Uniformity and Inexact Version of a Proximal Method for Metrically Regular Mappings

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Abstract. We study stability properties of a proximal point algorithm for solving the inclusion $0 \in T(x)$ when T is a set-valued mapping that is not necessarily monotone. More precisely we show that the convergence of our algorithm is uniform, in the sense that it is stable under small perturbations whenever the set-valued mapping T is metrically regular at a given solution. We present also an inexact proximal point method for strongly metrically subregular mappings and show that it is super-linearly convergent to a solution to the inclusion $0 \in T(x)$.

Key words: proximal point algorithm, set-valued mapping, metric regularity, strong subregularity, strong regularity.

AMS 2000 Subject Classification: 49J53, 49J40, 90C48.

1 Introduction

In [1] we studied the convergence of a general version of the proximal point algorithm for solving the inclusion

$$0 \in T(x), \tag{1.1}$$

where T is a set-valued mapping acting from a Banach space X to the subsets of a Banach space Y. Such an inclusion is an abstract model for a wide variety of variational problems including complementarity problems, systems of non linear equations and variational inequalities. In particular it may characterize optimality or equilibrium. Choosing a sequence

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of Lipschitz continuous functions $g_n : X \to Y$ with $g_n(0) = 0$ we considered the algorithm given by the recursion:

$$0 \in g_n(x_{n+1} - x_n) + T(x_{n+1}) \quad \text{for } n = 0, 1, 2, \ldots \tag{1.2}$$

If $g_n(u) = \lambda_n u$ and then we assume $Y = X$ a Hilbert space, we obtain the classical proximal point algorithm

$$0 \in \lambda_n(x_{n+1} - x_n) + T(x_{n+1}) \quad \text{for } n = 0, 1, 2, \ldots \tag{1.3}$$

Algorithm (1.3) seems to have been applied for the first time to convex optimization by Martinet [14] and has been thoroughly explored in a subsequent paper by Rockafellar [20] in the general framework of maximal monotone inclusions. In particular, Rockafellar (see [20, Theorem 1]) showed that when x_{n+1} is an approximate solution of (1.3) and T is maximal monotone, then for a sequence of positive scalars λ_n the iteration (1.3) generates a sequence x_n which is weakly convergent to a solution to (1.1) for any starting point $x_0 \in X$.

We proved in [1] that if $\bar{x}$ is a solution of (1.1) and the mapping T is *metrically regular* at $\bar{x}$ for 0 and such that its graph is locally closed at $(\bar{x}, 0)$, then, for any sequence of functions g_n that are Lipschitz continuous in a neighborhood O of the origin, the same for all n, and whose Lipschitz constants λ_n have supremum that is bounded by half the reciprocal of the modulus of regularity of T, there exists a neighborhood U of $\bar{x}$ such that for each initial point $x_0 \in U$ one can find a sequence x_n satisfying (1.2) which is linearly convergent to $\bar{x}$ in the norm of X. Moreover, if the functions g_n have their Lipschitz constants λ_n convergent to zero, then, among the sequences obtained by (1.2) with $x_0 \in U$ there exists at least one which is superlinearly convergent to $\bar{x}$. We also studied the convergence of the proximal point algorithm (1.2) under different regularity assumptions for the mapping T, indeed, we considered the cases when T is *strongly metrically subregular* and *strongly metrically regular*. The purpose of this paper was to propose a proximal point method for non-monotone mappings. Other contributions on this topic ([11, 17], *etc.*) have already been cited in [1].

In many situations the mapping T happens to be monotone (as for instance for the subdifferential of a lower semicontinuous convex function). Nevertheless, there is an interest in considering and studying such a method without monotonicity. First, because monotonicity forces us to work with mappings acting between a space and its dual, which usually yields to restrict the algorithm for mappings on a Hilbert space. Metric regularity does not require this, and consequently, allows us to work with mappings acting between two different Banach spaces. Second, because in some cases monotonicity turns out to be rather a strong assumption, excluding several mappings that are metrically regular. As a simple example, consider a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$. From [21, Proposition 12.3], f is monotone if and only if the Jacobian $\nabla f(x)$ is positive-semidefinite at each x. On the other hand, a consequence of the Lyusternik-Graves theorem (see Theorem 2.2, Section 2) is that f is metrically regular at some point $\bar{x}$ if and only if $\nabla f(\bar{x})$ is surjective. Therefore it covers several cases that do not verify the monotonicity condition.

An interesting class of examples is given by the so-called *feasibility problems*: let X and Y be Banach spaces and let $K \subset Y$ be a closed convex cone. Considering a function $f : X \to Y$, a closed convex set $C \subset X$ and a point $y \in Y$, the feasibility problem consists in finding $x \in C$ such that

$$y \in f(x) + K.$$

The latter can be rewritten in the form $y \in F(x)$ where

$$F(x) = \begin{cases} f(x) + K & \text{if } \; x \in C, \\ \emptyset & \text{otherwise.} \end{cases}$$

According to [3], when f is strictly differentiable, the mapping F is metrically regular at $\bar{x}$ for $\bar{y}$ (where $(\bar{x}, \bar{y})$ is in the graph of F) if and only if the mapping L, given by

$$L(x) = \begin{cases} f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + K & \text{if } \; x \in C, \\ \emptyset & \text{otherwise,} \end{cases}$$

is metrically regular at $\bar{x}$ for $\bar{y}$. Since L has closed and convex graph, from the works of Ursescu [24] and Robinson [18] we know that the mapping L is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{y}$ lies in the interior of the range of L. An easy computation shows that this holds true whenever $\bar{y} - f(\bar{x})$ is in the interior of K while the monotonicity of F requires (at least) f to be a monotone mapping.

In this paper, following the work of Dontchev [5], for given $y \in Y$ we study the uniformity of the method

$$y \in g_n(x_{n+1} - x_n) + T(x_{n+1}) \quad \text{for } n = 0, 1, 2, \ldots \tag{1.4}$$

We consider the perturbed inclusion $y \in T(x)$ and we show that the attraction region (*i.e.*, the ball in which the initial guess x_0 can be taken arbitrarily) does not depend on small variations of the perturbation parameter y near 0. More precisely, we prove that whenever T is metrically regular at $\bar{x}$ for 0, where $\bar{x}$ is a solution to the inclusion (1.1), there exist neighborhoods U of $\bar{x}$ and V of 0 such that for every elements $x_0 \in U$ and $y \in V$ there is a sequence x_n generated by (1.4) starting from x_0 and converging to a solution x such that $y \in T(x)$. When T happens to be strongly metrically regular, we obtain the uniqueness of the sequence x_n in U. Such statements enhance the results obtained in [1].

