IN CONTINUOUS LINEAR OPTIMIZATION
VIA PARTITIONS OF THE SPACE OF PARAMETERS

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Abstract

In this paper we consider the parameter space of continuous linear optimization problems with a given decision space and a given index set. We consider different partitions of this space, on the basis of the primal, the dual, and the primal-dual status of each parameter. We define ill-posedness and relative ill-posedness w.r.t. a given set and absolute ill-posedness w.r.t. a given family of sets. These concepts are characterized for the elements of the partitions considered in this paper.

1. Introduction. Let $n \in \mathbb{N}$, $n \geq 2$, and a non-empty compact Hausdorff topological space T be given. We associate with each triple

$$\pi := (a, b, c) \in \Pi := \mathcal{C}(T)^n \times \mathcal{C}(T) \times \mathbb{R}^n$$

a primal optimization problem called *primal*,

$$\begin{aligned} P : \quad & \text{Min} \quad c'x \\ & \text{s.t.} \quad a'_t x \geq b_t, \quad t \in T, \end{aligned}$$

with space of variables \mathbb{R}^n , and its (Haar's) *dual* problem

$$\begin{aligned} D : \quad & \text{Max} \quad \sum_{t \in T} \lambda_t b_t \\ & \text{s.t.} \quad \sum_{t \in T} \lambda_t a_t = c, \\ & \quad \lambda_t \geq 0, \quad t \in T, \end{aligned}$$

whose space of variables is the linear space of all functions $\lambda : T \mapsto \mathbb{R}$ such that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices. If $|T| < \infty$ and we consider T

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equipped with the discrete topology, P and D are ordinary linear programming (LP) problems. Otherwise, P and D are continuous linear semi-infinite programming (LSIP) problems as far as either the number of variables or the number of constraints (but not both) is infinite. Interesting applications of continuous LSIP can be found in ([7], Chapters 1–2), [6], and the references therein. Π is called the *parameters space*, and it can be the result of all possible perturbations performed on a given continuous problem provided the structure of the problem is preserved. In particular, $\Pi := \mathbb{R}^{n|T|+|T|+n}$ when T is finite.

We denote by Π_C^P (Π_I^P) the set of parameters providing a consistent (inconsistent, respectively) primal problem. $\{\Pi_C^P, \Pi_I^P\}$ is the *binary primal partition* of Π . Replacing “primal” by “dual” we get the *binary dual partition* of Π , $\{\Pi_C^D, \Pi_I^D\}$. The *binary primal-dual partition* of Π is formed by the four crossed intersections $\Pi_{CC} := \Pi_C^P \cap \Pi_C^D$, $\Pi_{CI} := \Pi_C^P \cap \Pi_I^D$, $\Pi_{IC} := \Pi_I^P \cap \Pi_C^D$, and $\Pi_{II} := \Pi_I^P \cap \Pi_I^D$.

We denote by $v^P(\pi)$ ($v^D(\pi)$) the optimal value of P (D), defining as usual $v^P(\pi) = +\infty$ ($v^D(\pi) = -\infty$, respectively) when the corresponding problem is inconsistent. A problem is bounded when its optimal value is a real number. Given $\pi := (a, b, c) \in \Pi$, since P can be either inconsistent (IC) or bounded (B) or unbounded (UB), we can classify π in one of the elements of the *ternary primal partition* $\{\Pi_I^P, \Pi_B^P, \Pi_U^P\}$. Similarly, π can be classified in one of the elements of the *ternary dual partition* $\{\Pi_I^D, \Pi_B^D, \Pi_U^D\}$. The *ternary primal-dual partition* is formed by the non-empty pairwise intersections of the elements of the primal and the dual partitions. The elements of the ternary primal-dual partition are codified as shown in Table 1, where the set in each cell is the intersection of the entries of its column and its row.

T a b l e 1

Π	Π_I^P	Π_B^P	Π_U^P
Π_I^D	Π_{II}	Π_{BI}	Π_{UI}
Π_B^D	Π_{IB}	Π_{BB}	
Π_U^D	Π_{IU}		

The null element of Π belongs to $\Pi_{BB} := \Pi_B^P \cap \Pi_B^D$, the set of parameters with bounded associated problems. Each element of the primal-dual partition corresponds to a duality state ([1] and [10] have analysed the role played by the duality states in LP and LSIP).

