Mathematical Reasoning Fostered by (Fostering) Transformations of Rational Number Representations

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ABSTRACT

In this article we aim to understand the transformations of rational number representations carried out by students and their mathematical reasoning processes. We report part of a Design Based Research, within which an intervention was carried out in a class with 25 students and their teacher, in grades 3 and 4. We analyze six classroom episodes in three main moments concerning the construction of understanding of rational numbers. The results indicate that both transformations and mathematical reasoning processes have an intricate and bidirectional relation, one fostering the other. Students carried out treatments and conversions, including conversions between representations as well as conversions by compositions of different representations. Regarding mathematical reasoning processes, students formulated solving strategies, conjectures and justifications. We also conclude that the social interactions within the class were crucial for the students both doing the transformations and engaging in mathematical reasoning processes.

Keywords: Rational Numbers. Transformations. Mathematical Reasoning. Representations. Elementary School.

Raciocínio Matemático Promovido por (Promovendo) Transformações de Representações de Números Racionais

RESUMO

Neste artigo temos por objetivo compreender as transformações de representações de números racionais realizadas por alunos e os seus processos de raciocínio. Relatamos parte de uma Investigação Baseada em Design, que inclui uma intervenção onde participaram 25 alunos de uma turma e respectiva professora, no 3.º e 4.º anos de escolaridade. Analisamos seis episódios

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em três momentos relativos à construção da compreensão dos números racionais. Os resultados mostram que as transformações de representações e os processos de raciocínio têm uma relação intrincada e bidirecional. Os alunos realizam tratamentos e conversões, realizando conversões entre representações e também conversões envolvendo composições de diferentes representações. Relativamente aos processos de raciocínio, os alunos formulam estratégias de resolução, conjeturas e justificações. As conclusões apontam ainda a importância das interações sociais para os alunos realizarem transformações e para se envolverem em processos de raciocínio matemáticos.

Palavras-chave: Números racionais. Transformações. Raciocínio Matemático. Representações. Ensino Básico.

INTRODUCTION

To be able to understand rational numbers one has to move flexibly around a world of contexts, meanings, operations, and representations (Lamon, 2001). When learning rational numbers[1], students realize that the same rational number can be expressed in different symbolic representations, such as decimal number[2], percentage or fraction. Moving across different representations of a concept provides the recognition that each representation presents a different perspective of the concept, and our insight develops as we increase the perspectives that we take (Tripathi, 2008). Post, Cramer, Behr, Lesh, and Harel (1993) highlight the role of representations in understanding rational numbers, relating this to flexibility with transformations between and within rational number representations. Thus, students' flexibility in making transformations involving different representations can show their understanding of the rational numbers involved (Post, Wachsmuth, Lesh, & Behr, 1985). The transformations among the representations of rational numbers are central in the mathematical activity (Duval, 2006), and its analysis provides a lens to access students' mathematical reasoning processes.

Mathematical reasoning is crucial in learning mathematics at all school levels (NCTM, 2007). Starting from the early school years, students should engage in mathematical experiences that entail different mathematical reasoning processes, which should be conceptualized according to the different school years (Stylianides, 2007a). For instance, the study carried out by Stylianides (2007b) provides evidence that students from grade 3 engage in conjecturing, stating that numerical expressions that equal 10 are infinite because one can always add and then subtract the same number, then adding 10. These students, with the teacher's help, were also able to rewrite the statement using symbols (such as $x - x + 10 = 10$), which could also be interpreted as a proof of the initial conjecture. Similarly, Komatsu (2010) analyzed grade 5 students' conjectures and justifications, which the author identified as proofs, stressing that students starting from even earlier ages, can conjecture and improve their conjectures and justifications, highlighting the role of counterexamples to trigger that refinement.

[1] In this paper, we use the term "rational numbers" to designate positive rational numbers.
[2] We use the term "decimal numbers" to identify positive rational numbers written according to the decimal system notation, using the decimal point.

Considering the crucial role of transformations among different representations for understanding rational numbers as well as of engaging in mathematical reasoning processes for mathematical activity, in this paper we aim to understand the transformations of representations of rational numbers carried out by students and the mathematical reasoning processes that they show.

THEORETICAL BACKGROUND

Representations and Transformations

In a broader sense, representation can be understood as ". . . a configuration of signs, characters, icons, or objects that can somehow stand for, or 'represent' something else" (Goldin, 2003, p.276). This "something else" refers to an idea that goes beyond the representation, which means that without knowing the meaning of what is represented, the representation is unintelligible (Ponte & Serrazina, 2000). A representation describes aspects of the structure of the mathematical object that is represented as well as connections between the object and other ideas (Tripathi, 2008). Thus, a representation cannot be understood as self-contained, because its interpretation implies a connection between the representation and the mathematical object, knowing the circumstances in which the representation can be used, and knowing how the representation can be used to reach particular goals (Ponte & Serrazina, 2000).