Finally, when T is strongly metrically subregular at $\bar{x}$ for 0 where $\bar{x}$ is a solution to (1.1), we present an inexact proximal point method for solving (1.1) given by the iteration

$$e_n \in g_n(x_{n+1} - x_n) + T(x_{n+1}) \quad \text{for } n = 0, 1, 2, \ldots \tag{1.5}$$

where e_n is a so-called *error sequence*. We show that any sequence generated by (1.5) and whose elements are sufficiently close to $\bar{x}$ converges linearly to $\bar{x}$. We prove that the convergence is actually superlinear whenever the functions g_n are chosen such that the sequence of their Lipschitz constants λ_n converges to 0. Moreover, as in [20, Theorem 2], we show that any bounded sequence generated by (1.5) converges to a solution to (1.1) whenever the mapping T^{-1} is *Lipschitz continuous* at 0.

The content of this paper is as follows. In Section 2, we present some background material on metric regularity. Results on the uniformity of the proximal point method for both metrically regular and strongly metrically regular mappings are developed in Section 3. The last section is devoted to the study of an inexact proximal point algorithm, for strongly metrically subregular mappings, which is proved to converge to a solution to (1.1).

2 Background material

Throughout, X and Y are Banach spaces, let F be a set-valued mapping from X into the subsets of Y, indicated by $F : X \rightrightarrows Y$. Here $\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ is the graph of F. We denote by $d(x, C)$ the distance from a point x to a set C, that is, $d(x, C) = \inf_{y \in C} \|x - y\|$ while $I\!B_r(a)$ denotes the closed ball of radius r centered at a and F^{-1} is the inverse of F defined as $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

Our study is organized around three key notions: metric regularity, strong metric regularity and strong metric subregularity.

Definition 2.1. *A mapping $F : X \rightrightarrows Y$ is said to be metrically regular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exist some positive constants κ, a and b such that*

$$(2.1) \qquad d(x, F^{-1}(y)) \leq \kappa d(y, F(x)) \text{ for all } x \in I\!B_a(\bar{x}),\ y \in I\!B_b(\bar{y}).$$

The infimum of κ for which (2.1) holds is the *regularity modulus* denoted $\operatorname{reg} F(\bar{x}\,|\,\bar{y})$; the case when F is not metrically regular at $\bar{x}$ for $\bar{y}$ corresponds to $\operatorname{reg} F(\bar{x}\,|\,\bar{y}) = \infty$. The inequality (2.1) has direct use in providing an estimate for how far a point x is from being a solution to the variational inclusion $F(x) \ni y$; the expression $d(y, F(x))$ measures the residual when $F(x) \not\ni y$. Smaller values of κ correspond to more favorable behavior. The metric regularity of a mapping F at $\bar{x}$ for $\bar{y}$ is known to be equivalent to the Aubin continuity of the inverse F^{-1} at $\bar{y}$ for $\bar{x}$ (see, *e.g.,* [21]). Recall that a set-valued map $\Gamma : Y \rightrightarrows X$ is Aubin continuous at $(\bar{y}, \bar{x}) \in \operatorname{gph} \Gamma$ (see [2]) if there exist positive constants κ, a and b such that

$$(2.2) \qquad e(\Gamma(y') \cap I\!B_a(\bar{x}), \Gamma(y)) \leq \kappa \|y' - y\| \text{ for all } y, y' \in I\!B_b(\bar{y}),$$

where $e(A, B)$ denotes the excess from a set A to a set B and is defined as $e(A, B) = \sup_{x \in A} d(x, B)$. For more details on metric regularity and applications to variational problems one can refer to [8, 10, 15] and the monographs [16, 21].

An important result in the theory of metric regularity is the Lyusternik-Graves theorem which roughly says that the metric regularity is stable under perturbations of order higher than one. The following statement, which is from [7], is the general form of this theorem. We use the following convention: we say that a set $C \subset X$ is locally closed at $z \in C$ if there exists $a > 0$ such that the set $C \cap I\!B_a(z)$ is closed.

Theorem 2.2. *Let $F : X \rightrightarrows Y$ be a mapping with locally closed graph at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Let F be metrically regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa > 0$. Consider a function $g : X \to Y$ which is Lipschitz continuous at $\bar{x}$ with Lipschitz constant λ such that $\lambda < \kappa^{-1}$. Then the mapping $F + g$ is metrically regular at $\bar{x}$ for $\bar{y} + g(\bar{x})$ with constant $\kappa/(1 - \kappa\lambda)$.*

In order to introduce the next regularity property, we need the notion of graphical localization. A *graphical localization* of a mapping $F : X \rightrightarrows Y$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a mapping $\tilde{F} : X \rightrightarrows Y$ such that $\operatorname{gph} \tilde{F} = (U \times V) \cap \operatorname{gph} F$ for some neighborhood $U \times V$ of $(\bar{x}, \bar{y})$.

Definition 2.3. *A mapping $F : X \rightrightarrows Y$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if the metric regularity condition in Definition 2.1 is satisfied by some κ and some neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that, in addition, the graphical localization of F^{-1} with respect to U and V is single-valued.*

Strong regularity implies metric regularity by definition. Nevertheless, in some cases, metric regularity and strong regularity are equivalent. Indeed, this equivalence holds for mappings of the form of the sum of a smooth function and the normal cone mapping over a polyhedral convex set (see, *e.g.*, [6]). Moreover, for any set-valued mapping that is *locally monotone* near the reference point metric regularity at that point implies, and hence is equivalent to, strong regularity. This is a consequence of a deeper result by Kenderov [12, Proposition 2.6] regarding single-valuedness of lower semicontinuous monotone mappings. Furthermore, see [21, Proposition 12.54], any maximal monotone mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ which is strongly monotone is strongly regular at the unique solution of $T(x) \ni 0$.

Now, we present briefly the last notion of regularity we will consider in this work.

Definition 2.4. *A mapping $F : X \rightrightarrows Y$ is strongly subregular at $\bar{x}$ for $\bar{y}$ if $F(\bar{x}) \ni \bar{y}$ and there exist positive constants κ, a and b such that*

$$\|x - \bar{x}\| \leq \kappa d(\bar{y}, F(x) \cap \mathbb{B}_b(\bar{y})) \quad \textit{for all } x \in \mathbb{B}_a(\bar{x}). \tag{2.3}$$

This property is equivalent to the "local Lipschitz property at a point" of the inverse mapping, a property first formally introduced in [4] where a stability result parallel to the Lyusternik-Graves theorem was proved. According to Robinson [19, Proposition 1], a mapping F acting in finite dimensions whose graph is the union of polyhedral sets is upper-Lipschitz at any point. Therefore F is strongly subregular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{x}$ is an isolated point of $F^{-1}(\bar{y})$.

3 Uniform convergence of the proximal point method

To prove our main result we need the following Lemma.

