Constructive analytic solutions of mixed problems for the bidimensional diffusion equation with delay

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ABSTRACT

The aim of this work is to obtain constructive analytic solutions of mixed problems for the bidimensional diffusion equation with delay of the type

$$u_t(t,x,y)= = a^2\left(u_{xx}(t,x,y)+u_{yy}(t,x,y)\right)+b^2\left(u_{xx}(t-\tau,x,y)+u_{yy}(t-\tau,x,y)\right)$$
$$t>\tau,\quad 0\le x\le l_1,\quad 0\le y\le l_2.$$

A separation of variables method is used to develop an exact theoretical series solution, which can be truncated to obtain a continuous numerical solution with prescribed accuracy in bounded domains.

INTRODUCTION

Partial differential equations with delay constitute useful modelling tools in many scientific and technical problems which exhibit different types of lags, retarded responses or hereditary effects, as in the dynamics of populations when age structure or species with different life-forms are to be considered (see [1], and references therein).
In this work we consider mixed problems for the bidimensional diffusion equation with delay

(1) $$u_t(t,x,y)= = a^2\left(u_{xx}(t,x,y)+u_{yy}(t,x,y)\right)+b^2\left(u_{xx}(t-\tau,x,y)+u_{yy}(t-\tau,x,y)\right)$$
$$t>\tau,\quad 0\le x\le l_1,\quad 0\le y\le l_2,$$

with initial values

(2) $$u(t,x,y)=\varphi(t,x,y),\quad 0<t<\tau,\quad 0\le x\le l_1,\quad 0\le y\le l_2,$$

and Dirichlet type boundary conditions

(3) $$u(t,0,y)=u(t,l_1,y)=u(t,x,0)=u(t,x,l_2)=0,$$
$$0<t,\quad 0\le x\le l_1,\quad 0\le y\le l_2.$$

Following the work in [2], we use the method of separation of variables to obtain solutions of (1) of the form

$$u(t,x,y)=F(t)G(x)H(y),$$

and we are led to propose the infinite series solution of (1)-(3)

$$u(t,x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}F_{nm}(t)\sin\left(\frac{n\pi x}{l_1}\right)\sin\left(\frac{m\pi y}{l_2}\right).$$

Here, $F_{nm}(t)$ is the solution of the delay differential equation-initial value problem

(4) $$F'_{nm}(t)=-\lambda_{nm}^2\left(a^2F_{nm}(t)+b^2F_{nm}(t-\tau)\right),\quad t>\tau,$$

(5) $$F_{nm}(t)=B_{nm}(t),\quad 0\le t\le\tau,$$

where $B_{nm}(t)$ are the coefficients of the Fourier series of $\varphi(t,x,y)$,

$$\varphi(t,x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}B_{nm}(t)\sin\left(\frac{n\pi x}{l_1}\right)\sin\left(\frac{m\pi y}{l_2}\right),$$

i.e.,

$$B_{nm}(t)=\frac{4}{l_1l_2}\int_0^{l_1}\int_0^{l_2}\varphi(t,x,y)\sin\left(\frac{n\pi x}{l_1}\right)\sin\left(\frac{m\pi y}{l_2}\right)dxdy.$$

EXACT AND APPROXIMATE SOLUTIONS

Using the expression for the solution of (4)-(5) given in [2], we can obtain a formal series solution of the problem (1)-(3).
Writing

$$c=\frac{b}{a},\quad d_{1n}=\frac{n\pi}{l_1},\quad d_{2m}=\frac{m\pi}{l_2},\quad \lambda_{nm}^2=d_{1n}^2+d_{2m}^2,\quad Q(k,t)=\frac{\Gamma(k,t)}{\Gamma(k)},$$

where $\Gamma(k)$ and $\Gamma(k,t)$ are the gamma and the complementary incomplete gamma functions, we have