We consider Π as a metric space equipped with the following distance: given two parameters $\pi^1 = (a^1, b^1, c^1)$ and $\pi^2 = (a^2, b^2, c^2)$,

$$(1) \quad d(\pi^1, \pi^2) := \max \left\{ \|c^1 - c^2\|_\infty, \max_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\|_\infty \right\}.$$

In fact, it can be easily seen that Π is also a Banach space with the usual sup norm. Throughout the paper the elements of Π will be distinguished by means of upperscripts, and the same (either as subscripts or as superscripts) applies for their corresponding objects: $\pi^r = (a^r, b^r, c^r)$, D_r , P_r , and so on.

In general LSIP, the functions $a : T \mapsto \mathbb{R}^n$ and $b : T \mapsto \mathbb{R}$ are not necessarily continuous, so that the space of parameters is $(\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$. Replacing “max” with

“sup” in (1), we get a pseudometric, which generates the topology of uniform convergence in this space and for which our parameters space Π is a topological subspace.

Let us observe that the recent paper [4] provides characterizations of the interior, the boundary and the exterior of the sets Π_C^D and Π_{BB} in general LSIP. Obviously, these characterizations become sufficient conditions in the context of continuous LSIP.

In this paper we consider three different types of ill-posed parameters, two of them relative to a given subset of Π and the third one relative to a family of subsets of Π . The individual sets will be the elements of the binary and the ternary partitions defined above whereas the families will be the stable elements in each partition (in the sense that they have non-empty interior). The way to do this consists of characterizing the elements of these partitions together with their respective interiors.

2. Analysing the partitions. Given a non-empty set $X \subset \mathbb{R}^p$, $\text{cone } X$ denotes the conical convex hull of $X \cup \{0_p\}$ (0_p denotes here the zero vector of \mathbb{R}^p). From the topological side, if X is a subset of any topological space, $\text{int } X$, $\text{cl } X$ and $\text{bd } X$ represent the interior, the closure and the boundary of X , respectively.

We associate with $\pi = (a, b, c)$ the *first moment cone* of π , $M := \text{cone}\{a_t, t \in T\}$, and the *characteristic cone*, $K := \text{cone}\{(a_t, b_t), t \in T; (0_n, -1)\}$. If π satisfies the *Slater condition* (SC in brief), i.e., there exists $\bar{x} \in \mathbb{R}^n$ such that $a'_t \bar{x} > b_t$ for all $t \in T$, then K is closed.

The next result summarizes well-known results on the primal and dual binary partitions ($\{\Pi_C^P, \Pi_I^P\}$ and $\{\Pi_C^D, \Pi_I^D\}$). The proof can be found in ([8], Theorem 3.1), [7] and ([9], Theorem 5).

Lemma 1. The elements of the binary primal and dual partitions are neither open nor closed. Moreover, the following statements are true:

- (i) $\pi \in \Pi_C^P$ if and only if $(0_n, 1) \notin \text{cl } K$. In particular, $\pi \in \text{int } \Pi_C^P$ if and only if P satisfies SC.
- (ii) $\pi \in \Pi_I^P$ if and only if $(0_n, 1) \in \text{cl } K$. In particular, $\pi \in \text{int } \Pi_I^P$ if and only if $(0_n, 1) \in \text{int } K$.
- (iii) $\pi \in \Pi_C^D$ if and only if $c \in M$. In particular, $\pi \in \text{int } \Pi_C^D$ if and only if $c \in \text{int } M$.
- (iv) $\pi \in \Pi_I^D$ if and only if $c \notin M$. In particular, $\pi \in \text{int } \Pi_I^D$ if and only if there exists $y \in \mathbb{R}^n$ such that $c'y < 0$ and $a'_t y > 0$ for all $t \in T$.