Lemma 3.1. *Consider a mapping $T : X \rightrightarrows Y$ with locally closed graph at $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ such that T is metrically regular at $\bar{x}$ for $\bar{y}$ with constant $\kappa > 0$. Let $g : X \to Y$ be a function, such that $g(0) = 0$, which is Lipschitz continuous in a neighborhood O of 0 with a Lipschitz constant λ satisfying $\lambda < 1/(3\kappa)$. Let $P_x = [g(\cdot - x) + T(\cdot)]^{-1}$ for $x \in X$. Then there exist* *positive constants α and β such that for every $x \in I\!B_\alpha(\bar{x})$, $y', y'' \in I\!B_\beta(\bar{y})$, $x' \in P_x(y') \cap I\!B_\alpha(\bar{x})$ and $\varepsilon > 0$ there exists $x'' \in P_x(y'')$ such that*

$$\|x' - x''\| \leq (\kappa/(1 - 3\lambda\kappa) + \varepsilon)\|y' - y''\|.$$

Proof. The mapping $g(\cdot - \bar{x})$ is Lipschitz continuous on $O + \bar{x}$. Applying the Lyusternik-Graves theorem (see Theorem 2.2), we obtain that $[P_{\bar{x}}]^{-1}$ is metrically regular at $\bar{x}$ for $\bar{y}$, with constant $\kappa/(1 - \kappa\lambda)$. Therefore there exist positive constants a and b such that $P_{\bar{x}}$ is Aubin continuous at $(\bar{y}, \bar{x})$ with respect to a, b and $M := \kappa/(1 - \kappa\lambda)$.

Make a and b smaller if necessary so that $\operatorname{gph} T$ is closed relative to $I\!B_a(\bar{x}) \times I\!B_{b+\lambda a}(\bar{y})$. Choose $\alpha > 0$ such that $I\!B_{2\alpha}(0) \subset O$. Make α smaller if necessary and take some positive scalars β and γ such that

$$2\lambda(M + \gamma) < 1, \quad 2\alpha \leq a, \quad \beta + \alpha\lambda \leq b, \quad \text{and} \quad 2(M + \gamma)\beta/(1 - 2\lambda(M + \gamma)) \leq \alpha.$$

Let $x \in I\!B_\alpha(\bar{x})$, let $y', y'' \in I\!B_\beta(\bar{y})$ with $y' \neq y''$ (otherwise, there is nothing to prove) and let $x_1 \in P_x(y') \cap I\!B_\alpha(\bar{x})$. Then $y' \in g(x_1 - x) + T(x_1)$ and therefore

$$x_1 \in P_{\bar{x}}(y' - g(x_1 - x) + g(x_1 - \bar{x})) \cap I\!B_a(\bar{x}).$$

Since $\|x - x_1\| \leq \|x - \bar{x}\| + \|\bar{x} - x_1\| \leq 2\alpha$, $x_1 - x \in O$ and we have

$$\begin{aligned} \|y' - g(x_1 - x) + g(x_1 - \bar{x}) - \bar{y}\| &\leq \|y' - \bar{y}\| + \|g(x_1 - x) - g(x_1 - \bar{x})\| \\ &\leq \beta + \lambda\|x - \bar{x}\| \leq \beta + \lambda\alpha \leq b. \end{aligned}$$

Obviously, the same result holds for y''. From the Aubin continuity of $P_{\bar{x}}$,

$$d(x_1, P_{\bar{x}}(y'' - g(x_1 - x) + g(x_1 - \bar{x})) \leq M\|y' - y''\| < (M + \gamma)\|y' - y''\|,$$

and then there exists

$$x_2 \in P_{\bar{x}}(y'' - g(x_1 - x) + g(x_1 - \bar{x}))$$

such that

$$\|x_2 - x_1\| \leq (M + \gamma)\|y' - y''\|.$$

Thus, $\|x_2 - \bar{x}\| \leq \|x_2 - x_1\| + \|x_1 - \bar{x}\| \leq 2\beta(M + \gamma) + \alpha \leq 2\alpha$ and we get $x_2 \in I\!B_a(\bar{x})$.

Proceeding by induction, suppose that we have found $x_2, \ldots, x_n$ for $n \geq 2$ with

$$x_i \in P_{\bar{x}}(y'' - g(x_{i-1} - x) + g(x_{i-1} - \bar{x})) \cap I\!B_a(\bar{x})$$

and

$$\|x_i - x_{i-1}\| \leq (2\lambda)^{i-2}(M + \gamma)^{i-1}\|y' - y''\|,$$

for $i = 2, \ldots, n$. Since

$$\|y'' - g(x_{n-1} - x) + g(x_{n-1} - \bar{x}) - \bar{y}\| \leq \beta + \lambda\|x - \bar{x}\| \leq \beta + \alpha\lambda \leq b$$

and

$$\|y'' - g(x_n - x) + g(x_n - \bar{x}) - \bar{y}\| \leq \beta + \lambda\|x - \bar{x}\| \leq b,$$

from the Aubin continuity of $P_{\bar{x}}$, there exists

$$x_{n+1} \in P_{\bar{x}}(y'' - g(x_n - x) + g(x_n - \bar{x})),$$

with

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (M+\gamma)\| - g(x_n - x) + g(x_n - \bar{x}) + g(x_{n-1} - x) - g(x_{n-1} - \bar{x})\| \\ &\leq (M+\gamma)(\|g(x_n - x) - g(x_{n-1} - x)\| + \|g(x_n - \bar{x}) - g(x_{n-1} - \bar{x})\|) \\ &\leq 2\lambda(M+\gamma)\|x_n - x_{n-1}\| \leq (2\lambda)^{n-1}(M+\gamma)^n\|y' - y''\|.\end{aligned}$$

Since $2\lambda(M+\gamma) < 1$ we have,

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \sum_{j=2}^{n+1} \|x_j - x_{j-1}\| + \|x_1 - \bar{x}\| \\ &\leq (M+\gamma)\|y' - y''\| \sum_{j=2}^{n+1} (2\lambda(M+\gamma))^{j-2} + \alpha \\ &\leq 2(M+\gamma)\beta/(1 - 2\lambda(M+\gamma)) + \alpha \leq 2\alpha \leq a.\end{aligned}$$