In this article, we particularly focus on symbolic representations of rational numbers: decimal number, percentage, and fraction. Each of these representations is embedded with particular rules and assumptions that enable the transition among representations (Goldin, 2003). Duval (2006) states that mathematical activity consists of two types of transformations among representations: treatments and conversions. Treatments are transformations carried out within the same kind of representation, e.g., when $\frac{2}{8}$ is transformed into $\frac{1}{4}$. Baturo (2000) offers a categorization of transformations involving decimal numbers that helps to understand treatments within this representation. The author describes three types of transformations that entail the ability to change the perception of the measurement unit, which she names as reunitising strategies: partitioning to make smaller units (6 tenths=60 hundredths); grouping to make larger units (60 hundredths=6 tenths); and regrouping (6 tenths=5 tenths+10 hundredths).

Even though Baturo (2000) relates these transformations to decimal numbers, this categorization can be extended to percentage and fraction, which can lead to conversions, the second type of transformations identified by Duval (2006). For instance, 40% can be transformed into 4x10%, in which the 10% becomes the measurement unit (partitioning), or $\frac{1}{2}$ x80% if one takes 80% as the measurement unit (grouping). The same 40% can also be regrouped as 50%–10%. In the same way, $\frac{2}{4}$ can be partitioned into $\frac{4}{8}$, with the unit of measurement of $\frac{1}{4}$ in the former being partitioned into $\frac{1}{8}$ in the latter. On the other hand, $\frac{2}{4}$ can be transformed into $\frac{1}{2}$, in which the measurement unit is now $\frac{1}{2}$. And $\frac{2}{4}$ can also be regrouped as, for example, $\frac{1}{4}+\frac{2}{8}$.

Conversions are transformations more complex than treatments. In a conversion one has to perceive the same mathematical object in two representations that can greatly differ in their notation characteristics or representation content (Duval, 2006). As we indicated, conversions are highly important in mathematical activity, because it is with the approach to the same object through different perspectives, each reflecting a different aspect of what is represented, that one can fully understand it (Duval, 2006; Tripathi, 2008). Therefore, using several representations of the same object supports students' in integrating them as part of the same number domain (Owen & Super, 1993; Wang & Siegler, 2013).

The non-congruence among decimal, percentage and fraction symbolic representations of rational numbers adds cognitive complexity to the conversions (Duval, 2006). For example, transforming 0,75 into $\frac{3}{4}$ or 75% implies the recognition of the same rational number expressed in different representations that are written according to different rules and even have different representation notation: the decimal point, the fraction bar, and the percentage symbol.

Duval (2006) states that "Mathematical comprehension begins when coordination of registers starts up" (p.126). Recognizing the same mathematical construct across different representations is not an occasional occurrence, but the outcome of a global coordination of representations, which in turn empowers mathematical reasoning (Duval, 2006).

Mathematical Reasoning

Mathematical reasoning, accessible through representations used by students, is at the core of mathematical learning (Russell, 1999; NCTM, 2007). It is prompted by the search to understand "why something works" and the drive to discover mathematical connections, which are crucial to develop a deeper understanding of mathematics (Lannin, Ellis, & Elliot, 2011; NCTM, 2007).

Oliveira (2008) describes mathematical reasoning as a set of processes through which new knowledge is drawn from previous knowledge. These processes include formulating questions and solving strategies, conjecturing, generalizing and testing generalizations and justifying them (Lannin et al., 2011; Mata-Pereira & Ponte, 2017).

Conjecturing is a process that involves reasoning about mathematical relationships, developing statements, named as conjectures, which require further exploration to ascertain if they are true or not true (Lannin et al., 2011). These conjectures, either spoken or unspoken (Lannin et al., 2011), can refer to particular cases as in the example "$\frac{1}{7}$ is smaller than $\frac{1}{5}$ because the first has a larger denominator". Such statement requires that students seek to understand if it is always true or not. In fact, striving to understand if a specific conjecture is true, implies an important transition from considering individual mathematical objects (such as "this fraction") to consider a class of objects (as in "all rational numbers in fraction representation") (NCTM, 2007). Thus, to show if a conjecture is true or not, students move away from looking at the singularity of the involved objects, to seek commonalities among different cases. By doing so, students develop general conjectures, that is, generalizations, that lead them to call upon and further develop concepts, symbols, and representations (Mata-Pereira & Ponte, 2012).