From Lemma 1 it is possible to characterize the elements of the binary primal-dual partition and their corresponding topological interiors.

Theorem 2. The elements of the binary primal-dual partition are neither open nor closed and satisfy the following statements:

- (i) $\pi \in \Pi_{CC}$ if and only if $(0_n, 1) \notin \text{cl } K$ and $c \in M$. In particular, $\pi \in \text{int } \Pi_{CC}$ if and only if SC holds and $c \in \text{int } M$. Moreover, $\text{int } \Pi_{CC}$ is dense in Π_{CC} if and only if $|T| \geq n$.
- (ii) $\pi \in \Pi_{CI}$ if and only if $(0_n, 1) \notin \text{cl } K$ and $c \notin M$. In particular, $\pi \in \text{int } \Pi_{CI}$ if and only if SC holds and there exists $y \in \mathbb{R}^n$ such that $c'y < 0$ and $a'_t y > 0$ for all $t \in T$. Moreover, $\text{int } \Pi_{CI}$ is dense in Π_{CI} .
- (iii) $\pi \in \Pi_{IC}$ if and only if $(0_n, 1) \in \text{cl } K$ and $c \in M$. In particular, $\pi \in \text{int } \Pi_{IC}$ if and only if $(0_n, 1) \in \text{int } K$ and $c \in \text{int } M$. Moreover, $\text{int } \Pi_{IC}$ is dense in Π_{IC} if and only if $|T| \geq n + 1$.

(iv) $\pi \in \Pi_{II}$ if and only if $(0_n, 1) \in \text{cl } K$ and $c \notin M$. Moreover, $\text{int } \Pi_{II} = \emptyset$.

The next theorem constitutes the counterpart of Theorem 2 for the ternary primal-dual partition (statements (i), because $\Pi_{BB} = \Pi_{CC}$, and (iv) are obviously redundant). Its proof is based on the following lemma.

Lemma 3. The following statements are true:

- (i) $\pi \in \Pi_B^P$ if and only if $(0_n, 1) \notin \text{cl } K$ and $(\{c\} \times \mathbb{R}) \cap \text{cl } K \neq \emptyset$. Moreover, $\text{int } \Pi_B^P = \text{int } \Pi_{BB}$.
- (ii) $\pi \in \Pi_U^P$ if and only if $(0_n, 1) \notin \text{cl } K$ and $(\{c\} \times \mathbb{R}) \cap \text{cl } K = \emptyset$. Moreover, $\text{int } \Pi_U^P = \text{int } \Pi_{UI}$.
- (iii) $\pi \in \Pi_B^D$ if and only if $c \in M$ and $\{c\} \times \mathbb{R} \not\subseteq K$. Moreover, $\text{int } \Pi_B^D = \text{int } \Pi_{BB}$.
- (iv) $\pi \in \Pi_U^D$ if and only if $c \in M$ and $\{c\} \times \mathbb{R} \subset K$. Moreover, $\text{int } \Pi_U^D = \text{int } \Pi_{IU}$.

Theorem 4. The elements of the ternary primal-dual partition are neither open nor closed and satisfy the following statements:

