ANALYTIC ASPECTS OF THE RIEMANN ZETA AND MULTIPLE ZETA VALUES

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This manuscript contains two parts. The first part contains fast converging series representations involving $\zeta(2n)$ for Apery's constant $\zeta(3)$. These representations are obtained via Clausen acceleration formulae. Moreover, we also find evaluations for more general rational zeta series involving $\zeta(2n)$ and binomial coefficients.

The second part will be devoted to the multiple zeta and special Hurwitz zeta values (multiple t-values). In this part, using a new approach involving integer powers of arcsin which come from particular values of the Gauss hypergeometric function, we are able to provide new proofs for the evaluations of $\zeta(2, 2, \ldots, 2)$, and $t(2, 2, \ldots, 2)$. Moreover, we are able to evaluate $\zeta(2, 2, \ldots, 2, 3)$, and $t(2, 2, \ldots, 2, 3)$ in terms of rational zeta series involving $\zeta(2n)$. On the other hand, using properties of the Clausen functions we can express these rational zeta series as a finite $\mathbb{Q}$-linear combinations of powers of π and odd zeta values. In particular, we deduce the famous formula of Zagier for the Hoffman elements in a special case.

Zagier's formula is a remarkable example of both strength and the limits of the motivic formalism used by Brown in proving Hoffman's conjecture where the motivic argument does not give us a precise value for the special multiple zeta values $\zeta(2, 2, \ldots, 2, 3, 2, 2, \ldots, 2)$ as rational linear combinations of products $\zeta(m)\pi^{2n}$ with m odd. In [75] the formula is proven indirectly by computing the generating functions of both sides in closed form and then showing that both are entire functions of exponential growth and that they agree at sufficiently many points to force their equality.

Keywords: Riemann zeta function, rational zeta series, multiple zeta values, multiple special Hur-

witz zeta values, Gauss hypergeometric function, Apery's constant, Zagier's formula for Hoffman elements.

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1.0 AN OVERVIEW OF THE THESIS

1.1 INTRODUCTION

The present thesis is in the areas of number theory, analysis and special functions. Most of this manuscript is centered around special values of L-functions and multiple zeta functions which play an important role at the interface of analysis, number theory, geometry and physics with applications ranging from periods of mixed Tate motives to evaluating Feynman integrals in quantum field theory. I employ methods from real analysis and special functions. Let me describe briefly the novel results. In the first part of the thesis, we derive some new fast converging series representations involving $\zeta(2n)$ for Apery's constant $\zeta(3)$. Moreover, we are able to prove some new rational zeta series. These formulas are explicit in terms of the even and odd values of the Riemann zeta and Dirichlet bets functions. In particular cases, we recover some well-known representations of π. In the second part, we give an new approach for the evaluation of certain multiple zeta values involving Hoffman elements. This approach relies on special values of the Gauss hypergeometirc function. In particular, we are able to give an elementary and direct proof of Zagier's formula. Similar results for multiple t-values are also obtained.

The objects I study are the Riemann zeta and multiple zeta functions and their special values. They are defined by the following,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \mathrm{Re}(s) > 1,$$

and

$$\zeta(s_1, s_2, \ldots, s_r) = \sum_{1\le n_1<n_2<\ldots<n_r} \frac{1}{n_1^{s_1} n_2^{s_2} \ldots n_r^{s_r}}, \mathrm{Re}(s_r) > 1, \sum_{j=1}^{r} \mathrm{Re}(s_j) > r,$$

for all $j = 1, 2, \ldots, r$.

Besides of their mathematical relevance, the Riemann (single) zeta and multiple zeta functions and their special values have applications to values of Feynman integrals in quantum field theory. Initially, the Riemann (single) and double zeta values were defined by Euler long time ago with his sensational proof of $\zeta(2) = \frac{\pi^2}{6}$. This was followed by a generalization given by the same Euler in 1740,

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \cdot \frac{2^{2k-1} B_{2k}}{(2k)!} \cdot \pi^{2k},$$

where B_n are the Bernoulli numbers which arise in the Taylor series expansion of $\dfrac{z}{e^z - 1} = \sum_{n=0}^{\infty} \dfrac{B_n}{n!} z^n, |z| < 2\pi$.

On the other hand, when it comes to odd zeta values $\zeta(3), \zeta(5), \ldots, \zeta(2n+1)$ very little is known. The current status of the known results are that $\zeta(3)$ is irrational (Apery [7], 1979) and that there infinitely many irrational numbers among odd zeta values (Ball and Rivoal [9], 2001). Also, another important result due to W. Zudilin [79] goes back to 2002. He proved that at least one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

However, we have the following integral representation,

$$\zeta(2k+1) = (-1)^{k+1} \frac{(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(x) \cot(\pi x) dx,$$

where $B_k(x)$ are the Bernoulli polynomials which are defined by $\dfrac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \cdot \dfrac{z^n}{n!}$.

Moreover, the odd zeta values can be represented as rational zeta series involving $\zeta(2n)$. Two such examples are given by Euler's representations of Apery's constant and reproved by Ewell [46],

$$\zeta(3) = \frac{\pi^2}{7}\left(1 - 4\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}\right),$$

and

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.$$

A natural question can be asked in this context: is it possible to establish the irrationality of $\zeta(3)$ using the above formulas? If so, this could be extended to the values $\zeta(2k+1), k > 1$. Moreover, Ewell [47] prove that the values of the Riemann zeta function at an integer can be represented by the formula

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{n=0}^{\infty}(-1)^n A_{2n}(r-2)\frac{\pi^{2n}}{(2n+2)!},$$

where $A_{2n}(r)$ is an explicit complicated rational constant involving Bernoulli numbers.

One of the main goals would be to prove the following

Transcendence conjecture. *The numbers* $\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)$ *are algebraically independent over* $\mathbb{Q}$.

In other words, we want to understand polynomial relations in odd zeta values. It will turn out that this is equivalent with understanding $\mathbb{Q}$-linear relations among other objects called multiple zeta values which will be defined below.

In 1992, M. Hoffman [50] and independently D. Zagier [74] generalized Euler's single and double zeta values to multiple zeta values which sometimes are called Euler-Zagier sums. For $k_1, k_2, \ldots, k_{r-1} \geq 1$ and $k_r \geq 2$, we define

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{1 \leq n_1 < n_2 < \ldots < n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_r^{k_r}},$$

where we fix the weight $k = k_1 + k_2 + \ldots + k_r$ and the depth (length) r.

For example, there are 2^{13} such numbers in weight 15, but they form a vector space over $\mathbb{Q}$ of dimension at most 28. Multiple zeta values satisfy many relations, For example, the simplest is the product of two single zeta values, which gives us the simplest example of what we call *stuffle* product,

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).$$

Similarly, for triple zeta values, we have the following identity,

$$\zeta(k_1, k_2)\zeta(k_3) = \zeta(k_1, k_2, k_3) + \zeta(k_3, k_1, k_2) + \zeta(k_1, k_3, k_2) + \zeta(k_1 + k_3, k_2) + \zeta(k_1, k_2 + k_3).$$

For $M = \mathbb{C} - \{0, 1\}$ let us consider the 1-forms $\omega_0 = \dfrac{dz}{z}$ and $\omega_1 = \dfrac{dz}{1-z}$. The iterated integral

$$\int_0^1 \omega_{e_1} \ldots \omega_{e_r}, e_j \in \{0, 1\}$$

converges if and only if $e_1 = 1$ and $e_0 = 0$. In this sense, M. Kontsevich observed that all multiple zeta values are periods,

$$\zeta(k_1, k_2, \ldots, k_r) = \int_0^1 \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{k_1 - 1} \cdot \ldots \cdot \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{k_r - 1}.$$

The simplest example is the following integral representation of $\zeta(2) = \displaystyle\int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$. If we multiply the integral representation of two single zeta values, we have

$$\zeta(2)\zeta(2) = 4\zeta(1, 3) + 2\zeta(2, 2).$$

Now, we turn to some important evaluations of multiple zeta values. Since $\zeta(2) = \frac{\pi^2}{6}$ and from the stuffle relations, $\zeta(2, 2) = \frac{\pi^4}{120}$, one can ask what is the exact value for the multiple zeta values $\zeta(\underbrace{2, 2, \ldots, 2}_{n})$? This was answered by Hoffman [50] and Zagier [73] in 1992.

Theorem 1.1.1. *The following equality holds true*

$$\zeta(\underbrace{2, 2, \ldots, 2}_{n}) = \frac{\pi^{2n}}{(2n + 1)!}.$$

The proof given by Hoffman [50] relies on combinatorial methods. Zagier's argument is simpler and it uses generating functions and Euler's formula for the infinite sine product. Another important reslt was to extend Euler's remarkable equality for double zeta values, $\zeta(1,2) = \zeta(3)$. Indeed, this is due to Hoffman [50],

Theorem 1.1.2. *The following inequality is valid*

$$\zeta(\underbrace{1,2,\ldots,1,2}_{n}) = \zeta(\underbrace{3,3,\ldots,3}_{n})$$

Another important evaluations will be discussed in Chapter 4.

Let us denote by $\mathcal{Z}$ the $\mathbb{Q}$-vector space spanned by all multiple zeta values. It is not hard to see that $\mathcal{Z}$ has the structure of an algebra.

In 2011, F. Brown [24] proved the following result

Theorem 1.1.3. *The periods of mixed Tate motives over* $\mathbb{Z}$ *lie in* $\mathcal{Z}\left[\frac{1}{2\pi i}\right]$.

It was known that multiple zeta values are examples of periods of mixed Tate motives over $\mathbb{Z}$ from previous work of Deligne and Goncharov [43], but it turns out that also the converse is true in some sense.