The generalizations can take different forms, other than algebraic, such as natural oral language, and are not always valid (Lannin et al., 2011). For example, students often generalize the role of zero when used to write decimal numbers, such as used to write whole numbers, disregarding its place value when placed in the leftmost place in the non-whole part of decimal numbers (e.g., 0,08 represents the same number as 0,8).

Ellis (2011) emphasizes that the process of generalizing evolves through collaboration. The author describes generalizing as an activity where people, that share a specific sociomathematical context, are involved in at least one of three actions: (i) identifying commonalities across cases; (ii) extending one's reasoning beyond the initial case; or (iii) drawing broader results from particular cases.

Conjecturing can thus be an entry point into mathematical reasoning (Lannin et al., 2011) leading to generalizations that, in turn, lead to justification. A justification is understood by Kilpatrick, Swafford and Findel (2001) as "sufficient reason for" (p.130), and is intended to both convince oneself and others (Lannin et al., 2011). The justification not only shows that a statement is true but also provides reasons why it is true or valid in every possible case. It is desirable that students become progressively aware of the need to justify and of what makes a justification valid, rejecting statements based on authority (of a teacher, a colleague or a textbook), perception or examples of particular cases (Lannin et al., 2001). By engaging in justification processes, students revisit their mathematical ideas, leading not only to a solid understanding of those ideas but also to the development of new ideas (Whitenack & Yackel, 2008). Thus, by engaging in justification processes students not only develop their reasoning skills but also their conceptual understanding (Kilpatrik et al., 2001). Through students' justifications, one can access, on one hand, how broad they see their own generalizations and, on the other hand, what is students' understanding of what is socially accepted as a justification (Lannin, 2005).

Students from different school years can engage in mathematical reasoning processes. This means that conjectures, generalizations, and how these conjectures are tested and justified will assume different forms throughout the school years. Stylianides (2007a), focusing on the notion of proof in school mathematics, conceptualizes it considering two principles that apply not only to proof but also to other concepts, (i) the intellectual-honesty principle; and (ii) the continuum principle.

The intellectual-honesty principle states that a concept should be conceptualized as it is honest to mathematics as a discipline but also honest to students as learners. This means that we have to consider, on one hand, the formal aspect of that particular concept in the mathematics field and, on the other hand, the conceptual understanding of students or that is accessible to them (Stylianides, 2007a). The continuum principle refers to the continuity of the notion of the concept throughout the school years, assuring that the experience that students have when working with that particular concept is cohesive throughout the years, yet at different levels (Stylianides, 2007a).

These two guiding principles are particularly useful when we are considering mathematical reasoning processes at elementary school level, helping to conceptualize the notion of conjecture, generalization and justification in ways not restricted to its formal forms. With these principles in mind, and in the specific case of justifying process, the justifications of elementary school students can rely on examples or can be explanations of how they solve a problem. Even though these do not constitute valid forms of justification in mathematics, in the sense that they do not show how a particular mathematical relationship is valid in every possible case (Lannin et al., 2011), they provide "sufficient reason for" (Kilpatrik et al., 2001) in the early school years.

METHODOLOGY

In this paper, we report part of a broader study that follows a design research methodology, specifically a classroom design study (Cobb, Jackson, & Dunlap, 2016). The participants were 25 students in a school in Lisbon, their teacher and the researcher (first author). Among the principles that guided the intervention, we highlight (i) frequent work with transformations among representations of rational numbers, (ii) attention to planned and unplanned situations that suggested the use of whole number knowledge leading to invalid generalizations; and (iii) establishment of a learning environment where students were encouraged and felt confident to share and discuss their mathematical ideas. The intervention was carried out in grade 3[3] (February to June 2014), and the last tasks were carried out at grade 4. It, generally, took place once a week in a 90 minutes lesson involving a total of 16 weeks over two school years.

The tasks were proposed by the researcher and were discussed and changed whenever necessary in meetings with the teacher. Lesson plans were elaborated by the researcher and included suggestions to support teacher inquiry, possible students' answers and solutions, and possible difficulties. Most of the lessons followed a common organization, having an initial moment to launch the task, a second moment to solve the task in small groups, and a third moment of whole-class discussion. The researcher participated in these moments, closely following the teacher and the students, participating in the lesson when pertinent.