- (i) $\pi \in \Pi_{BB}$ if and only if $(0_n, 1) \notin \text{cl } K$ and $c \in M$. In particular, $\pi \in \text{int } \Pi_{BB}$ if and only if SC holds and $c \in \text{int } M$. Moreover, $\text{int } \Pi_{BB}$ is dense in Π_{BB} if and only if $|T| \geq n$.
- (ii) $\pi \in \Pi_{UI}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \notin M$ and $(\{c\} \times \mathbb{R}) \cap \text{cl } K = \emptyset$. In particular, $\pi \in \text{int } \Pi_{UI}$ if and only if there exists $y \in \mathbb{R}^n$ such that $c'y < 0$ and $a'_t y > 0$ for all $t \in T$. Moreover, $\text{int } \Pi_{UI}$ is dense in Π_{UI} .
- (iii) $\pi \in \Pi_{IU}$ if and only if $(0_n, 1) \in \text{cl } K$, $c \in M$ and $\{c\} \times \mathbb{R} \subset K$. In particular, $\pi \in \text{int } \Pi_{IU}$ if and only if $(0_n, 1) \in \text{int } K$. Moreover, $\text{int } \Pi_{IU}$ is dense in Π_{IU} if and only if $|T| \geq n + 1$.
- (iv) $\pi \in \Pi_{II}$ if and only if $(0_n, 1) \in \text{cl } K$ and $c \notin M$. Moreover, $\text{int } \Pi_{II} = \emptyset$.
- (v) $\pi \in \Pi_{BI}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \notin M$ and $(\{c\} \times \mathbb{R}) \cap \text{cl } K \neq \emptyset$. In particular, $\Pi_{BI} = \emptyset$ if $|T| < \infty$. Moreover, $\text{int } \Pi_{BI} = \emptyset$.
- (vi) $\pi \in \Pi_{IB}$ if and only if $(0_n, 1) \in \text{cl } K$, $c \in M$ and $\{c\} \times \mathbb{R} \not\subseteq K$. In particular, $\Pi_{IB} = \emptyset$ if $|T| < \infty$. Moreover, $\text{int } \Pi_{IB} = \emptyset$.
- (vii) $\bigcup_{i \in \{BB, UI, IU\}} \text{int } \Pi_i$ is a dense subset of Π .

From Theorem 4 it can be shown that

$$(2) \quad \bigcup_{i \in \{I, B, U\}} \text{int } \Pi_i^P = \bigcup_{i \in \{I, B, U\}} \text{int } \Pi_i^D = \bigcup_{i \in \{BB, UI, IU\}} \text{int } \Pi_i.$$

Example 5. Let $n = 2$, $T = \{1, 2, 3\}$ and $\pi := (a, b, c)$ such that $(a_1, b_1) = (0, 0, -1)$, $(a_2, b_2) = (1, 1, -1)$, $(a_3, b_3) = (-1, 1, -1)$, and $c = (0, -1)$. By Theorem 4(ii), $\pi \in \Pi_{UI}$. Now consider the sequence $\{\pi^r\}_{r=1}^\infty \subset \Pi$, where π^r is obtained from π by replacing $a_1 = (0, 0)$ with $a_1^r = (0, -\frac{1}{r})$. It is easy to see, applying Theorem 4(i), that $\pi^r \in \Pi_{BB} \forall r \in \mathbb{N}$ and $\lim_{r \rightarrow \infty} \pi^r = \pi$. Thus, $\pi \in \Pi_{UI} \cap \text{bd} \Pi_{BB}$. Hence,

$$(\text{int} \Pi_{BB}) \cup (\text{int} \Pi_{UI}) \not\subseteq \text{int}(\Pi_{BB} \cup \Pi_{UI}).$$

3. Ill-posedness in continuous linear optimization. A mathematical programming problem is called *ill-posed in the feasibility sense* if arbitrarily small perturbations provide both consistent and inconsistent problems ([12], [5] and [2] give formulae for the distance to ill-posedness in ordinary LP, in conic LP, and in general LSIP, respectively). In continuous linear optimization, the set of *well-posed problems in primal (dual) feasibility sense* is then the union of topological interiors $(\text{int} \Pi_C^P) \cup (\text{int} \Pi_I^P)$ ($(\text{int} \Pi_C^D) \cup (\text{int} \Pi_I^D)$, respectively), whereas the set of *ill-posed problems in primal (dual) feasibility sense* is $\text{bd} \Pi_C^P$ ($\text{bd} \Pi_C^D$, respectively). The open sets $\text{int} \Pi_C^P$, $\text{int} \Pi_I^P$, $\text{int} \Pi_C^D$, and $\text{int} \Pi_I^D$ are characterized in Lemma 1. In [3] a general LSIP problem is called *ill-posed in the solvability sense* if arbitrarily small perturbations provide both solvable and non-solvable problems. In fact, it is shown that this set is the boundary of the set of parameters which have a finite primal optimal value. In our context of continuous linear optimization this set is $\text{bd} \Pi_{BB}$. In the same vein, the set of well-posed problems w.r.t. the primal (dual) ternary partition is $\bigcup_{i \in \{I, B, U\}} \text{int} \Pi_i^P$ ($\bigcup_{i \in \{I, B, U\}} \text{int} \Pi_i^D$,

respectively).