Therefore, the induction step is complete and since x_n is a Cauchy sequence there exists $x'' \in I\!B_a(\bar{x})$ such that x_n converges to x''. From

$$x_n \in P_{\bar{x}}(y'' - g(x_{n-1} - x) + g(x_{n-1} - \bar{x})),$$

we get

$$y'' \in g(x_{n-1} - x) - g(x_{n-1} - \bar{x}) + g(x_n - \bar{x}) + T(x_n).$$

Passing to the limit when n goes to ∞ and using the local closedness of T, we obtain

$$y'' \in g(x'' - x) + T(x''),$$

that is, $x'' \in P_x(y'')$. Moreover,

$$\begin{aligned}\|x_1 - x''\| &\leq \limsup_{n\to\infty} \sum_{i=2}^{n} \|x_i - x_{i-1}\| \\ &\leq \limsup_{n\to\infty} \sum_{i=2}^{n} (2\lambda)^{i-2}(M+\gamma)^{i-1}\|y' - y''\| \\ &\leq (M+\gamma)/(1 - 2\lambda(M+\gamma))\|y' - y''\| \\ &= (\kappa + \gamma(1-\lambda\kappa))/(1 - 3\lambda\kappa - 2\lambda\gamma(1-\lambda\kappa))\|y' - y''\|.\end{aligned}$$

An easy computation shows that, whenever γ is sufficiently small, one has

$$(\kappa + \gamma(1-\lambda\kappa))/(1 - 3\lambda\kappa - 2\lambda\gamma(1-\lambda\kappa)) \leq \kappa/(1 - 3\lambda\kappa) + \varepsilon,$$

which completes the proof. $\square$

We will need the following remark to prove our main result.

Remark 1. According to the proof of the Lyusternik-Graves theorem given by Dontchev *et al.* [7, Theorem 3.3], the constants a and b obtained at the very beginning of the proof of Lemma 3.1 only depend on the growth constant κ, the scalar λ and the neighborhoods with respect to which $\operatorname{gph} T$ is locally closed and g is Lipschitz continuous. More precisely, Dontchev *et al.* proved that a can be chosen such that $0 < a < \frac{1}{16}\rho(1-\kappa\lambda)\min\{1,\kappa\}$ and $0 < b < \frac{1}{16\kappa}\rho(1-\kappa\lambda)\min\{1,\kappa\}$ where ρ is such that $\operatorname{gph} T$ is closed relative to $\mathbb{B}_\rho(\bar{x}) \times \mathbb{B}_\rho(\bar{x})$ and g is Lipschitz continuous with constant λ on $\mathbb{B}_\rho(0)$.

It is also interesting to point out the following:

Remark 2. Lemma 3.1 reveals in particular that the mapping P_x is Aubin continuous at $\bar{y} + g(\bar{x} - x)$ for $\bar{x}$ with constant $\kappa/(1 - 3\lambda\kappa)$, for all x in a neighborhood of $\bar{x}$. Indeed, let a and b be positive numbers such that $b + \lambda a \leq \beta$ and $a \leq \alpha$. Let $x \in \mathbb{B}_a(\bar{x})$ and $y \in \mathbb{B}_b(\bar{y}+g(\bar{x}-x))$. Then $\|y-\bar{y}\| \leq \|y-\bar{y}-g(\bar{x}-x)\|+\|g(\bar{x}-x)\| \leq b+\lambda\|\bar{x}-x\| \leq b+\lambda a$, and therefore P_x is Aubin continuous at $(\bar{y} + g(\bar{x} - x), \bar{x})$ with neighborhoods $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y} + g(\bar{x} - x))$ and constant $\kappa/(1 - 3\lambda\kappa)$, for all $x \in \mathbb{B}_a(\bar{x})$.

We are now able to prove that the metric regularity of T implies uniform convergence of the proximal point method.

Theorem 3.2. *Consider a mapping $T : X \rightrightarrows Y$ with locally closed graph at $(\bar{x}, 0) \in \operatorname{gph} T$. Let T be metrically regular at $\bar{x}$ for 0 with constant $\kappa > 0$. Choose a sequence of functions $g_n : X \to Y$ with $g_n(0) = 0$ which are Lipschitz continuous in a neighborhood O of 0, the same for all n, with Lipschitz constants λ_n satisfying*

$$\lambda := \sup_n \lambda_n < 1/(4\kappa). \tag{3.1}$$

Then there exist positive constants a, b and α such that for every $x_0 \in \mathbb{B}_a(\bar{x})$ and $y \in \mathbb{B}_b(0)$ there exists a sequence x_n in $\mathbb{B}_\alpha(\bar{x})$ generated by the iteration (1.4) and starting from x_0 which is linearly convergent to a solution x such that $y \in T(x)$; moreover, if g_n is chosen so that $\lambda_n \to 0$, then the sequence is superlinearly convergent to x.

Proof. We apply Lemma 3.1 with $g := g_n$. Let α, β and $\theta := \kappa/(1-3\lambda\kappa)+\varepsilon$ be the constants in Lemma 3.1, where $\varepsilon > 0$ is chosen such that $\theta\lambda < 1$. Thanks to the Remark 1, we can assume without loss of generality that the constants α and β are the same for all n since they only depend on κ, λ and the neighborhood O. Since T^{-1} is Aubin continuous at $(0, \bar{x})$ with modulus k there is some $\delta > 0$ and some neighborhood U of $\bar{x}$ such that

$$e(T^{-1}(y') \cap U, T^{-1}(y)) \leq \kappa\|y' - y\| \quad \text{for any} \quad y, y' \in \mathbb{B}_\delta(0).$$

Then, since $\bar{x} \in T^{-1}(0) \cap U$, we have

$$d(\bar{x}, T^{-1}(y)) \leq \kappa\|y\| < (\kappa + \varepsilon)\|y\| \quad \text{for any} \quad y \in \mathbb{B}_\delta(0) \setminus \{0\},$$

and thus, for every $y \in \mathbb{B}_\delta(0) \setminus \{0\}$ there exists $x \in T^{-1}(y) \cap \mathbb{B}_{(\kappa+\varepsilon)\|y\|}(\bar{x})$. Let $P_z^n = [g_n(\cdot - z) + T(\cdot)]^{-1}$ for $z \in X$. Choose a and b positive such that $\mathbb{B}_{(\kappa+\varepsilon)b+a}(0) \subset O$ and

$$(\kappa + \varepsilon)b \leq \alpha, \quad b \leq \delta, \quad b + \lambda_0((\kappa + \varepsilon)b + a) \leq \beta \quad \text{and} \quad \theta\lambda_0((\kappa + \varepsilon)b + a) + (\kappa + \varepsilon)b \leq \alpha.$$

Let $x_0 \in I\!B_a(\bar{x}), y \in I\!B_b(0)$, and let $x \in T^{-1}(y) \cap I\!B_{(\kappa+\varepsilon)\|y\|}(\bar{x})$ (if $y = 0$, take $x = \bar{x}$). Then

$$\|x - \bar{x}\| \leq (\kappa + \varepsilon)b \leq \alpha.$$

Note that

$$x \in P_{x_0}^0(y + g_0(x - x_0)) \cap I\!B_\alpha(\bar{x}),$$

and

$$\|y + g_0(x - x_0)\| \leq b + \lambda_0\|x - x_0\| \leq b + \lambda_0((\kappa + \varepsilon)b + a) \leq \beta.$$