The main data sources were records of all the students' written work, along with participant observation by the researcher supported by audio/video recordings and field notes. In the data presented, the students will be referred to with fictitious names. For the data analysis, the audio recordings were fully transcribed, complemented with information gathered from the video recordings, students' records and researcher notes. The transcriptions were coded according to the type of transformations carried out by the students. We followed the two types of transformations as described by Duval (2006),

[3] With 8-year-old students.

and within which we distinguished two subtypes, that arose from our reinterpretation of Baturo (2000) reunitising strategies (presented in Table 1).

Table 1. *Categories of Transformations used in the Analysis Process.*

Categories	Sub-categories	Example
Treatments	Within the same representation	$\frac{1}{3}=\frac{2}{6}$
	Composition within the same representation	0,52=0,5+0,02
Conversions	Between different representations	40% = 0,4
	Composition of different representations	Through additive relations: $0{,}52 = \frac{1}{2} + 0{,}02$ Through multiplicative relations: $0{,}01 = \frac{1}{10} \times 0{,}1$

Source: This research.

Regarding students' mathematical reasoning, we considered formulating solving strategies, conjecturing, generalizing, testing generalizations and justifying (Ellis, 2011; Lannin et al., 2011; Mata-Pereira & Ponte, 2012, 2017). We organized the episodes in three main sections, trying to reflect three different moments of students' understanding of rational numbers.

RESULTS

Starting…

In the first task carried out in the intervention, in grade 3, three water bottles with different capacities were given to each group of students. The bottles had the label covered: the 1 *l* bottle had a blue label, the 0,5 *l* had a pink label and the 0,25 *l* bottle had a red label. Students were asked to establish connections among the different bottle capacities, before and after they tried to fill in each bottle with the water contained in the other bottles. Finally, the students were asked to attach each tag to its bottle. All groups had the same tags "0,25 *l*" and "1 *l*", half of the groups had a tag with the representation "0,5 *l*" and the other half "0,50 *l*".

After establishing relations among the different bottles capacities, this excerpt shows how students, in one of the groups, tried to relate the symbolic representation "0,5 *l*" to one of the bottles:

> Dinis: Red [referring to the red label bottle – 0,25 *l*] equal to five liters…
>
> . . .

Rute: Liters… Five liters??

Dinis: No, no...

Bárbara: A quarter of the liter.

Rute: No, this is not even a liter!

André: That's right!

Bárbara: One quarter of a liter.

Rute: I don't know how to say, but it's not a liter, it's…

André: Zero comma five "millilimiters"… (confusion between millimeters and milliliters)

Rute: It's half!

The students seemed to recognize that the representation "0,5 *l*" relates to a number smaller than the unit, 1 *l*. In this initial stage, the students focused their attention on the specificities of the representation. This was evident when Bárbara related "one quarter of a liter" to the red bottle, because the group had concluded that four red label bottles would fill completely the 1 *l* bottle, but she did not establish any relation between "one quarter of a liter" and "0,25 *l*". Rute made a conversion from the symbolic representation (0,5) to oral language ("half"), however, this notion of half seemed to be disconnected of the context of the bottles, since she did not relate it with the 0,5 *l* capacity bottle.

The students of this group related the tags with the bottles by interpreting the non-whole part of decimal numbers as whole numbers:

Researcher (R): Why do you think that [0,5 *l* tag] is there [red label bottle – 0,25 *l*]?

Rute: It's because this [bottle] is the smallest and that is the smallest number.

. . .

Dinis: [both] The pink bottle (0,5 *l*) as the tag [0,25 *l*] they are both in the middle.

. . .

Rute: Because this bottle (1 *l*) it's the biggest and this [1 *l*] it's the largest number here.

The students related "0,25 *l*" to the medium bottle and "0,5 *l*" to the smaller bottle because 25 represents a higher number than 5. Based on the particular cases of the tags provided to the group, the students seemed to generalize that the non-whole part of the decimal representation of rational numbers could be interpreted and ordered, as if they were whole numbers. It was based on this generalization that students justified how they organized the tags.

In the whole-class discussion, the students from another group that had the tag "0,50 *l*" justified how they related it to "0,25 *l*":

António: Because one liter is one hundred… It's one hundred percent.

R: OK, good. So half…

> António: It's fifty percent.
>
> Fábio: Then half of those fifty percent will be twenty five percent.

Considering the unit as 100%, the students' justification was based in a conversion of 0,50 to 50%, probably due to the similarity among the decimal number system used in the writing both representations. This conversion supported the conversion of 0,25 as 25%, half of 50%. Thus, the student concluded that 0,25 would be half of 0,50.