A natural question to ask is what is a basis for $\mathcal{Z}$ as a $\mathbb{Q}$-vector space? The natural thing to try at first is to add a basis for a general family of even zeta values, add odd zeta values, continue with multiple zeta values in low depth (2, 3) with odd arguments and so on. Unfortunately, this approach fails because of linear dependence relations over $\mathbb{Q}$.

We can circumvent this problem with the following "high depth" result which was conjectured by M. Hoffman [51] in 1997 and settled by F. Brown [24] in 2012.

Theorem 1.1.4. *Every multiple zeta value of weight k can be expressed as a $\mathbb{Q}$-linear combination of multiple zeta values of the same weight involving* 2*'s and* 3*'s.*

In other words, every multiple zeta value of weight k is a $\mathbb{Q}$-linear combination of $\zeta(k_1, k_2, \ldots, k_r)$, where $k_i \in \{2,3\}$ and $\sum_{i=1}^{r} k_i = k$. The arguments used by Brown in proving the above theorem are purely motivic and it used motivic multiple zeta values. In fact, Brown showed that the multiple zeta values involving Hoffman elements,

$$H(r,s)=\zeta(\underbrace{2,2,\dots,2}_{r},3,\underbrace{2,2,\dots,2}_{s})$$

can be expressed as a $\mathbb{Q}$-linear combination of products $\pi^{2m}\zeta(2n+1)$, with $m+n=r+s+1$.

The next result gives an explicit formula for $H(r,s)$ which are part of Hoffman's conjectural basis. This result confirms the 2-adic properties of the coefficients required for Brown's proof. The exact coefficients for $H(r,s)4$ were found by D. Zagier [75] in 2012 who proved the following

Theorem 1.1.5. *For all integers $r, s \geq 0$, we have*

$$H(r,s)=2\sum_{k=1}^{r+s+1}(-1)^k\left[\binom{2k}{2r+2}-\left(1-\frac{1}{2^{2k}}\right)\binom{2k}{2s+1}\right]\zeta(2k+1)\zeta(\underbrace{2,2,\dots,2}_{r+s+1-k}).$$

1.2 RESULTS

1.2.1 Rational zeta series representations involving $\zeta(2n)$ for Apery's constant $\zeta(3)$

This is based on a joint work with Derek Orr [61]. We study some rational zeta series representations involving $\zeta(2n)$ for Apery's constant $\zeta(3)$. We also derive some formulas for these type of rational series involving values of the Riemann zeta and Dirichlet beta functions.

The central part in our derivation of rational zeta series for Apery's constant is played by the following transcendental function which was introduced in 1832 by T. Clausen,

$$\mathrm{Cl}_2(\theta):=-\int_0^\theta \log\left(2\sin\frac{t}{2}\right)dt=\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^2}.$$

This integral was considered for the first time by Clausen in 1832 ([39]), and it was investigated later by many authors [21, 31, 32, 33, 35, 36, 49, 52, 71, 72]. The Clausen functions are very important in mathematical physics and it is intimately connected with the polylogarithm, polygamma function and ultimately with the Riemann zeta function.

An important tool in our paper is the following result which is known as *Clausen acceleration formula*,

$$\mathrm{Cl}_2(\theta) = \theta - \theta \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)} \left(\frac{\theta}{2\pi}\right)^{2n}, |\theta| < 2\pi.$$

As it has been already highlighted in [13], the above formula is useful in evaluating to high precision the volume of hyperbolic 3-manifolds. In our paper, we give a new proof to this formula based on some ideas from [4] and we use this as well as other related acceleration formula to prove the following

Theorem 1.2.1. *We have the following series representations*

$$\zeta(3) = \frac{4\pi^2}{35}\left(\frac{3}{2} - \log\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n}\right),$$

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{8\pi^2}{9}\left(\frac{4}{3} - \log\left(\frac{\pi}{2}\right) + 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n}\right),$$

and

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{16\pi^2}{27}\left(\frac{1}{2} + \frac{3G}{\pi} - 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n}\right),$$

where G is the Catalan constant, and $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$ is the Dirichlet beta function.

Moreover, using similar techniques involving special functions, we also prove the following results for rational series representations involving $\zeta(2n)$ and binomial coefficients:

Theorem 1.2.2. *The following representation is true*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m} & m \text{ odd}, \\[2ex] \dfrac{1}{m}\left(2\zeta(m)\left(1 - \dfrac{1}{2^m}\right) - 1\right) & m \text{ even}. \end{cases}$$

Theorem 1.2.3. *We have the following series representation*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m}\left(1-\beta(m)\right) & m \text{ odd}, \\[2ex] \dfrac{1}{m}\left(\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m \text{ even}, \end{cases}$$

1.2.2 Another look at the evaluation of some multiple zeta and special Hurwitz zeta values

This work is based on the project [59] which involves evaluation of multiple zeta and Hurwitz zeta values. Some of these evaluations are directly connected to Zagier's formula for $H(r, s)$, where I obtain a different and direct proof for the special case $H(r, 0)$.

In analogy with the multiple zeta values, one can define the multiple Hurwitz zeta values and multiple t-values,

$$\zeta(k_1, k_2, \ldots, k_r; a_1, a_2, \ldots, a_r) = \sum_{1\le n_1<n_2<\ldots<n_r} \frac{1}{(n_1+a_1)^{k_1}(n_2+a_2)^{k_2}\ldots(n_r+a_r)^{k_r}}$$

and

$$t(k_1, k_2, \ldots, k_r) = 2^{-(k_1+k_2+\ldots+k_r)}\zeta(k_1, k_2, \ldots, k_r; -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}) =$$

$$= \sum_{1\le n_1<n_2<\ldots<n_r} \frac{1}{(2n_1-1)^{k_1}(2n_2-1)^{k_2}\ldots(2n_r-1)^{k_r}}.$$

My approach uses special functions such as Gauss hypergeometric function which is defined for $|x| < 1$ by the power series

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{x^n}{n!},$$

where $(q)_n = q(q+1)\cdot\ldots\cdot(q+n-1)$ is the Pochhammer symbol. The even and odd integer powers from the Taylor series of arcsin can be obtained by comparing the coefficients of like powers of λ in the formulas:

$$\cos(\lambda \arcsin(x)) = {}_2F_1\left(\frac{\lambda}{2}, -\frac{\lambda}{2}; \frac{1}{2}; x^2\right),$$

$$\sin(\lambda \arcsin(x)) = \lambda x \cdot {}_2F_1\left(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}; \frac{1}{2}; x^2\right).$$

This implies some formulas for the Taylor series for integer powers of arcsin function. More details are given in Chapter 5.

Hoffman-Zagier evaluation for $\zeta(2,2,\ldots,2)$ (Theorem 1.3) follows easily from these Taylor series expansion.

Theorem 1.2.4. *We have the following evaluations*

$$\zeta(\underbrace{2,2,\ldots,2}_{r}) = \frac{\pi^{2r}}{(2r+1)!}.$$

Theorem 1.2.5. *We have the following evaluation*

$$t(\underbrace{2,2,\ldots,2}_{r}) = \frac{\pi^{2r}}{2^{2r}(2r)!}.$$

Using the same ideas, I was able to derive the following results,

Theorem 1.2.6. *We have the following evaluations*

$$H(r-1,0) := \zeta(\underbrace{2,2,\ldots,2}_{r-1},3) = -4(2r+1)\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}}\cdot\zeta(\underbrace{2,2,\ldots,2}_{r}).$$

Theorem 1.2.7.

$$T(r) := t(\underbrace{2,2,\ldots,2}_{r},3) = -4(r+1)\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2r+1)(2n+2r+2)2^{2n}}\cdot t(\underbrace{2,2,\ldots,2}_{r}).$$

The above theorem gives an evaluation of Hoffman elements for multiple zeta and t-values in terms of rational zeta series involving $\zeta(2n)$. This is related to Zagier's formula for the special case $s = 0$ which reads as following

Corollary 1.2.8. *We have*

$$H(r-1,0) = 2\sum_{k=1}^{r}(-1)^k\left[\binom{2k}{2r} - \left(1-\frac{1}{2^{2k}}\right)2k\right]\zeta(2k+1)\zeta(\underbrace{2,2,\ldots,2}_{r-k}).$$

To derive the result from above, we have to prove the equality

$$\sum_{k=1}^{r}(-1)^k \cdot c_{r-1,0}^{k}\zeta(2k+1)\zeta(\underbrace{2,2,\ldots,2}_{r-k}) =$$

$$= -2(2r+1)\sum_{n=0}^{\infty}\frac{\zeta(2n)}{2^{2n}(2n+2r)(2n+2r+1)} \cdot \zeta(\underbrace{2,2,\ldots,2}_{r}),$$

where $c_{r-1,0}^{k} = \binom{2k}{2r} - \left(1-\frac{1}{2^{2k}}\right)2k$, where $r \geq 1$ is an integer. This issue is solved by the following missing ingredient,

Lemma 1.2.9. *We have the following equality*

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)2^{2n}} = \log 2 + \sum_{m=1}^{[p/2]}\frac{p!(-1)^m(4^m-1)\zeta(2m+1)}{(p-2m)!(2\pi)^{2m}} + \delta_{[p/2],p/2}\frac{p!(-1)^{p/2}\zeta(p+1)}{\pi^p}.$$

Theorem 2.2 and elementary algebraic manipulations of Lemma 2.3 give us Zagier's formula in the special case $s = 0$. It is somewhat surprising that from a rational zeta series involving $\zeta(2n)$ one can actually derive a formula involving $\mathbb{Q}$-linear combinations of odd zeta values. On the other hand, I do believe that a modification of the above argument could lead to a direct and completely analytic proof for Zagier's formula in the general case for $H(r, s)$.