Hence, from Lemma 3.1, there exists an $x_1 \in P_{x_0}^0(y)$, that is,

$$y \in g_0(x_1 - x_0) + T(x_1),$$

such that

$$\|x - x_1\| \leq \theta\|g_0(x - x_0)\| \leq \theta\lambda_0\|x - x_0\|.$$

Then

$$\begin{aligned}\|x_1 - \bar{x}\| &\leq \|x_1 - x\| + \|x - \bar{x}\| \leq \theta\lambda_0\|x - x_0\| + (\kappa + \varepsilon)b \\ &\leq \theta\lambda_0((\kappa + \varepsilon)b + a) + (\kappa + \varepsilon)b \leq \alpha.\end{aligned}$$

Suppose that for some integer $n \geq 1$, the points $x_1, \ldots, x_n$ have been obtained by the method (1.4), and satisfy

$$x_i \in P_{x_{i-1}}^{i-1}(y) \cap I\!B_\alpha(\bar{x}) \quad \text{and} \quad \|x - x_i\| \leq \theta\lambda_{i-1}\|x - x_{i-1}\| \quad \text{for} \quad i = 1, \ldots, n.$$

Observe that

$$x \in P_{x_n}^n(y + g_n(x - x_n)) \cap I\!B_\alpha(\bar{x}),$$

and since $\theta\lambda < 1$,

$$\begin{aligned}\|y + g_n(x - x_n)\| &\leq b + \lambda_n\|x - x_n\| \leq b + \lambda_n\theta^n\lambda_{n-1}\ldots\lambda_0\|x - x_0\| \\ &\leq b + \lambda_0((\kappa + \varepsilon)b + a) \leq \beta.\end{aligned}$$

Thus, applying Lemma 3.1, there exists an $x_{n+1} \in P_{x_n}^n(y)$, that is,

$$y \in g_n(x_{n+1} - x_n) + T(x_{n+1}),$$

such that

$$\|x - x_{n+1}\| \leq \theta\|g_n(x - x_n)\| \leq \theta\lambda_n\|x - x_n\|.$$

Further,

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \|x_{n+1} - x\| + \|x - \bar{x}\| \leq \theta\lambda_n\|x - x_n\| + (\kappa + \varepsilon)b \\ &\leq (\theta\lambda_n)(\theta\lambda_{n-1})\ldots(\theta\lambda_0)\|x - x_0\| + (\kappa + \varepsilon)b \\ &\leq \theta\lambda_0((\kappa + \varepsilon)b + a) + (\kappa + \varepsilon)b \leq \alpha,\end{aligned}$$

and then, the induction step is complete. Therefore, we have obtained a sequence x_n generated by (1.4) with

$$\|x - x_n\| \leq \theta\lambda_{n-1}\|x - x_{n-1}\| \quad \text{for all} \quad n \geq 1,$$

that is, x_n converges linearly to x (the convergence is superlinear whenever $\lambda_n \to 0$). $\square$

We show now that if T is strongly metrically regular then the sequence which existence is proved in Theorem 3.2 is unique.

Theorem 3.3. *Consider a mapping $T : X \rightrightarrows Y$ with locally closed graph at $(\bar{x}, 0) \in \operatorname{gph} T$. Let T be strongly metrically regular at $\bar{x}$ for 0 with constant $\kappa > 0$. Choose a sequence of functions $g_n : X \to Y$ with $g_n(0) = 0$ which are Lipschitz continuous in a neighborhood O of 0, the same for all n, with Lipschitz constants λ_n satisfying the inequality (3.1). Then the conclusion of Theorem 3.2 holds and, in addition, the sequence x_n is unique in $I\!B_\alpha(\bar{x})$.*

Proof. Take α as in the proof of Theorem 3.2. Make α smaller if necessary and let $\rho > 0$ such that T is strongly regular at $\bar{x}$ for 0 with neighborhoods $I\!B_\alpha(\bar{x})$ and $I\!B_\rho(0)$. Now take $\hat{\alpha} := \min\{\alpha/3, \kappa\rho\}$ and repeat the proof using $\hat{\alpha}$ instead of α. Then we obtain some positive constants a and b such that for every $x_0 \in I\!B_a(\bar{x})$ and $y \in I\!B_b(0)$ there exists a sequence x_n in $I\!B_{\hat{\alpha}}(\bar{x})$ generated by the iteration (1.4) and satisfying the same properties as in Theorem 3.2. Make b smaller if necessary so that $b \leq \rho/2$. In order to complete the proof, we just have to show that the sequence is unique. Assume that, from a given x_n, there exist two points x_{n+1} and z_{n+1} in $I\!B_{\hat{\alpha}}(\bar{x})$ which are obtained by (1.4). Since

$$\|y - g_n(z_{n+1} - x_n)\| \leq b + \lambda_n \|z_{n+1} - x_n\| \leq b + 2\lambda_n \hat{\alpha} \leq \frac{\rho}{2} + \frac{1}{2\kappa}\hat{\alpha} \leq \rho,$$

then, from the single-valuedness of the graphical localization of T^{-1} with respect to $I\!B_\alpha(\bar{x})$ and $I\!B_\rho(0)$, we obtain

$$z_{n+1} = T^{-1}(y - g_n(z_{n+1} - x_n)) \cap I\!B_\alpha(\bar{x}).$$

Moreover, we have $z_{n+1} \in I\!B_{2\hat{\alpha}}(x_{n+1}) \subset I\!B_\alpha(\bar{x})$, and hence

$$z_{n+1} = T^{-1}(y - g_n(z_{n+1} - x_n)) \cap I\!B_{2\hat{\alpha}}(x_{n+1}).$$

Since $y - g_n(x_{n+1} - x_n) \in T(x_{n+1})$, we get

$$\begin{aligned}
\|x_{n+1} - z_{n+1}\| &= d(x_{n+1}, T^{-1}(y - g_n(z_{n+1} - x_n)) \cap I\!B_{2\hat{\alpha}}(x_{n+1})) \\
&= d(x_{n+1}, T^{-1}(y - g_n(z_{n+1} - x_n))) \\
&\leq \kappa d(T(x_{n+1}), y - g_n(z_{n+1} - x_n)) \\
&\leq \kappa \|g_n(x_{n+1} - x_n) - g_n(z_{n+1} - x_n)\| \\
&\leq \kappa\lambda_n \|x_{n+1} - z_{n+1}\| < \|x_{n+1} - z_{n+1}\|,
\end{aligned}$$

which is absurd. ◻

Let x_0 be a solution to the perturbed inclusion

$$y_0 \in T(x), \tag{3.2}$$

where $y_0 \in Y$ is a perturbation vector close to 0. Then, we prove in the next theorem that one can find neighborhoods U of $\bar{x}$ and V of 0 such that for every $x_0 \in U$ and $y_0, \delta_0 \in V$ such that x_0 is a solution to (3.2), there is a sequence x_n generated by the proximal point iteration (1.4) and starting from x_0 which is convergent to a solution to $y := y_0 + \delta_0 \in T(x)$ and such that all the elements of x_n stay at distance from x_0 proportional to the variation δ_0 of y_0.