After, Rute shared that she realized that her group did not organize the tags correctly:

> Rute: We did it wrong, but now we understood.
>
> . . .
>
> Rute: Because after what that group said, I thought that five was half, and in the other work [filling the bottles]… We thought that this one [0,5 *l* bottle] was half of this one [1 *l* bottle] so I realized that this one [0,5 *l* bottle] was the one that was zero comma five.
>
> . . .
>
> Rute: The red [bottle] we conclude one quarter in the other [question], so twenty five, plus twenty five, plus twenty five, plus twenty five equals one hundred.
>
> Teacher (T): Well done, good.
>
> Bárbara: Which is the one hundred percent that António was talking about.

By converting decimal representation to percentage, as shared by António and Fábio, Rute as well as the other members of the group were able to relate the connections made previously among the different bottles with the 0,25 *l* and 0,5 *l* tags. These conversions helped her group to abandon their previous generalization.

Then, the discussion focused on the meaning of 0,5 and 0,50. Even though the students had use these numerals to identify the capacity of the pink label bottle, it was not clear that both represented the same number. Rute and Tomás shared with their colleagues why they considered both numbers equivalent:

> Rute: Because zero comma five represents half, as well as zero comma fifty.
>
> R: How do you know that?
>
> Rute: Because fifty is half of one hundred.
>
> Tomás: Because fifty is half of one hundred and five is half of ten.

Mobilizing whole number knowledge, the students considered the non-whole part of decimal numerals as whole numbers and justified the equivalence of representations by establishing the ratio between 5 and 10, and 50 and 100, identifying the proportion between the ratios as half.

Still in grade 3, in another task, the students were asked to discuss the question "Is 0,67 bigger than 0,9?" The students worked in pairs and, after discussing the question, they needed to justify their answer. Bárbara and Rute, that worked together, considered that 0,9 would be bigger than 0,67:

> Rute: I think that zero comma nine is bigger than zero comma sixty seven because zero comma sixty seven or sixty seven hundredths, only has hundredths. Nine tenths has nine… has nine tenths.
>
> Bárbara: Now me, I think that zero comma sixty seven is bigger… Is smaller (corrects) than zero comma nine because zero comma nine is the same as nine tenths, and sixty… and zero comma sixty seven is the same as zero comma… It's the same as sixty seven hundredths. And nine tenths is bigger than the hundredths, because it is divided into larger parts.
>
> . . .
>
> Bárbara: It's tenths! [in 0,9] And a tenth is a bigger unit than a hundredth! So, this [0,9] is bigger.
>
> . . .
>
> Rute: I think that zero comma… That nine tenths are higher than sixty seven hundredths because sixty seven hundredths only has hundredths and doesn't have tenths, like nine tenths.

Both students agreed that 0,9 represents the number with greater magnitude, and their justifications, even though expressed in different ways, are based in the same implicit conjecture. Both seem to conjecture that numerals with more digits in the non-whole part represent smaller magnitudes than numerals with fewer digits. This conjecture, although invalid, results from understanding the relation between the unit and its parts. The further the unit is divided, the smaller are the parts that result from that division. In this particular case, with the numerals 0,9 and 0,67, it led students to a correct answer, however, with different numerals it could lead to errors.

In the whole-class discussion, the research and the teacher instigated students to consider more cases. The comparison between 0,581 and 0,45 was proposed to the class. Rute and Bárbara identified 0,45 as the larger of the two numerals and justified it in the same manner:

> Rute: I think that it's forty… Forty five hundredths, forty five hundredths because… It is divided into fewer parts.
>
> . . .
>
> Bárbara: Because forty five hundredths is divided into… Fewer parts and those parts are bigger than… The others [the thousandths].

Another student, Jorge, added "The five hundred and eighty one thousandths are divided into more parts, each part is worth less than one part of the forty five". This justification seemed to be convincing for their colleagues. As no one presented other justification or a counterexample, the researcher asked students to use the 10x10 grid (a model that was already known by students) to represent both numerals in order to test if 0,45 was larger than 0,581. Jorge immediately corrected his initial statement:

> Jorge: Now I realized my mistake, it's because five hundredth and eighty one, that is greater, are less… the part are worth less but they are more!

By converting the symbolic representations of the numerals to the iconic representation of the 10x10 grid, Jorge realized that even though it is true that one thousandth is a smaller unit than one hundredth, or one tenth, this was not sufficient to state that 0,45 is larger than 0,581, as this last number corresponded to more of the smaller units. This conversion between representations, together with Jorge's justification, seemed to have convinced the students in the class, including Rute and Bárbara.