Similarly, with the above corollary, we also have a Zagier type formula for multiple t-values,

Corollary 1.2.10.

$$T(r) = 2\sum_{k=1}^{r+1}(-1)^k\frac{1}{4^k(2(r-k+1)+1)}\left[\left(1-\frac{1}{4^k}\right)2k - (2r+1)(2r+2)\binom{2k}{2r+2}\right]t(\underbrace{2,2,\ldots,2}_{r-k+1})\zeta(2k+1).$$

1.3 ORGANIZATION OF THE THESIS

In Chapter 1, we present the overview of the thesis where we give an account of the research area and explain the main results obtained.

In Chapter 2, we introduce the Riemann zeta function and other L-functions and their special values. We also introduce Clausen transcendental functions and their connections to the values of the L-functions. We also give the proofs for the values $\zeta(2)$ using the Taylor series for $\arcsin^2$, and for $\zeta(2k)$ from the paper [4]. Last but not least, we introduce rational zeta series and we display some representations for Apery's constant $\zeta(3)$, and other odd zeta values.

Chapter 3 is organized as follows. In the first subsection (2.1) of the next section, we give a new proof for the Clausen acceleration formula. This formula serves as an ingredient for Theorem 2.2 where we provide some "fast" converging series representations for Apery's constant $\zeta(3)$. On the other hand, in the second subsection, we produce some rational representations involving $\zeta(2n)$ and binomial coefficients and we also provide some interesting series summations as corollaries.

Chapter 4 is dedicated to multiple zeta and Hurwitz zeta functions and their special values. First, we introduce the Gauss hypergeometric function and we relate some of its special values to the Taylor series for integer powers of the arcsin function. Also, we display some of the classical evaluations of multiple zeta values. Last, and not least, we present Zagier's formula involving Hoffman elements and its consequences.

Chapter 5 describes a new method for evaluating multiple zeta values involving 2's and 3's. We derive the classical evaluation of Hoffman and Zagier for the simplest family of MZVs involving Hoffman elements, and we also prove the famous Zagier formula in a special case. Similar results are obtained for the special multiple Hurwitz zeta values.

Chapter 6 describes future research projects that arise from this manuscript.

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Pluteam pe-un râu. Sclipiri bolnave
Fantastic trec din val în val,
În urmă-mi noaptea de dumbrave,
Nainte-mi domnul cel regal.

Căci pe-o insulă în farmec
Se nalţă negre, sfinte bolţi
Şi luna murii lungi albeşte,
Cu umbră împle orice colţ

Mă urc pe scări, intru-n lăuntru,
Tăcere-adâncă l-al meu pas.
Prin întuneric văd înalte
Chipuri de sfinţi p-iconostas...

Şi ochii mei în cap îngheaţă
Şi spaima-mi sacă glasul meu.
Eu îi rup vălul de pe faţă...
Tresar - încremenesc - sunt eu.

Mihai Eminescu, Vis, 1876

2.0 RIEMANN ZETA VALUES AND RATIONAL ZETA SERIES REPRESENTATIONS

2.1 L-FUNCTIONS AND THEIR SPECIAL VALUES

2.1.1 The Riemann and Hurwitz zeta, and other related functions

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{Re}\, s > 1.$$

The uniform and absolute convergence of $\zeta(s)$ is easy to see. Indeed, since $s = x + y \cdot i, x > 1$, we have that

$$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{|n^s|} = \sum_{n=1}^{\infty} \frac{1}{e^{|s \log n|}} = \sum_{n=1}^{\infty} \frac{|e^{-iy \log n}|}{n^x} = \sum_{n=1}^{\infty} \frac{1}{n^x},$$

which is convergent.

Moreover, one can extend $\zeta(s)$ from $\text{Re}\, s > 1$ to $\text{Re}\, s > 0$ in the following way:

$$(1 - 2^{1-s})\zeta(s) = \left(1 - 2 \cdot \frac{1}{2^s}\right)\left(\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots\right) =$$

$$= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \dots\right) =$$

$$= \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = \eta(s).$$

Therefore, we obtained another formula for the Riemann zeta function,

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \operatorname{Re} s > 0, s \neq 1.$$

On the other hand, since we have the equality $n^{-s}\Gamma(s) = \int_0^\infty u^{s-1} e^{-nu} du$, it is not hard to see that $\zeta(s)$ has the integral representation,

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Another very important property of $\zeta(s)$ is that it can be extended through analytic continuation to the whole complex plane where $s = 1$ is a simple pole. There are many ways for this procedure. For example, one way is through the integral representation provided above and contour integrals. Another way is by applying the Euler-Maclaurin summation formula for the function $\frac{1}{n^{-s}}$.

In 1859, Riemann came with a revolutionary idea by showing that $\zeta(s)$ satisfies the following functional equation

$$\xi(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) =: \xi(1-s).$$

Riemann's main idea was to introduce the functional,

$$\phi(s) := \int_1^\infty (\theta(t) - 1) t^{s/2} \frac{dt}{t} + \int_0^1 \left(\theta(t) - \frac{1}{\sqrt{t}}\right) t^{s/2} \frac{dt}{t},$$

where $\theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}, t > 0$ is the Jacobi theta function. Using Mellin transform of $\theta(t)$, properties of Euler's gamma function $\Gamma(s)$ and Poisson summation formula, Riemann found the following analytic continuation to the whole complex plane with a simple pole at $s = 1$,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left(\frac{1}{2}\phi(s) - \frac{1}{s} - \frac{1}{s-1}\right), \operatorname{Re} s > 1.$$

By Euler's reflection formula, $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin(\pi s)}$, and Legendre's duplication formula, $\Gamma(s)\Gamma(s+1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s)$, one can rewrite Riemann's functional equation in the following equivalent forms:

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s), \mathrm{Re}\, s < 1$$

and

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s), \mathrm{Re}\, s > 0.$$

Obviously,

$$\zeta(-2k) = 2(2\pi)^{-2k-1}\sin(\pi(-2k)/2)\Gamma(1+2k)\zeta(1+2k) = 0, k = 1, 2, \ldots.$$

The above assertion does not tell us what happens with the zeros of $\zeta(s)$ in the strip $0 < \mathrm{Re}\, s < 1$. This is given by the following conjecture:

Riemann Hypothesis. *All nontrivial zeros of* $\zeta(s)$ *are on the line* $\mathrm{Re}\, s = \frac{1}{2}$.

In 1882, Hurwitz defined the following "shifted" zeta function,

$$\zeta(s;a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \mathrm{Re}\, s > 1, 0 < a \leq 1.$$

It is clear that $\zeta(s;1) = \zeta(s)$. In the spirit of the Riemann zeta function, $\zeta(s;a)$ has an integral representation given by

$$\zeta(s;a) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}} dx, \mathrm{Re}\, s > 1, 0 < a \leq 1,$$

and it can be extended through analytic continuation to the whole complex plane except $s = 1$, a simple pole with residue 1. At $s = 1$ it has a simple pole with residue 1. The constant term is given by

$$\lim_{s \to 1}\left(\zeta(s,a) - \frac{1}{s-1}\right) = -\frac{\Gamma'(a)}{\Gamma(a)} = -\Psi(a),$$

where Ψ is the digamma function. Also, the Hurwitz zeta function is related to the *polygamma function*,

$$\Psi_m(z) = (-1)^{m+1} m! \zeta(m+1, z).$$

Moreover, it satisfies the following functional equation:

$$\zeta(1-s;a) = \frac{\Gamma(s)}{(2\pi)^s} \left(e^{\frac{-\pi i s}{2}} F(a,s) + e^{\frac{\pi i s}{2}} F(-a,s) \right),$$

where $F(x, s)$ is the periodic zeta function given by $F(x,s) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n x}}{n^s}$.

Similarly, one can define the Dirichlet eta and beta functions by

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s}, \mathrm{Re}\, s > 0, s \neq 1,$$

and

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \mathrm{Re}\, s > 0.$$

Alternatively, one can express the beta function in terms of Hurwitz zeta function by the following formula:

$$\beta(s) = \frac{1}{4^s} \left(\zeta(s, 1/4) - \zeta(s, 3/4) \right).$$

Equivalently, $\beta(s)$ has the following integral representation:

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1} e^{-t}}{1 + e^{-2t}} dt.$$

2.1.2 Special values of Riemann zeta and other L-functions

Let us recall what happens when we evaluate $\zeta(s)$ at integers. First, let us start with Euler's result from 1734 which asserts that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Later, in 1740, the same Euler proved the following generalization:

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k+1} \frac{2^{2k-1} B_{2k}}{(2k)!} \cdot \pi^{2k},$$

where B_k are the Bernoulli numbers and they are given by the Taylor series expansion

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \cdot z^k, |z| < 2\pi.$$

The key ingredient in the classical proof of Euler's formula is the following cotangent identity which is also due to Euler:

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n \geq 1} \frac{2x}{x^2 - n^2}.$$

Expanding the quotient inside the sum sign as a geometric series and interchanging the order of summation, we obtain the following identity:

$$\pi \cot(\pi x) = \frac{1}{x} - 2 \sum_{k \geq 1} \zeta(2k) x^{2k-1}.$$

Remarks.

- Euler's formula implies the following equality of subrings of $\mathbb{R}$:

$$\mathbb{Q}[\zeta(2), \zeta(4), \ldots] = \mathbb{Q}[\pi^2].$$

- Thanks to the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

  one can deduce the values of $\zeta(s)$ at negative integers: $\zeta(-k) = -\dfrac{B_{k+1}}{k+1}$ for all integers $k \geq 1$. In particular, $\zeta(-2k) = 0$ for all $k \geq 1$; we call these values the trivial zeros of the function ζ. Also, we have $\zeta(0) = -\frac{1}{2}$ and $\zeta(-1) = -\frac{1}{12}$.