(6)
$$\begin{aligned}
u(t,x,y)&=\\
&=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{p}(-1)^{k-1}c^{2(k-1)}\left(B_{nm}(\tau)+c^2B_{nm}(0)\right)\sin(d_{1n}x)\sin(d_{2m}y)Q\left(k,\lambda_{nm}^2a^2(t-k\tau)\right)\\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{p-1}(-1)^{k-1}c^{2k}\sin(d_{1n}x)\sin(d_{2m}y)\int_0^{\tau}B'_{nm}(s)Q\left(k,\lambda_{nm}^2a^2(t-k\tau-s)\right)ds\\
&+(-1)^{p-1}c^{2p}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sin(d_{1n}x)\sin(d_{2m}y)\int_0^{t-p\tau}B'_{nm}(s)Q\left(p,\lambda_{nm}^2a^2(t-p\tau-s)\right)ds\\
&+(-1)^{p}c^{2p}\varphi(t-p\tau,x,y).
\end{aligned}$$

With some regularity conditions on the initial function, the convergence and regularity properties of the series solution (6) can be proven. By truncating this series with an appropriate number of terms, a continuous numerical solution with *a priori* error bounds can be computed.

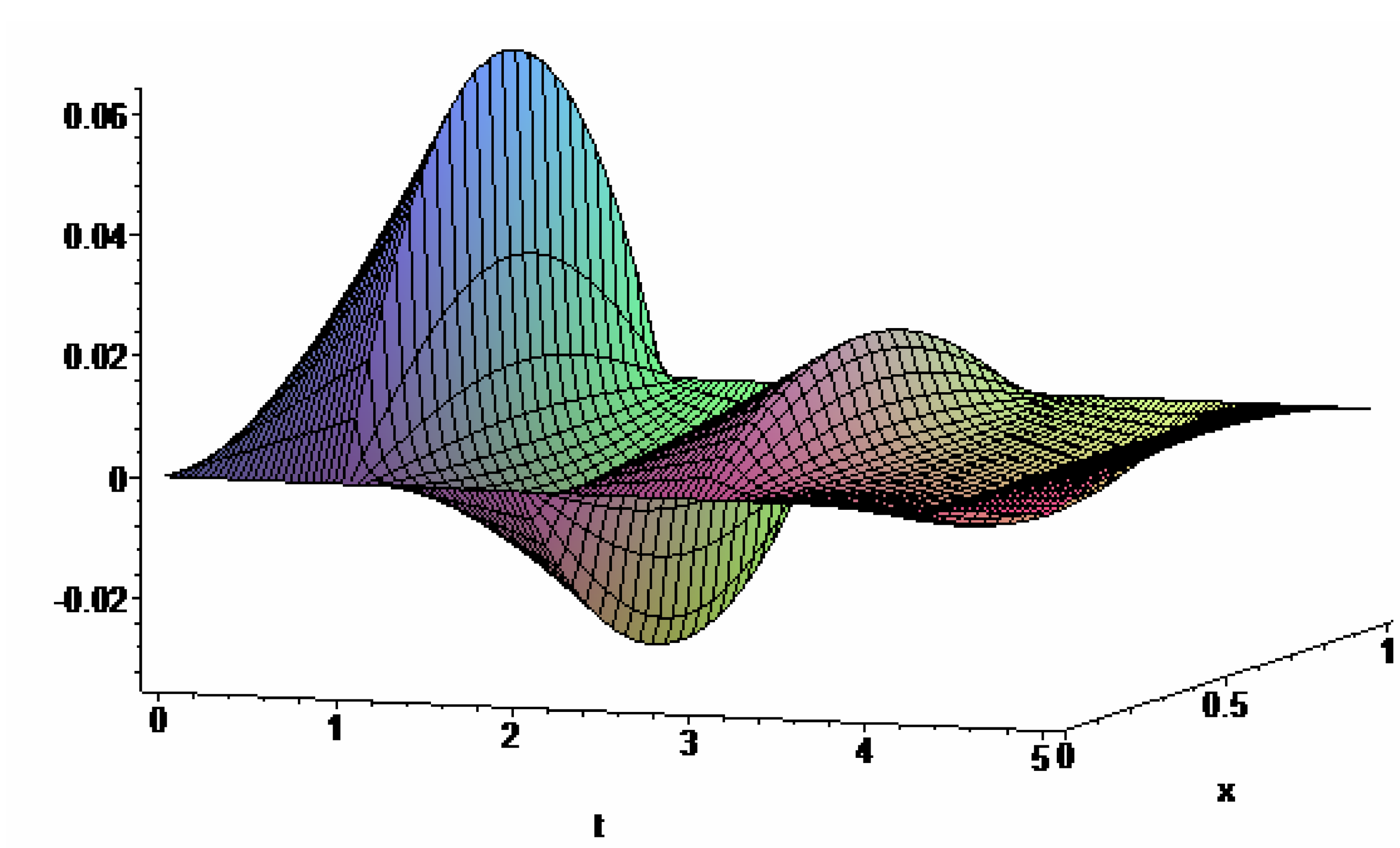

Fig. 1. Numerical evaluation of a truncated series solution (N = 20 terms) for the problem (1) – (3) with parameters a = 1, c = 0.75, $\tau=1$ and initial function $\varphi(t,x,y)=tx(1-x)y(1-y)$.

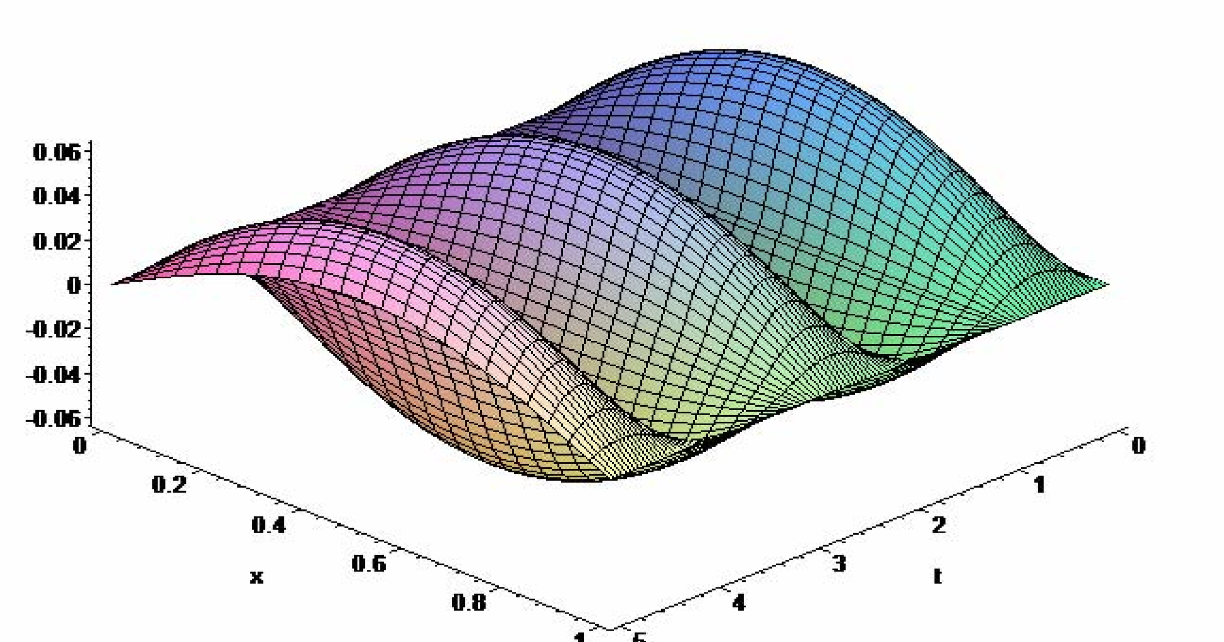

Fig. 2. Numerical evaluation of a truncated series solution (N = 20 terms) for the problem (1) – (3) with parameters a = 1, c = 1, $\tau=1$ and initial function $\varphi(t,x,y)=\sin(\pi t)x(1-x)y(1-y)$.

REFERENCES

[1] Wu, J. *Theory and Application of Partial Functional Differential Equations*. Springer Verlag, New York, 1996.

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