Heading to a Shared Meaning

In a different task, also solved in grade 3, a discussion took place about the result of 5,7 plus 0,003. Frederico answered that the result would be 5,703 reading it as "five thousand, seven hundred and three thousandths". Probably he did a treatment of the representation in order to be easier to reach the result, but when asked to explain how he could be sure of such a transformation, he did not present a valid justification:

> Frederico: If we read the number like a whole number, without comma, without anything, it would be, at least for me, five thousand, seven hundred and three thousandths.
>
> Several students agreed with him, and the justifications presented were similar to the one that Mário shared:
>
> Mário: If we want to read the number without comma we… because we are going to read the number we say the last position of the number, I think…

Still, the teacher and researcher continued to ask how students could be sure that it was the same, in order for students to present what could be considered, by all, as a justification. Catarina then said she could justify why:

> Catarina: I think I can explain it with towels (referring to the 10x10 grid). First we painted five towels, because it was five units. . . Then, on another towel we would

paint seven columns, because it is seven tenths . . . and then we painted the [three] thousandths and it was the same . . .

R: How many thousandths have five units?

Catarina: Five thousand!

To justify, Catarina did the conversion between the symbolic representation of 5,703 and the iconic representation using the 10x10 grid. Also, by doing so, she presented a justification that convinced her colleagues why 5,703 could be read and understood as 5703 thousandths.

The following task was solved in grade 4. It included comparisons already made by students in previous tasks (including some from grade 3), along with statements made by students as they solved the tasks. The goal of this task was for students, in pairs, to analyze the statements and evaluate their validity as justifications.

Most students begun by focusing the decimal numbers' comparisons, identifying the larger or smaller number and justifying why, without, in fact, analyzing the statements presented. The first statement presented referred to a situation that was already addressed on this analysis: "0,67 is smaller than 0,9 – Sixty seven hundredths only has hundredths and zero comma nine are tenths. And a tenth is a bigger unit than a hundredth because it is divided into bigger parts, so 0,9 is bigger."

In the whole-class discussion, Tomás and Manuel shared with their colleagues their analysis of this statement:

Tomás: We think that their strategy is wrong… Because if we were to think that, the sixty seven… We gave an example: sixty seven hundredths and six tenths. If we think like that the sixty seven hundredths had hundredths so the six tenths is bigger, but it's not…

Manuel: To them it would be, with this strategy. But it doesn't work in all cases.

The students used a counterexample to refute the validity of the justification, clearly stating that the justification would not held in all possible cases. The counterexample was very useful to other students understand that even though the statement did not seem to be incorrect, if they tested it with other cases, they would realize that it was not valid.

Another statement presented in the task was "2,005 < 2,7 – If we transform this [2,005] it can also be the same as 2,5". Most of the students disagreed with the statement based on the treatment of 5 tenths to 500 thousandths, as Fábio referred:

Fábio: . . . If we now hide the two, there is no two, I can transform five tenths in five hundred thousandths and five hundred is bigger than five. So, two thousand and five hundred thousandths is bigger than two thousand and five thousandths.

The teacher challenged the students to explain this idea in a different way, to prevent them to think about adding zeros to the numeral in order to compare only digits, instead of understanding the relations between the different units in the numbers:

> T: Why do you have the need to put two zeros here [2,5]? To have the same number of orders than this [2,005]? Is that why? Can't you compare them in another way?
> Manuel: We can compare because is… is simple. We only have to do, for example, 2,5 minus 2,005 that is… If it is zero it's because it is the same [number]…

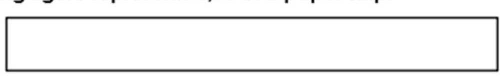
To refute the statement presented in the task, Manuel generalized that if the numbers were equal, the difference between them is zero. He started by considering the particular case of 2,5 and 2,005, but he ended the statement saying "if it is zero it's because it is the same", drawing this broader conclusion from the particular case of the initial pair of numbers, that he knew that were not equal.

Moving towards a Flexible Understanding…

The paper strip task (see Figure 1), also solved by students in grade 4, implied the reconstruction of the unit and the constructions of parts bigger and smaller than the unit, considering symbolic representations of rational numbers and the iconic representation of the bar, without using the ruler.

Finding the paper strip

The following figure represents 0,75 of a paper strip.



a) What will the complete paper strip look like? Draw it.

b) Represent 50%, $\frac{4}{3}$, 1,125 and 20% of that paper strip.

Figure 1. Prompt given in the task "Finding the paper strip". (Research task, adapted from Menezes, Rodrigues, Tavares and Gomes, 2008).