  Question: *What can we say about $\zeta(s)$ when s is odd?*

Unfortunately, not too much is known, We cannot even find a closed formula for $\zeta(2n+1)$ in terms of π. This led to the following:

Transcendence conjecture. *The numbers $\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)$ are algebraically independent over $\mathbb{Q}$. In other words, for each $k \geq 0$ and each nonzero polynomial $P \in \mathbb{Z}[x_0, x_1, \ldots, x_k]$, we have*

$$P(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2k+1)) \neq 0.$$

At this point, this conjecture seems completely out of reach! In 1882, Lindemann proved that π is transcendental. If we combine this with Euler's formula, it follows that $\zeta(2k)$ are also transcendental. At this point, we do not know whether $\zeta(3)$ is transcendental or not. We only know that $\zeta(3)$ is irrational. This result was proved by R. Apery [7] back in 1979.

In 2001, Ball and Rivoal [9] proved that there exist infinitely many irrational numbers among $\zeta(2k+1)$. In fact, they proved even more:

$$\dim_{\mathbb{Q}}\langle 1, \zeta(3), \zeta(5), \ldots, \zeta(2n+1), \ldots\rangle \geq \frac{1}{3}\log n.$$

In 2002, W. Zudilin [79] proved that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9)$ and $\zeta(11)$ is irrational.

In other words, we have the following:

- $\zeta(-2k) = 0$ for $k = 1, 2, \ldots$ (trivial zeros)

- $\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$; with $\zeta(-1) = -\frac{1}{12}$
- The values $\zeta(2k)$, for $n = 1, 2, \ldots$ have been found by Euler in 1740
- The values $\zeta(1-2k)$, for $k = 1, 2, \ldots$ can be evaluated in terms of $\zeta(2k)$. In fact, we have

$$\zeta(1-2k) = 2(2\pi)^{2k}(-1)^k(2k-1)!\zeta(2k).$$

- There is a mystery about $\zeta(2k+1)$ values; we know $\zeta(3) \notin \mathbb{Q}$ thanks to Apery;
- $\zeta(0) = -\frac{1}{2}$
- $\zeta(1)$ does not exist, but one has the following

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma.$$

Regarding integer the values of the Hurwitz zeta and Dirichlet beta functions one can summarize as follows:

- $\zeta(-k; a) = -\dfrac{B_{k+1}(a)}{k+1}$, where $B_k(a)$ is the Bernoulli polynomial;
- $\beta(2) = G$ (Catalan's constant)
- $\beta(3) = \frac{\pi^3}{32}$
- $\beta(2k+1) = \dfrac{(-1)^k E_{2k}\pi^{2k+1}}{4^{k+1}(2k)!}$ where E_k are the Euler numbers in the Taylor series

$$\frac{2}{e^t + e^{-t}} = \sum_{k=0}^{\infty} \frac{E_k}{k!} t^k.$$

- Other special values include $\beta(0) = \frac{1}{2}, \beta(1) = \frac{\pi}{4}, \beta(-k) = \frac{E_k}{2}$.

Also, let us mention that the Bernoulli and Euler numbers are involved in the Taylor series for the tangent, cotangent, secant and cosecant functions:

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}, |x| < \frac{\pi}{2} \tag{2.1}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1}, |x| < \pi \tag{2.2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, |x| < \frac{\pi}{2} \tag{2.3}$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(2^{2n-1} - 1) B_{2n}}{(2n)!} x^{2n-1}, |x| < \pi \tag{2.4}$$

2.1.3 Clausen transcendental functions

The *Clausen function (Clausen integral)*, introduced by Thomas Clausen [39] in 1832, is a transcendental special function of single variable and it is defined by

$$\mathrm{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} = -\int_0^{\theta} \log\left(2 \sin\left(\frac{x}{2}\right)\right)\, dx.$$

It is intimately connected with the polylogarithm, inverse tangent integral, polygamma function, Riemann zeta function, Dirichlet eta function, and Dirichlet beta function.

Some well-known properties of the Clausen function include periodicity in the following sense:

$$\mathrm{Cl}_2(2k\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\, \mathrm{Cl}_2(\theta).$$

Moreover, it is quite clear from the definition that $\mathrm{Cl}_2(k\pi) = 0$ for k integer. For example, for $k = 1$ we deduce

$$\int_0^{\pi} \log\left(2 \sin\left(\frac{x}{2}\right)\right)\, dx = 0, \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2} \log 2.$$

By periodicity we have $\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = -\,\mathrm{Cl}_2\left(\frac{3\pi}{2}\right) = G$, where G is the Catalan constant defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159...$$

More generally, one can express the above integral as the following:

$$\int_0^{\theta} \log(\sin x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(\cos x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta) - \theta \log 2,$$

$$\int_0^\theta \log(1 + \cos x)\, dx = 2\,\mathrm{Cl}_2(\pi - \theta) - \theta \log 2,$$

and

$$\int_0^\theta \log(1 + \sin x)\, dx = 2G - 2\,\mathrm{Cl}_2\left(\frac{\pi}{2} + \theta\right) - \theta \log 2.$$

Another important property of Cl_2 is the following duplication formula:

$$\mathrm{Cl}_2(2\theta) = 2\,\mathrm{Cl}_2(\theta) - 2\,\mathrm{Cl}_2(\pi - \theta).$$

The Clausen integral can be evaluated to a high precision by the following acceleration formula:

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 1 - \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n} n(2n+1)} \theta^{2n}, |\theta| < 2\pi.$$

The higher order Clausen functions are

$$\mathrm{Cl}_{2m} = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^{2m+1}}, \mathrm{Cl}_{2m+1}(\theta) = \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^{2m+1}}.$$

Using the properties of the Riemann zeta function, we have the following particular values:

$$\mathrm{Cl}_{2m}(\pi) = 0, \mathrm{Cl}_{2m+1}(\pi) = -\frac{(4^m - 1)\zeta(2m+1))}{4^m},$$

and

$$\mathrm{Cl}_{2m}\left(\frac{\pi}{2}\right) = \beta(2m), \mathrm{Cl}_{2m+1}\left(\frac{\pi}{2}\right) = -\frac{(4^m - 1)\zeta(2m+1)}{2^{4m+1}},$$

where $\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$, Re$s > 0$ is the Dirichlet beta function.

Moreover,

$$\frac{d}{d\theta}\operatorname{Cl}_{2m}(\theta) = \operatorname{Cl}_{2m-1}(\theta), \frac{d}{d\theta}\operatorname{Cl}_{2m+1}(\theta) = -\operatorname{Cl}_{2m}(\theta),$$

and

$$\int_0^{\theta} \operatorname{Cl}_{2m}(x)dx = \zeta(2m+1) - \operatorname{Cl}_{2m+1}(\theta), \int_0^{\theta} \operatorname{Cl}_{2m-1}(x)dx = \operatorname{Cl}_{2m}(\theta).$$

This implies the following equalities:

$$\operatorname{Cl}_1(\theta) = -\log\left(2\sin\frac{\theta}{2}\right), |\theta| < 2\pi,$$

$$\operatorname{Cl}_3(\theta) = \zeta(3) - \int_0^{\theta} \operatorname{Cl}_2(t)dt,$$

$$\operatorname{Cl}_4(\theta) = \int_0^{\theta} \operatorname{Cl}_3(x)dx = \theta\zeta(3) - \int_0^{\theta}\int_0^{x} \operatorname{Cl}_2(t)dtdx = \theta\zeta(3) - \int_0^{\theta}(\theta - t)\operatorname{Cl}_2(t)dt,$$

$$\operatorname{Cl}_5(\theta) = \zeta(5) - \int_0^{\theta} \operatorname{Cl}_4(x)dx = \zeta(5) - \frac{1}{2}\zeta(3)\theta^2 + \frac{1}{2}\int_0^{\theta}(\theta - t)^2\operatorname{Cl}_2(t)dt.$$

By induction, one can show that

$$\operatorname{Cl}_m(\theta) = (-1)^{\left[\frac{m-1}{2}\right]}\sum_{k=1}^{\left[\frac{m-1}{2}\right]} \frac{(-1)^k\theta^{m-2k-1}}{(m-2k-1)!}\zeta(2k+1) + \frac{(-1)^{\left[\frac{m-1}{2}\right]}}{(m-3)!}\int_0^{\theta}(\theta - t)^{m-3}\operatorname{Cl}_2(t)dt, m \geq 3.$$

2.1.4 A quick proof of Euler's $\zeta(2) = \frac{\pi^2}{6}$

This is proof is due originally to Euler [8]. Let us consider the power series expansion for $\arcsin^2(x)$ near $x = 0$. Let us consider the following initial value problem

$$(1 - x^2)y'' - xy' - 2 = 0, y(0) = y'(0) = 0.$$

It is quite clear that the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = \arcsin^2(x)$ satisfies the ODE above and since the coefficients are variable it is natural to look for a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$. Differentiation term by term yields

$$y'(x) = \sum_{n=0}^{\infty} (n + 1)a_{n+1}x^n$$

and

$$y''(x) = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n.$$

Since $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$, our initial value problem will be equivalent to

$$2a_2 + 6a_3x + \sum_{n=2}^{\infty} ((n + 2)(n + 1)a_{n+2} - n^2 a_n)x^n = 2.$$

Clearly, $a_2 = 1$, $a_3 = 0$, and for $n \geq 2$, $a_{2n+1} = 0$, $a_{n+2} = \frac{n^2}{(n + 2)(n + 1)} a_n$.

We easily obtain that for $n \geq 2$, $a_{2n+1} = 0$ and $a_{2n} = \frac{(2^{n-1}(n - 1)!)^2}{(2n)(2n - 1) \ldots 4 \cdot 3} = \frac{1}{2} \frac{2^{2n}}{n^2 \binom{2n}{n}}$.