Theorem 3.4. *Consider a mapping $T : X \rightrightarrows Y$ with locally closed graph at $(\bar{x}, 0) \in \operatorname{gph} T$. Let T be metrically regular at $\bar{x}$ for 0 with constant $\kappa > 0$. Choose a sequence of functions $g_n : X \to Y$ with $g_n(0) = 0$ which are Lipschitz continuous in a neighborhood O of 0, the same for all n, with Lipschitz constant $\lambda < 1/4\kappa$. Then there exist positive constants a, b and α such that for every $x_0 \in I\!B_a(\bar{x}), y \in I\!B_b(0), y_0 \in T(x_0) \cap I\!B_b(0)$ and $\varepsilon > 0$, there exists a sequence x_n in $I\!B_\alpha(\bar{x})$ generated by the iteration (1.4) and starting from x_0 which is convergent to a solution x to the inclusion $y \in T(x)$ and satisfying for all n*

$$\|x_n - x_0\| \le (\kappa/(1 - 4\lambda\kappa) + \varepsilon)\|y - y_0\|.$$

Proof. Let α, β and $\theta := \kappa/(1 - 3\lambda\kappa) + \gamma$ be the constants in Lemma 3.1, where γ is a positive number such that $\theta\lambda < 1$. Define $P_x^n = [g_n(\cdot - x) + T(\cdot)]^{-1}$ for $x \in X$. Make α and β smaller if necessary so that $I\!B_{2\alpha}(0) \subset O$ and $\operatorname{gph} T$ is closed relative to $I\!B_\alpha(\bar{x}) \times I\!B_\beta(0)$. Choose a and b positive such that

$$2\theta b/(1 - \theta\lambda) + a \le \alpha \quad \text{and} \quad 3b \le \beta.$$

Let $y \in I\!B_b(0)$, $x_0 \in I\!B_a(\bar{x})$ and $y_0 \in T(x_0) \cap I\!B_b(0)$. Then $x_0 \in P_{x_0}^0(y_0) \cap I\!B_\alpha(\bar{x})$. From Lemma 3.1, there exists an $x_1 \in P_{x_0}^0(y)$ such that $\|x_1 - x_0\| \le \theta\|y - y_0\|$, and therefore

$$\|x_1 - \bar{x}\| \le \|x_1 - x_0\| + \|x_0 - \bar{x}\| \le 2\theta b + a \le \alpha.$$

Note that

$$x_1 \in P_{x_1}^1(y - g_0(x_1 - x_0)) \cap I\!B_\alpha(\bar{x})$$

and

$$\|y - g_0(x_1 - x_0)\| \le b + \lambda\|x_1 - x_0\| \le b + 2\lambda\theta b \le 3b \le \beta.$$

Thus, from Lemma 3.1, there exists an $x_2 \in P_{x_1}^1(y)$ such that

$$\|x_2 - x_1\| \le \theta\|g_0(x_1 - x_0)\| \le \theta\lambda\|x_1 - x_0\|.$$

Besides,

$$\begin{aligned}\|x_2 - \bar{x}\| &\le \|x_2 - x_1\| + \|x_1 - x_0\| + \|x_0 - \bar{x}\| \\ &\le (1 + \theta\lambda)\|x_1 - x_0\| + \|x_0 - \bar{x}\| \\ &\le 2\theta b/(1 - \theta\lambda) + a \le \alpha.\end{aligned}$$

Suppose that for some integer $n \ge 2$, the points $x_2, \dots, x_n$ are obtained by the iteration (1.4), that is, $x_i \in P_{x_{i-1}}^{i-1}(y) \cap I\!B_\alpha(\bar{x})$ and

$$\|x_i - x_{i-1}\| \le (\theta\lambda)^{i-1}\|x_1 - x_0\|, \quad \text{for} \quad i = 2, \dots, n.$$

Then, since $\theta\lambda < 1$,

$$\begin{aligned}\|y - g_{n-1}(x_n - x_{n-1})\| &\le b + \lambda\|x_n - x_{n-1}\| \\ &\le b + \lambda(\theta\lambda)^{n-1}\|x_1 - x_0\| \\ &\le b + 2b(\theta\lambda)^n \le 3b \le \beta.\end{aligned}$$

Further,

$$x_n \in P^n_{x_n}(y - g_{n-1}(x_n - x_{n-1})) \cap I\!B_\alpha(\bar{x}).$$

Hence, applying Lemma 3.1, there exists $x_{n+1} \in P^n_{x_n}(y)$ such that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \theta\|g_{n-1}(x_n - x_{n-1})\| \leq \theta\lambda\|x_n - x_{n-1}\| \\ &\leq (\theta\lambda)^n\|x_1 - x_0\|.\end{aligned}$$

In addition,

$$\begin{aligned}\|x_{n+1} - \bar{x}\| &\leq \sum_{j=2}^{n+1}\|x_j - x_{j-1}\| + \|x_1 - x_0\| + \|x_0 - \bar{x}\| \\ &\leq \sum_{j=1}^{n+1}(\theta\lambda)^{j-1}\|x_1 - x_0\| + \|x_0 - \bar{x}\| \\ &\leq 2\theta b/(1 - \theta\lambda) + a \leq \alpha,\end{aligned}$$

and the induction step is complete. Thus, there exists a sequence x_n generated by (1.4) which is a Cauchy sequence, and therefore it converges to some $x \in I\!B_\alpha(\bar{x})$. Since $x_{n+1} \in P^n_{x_n}(y)$ one has

$$y \in g_n(x_{n+1} - x_n) + T(x_{n+1}). \tag{3.3}$$

Passing to the limit and using the local closedness of T we obtain that $y \in T(x)$. Moreover,

$$\begin{aligned}\|x_n - x_0\| &\leq \sum_{i=1}^{n}\|x_i - x_{i-1}\| \leq \sum_{i=1}^{n}(\theta\lambda)^{i-1}\|x_1 - x_0\| \\ &\leq \theta/(1 - \theta\lambda)\|y - y_0\| \\ &= (\kappa + \gamma(1 - 3\lambda\kappa))/(1 - 4\lambda\kappa - \lambda\gamma(1 - 3\lambda\kappa))\|y - y_0\|.\end{aligned}$$