On the other hand, [11] defines a conic programming problem to be *ill-posed (in primal-dual feasibility sense)* when it lays on the boundary of the set of consistent problems whose corresponding dual is also consistent. This class of primal-dual ill-posed parameters is, in our setting, $\text{bd} \Pi_{BB}$. The interior of Π_{BB} was characterized in [13]. Now the set of well-posed problems w.r.t. the primal-dual ternary partition is

$$\bigcup_{i \in \{BB, UI, IU\}} \text{int} \Pi_i.$$

First we recall the usual definition of ill-posedness in an arbitrary topological space.

Definition 6. Let X be a topological space and $\emptyset \neq A \subseteq X$. We say that the point $y \in X$ is ill-posed w.r.t. to set A if each neighbourhood of y contains points of A and its complement.

We denote the set of ill-posed points w.r.t. the set A with A^{IP} . Obviously, $A^{IP} = \text{bd} A = (\text{cl} A) \setminus (\text{int} A)$. Moreover, $(X \setminus A)^{IP} = A^{IP}$.

Definition 7. Let X be a topological space and $\emptyset \neq A \subseteq X$. We say that the point $y \in A$ is relatively ill-posed w.r.t. to set A if each neighbourhood of y contains points of the complement of A .

We denote the set of relatively ill-posed points w.r.t. the set A with A^{RIP} . Obviously, $A^{RIP} = A \setminus \text{int} A = A \cap A^{IP}$. This definition makes sense, because f.i. sometimes we need only the ill-posed points which belong to the set A . Observe that A is open (closed) if and only if $A^{RIP} = \emptyset$ ($A^{RIP} = A^{IP}$, respectively).

Definition 8. Let X be a topological space and let $\mathcal{A} = \{A_i, i \in I\}$ be a family of pairwise disjoint subsets of X such that $\text{int } A_i \neq \emptyset \forall i \in I$ and $\bigcup_{i \in I} A_i$ is a dense subset of X . We say that the point $y \in X$ is absolutely ill-posed w.r.t. to \mathcal{A} if $y \in X \setminus \left(\bigcup_{i \in I} \text{int } A_i \right)$.

We denote the set of absolutely ill-posed points w.r.t. \mathcal{A} as \mathcal{A}^{AIP} . In particular, if A is a dense subset of X , $\{A, X \setminus A\}^{AIP} = A^{IP}$.

Let us go back to the linear continuous optimization. The proofs of the next characterizations of the relatively ill-posed problems w.r.t. the elements of the partitions of Π considered in this paper follow from Lemma 3 and Theorems 2 and 4:

$\pi \in (\Pi_C^P)^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$ and SC fails.

$\pi \in (\Pi_I^P)^{RIP}$ if and only if $(0_n, 1) \in \text{bd } K$.

$\pi \in (\Pi_C^D)^{RIP}$ if and only if $c \in M \setminus \text{int } M$.

$\pi \in (\Pi_I^D)^{RIP}$ if and only if $c \notin M$ and $c'y \geq 0$ is a consequence of $\{a'_t y > 0, t \in T\}$.

$\pi \in (\Pi_B^P)^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $(\{c\} \times \mathbb{R}) \cap \text{cl } K \neq \emptyset$ and SC fails if $c \in \text{int } M$.

$\pi \in (\Pi_U^P)^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $(\{c\} \times \mathbb{R}) \cap \text{cl } K = \emptyset$ and $c'y \geq 0$ is a consequence of $\{a'_t y > 0, t \in T\}$.

$\pi \in (\Pi_B^D)^{RIP}$ if and only if $c \in M$, $\{c\} \times \mathbb{R} \not\subset K$ and SC fails if $c \in \text{int } M$.

$\pi \in (\Pi_U^D)^{RIP}$ if and only if $c \in M$, $\{c\} \times \mathbb{R} \subset K$ and $(0_n, 1) \notin \text{int } K$.