Therefore, this implies

$$y(x) = \arcsin^2(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}}, |x| \leq 1.$$

This series converges at $x = 1$ by Raabe's test, and for $x \in (-1, 1)$ it is uniformly convergent by the Weierstrass M-test.

Now, substitute $x = \sin t$, $0 < t < \frac{\pi}{2}$, and we get

$$t^2 = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2\binom{2n}{n}} \sin^{2n} t.$$

Integrating from 0 to $\frac{\pi}{2}$, we have

$$\frac{\pi^3}{24} = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2\binom{2n}{n}} \int_0^{\frac{\pi}{2}} \sin^{2n} t.$$

On the other hand, Wallis' formula (integral form) tells us that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} t dt = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$, and thus we finally obtain Euler's celebrated

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Remark. In [30], Choe gave a similar proof by expanding $\arcsin x$ near $x = 0$.

2.1.5 An elementary proof of Euler's formula for $\zeta(2k)$

We present an elementary proof of Euler's formula for $\zeta(2k)$ using the Taylor series for $\tan x$. Our presentation will follow [4]. But, first, let us recall the Taylor series for the tangent function:

$$\tan x = \sum_{n=0}^{\infty} \frac{(-4)^n(1-4^n)B_{2n}}{(2n)!} x^{2n-1}, |x| < \frac{\pi}{2}.$$

Integrating from 0 to $\frac{\pi}{2}$ and using the above formula, we have that for $|x| < \frac{\pi}{2}$ and $|y| < 1$,

$$\int_0^{\frac{\pi}{2}} \tan(xy)dx = \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}(xy)^{2n-1}dx =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} y^{2n-1} \int_0^{\frac{\pi}{2}} x^{2n-1}dx =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} \cdot \frac{\pi^{2n}}{2n2^{2n}} y^{2n-1}.$$

On the other hand, using integration by parts, together with the infinite product formula for the cosine function and Fubini's theorem, we have

$$\int_0^{\frac{\pi}{2}} \tan(xy)dx = -\frac{\log(\cos(xy))}{y}\Big|_0^{\frac{\pi}{2}} = -\frac{-\log(\cos(\pi y/2))}{y} = \frac{-\log\left(\prod_{n=1}^{\infty}\left(1-\frac{\left(\frac{\pi y}{2}\right)^2}{\pi^2\left(n-\frac{1}{2}\right)^2}\right)\right)}{y} =$$

$$= \frac{-\log\left(\prod_{n=1}^{\infty}\left(1-\frac{y^2}{(2n-1)^2}\right)\right)}{y} = \frac{-\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\cdot\frac{(-1)^k}{(2n-1)^{2k}}y^{2k}}{y} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{2k+1}}{k}\cdot\frac{y^{2k-1}}{(2n-1)^{2k}} =$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\cdot\frac{y^{2k-1}}{(2n-1)^{2k}} = \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}\cdot\frac{y^{2k-1}}{k}\right).$$

Now, identifying the coefficients in the two series expansions, we obtain

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}} = \frac{(-1)^k(1-4^k)B_{2k}}{(2k)!}\cdot\frac{\pi^{2k}}{2}.$$

In this moment, we only need to observe that

$$\sum_{n=1}^{\infty}\frac{1}{n^{2k}} - \sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}$$

or equivalently

$$\zeta(2k) - \frac{1}{2^{2k}}\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}},$$

and this gives us

$$\left(1-\frac{1}{4^k}\right)\zeta(2k) = \frac{(-1)^k(1-4^k)B_{2k}}{(2k)!}\cdot\frac{\pi^{2k}}{2},$$

and Euler's formula follows immediately.□

2.2 RATIONAL ZETA SERIES REPRESENTATIONS INVOLVING $\zeta(2N)$ FOR APERY CONSTANT $\zeta(3)$

2.2.1 Classical rational zeta series involving $\zeta(2n)$

A classical problem which goes back to Goldbach and Bernoulli asserts that

$$\sum_{\omega\in S}(\omega-1)^{-1}=1,$$

where $S=\{n^k : n,k\in\mathbb{Z}_{\geq 0}-\{1\}\}$. In terms of the Riemann zeta function $\zeta(s)$, the above problem reads as,

$$\sum_{n=2}^{\infty}(\zeta(n)-1)=1.$$

Also, there are other representations for $\log 2$ and γ (Euler-Mascheroni constant) such as,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n}=\log 2,$$

and

$$\sum_{n=2}^{\infty}\frac{\zeta(n)-1}{n}=1-\gamma.$$

For instance, one way to generate rational zeta series involving $\zeta(2n)$ is by looking at the cotangent power series formula in the form:

$$\sum_{n=1}^{\infty}\zeta(2n)x^{2n}=\frac{1}{2}(1-\pi x\cot(\pi x)),|x|<1.$$

Dividing by x and integrating once, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n}x^{2n}=\log\left(\frac{\pi x}{\sin(\pi x)}\right),|x|<1.$$

For $x = \frac{1}{2}$ and $x = \frac{1}{4}$ in the above formulas, we obtain the following representations:

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{2n}} = \frac{1}{2} \tag{2.5}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2^{4n}} = \frac{4-\pi}{8} \tag{2.6}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n2^{2n}} = \log \pi - \log 2 \tag{2.7}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n2^{4n}} = \log \pi - \frac{3}{2} \log 2 \tag{2.8}$$

Moreover, integrating from 0 to $\frac{1}{2}$ the last power series equality, we derive

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)2^{2n}} = \log \pi - 1 \tag{2.9}$$

which can be found in [69]. These results and many others will be generalized in Chapter 3.

2.2.2 Euler's rational zeta series representation for $\zeta(3)$

Another remarkable result involving the Riemann's zeta function is the following series representation for Apery's constant [46],

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}} = \frac{\pi^2}{7}\left(1 - 4\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}\right) \tag{2.10}$$

Since similar ideas will be used in proving other results in Chapter 5, in what follows, we present a proof of a similar formula which is also due to Euler [12],

Theorem 2.2.1. *We have the following representation for Apery's constant,*

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}. \tag{2.11}$$

Proof. The ideas will be similar with the proof of Euler's $\zeta(2) = \frac{\pi^2}{6}$ from subsection 2.1.4. Indeed, let us recall the Taylor series for $\arcsin^2 x$ near $x = 0$,

$$\arcsin^2(x) = \frac{1}{2}\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}}, |x| \leq 1.$$

Dividing by x and integrating from 0 to $\sin t$ we obtain

$$\int_0^{\sin t} \frac{\arcsin^2 x}{x} dx = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n^3\binom{2n}{n}} \sin^{2n} t.$$

By the substitution $x = \sin u$ in the integral, we have

$$\int_0^t u^2 \cot u du = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n^3\binom{2n}{n}} \sin^{2n} t.$$

Now, by using the power series expansion of the cotangent, we have

$$u \cot u = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n}, |u| < \pi.$$

This gives us

$$\int_0^t u^2 \cot u du = -2\int_0^t \sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} u^{2n+1} du = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n^3\binom{2n}{n}} \sin^{2n} t,$$

which is equivalent to

$$-2\sum_{n=0}^{\infty} \frac{\zeta(2n)t^{2n+2}}{\pi^{2n}(2n+2)} = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{2n^3\binom{2n}{n}} \sin^{2n} t.$$

Finally, integrating from 0 to $\frac{\pi}{2}$, and using Wallis' integral formula, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} t dt = \frac{\pi}{2^{2n+1}}\binom{2n}{n}$, we arrive at our result. □

2.2.3 Other fast converging rational zeta series for $\zeta(3)$ and $\zeta(2n+1)$

In [47], J. Ewell gave a further generalization of Euler's representations of Apery's constant $\zeta(3)$,

Theorem 2.2.2. *The following representation for $\zeta(r)$ (for any integer $r > 2$) is valid*

$$\zeta(r) = \frac{2^{r-2}}{2^r - 1}\pi^2 \sum_{n=0}^{\infty}(-1)^n A_{2n}(r-2)\frac{\pi^{2n}}{(2n+2)!},$$

where the coefficients $A_{2n}(r)$ are given by

$$A_{2n}(r) = \sum \frac{\binom{2n}{2i_1, 2i_2, \ldots, 2i_r}}{(2i_1+1)(2(i_1+i_2)+1)\ldots(2(i_1+i_2+\ldots i_{r-1})+1)}.$$

Here the sum is taken over all r-tuples $(i_1, i_2, \ldots, i_r)$od nonnegative integers whose sum is n, $\binom{2n}{2i_1, 2i_2, \ldots, 2i_r}$ is multinomial coefficient, and B_{2i} are the Bernoulli numbers.

Another generalization of Euler's results from [46] was given by D. Cvijovic and J. Klinowski [42] for $\zeta(2k+1)$. The result is based on the functional equation for $\zeta(s)$.

Theorem 2.2.3. *Let $\zeta(s)$ be the Riemann zeta function and n a positive integer. We then have*

$$\zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\left[\sum_{k=1}^{n-1}(-1)^{k-1}\frac{k\zeta(2k+1)}{\pi^{2k}(2n-2k)!} + \sum_{k=0}^{\infty}\frac{\zeta(2k)(2k)!}{2^{2k}(2k+2n)!}\right],$$

where the finite sum on the right-harnd side is 0 *when* $n = 1$.

For other similar result, one can consult [2, 3, 35, 32, 33, 64].

3.0 SERIES REPRESENTATIONS FOR APERY CONSTANT $\zeta(3)$ INVOLVING VALUES $\zeta(2N)$

3.1 INTRODUCTION AND PRELIMINARIES

Let us recall that, in 1734, Leonard Euler produced a sensation when he proved the following formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Later, in 1740, Euler generalized the above formula for even positive integers:

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k} 2^{2k-1} \pi^{2k}}{(2k)!}. \tag{3.1}$$

An elementary proof of (1) can be found in [6]. In [4], de Amo, Carrillo and Sanchez produced another proof of Euler's formula (1) using the Taylor series expansion of the tangent function and Fubini's theorem. We have already presented the proof in full detail in the previous chapter 2.1.5.