Making γ smaller if necessary, we complete the proof. □

4 An inexact proximal point method

Since solving the inclusion (1.2) may be as hard as solving the original problem of finding a solution to the inclusion $T(x) \ni 0$ it is essential, from a practical point of view, to replace (1.2) with a looser relation. In the case when $T : H \to H$ is a maximal monotone operator acting on a Hilbert space H, Rockafellar [20] considered two criteria for an approximate proximal point method. One of them was of the following form:

$$\|x_{n+1} - (I + \lambda_n^{-1}T)^{-1}x_n\| \leq d_n\|x_{n+1} - x_n\|, \quad \sum_{k=0}^{\infty} d_n < \infty. \tag{4.1}$$

Moreover, it was established (see [20, Proposition 3]) that the following condition (more suitable for implementations) implies relation (4.1):

$$d(0, \lambda_n(x_{n+1} - x_n) + T(x_{n+1})) \leq d_n \lambda_n \|x_{n+1} - x_n\|, \quad \sum_{k=0}^{\infty} d_n < \infty. \tag{4.2}$$

Using criteria (4.1) and (4.2) Rockafellar obtained the linear convergence of the sequence x_n to a solution of $T(x) \ni 0$ whenever the mapping T^{-1} is Lipschitz continuous at 0.

In the last decade, many authors have considered variants of the inexact proximal point method proposed by Rockafellar. We mention here a hybrid approximate proximal point method, involving an extra projection step due to Solodov and Svaiter [22], more recently, a relaxed proximal point method by Yang and He [25] and, the same year, a hybrid algorithm for finding a zero of a maximal monotone operator combining extragradients and proximal methods proposed by Humes and Silva [9]. Other studies are also of interest: see, e.g., [13, 23] and the references therein. All these contributions deal with maximal monotone operators in a Hilbert space. We propose here an inexact proximal point method for metrically regular mappings acting in Banach spaces.

We consider the following inexact proximal point algorithm: having a current iterate $(x_n, e_n) \in X \times Y$, find (x_{n+1}, e_{n+1}) such that

$$e_{n+1} \in g_n(x_{n+1} - x_n) + T(x_{n+1}), \tag{4.3}$$

where e_n is a so-called *error sequence* satisfying

$$\|e_{n+1}\| \leq \varepsilon_n \lambda_n \|x_{n+1} - x_n\| \quad \text{with} \quad \varepsilon_n \downarrow 0. \tag{4.4}$$

Of course, any sequence x_n generated by the exact proximal point method (1.2) satisfies conditions (4.3) and (4.4) with $e_n = 0$ for all n. When $g_n(x) = \lambda_n x$ with $\lambda_n > 0$ the inclusions (4.3) and (4.4) reduce to

$$e_{n+1} \in \lambda_n(x_{n+1} - x_n) + T(x_{n+1}) \quad \text{with} \quad \|e_{n+1}\| \leq \varepsilon_n \lambda_n \|x_{n+1} - x_n\|. \tag{4.5}$$

Under the stronger assumption $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, we obtain Rockafellar's criterion (4.2). On the other hand, if the sequence x_n generated by (4.2) belongs to a neighborhood of $(\bar{x}, 0)$ where $\operatorname{gph} T$ is locally closed, then it is easy to find a sequence $e_n \in Y$ such that (x_n, e_n) satisfies (4.5).

Theorem 4.1. *Consider a mapping $T : X \rightrightarrows Y$, let $\bar{x}$ be a solution of the inclusion $0 \in T(x)$ and let T be strongly subregular at $\bar{x}$ for 0 with a constant κ. Choose a sequence of functions $g_n : X \to Y$ with $g_n(0) = 0$ which are Lipschitz continuous in a neighborhood O of 0, the same for all n, with a Lipschitz constant λ_n and such that $\kappa\lambda_n(1+\varepsilon_n) < 1$ for $n = 0, 1, 2, \ldots$ Then there exists a positive scalar a such that for every sequence x_n generated by the iteration (4.3) and whose elements are in $\mathbb{B}_a(\bar{x})$ for all n we have*

$$\|x_{n+1} - \bar{x}\| \leq \frac{\kappa\lambda_n(1 + \varepsilon_n)}{1 - \kappa\lambda_n(1 + \varepsilon_n)} \|x_n - \bar{x}\| \quad \textit{for } n = 0, 1, 2, \ldots \tag{4.6}$$

If, in addition, $\sup_n \kappa\lambda_n(1+\varepsilon_n) < 1/2$*, the sequence* x_n *is linearly convergent to* $\bar{x}$*. Moreover, if* g_n *is chosen so that* $\lambda_n \to 0$*, then* x_n *is superlinearly convergent to* $\bar{x}$*.*

The proof of Theorem 4.1 is very similar to the one of Theorem 4.2 in [1]. We state it anyway for the convenience of the reader.

Proof. Take a and b positive scalars such that $I\!B_{2a}(0) \subset O$ and the mapping T is strongly metrically subregular at $\bar{x}$ for 0 with constant κ and neighborhoods $I\!B_a(\bar{x})$ and $I\!B_b(0)$. Adjust a if necessary so that $2\lambda_n a(1+\varepsilon_n) \le b$ for all n. Then, by the definition of the strong subregularity, we have

$$\|x - \bar{x}\| \le \kappa\, d(0, T(x) \cap I\!B_b(0)) \quad \text{for all } x \in I\!B_a(\bar{x}).$$

Now, suppose that (4.3) generates a sequence x_n such that $x_n \in I\!B_a(\bar{x})$ for all n. Then, using relation (4.4), we obtain

$$\begin{aligned} \|e_{n+1} - g_n(x_{n+1} - x_n)\| &= \|e_{n+1} + g_n(0) - g_n(x_{n+1} - x_n)\| \\ &\le \lambda_n(1+\varepsilon_n)\|x_{n+1} - x_n\| \\ &\le 2\lambda_n a(1+\varepsilon_n) \le b. \end{aligned}$$

Hence $e_{n+1} - g_n(x_{n+1} - x_n) \in T(x_{n+1}) \cap I\!B_b(0)$ and the strong subregularity of the mapping T at $\bar{x}$ for 0 yields

$$\begin{aligned} \|x_{n+1} - \bar{x}\| &\le \kappa\|e_{n+1} - g_n(x_{n+1} - x_n)\| \le \kappa\lambda_n(1+\varepsilon_n)\|x_{n+1} - x_n\| \\ &\le \kappa\lambda_n(1+\varepsilon_n)(\|x_{n+1} - \bar{x}\| + \|x_n - \bar{x}\|) \end{aligned}$$

This gives us (4.6). The remaining two statements follow directly from (4.6). □

We are able to state a result similar to [20, Theorem 2] established by Rockafellar. We prove that when T^{-1} is *Lipschitz continuous* at 0 any bounded sequence generated by (4.3) converges to a solution to (1.1). Contrary to Theorem 4.1, the whole sequence x_n does not need to be close to a solution of the inclusion (1.1).