We focus our analysis on the construction of the parts represented by 1,125 and 20%. In the whole-class discussion, Jorge shared the solving strategy that he followed, which allowed him to accurately represent 1,125:

> Jorge: . . . I did one unit, then I painted eleven tenths, then divided the second tenth in half. After I divided the part that was divided in half and did it again on the other

side [of the initial mark] as well . . . And I saw that because what we wanted was one thousand, one hundred and twenty five thousandths, I did here a thing to show that it was the one thousand, one hundred and twenty five thousandths.

Jorge's justification, along with his register (Figure 2), shows that he started by carrying out a treatment of 1,1 into eleven tenths. The student seemed to recognize that one tenth can be transformed into 100 thousandths, thus dividing in half the second tenth of the second unit, and writing down 1,150. Then, he divided each of the tenth's half again in half, resulting into parts that represented 25 thousandths. Finally, he identified 1,125 through a treatment of this representation into 1125 thousandths. Following this systematic strategy, Jorge was confident that he had done a rigorous representation of 1,125.

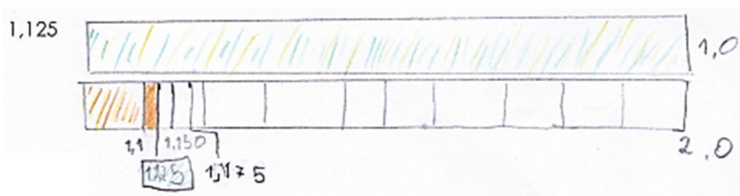


Figure 2. Representation of 1,125 by Jorge.

Another student, Hugo, traced a complete bar that he then divided into quarters (Figure 3). The student marked the bar with 0, 25, 50, 75 and 100. He erased the latter and wrote 1 instead. Even though he did not use the percentage symbol, he seemed to be using this representation. To mark 0,125, he drew one more quarter, from which he painted half. By doing this, Hugo showed that he recognized 0,125 as half of one quarter, that is, one eight of the unit, and because he seemed to call upon percentage, it was also possible that he related 0,125 with half of 25%, that is, 12,5%.

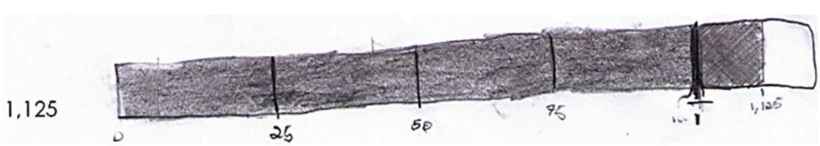


Figure 3. Representation of 1,125 by Hugo.

Manuel justified how he represented 20% (Figure 4), showing flexibility in using and transforming representations:

Manuel: Because I had already marked fifty percent, I thought that twenty five percent are half of the fifty percent. Then, I thought that, because I had also already used it to mark something else, I had already marked one tenth, half of that would be five [percent]. So I took five to those twenty five percent and more or less in that area I will get the twenty.

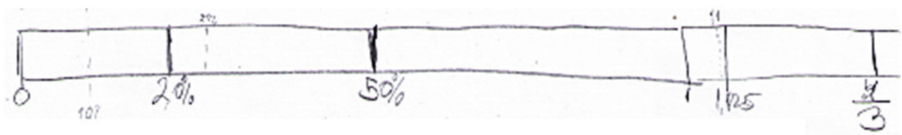


Figure 4. Manuel's written record.

Instead of drawing the requested parts, separately, Manuel used the bar model in a similar way as the number line. He justified how he located 20% using 50% as a reference. He then marked half of 50%, 25%, to which he needed to take off 5%. To have the perception of what would be 5% in the bar, he called upon one tenth already marked in the bar, doing a conversion between 10% that he had written and decimal representation, showing that he recognized 5% as half of one tenth. The latter relation ($5\%=\frac{1}{2}\times 0{,}1$) shows a conversion of 5% as a composition of fraction and decimal representation.

Finally, the following task was solved at the end of grade 4. Five tags were placed in the table with different representations of rational numbers: $\frac{1}{4}$; 0,025; 0,205; 20% and 0,002. We will focus on how Dinis ordered the numbers. He handled and moved the tags as he wished and was asked to sort the numbers represented by descending order of magnitudes. The researcher gave time for him to solve the task and then asked to justify how he compared the numbers.

Dinis readily organized the tags correctly, in silence, after which the researcher asked him to explain how he organized the numbers:

R: You were fast, how did you immediately see that one quarter was the larger?
Dinis: Because it is the same as two hundred and fifty thousandths.
R: OK. Then you have placed this… [0,205]
Dinis: That one was, sort of twenty and a half percent...
R: OK. Then this… [20%]
Dinis: Yes, twenty percent.
R: And how did you relate that percentage with this one? [$\frac{1}{4}$]
Dinis: That is twenty-five percent.