In this chapter, using similar ideas as in [4], but for other functions, we provide a new proof for the Clausen acceleration formula that will serve as an application in displaying some fast representations for Apery's constant $\zeta(3)$. Moreover, the Taylor series expansion for the secant and cosecant functions combined with some other integration techniques will give us some interesting rational series representations involving $\zeta(2n)$ and binomial coefficients. Also, we display some particular cases of such series. Some of them are well-known representations of π. For the sake of completeness we recall the Taylor series for the tangent, cotangent, secant and cosecant functions:

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1}, |x| < \frac{\pi}{2} \tag{3.2}$$

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^{n} 2^{2n} B_{2n}}{(2n)!} x^{2n-1}, |x| < \pi \tag{3.3}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^{n} E_{2n}}{(2n)!} x^{2n}, |x| < \frac{\pi}{2} \tag{3.4}$$

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2(2^{2n-1}-1)B_{2n}}{(2n)!} x^{2n-1}, |x| < \pi \tag{3.5}$$

where E_n are the Euler numbers, and B_n the Bernoulli numbers.

Recall the Riemann zeta function $\zeta(s)$ and the Hurwitz (generalized) function $\zeta(s,a)$ are defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \mathrm{Re}(s) > 1, \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \mathrm{Re}(s) > 0, s \neq 1, \end{cases} \tag{3.6}$$

and

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \mathrm{Re}(s) > 1; a \neq 0, -1, -2, \ldots \tag{3.7}$$

Both of them are analytic over the whole complex plane, except $s = 1$, where they have a simple pole. Also, from the two definitions above, one can observe that

$$\zeta(s) = \zeta(s,1) = \frac{1}{2^s - 1} \zeta\left(s, \frac{1}{2}\right) = 1 + \zeta(s,2).$$

Recall Clausen's function (or Clausen's integral, see [39]) $\mathrm{Cl}_2(\theta)$ which is defined by

$$\mathrm{Cl}_2(\theta) := \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} = -\int_0^{\theta} \log\left(2 \sin\left(\frac{x}{2}\right)\right) \, dx.$$

This integral was considered for the first time by Clausen in 1832 ([39]), and it was investigated later by many authors [21, 31, 32, 33, 35, 36, 49, 52, 71, 72]. The Clausen functions are very important in mathematical physics. Some well-known properties of the Clausen function include periodicity in the following sense:

$$\mathrm{Cl}_2(2k\pi \pm \theta) = \mathrm{Cl}_2(\pm\theta) = \pm\,\mathrm{Cl}_2(\theta).$$

Moreover, it is quite clear from the definition that $\mathrm{Cl}_2(k\pi) = 0$ for k integer. For example, for $k = 1$ we deduce

$$\int_0^{\pi} \log\left(2\sin\left(\frac{x}{2}\right)\right)\, dx = 0, \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx = -\frac{\pi}{2}\log 2.$$

By periodicity we have $\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = -\,\mathrm{Cl}_2\left(\frac{3\pi}{2}\right) = G$, where G is the Catalan constant defined by

$$G := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.9159...$$

More generally, one can express the above integral as the following:

$$\int_0^{\theta} \log(\sin x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(2\theta) - \theta\log 2,$$

$$\int_0^{\theta} \log(\cos x)\, dx = -\frac{1}{2}\,\mathrm{Cl}_2(\pi - 2\theta) - \theta\log 2,$$

$$\int_0^{\theta} \log(1 + \cos x)\, dx = 2\,\mathrm{Cl}_2(\pi - \theta) - \theta\log 2,$$

and

$$\int_0^{\theta} \log(1 + \sin x)\, dx = 2G - 2\,\mathrm{Cl}_2\left(\frac{\pi}{2} + \theta\right) - \theta\log 2.$$

The Dirichlet beta function is defined as

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

Alternatively, one can express the beta function in terms of Hurwitz zeta function by the following formula valid in the whole complex s-plane,

$$\beta(s) = \frac{1}{4^s}\left(\zeta(s, 1/4) - \zeta(s, 3/4)\right).$$

Clearly, $\beta(2) = G$ (Catalan's constant), $\beta(3) = \frac{\pi^3}{32}$, and $\beta(2n+1) = \dfrac{(-1)^n E_{2n} \pi^{2n+1}}{4^{n+1}(2n)!}$, where E_n are the Euler numbers mentioned above which are given by the Taylor series $\dfrac{2}{e^t + e^{-t}} = \displaystyle\sum_{n=0}^{\infty} \frac{E_n}{n!} t^n$.

3.2 MAIN RESULTS

3.2.1 A new proof for Clausen accelerationuon formula and some representations for $\zeta(3)$

We provide a new proof for the classical Clausen acceleration formula [33, 58].

Proposition 3.2.1. *We have the following representation for the Clausen function* $\mathrm{Cl}_2(\theta)$*,*

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 1 - \log|\theta| + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n} n(2n+1)} \theta^{2n}, |\theta| < 2\pi. \tag{3.8}$$

Proof. Integrating by parts the function $xy\cot(xy)$ we have

$$\int_0^{\frac{\pi}{2}} xy \cot(xy)\, dx = \frac{\pi}{2} \log\left(\sin\left(\frac{\pi y}{2}\right)\right) - \int_0^{\frac{\pi}{2}} \log(\sin(xy))\, dx$$

$$= \frac{\pi}{2} \log\left(\sin\left(\frac{\pi y}{2}\right)\right) - \frac{1}{2y} \int_0^{\pi y} \log\left(\sin\left(\frac{u}{2}\right)\right) du$$

$$= \frac{\pi}{2} \log\left(\sin\left(\frac{\pi y}{2}\right)\right) + \frac{\pi}{2} \log 2 + \frac{1}{2y} \mathrm{Cl}_2(\pi y)$$

$$= \frac{\pi}{2}\log\left(\frac{\pi y}{2}\prod_{k=1}^{\infty}\left(1-\frac{y^2}{4k^2}\right)\right)+\frac{\pi}{2}\log 2+\frac{1}{2y}\,\mathrm{Cl}_2(\pi y)$$

$$= \frac{\pi}{2}\log\left(\frac{\pi y}{2}\right)-\frac{\pi}{2}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left(\frac{y^2}{4k^2}\right)^n}{n}+\frac{\pi}{2}\log 2+\frac{1}{2y}\,\mathrm{Cl}_2(\pi y),$$

where we have used the product formula for sine, $\sin(\pi x) = \pi x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right)$.

On the other hand, using the formula $\theta\cot\theta = 1-2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\theta^{2n}$ we have

$$\int_0^{\frac{\pi}{2}} xy\cot(xy)\,dx = \frac{\pi}{2}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)\left(\frac{\pi}{2}\right)^{2n+1}}{\pi^{2n}(2n+1)}y^{2n}.$$

Therefore, we obtain

$$\frac{\pi}{2}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)\left(\frac{\pi}{2}\right)^{2n+1}}{\pi^{2n}(2n+1)}y^{2n} = \frac{\pi}{2}\log(\pi y)-\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}y^{2n}+\frac{1}{2y}\,\mathrm{Cl}_2(\pi y).$$

This implies that

$$\mathrm{Cl}_2(\pi y) = 2y\left(\frac{\pi}{2}-\pi\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n+1)4^n}y^{2n}-\frac{\pi}{2}\log(\pi y)+\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}y^{2n}\right),$$

which, after some computations, is equivalent to

$$\mathrm{Cl}_2(\pi y) = \pi y-\pi y\log(\pi y)+\pi y\sum_{n=1}^{\infty}\frac{\zeta(2n)(\pi y)^{2n}}{4^n n(2n+1)\pi^{2n}}.$$

Setting $\alpha = \pi y$, we obtain our result.□

Remarks. In particular case of $\theta = \frac{\pi}{2}$, using the fact that $\mathrm{Cl}_2(\frac{\pi}{2}) = G$, we obtain

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)16^n} = \frac{2G}{\pi}-1+\log\left(\frac{\pi}{2}\right). \tag{3.9}$$

In computations, the following accelerated peeled form [18] is used

$$\frac{\mathrm{Cl}_2(\theta)}{\theta} = 3 - \log\left(|\theta|\left(1 - \frac{\theta^2}{4\pi^2}\right)\right) - \frac{2\pi}{\theta}\log\left(\frac{2\pi+\theta}{2\pi-\theta}\right) + \sum_{n=1}^{\infty}\frac{\zeta(2n)-1}{n(2n+1)}\left(\frac{\theta}{2\pi}\right)^{2n}. \tag{3.10}$$

It is well-known that $\zeta(n) - 1$ converges to zero rapidly for large values of n. In [72], Wu, Zhang and Liu derive the following representation for the Clausen function $\mathrm{Cl}_2(\theta)$,

Proposition 3.2.2. *We have*

$$\mathrm{Cl}_2(\theta) = \theta - \theta\log\left(2\sin\frac{\theta}{2}\right) - \sum_{n=1}^{\infty}\frac{2\zeta(2n)}{(2n+1)(2\pi)^{2n}}\theta^{2n+1}. \tag{3.11}$$

Proof. For $0 \le \theta \le \pi$, let us consider again the Clausen integral,

$$\mathrm{Cl}_2(\theta) := -\int_0^{\theta}\log\left(2\sin\left(\frac{x}{2}\right)\right)\,dx.$$

Integrating by parts once, we have

$$\mathrm{Cl}_2(\theta) = -\theta\log\left(2\sin\frac{\theta}{2}\right) + \frac{\pi^2}{2}\int_0^{\frac{\theta}{\pi}} z\cot\left(\frac{\pi z}{2}\right)\,dz.$$

Employing again the Taylor series for the cotangent function, $\theta\cot(\theta) = 1 - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\theta^{2n}$, we have

$$\int_0^{\frac{\theta}{\pi}}\frac{z\pi}{2}\cot\left(\frac{\pi z}{2}\right)\,dz = \int_0^{\frac{\theta}{\pi}}\left(1 - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\left(\frac{z\pi}{2}\right)^{2n}\right)\,dz =$$

$$= \frac{\theta}{\pi} - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}\pi^{2n+1}(2n+1)}\theta^{2n+1}.$$

Therefore, we finally obtain formula (11),

$$\mathrm{Cl}_2(\theta) = \theta - \theta\log\left(2\sin\frac{\theta}{2}\right) - \sum_{n=1}^{\infty}\frac{2\zeta(2n)}{(2\pi)^{2n}(2n+1)}\theta^{2n+1}.$$

□

Remark. The same idea of integrating once by parts is displayed in [71], but instead the author uses the Chebyshev's expansion and properties of the Chebyshev's polynomials for $z\cot\left(\frac{\pi z}{2}\right)$ to obtain efficient numerical values of the Clausen integral.