We shall say that a set-valued mapping $F^{-1} : Y \rightrightarrows X$ is Lipschitz continuous at 0 (with modulus $\kappa > 0$) if there is a unique solution $\bar{x}$ to the inclusion $0 \in F(x)$ and for some $b > 0$ we have

$$\|x - \bar{x}\| \le \kappa\|y\| \quad \text{whenever} \quad x \in F^{-1}(y) \text{ and } y \in I\!B_b(0). \tag{4.7}$$

Relation (4.7) can be rewritten equivalently in the following form

$$\|x - \bar{x}\| \le \kappa d(0, F(x) \cap I\!B_b(0)) \quad \text{for all } x \in X.$$

Then it is easy to see that the Lipschitz continuity of F^{-1} at 0 forces the strong subregularity of F at $\bar{x}$ for 0.

Proposition 4.2. *Consider a mapping $T : X \rightrightarrows Y$, let $\bar{x}$ be the unique solution of the inclusion $0 \in T(x)$ and let T^{-1} be Lipschitz continuous at 0 with modulus κ. Choose a sequence of functions $g_n : X \to Y$ with $g_n(0) = 0$ which are Lipschitz continuous in X, with a Lipschitz constant λ_n such that $\lambda_n \to 0$. Then for every bounded sequence x_n generated by the iteration* (4.3) *we have*

$$\|x_{n+1} - \bar{x}\| \le \frac{\kappa\lambda_n(1+\varepsilon_n)}{1-\kappa\lambda_n(1+\varepsilon_n)}\|x_n - \bar{x}\| \quad \text{eventually,} \tag{4.8}$$

and the sequence x_n is superlinearly convergent to $\bar{x}$.

Proof. Let a be a positive scalar. Suppose that (4.3) generates a sequence x_n such that $\|x_n\| \le a$ for all n. Let κ and b be the constants in (4.7). Then, as in the proof of Theorem 4.1, since here both of the sequences λ_n and ε_n converge to 0 we have

$$\|e_{n+1} - g_n(x_{n+1} - x_n)\| \le 2a\lambda_n(1+\varepsilon_n) \le b, \quad \text{eventually.}$$

Keeping in mind that T^{-1} is Lipschitz continuous at 0 and using the fact that $x_{n+1} \in T^{-1}(e_{n+1} - g_n(x_{n+1} - x_n))$ we obtain the desired conclusions in Proposition 4.2. □

It is well known that any sequence generated by the original inexact proximal point method

$$x_n + e_{n+1} \in x_{n+1} + \lambda_n^{-1} T(x_{n+1}), \quad \text{where} \quad \|e_{n+1}\| \le \varepsilon_n \|x_{n+1} - x_n\| \quad \text{and} \quad \sum_{k=0}^{\infty} \varepsilon_n < \infty,$$

is bounded provided T is a maximal monotone operator from a Hilbert space onto itself and there exists at least one solution to $0 \in T(x)$. Thus, a natural question arises: does the regularity properties of T along with the existence of a solution to (1.1) ensure the boundedness of the sequences generated by (4.3)? The answer is no, as it is shown in the following counterexamples.

Consider the mapping $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by $T(x) = \{y \in \mathbb{R} \mid y \ge x\}$, for $x \in \mathbb{R}$. This mapping is metrically regular at 0 for 0 (but not strongly regular nor strongly subregular) and we can find an unbounded sequence generated by the iteration (1.3), and therefore verifying (4.3). Indeed, for this mapping, we have

$$d(x, T^{-1}(y)) = d(y, T(x)) = \begin{cases} 0 & \text{for} \quad x \le y, \\ x - y & \text{for} \quad x > y, \end{cases}$$

and therefore it verifies (2.1) with $\kappa = 1$. For this mapping the iteration (1.3) becomes

$$-\lambda_n(x_{n+1} - x_n) \ge x_{n+1},$$

or equivalently,

$$x_{n+1} \le \lambda_n/(1+\lambda_n)x_n. \tag{4.9}$$

Take $x_0 = \varepsilon$ for some $\varepsilon > 0$. Then for all $M < 0$, $x_1 = 2M$ satisfies (4.9), and therefore the sequence is unbounded for any starting point x_0 as close as we want to 0.

Consider now the function $T : \mathbb{R} \to \mathbb{R}$ plotted below (see Figure 1) which is defined by

$$T(x) = (-1)^p(2p+1)x + (-1)^{p+1}2p(p+1), \text{ when } x \in [p, p+1],$$

for all $p \in \mathbb{Z}$. This function is strongly metrically regular at 0 for 0 and we can also find a sequence generated by the algorithm (1.3), and consequently verifying (4.3), which is unbounded. Indeed, lets take $x_0 = \varepsilon$ for some $0 < \varepsilon < 1$ and choose some λ_0 such that $0 < \lambda_0 < 1$. The iteration (1.3) becomes

$$-\lambda_0(x_1 - \varepsilon) \in T(x_1),$$

that is, is the intersection of the straight line $-\lambda_0 x + \lambda_0 \varepsilon$ with the graph of T. Hence the points

$$x_1^n = (\lambda_0\varepsilon + (-1)^n 2n(n+1))/(\lambda_0 + (-1)^n(2n+1)) \in [n, n+1] \text{ for } n \in \mathbb{N}$$

satisfy the iteration (1.3). Thus the sequence is unbounded for any starting point x_0 as near as we desire to 0.

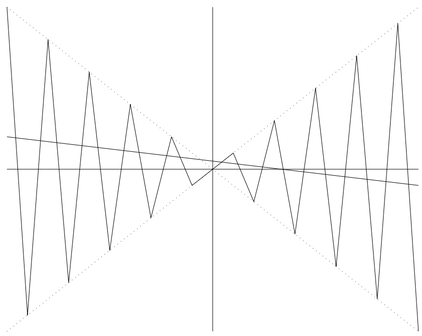

Figure 1: Plot of a strongly regular function which has an unbounded sequence.

Finally, the mapping $T : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$T(x) = \begin{cases} (-\infty, -1] & \text{for } x \neq 0, \\ (-\infty, -1] \cup \{0\} & \text{for } x = 0, \end{cases}$$

is strongly subregular at 0 for 0 for any $\kappa > 0$ (but not metrically regular). Indeed, for every positive $b < 1$,

$$d(0, T(x) \cap \mathbb{B}_b(0)) = \begin{cases} \infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

As in the previous examples, we can easily find an unbounded sequence generated by the algorithm (1.3).

Acknowledgment. We thank the referee for providing constructive comments and help in improving the contents of this paper.

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