Dinis started by converting the fraction representation ($\frac{1}{4}$) into decimal (0,250), but then he recurred to percentage to make sense of 0,205. He converted the number as a composition showing multiplicative relations, as "twenty and a half percent", in the sense that it is 20% + 0,5%. This conversion seems to show that, at one hand, Dinis understood 0,2; 0,20 or 0,200 as 20% and 0,005 as half of 1%. This latter transformation implies understanding 1% as 0,01 and consider it as the unit to which he relates 0,005 as the half.

Then, Dinis justified how he compared the decimal numbers that represented the smaller magnitudes (0,025 and 0,002):

> Dinis: This [0,025] is higher because is more… It had two percent and a half and this [0,002] had two tenths of one percent.

The justification of Dinis relied in conversions as compositions, in which he considered 1% as the reference unit. He composed 0,02 (in 0,025) as 2% and a half, again flexibly changing the unit into 1% or 0,01 and considered the remaining 0,005 has half of that unit. He did a similar composition for 0,002. He composed the number as "two tenths of the percent" (0,2x1%), implying the understanding that 1% can also be represented as 0.01, and then he perceived 0,002 as two tenths of one hundredth or 1% (0,002=0,2x0,01 or 0,002=0,2x1%). These conversions show a significant understanding of both the rational numbers expressed in these symbolic representations, and of the specificities of the three notations. By changing the unit to 1%, Dinis could organize in a systematic way each number as representing progressively smaller parts of the unit that he was considering.

DISCUSSION AND CONCLUSIONS

In this article, we aim to understand the transformations of rational number representations carried out by students and their mathematical reasoning processes, which were conceptualized according to Stylianides's guiding principles (2007a).

The conjectures identified in the episodes were implicit in the students' discussions. Since the conjectures emerged from the mathematical relationships established by the students (Lannin et al., 2011), they showed their knowledge about rational numbers. In the first episode, the conjecture that non-whole part of decimal numbers could be interpreted as whole numbers, derived from the fact that the students were just beginning to tackle the meaning of rational numbers in decimal representation. The conversions between rational numbers represented as percentage and fraction were used to refute that statement. In the second episode, the conjecture that decimal numbers that had more digits in their non-whole part would represent smaller magnitudes than the ones with fewer digits, showed an initial understanding of the relation between the unit and its parts, which is essential in interpreting rational numbers. Once again, conversions between representations, specifically between decimal numbers and the 10x10 grid iconic representation, was essential to demonstrate that the conjecture was false.

The third and fourth episodes illustrate the importance of discussing the validity of the statements that constitute a justification. As the students developed rational number understanding and engaged in mathematical reasoning processes, they became more aware of the statements that can validate a justification, identifying that they need to hold in all possible cases and recognizing that a counterexample is sufficient to invalidate a statement (Lannin et al., 2011). To do so, the students got involved in testing those statements, carrying mostly treatments. Also, the analysis of statements can lead students to other mathematical reasoning processes, such as generalizations, like the one that Manuel did regarding how to verify if two numbers are equal.

The final episodes illustrate how students were able to formulate solving strategies, which involved conversions between different representations but also as composition of different representations. The last one was also identified in Dinis' justification, in the final episode, in which he showed a flexible conceptualization of the unit. At this point, the students seemed to flexibly transform representations, showing that they consider all different representations as part of the rational number domain (Wang & Siegler, 2013). Additionally, and as Duval (2006) emphasizes, the relations established by students cannot be seen as occasional, but as an outgrowth of a global coordination of representations, rooted in a sound understanding of rational numbers.

We emphasize the role of the class activity throughout the episodes in both the transformations of representations and in the mathematical reasoning processes. The conversions done by some students helped others to also establish connections among different rational number representations. Regarding mathematical reasoning processes, when some students conjectured they did not further examine their statements, however together with their colleagues, they further developed justifications, testing and evaluating the statements, which underlines the fundamental role of social interactions (Ellis, 2011).

In conclusion, we identified treatments and conversions made by students and, within the conversions, we could also identify conversions between representations and conversions involving a composition of different representations, which showed a strong conceptualization of the unit and coordination of different representations. The transformations identified were closely related to the different mathematical reasoning processes showed by the students, namely formulating solving strategies, conjecturing and justifying. In fact, the transformations among rational numbers representations and the mathematical reasoning processes showed by the students are intimately related. Transformations of representations, on one hand, led students to engage in mathematical reasoning processes and, on the other hand, were called upon to support the students' mathematical reasoning processes. Thus, one fostered the other, being both crucial for the development of a sound rational number understanding.

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