It is interesting to see that integrating the above formula from 0 to $\pi/2$ we have the following representation for $\zeta(3)$ (see [35]),

$$\zeta(3)=\frac{4\pi^2}{35}\left(\frac{1}{2}+\frac{2G}{\pi}-\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(n+1)(2n+1)16^n}\right). \tag{3.12}$$

Also, in [67], Strivastava, Glasser and Adamchik derive series representations for $\zeta(2n+1)$ by evaluating the integral $\int_0^{\pi/\omega} t^{s-1}\cot t\,dt$, $s,\omega\geq 2$ integers in two different ways. One of the ways involves the generalized Clausen functions. When they are evaulated in terms of $\zeta(2n+1)$ one obtains the following formula for $\zeta(3)$,

$$\zeta(3)=\frac{2\pi^2}{9}\left(\log 2+2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+3)4^n}\right). \tag{3.13}$$

Another remarkable result which led to Apery's proof of the irrationality of $\zeta(3)$ is given by the rapidly convergent series

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3\binom{2n}{n}}. \tag{3.14}$$

Moreover, in [42], Cvijovic and Klinowski derive the following formula

$$\zeta(3)=-\frac{\pi^2}{3}\sum_{n=0}^{\infty}\frac{(2n+5)\zeta(2n)}{(2n+1)(2n+2)(2n+3)2^{2n}}. \tag{3.15}$$

This formula is related to the one found by Ewell [46],

$$\zeta(3)=-\frac{4\pi^2}{7}\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}. \tag{3.16}$$

The following result will provide some new representations for Apery's constant $\zeta(3)$. The main ingredients in the proof of this next result are Clausen acceleration formulae and Fubini's theorem.

Theorem 3.2.3. *We have the following series representations*

$$\zeta(3) = \frac{4\pi^2}{35}\left(\frac{3}{2} - \log\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n}\right), \tag{3.17}$$

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{8\pi^2}{9}\left(\frac{4}{3} - \log\left(\frac{\pi}{2}\right) + 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n}\right), \tag{3.18}$$

and

$$\zeta(3) = -\frac{64}{3\pi}\beta(4) + \frac{16\pi^2}{27}\left(\frac{1}{2} + \frac{3G}{\pi} - 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n}\right), \tag{3.19}$$

where G is the Catalan constant, and $\beta(s)$ is the Dirichlet beta function.

Proof. For (17), integrating (8) from 0 to $\pi/2$ and using Fubini's theorem, we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(y)\, dy &= \int_0^{\frac{\pi}{2}} (y - y\log y)\, dy + \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2\pi)^{2n} n(2n+1)} y^{2n+1}\, dy \\ &= \frac{\pi^2}{8} - \left(\frac{\pi^2}{8}\log\left(\frac{\pi}{2}\right) - \frac{1}{2}\int_0^{\frac{\pi}{2}} y\, dy\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)\left(\frac{\pi}{2}\right)^{2n+2}}{(2\pi)^{2n} n(2n+1)(2n+2)},\end{aligned}$$

which is equivalent to

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(y)\, dy = \frac{\pi^2}{8}\left(\frac{3}{2} - \log\left(\frac{\pi}{2}\right) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)16^n}\right).$$

Alternatively, we can integrate the Clausen function using its definition given in the introduction and changing the order of integration as follows:

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(y)\, dy = -\int_0^{\frac{\pi}{2}}\int_0^{y} \log\left(2\sin\left(\frac{x}{2}\right)\right) dx dy = -\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{2}} \log\left(2\sin\left(\frac{x}{2}\right)\right) dy dx$$

$$= -\int_0^{\frac{\pi}{2}} \frac{\pi}{2}\log\Big(2\sin\Big(\frac{x}{2}\Big)\Big)\,dx + \int_0^{\frac{\pi}{2}} x\log 2\,dx + \int_0^{\frac{\pi}{2}} x\log\Big(\sin\Big(\frac{x}{2}\Big)\Big)\,dx.$$

Using the definition of the Clausen function again and, after the substitution $x = 2u$, using the identity $\int_0^{\pi/4} u\log(\sin(u))\,du = \frac{35}{128}\zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32}\log 2$ (see [35]), we have

$$\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(y)\,dy = \frac{\pi}{2}\mathrm{Cl}_2\Big(\frac{\pi}{2}\Big) + \frac{\pi^2}{8}\log 2 + 4\Big(\frac{35}{128}\zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32}\log 2\Big) = \frac{35}{32}\zeta(3).$$

Setting the two results equal to each other and solving for $\zeta(3)$ gives us our result. For (18), we proceed similarly to the previous method. We will begin with

$$\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})\,dy = \int_0^{\pi^2/4} (\sqrt{y} - \sqrt{y}\log(\sqrt{y}))\,dy + \int_0^{\pi^2/4} \sum_{n=1}^{\infty} \frac{\zeta(2n)y^{n+1/2}}{n(2n+1)(2\pi)^{2n}}\,dy$$
$$= \frac{\pi^3}{12} - \Big(\frac{\pi^3}{12}\log\Big(\frac{\pi}{2}\Big) - \frac{1}{3}\int_0^{\pi^2/4} \sqrt{y}\,dy\Big) + \sum_{n=1}^{\infty} \frac{\zeta(2n)(\frac{\pi^2}{4})^{n+3/2}}{n(2n+1)(n+3/2)(2\pi)^{2n}},$$

which is equivalent to

$$\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})\,dy = \frac{\pi^3}{12}\Big(\frac{4}{3} - \log\Big(\frac{\pi}{2}\Big) + 3\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)(2n+3)16^n}\Big).$$

On the other hand, using the definition of the Clausen function and changing the order of integration, we see

$$\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})\,dy = -\int_0^{\pi^2/4}\int_0^{\sqrt{y}} \log\Big(2\sin\Big(\frac{x}{2}\Big)\Big)\,dxdy$$
$$= -\int_0^{\frac{\pi}{2}}\int_{x^2}^{\pi^2/4} \log\Big(2\sin\Big(\frac{x}{2}\Big)\Big)\,dydx = \frac{\pi^2}{4}\mathrm{Cl}_2\Big(\frac{\pi}{2}\Big) + \int_0^{\pi/2} x^2\log\Big(2\sin\Big(\frac{x}{2}\Big)\Big)\,dx$$
$$= \frac{\pi^2 G}{4} + \frac{1}{768}\Big(72\pi\zeta(3) - 192\pi^2 G + \psi_3\Big(\frac{1}{4}\Big) - \psi_3\Big(\frac{3}{4}\Big)\Big),$$

where ψ_3 is the trigamma function. Using the identity $\psi_n(z) = (-1)^{n+1}n!\zeta(n+1,z)$ where $\zeta(k,z)$ is the Hurwitz zeta function, and the relationship between the Hurwitz zeta function and the beta function, we have that

$$\int_0^{\pi^2/4} \mathrm{Cl}_2(\sqrt{y})\, dy = \frac{3\pi}{32}\zeta(3) + 2\beta(4).$$

Setting this result equal to the previous result of the integral and solving for $\zeta(3)$, we see (18) is indeed true. For (19), instead of integrating the Clausen function using (8), we will integrate it using (11). This gives us,

$$\int_0^{\frac{\pi^2}{4}} \mathrm{Cl}_2(\sqrt{y})\, dy = \int_0^{\frac{\pi^2}{4}} \sqrt{y}\, dy - \int_0^{\frac{\pi^2}{4}} \sqrt{y} \log\Big(2\sin\Big(\frac{\sqrt{y}}{2}\Big)\Big)\, dy$$

$$+2\int_0^{\frac{\pi^2}{4}} \sum_{n=1}^{\infty} \frac{\zeta(2n) y^{n+1/2}}{(2n+1)(2\pi)^{2n}}\, dy$$

$$= \frac{\pi^3}{12} - 2\int_0^{\frac{\pi}{2}} u^2 \log\Big(2\sin\Big(\frac{u}{2}\Big)\Big)\, du + 2\sum_{n=1}^{\infty} \frac{\zeta(2n)(\frac{\pi^2}{4})^{n+3/2}}{(2n+1)(n+3/2)(2\pi)^{2n}}$$

$$= \frac{\pi^3}{12} - \frac{1}{384}\Big(72\pi\zeta(3) - 192\pi^2 G + 3!4^4\beta(4)\Big) - \frac{\pi^3}{2}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n}.$$

Setting this result equal to the previous Clausen formula yields

$$\Big(\frac{3\pi}{32} + \frac{3\pi}{16}\Big)\zeta(3) = \frac{9\pi}{32}\zeta(3) = \frac{\pi^3}{12} + \frac{\pi^2 G}{2} - 6\beta(4) - \frac{\pi^3}{2}\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+3)16^n},$$

and from here, (19) follows. □

Remark. To prove (17), one could solve for G in (9) and plug it into (12) and rearrange. Also, to prove (19), one could solve for $\log\left(\frac{\pi}{2}\right)$ in (9) and plug that into (18) and rearrange. Further, if we integrate (10), we arrive at a rapidly converging series representation for $\zeta(3)$, that is

$$\zeta(3) = \frac{2\pi^2}{35}\Big(9 + 138\log 2 - 18\log 3 - 50\log 5 - 2\log\pi + 2\sum_{n=1}^{\infty} \frac{\zeta(2n) - 1}{n(2n+1)(n+1)16^n}\Big). \tag{3.20}$$

3.2.2 Rational series representations involving $\zeta(2n)$ and binomial coefficients

As it has been already been highlighted in [18] one can relate the rational ζ-series with various Dirichlet L-series. A rational ζ-series can be accelerated for computational purposes provided that one solves the exact sum

$$\sum_{n=2}^{\infty} \frac{q_n}{a^n},$$

where $a = 2, 3, 4, \ldots$.