$\pi \in \Pi_{CC}^{RIP} = \Pi_{BB}^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \in M$ and SC fails if $c \in \text{int } M$.

$\pi \in \Pi_{CI}^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \notin M$, and SC holds whenever $c'y \geq 0$ is a consequence of $\{a'_t y > 0, t \in T\}$.

$\pi \in \Pi_{IC}^{RIP}$ if and only if $(0_n, 1) \in \text{cl } K$, $c \in M$, and $(0_n, 1) \notin \text{int } K$ whenever $c \in \text{int } M$.

$\pi \in \Pi_{II}^{RIP} = \Pi_{II}$ if and only if $(0_n, 1) \in \text{cl } K$ and $c \notin M$.

$\pi \in \Pi_{BB}^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \in M$ and SC fails if $c \in \text{int } M$.

$\pi \in \Pi_{UI}^{RIP}$ if and only if $(0_n, 1) \notin \text{cl } K$, $c \notin M$, $(\{c\} \times \mathbb{R}) \cap \text{cl } K = \emptyset$ and $c'y \geq 0$ is a consequence of $\{a'_t y > 0, t \in T\}$.

$\pi \in \Pi_{IU}^{RIP}$ if and only if $(0_n, 1) \in \text{bd } K$, $c \in M$, and $\{c\} \times \mathbb{R} \subset K$.

Moreover, we have

$$\{\Pi_C^P, \Pi_I^P\}^{AIP} = (\Pi_C^P)^{IP} = (\Pi_I^P)^{IP} = (\Pi_C^P)^{RIP} \cup (\Pi_I^P)^{RIP}$$

and

$$\{\Pi_C^D, \Pi_I^D\}^{AIP} = (\Pi_C^D)^{IP} = (\Pi_I^D)^{IP} = (\Pi_C^D)^{RIP} \cup (\Pi_I^D)^{RIP}.$$

Finally, taking into account (2) and Theorem 4(vii), we conclude that the set of absolutely ill-posed problems w.r.t. the primal-dual, the primal, and the dual ternary

partitions coincide, and it can be expressed in different ways:

$$(3) \quad \{\Pi_{BB}, \Pi_{UI}, \Pi_{IU}\}^{AIP} = \left(\bigcup_{i \in \{BB, UI, IU\}} \text{int } \Pi_i \right)^{IP}$$

$$= \left(\bigcup_{i \in \{II, BI, IB\}} \Pi_i \right) \cup \left(\bigcup_{i \in \{BB, UI, IU\}} \Pi_i^{RIP} \right),$$

$$(4) \quad \{\Pi_I^P, \Pi_B^P, \Pi_U^P\}^{AIP} = \left(\bigcup_{i \in \{I, B, U\}} \text{int } \Pi_i^P \right)^{IP} = \bigcup_{i \in \{I, B, U\}} (\Pi_i^P)^{RIP},$$

and

$$(5) \quad \{\Pi_I^D, \Pi_B^D, \Pi_U^D\}^{AIP} = \left(\bigcup_{i \in \{I, B, U\}} \text{int } \Pi_i^D \right)^{IP} = \bigcup_{i \in \{I, B, U\}} (\Pi_i^D)^{RIP}.$$

Moreover, π is absolutely ill-posed w.r.t. any of the three ternary partitions (i.e., $\pi \in \{\Pi_{BB}, \Pi_{UI}, \Pi_{IU}\}^{AIP}$) if and only if π satisfies:

- 1) SC fails if $c \in \text{int } M$;
- 2) $c'y \geq 0$ is a consequence of $\{a'_t y > 0, t \in T\}$; and
- 3) $(0_n, 1) \notin \text{int } K$.

The parameter π in Example 5 satisfies these conditions, so that it is ill-posed in the sense of (3), (4), and (5). This example shows that

$$\{\Pi_{BB}, \Pi_{UI}, \Pi_{IU}\}^{AIP} \not\supseteq (\Pi_{BB} \cup \Pi_{UI} \cup \Pi_{IU})^{IP}.$$

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