In fact, it has been already highlighted in [18] we shall call *rational ζ-series* of a real number x, the following representation:

$$x = \sum_{n=2}^{\infty} q_n \zeta(n, m),$$

where q_n is a rational number and $\zeta(n, m)$ is the Hurwitz zeta function. For $m > 1$ integer, one has

$$x = \sum_{n=2}^{\infty} q_n \left(\zeta(n) - \sum_{j=1}^{m-1} j^{-n} \right).$$

In the particular case $m = 2$, one has the following series representations:

$$1 = \sum_{n=2}^{\infty} (\zeta(n) - 1)$$

$$1 - \gamma = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(n) - 1)$$

$$\log 2 = \sum_{n=2}^{\infty} \frac{1}{n} (\zeta(2n) - 1),$$

where γ is the Euler-Mascheroni constant. For other rational zeta series representations we recommend [3, 35, 67].

Theorem 3.2.4. *The following representation is true*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m} & m \,\text{odd}, \\ \\ \dfrac{1}{m}\left(2\zeta(m)\left(1-\dfrac{1}{2^m}\right)-1\right) & m \,\text{even}\,. \end{cases} \tag{3.21}$$

Proof. We start by integrating $xy\csc(xy)$ two different ways. Applying integration by parts, L'Hospital's rule, and properties of logarithm, we find

$$\int_0^{\frac{\pi}{2}} xy\csc(xy)\,dx = -\frac{\pi}{2}\left(\log\left(1+\cos\left(\frac{\pi y}{2}\right)\right) - \log\left(\sin\left(\frac{\pi y}{2}\right)\right)\right)$$

$$+\int_0^{\frac{\pi}{2}} \log(1+\cos(xy))\,dx - \int_0^{\frac{\pi}{2}} \log(\sin(xy))\,dx$$

$$= -\frac{\pi}{2}\frac{d}{d\alpha}\Big(2\,\mathrm{Cl}_2(\pi-\alpha) + \frac{1}{2}\,\mathrm{Cl}_2(2\alpha)\Big) + \frac{1}{y}\Big(2\,\mathrm{Cl}_2(\pi-\alpha) + \frac{1}{2}\,\mathrm{Cl}_2(2\alpha)\Big),$$

where $\alpha = \dfrac{\pi y}{2}$. Applying (8) to the result and simplifying, we get

$$\int_0^{\frac{\pi}{2}} xy\csc(xy)\,dx = \pi\sum_{n=1}^{\infty}\frac{\zeta(2n)(\frac{\pi}{2})^{2n}(2-y)^{2n}}{n(2\pi)^{2n}} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\zeta(2n)(\pi y)^{2n}}{n(2\pi)^{2n}} - \frac{\pi}{2}$$

$$+\frac{2\pi}{y}(1-\log\left(\frac{\pi}{2}\right) - \log(2-y)) + \frac{2}{y}\sum_{n=1}^{\infty}\frac{\zeta(2n)(\frac{\pi}{2})^{2n+1}(2-y)^{2n+1}}{n(2n+1)(2\pi)^{2n}} + \frac{1}{2y}\sum_{n=1}^{\infty}\frac{\zeta(2n)(\pi y)^{2n+1}}{(2\pi)^{2n}}.$$

On the other hand, we can apply Fubini's theorem and integrate its power series term by term. Thus, we will have

$$\int_0^{\frac{\pi}{2}} xy\csc(xy)\,dx = \int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(4^n-2)B_{2n}}{(2n)!}(xy)^{2n}\,dx$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(4^n-2)B_{2n}(\frac{\pi}{2})^{2n+1}}{(2n+1)(2n)!}y^{2n}.$$

Setting the two results equal to each other and simplifying more, we see

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\sum_{k=0}^{2n}\binom{2n}{k}\frac{(-1)^k}{2^k}y^k-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)4^k}y^{2k}+\frac{2}{y}(1-\log\pi)+2\sum_{k=1}^{\infty}\frac{1}{k2^k}y^{k-1}$$

$$-\frac{1}{2}+2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)4^n}\sum_{k=0}^{2n+1}\frac{(-1)^k}{2^k}\binom{2n+1}{k}y^{k-1}=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2^{2k-1}-1)B_{2k}\pi^{2k}}{4^k(2k+1)!}y^{2k}.$$

Now we group the coefficients on both sides. The odd powers of y (i.e., $2j-1$ for $j=1,2,3,...$), we find

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\binom{2n}{2j-1}\frac{(-1)^{2j-1}}{2^{2j-1}}+\frac{2}{(2j)2^{2j}}+2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)4^n}\frac{(-1)^{2j}}{2^{2j}}\binom{2n+1}{2j}=0.$$

Multiplying by 2^{2j-1}, we see

$$\frac{1}{2j}=\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\left(\binom{2n}{2j-1}-\frac{1}{2n+1}\binom{2n+1}{2j}\right)=\frac{2j-1}{2j}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\binom{2n}{2j-1}.$$

Setting $m=2j-1$, we arrive at the first part of the theorem. For the even powers of y (i.e., $2j$ for $j=1,2,3,...$), we arrive at the following:

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\frac{(-1)^{2j}}{2^{2j}}\binom{2n}{2j}-\frac{\zeta(2j)}{(2j+1)4^j}+\frac{2}{(2j+1)2^{2j+1}}$$

$$+2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n(2n+1)4^n}\frac{(-1)^{2j+1}}{2^{2j+1}}\binom{2n+1}{2j+1}=\frac{(-1)^{j+1}(2^{2j-1}-1)B_{2j}\pi^{2j}}{4^j(2j+1)!}.$$

Multiplying by 4^j and using (1) to replace the Bernoulli numbers by $\zeta(2j)$, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\left(\binom{2n}{2j}-\frac{1}{2n+1}\binom{2n+1}{2j+1}\right)=\frac{\zeta(2j)}{2j+1}-\frac{1}{2j+1}+\frac{\zeta(2j)}{(2j+1)}\left(1-\frac{2}{2^{2j}}\right).$$

Using the binomial identity $\binom{2n}{2j}-\frac{1}{2n+1}\binom{2n+1}{2j+1}=\frac{2j}{2j+1}\binom{2n}{2j}$, this simplifies to

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n4^n}\binom{2n}{2j}=\frac{1}{2j}\left(2\zeta(2j)\left(1-\frac{1}{2^{2j}}\right)\right).$$

Letting $m = 2j$ gives the final result of the theorem and thus, the proof is complete. □

Corollary 3.2.5. *([69, 12]) We have*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(2n+1)4^n} = \log \pi - 1. \tag{3.22}$$

Proof. This follows immediately by setting the coefficients of y^{-1} equal to each other on both sides. □

Corollary 3.2.6. *We have the following series representations*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{4^n} = \frac{1}{2}, \tag{3.23}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)(2n-2)}{4^n} = 1, \tag{3.24}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)}{4^n} = \frac{\pi^2}{8} - \frac{1}{2}, \tag{3.25}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n}{4^n} = \frac{\pi^2}{16}, \tag{3.26}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n^2}{4^n} = \frac{3\pi^2}{32}. \tag{3.27}$$

Proof. Define $\mathrm{F}(k) := \sum_{n=1}^{\infty} \frac{\zeta(2n)n^k}{4^n}$. Letting $m = 1, 3, 2$, we obtain (23), (24), and (25), respectively. Using (23), note that (25) can be rewritten as $2F(1) - F(0) = \frac{\pi^2}{8} - F(0)$, and so, (26) follows immediately. Finally, using (23) and (26), note that (24) can be rewritten as $4F(2) - 6F(1) + 2F(0) = 2F(0)$. From this, (27) follows immediately. □

Remark. Letting $m = 2k$ and $m = 2k - 1$ ($k = 1, 2, 3, ...$) in the even and odd parts of (21) respectively, we can add the two results to find

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\left(\binom{2n}{2k-1} + \binom{2n}{2k}\right) = \frac{\zeta(2k)}{k}\left(1 - \frac{1}{4^k}\right) + \frac{1}{2k-1} - \frac{1}{2k},$$

which is equivalent to

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\binom{2n+1}{2k} = \frac{\zeta(2k)}{k}\left(1 - \frac{1}{4^k}\right) - \frac{1}{2k(2k-1)}. \tag{3.28}$$

Theorem 3.2.7. *We have the following series representation*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n}\binom{2n}{m} = \begin{cases} \dfrac{1}{m}\left(1 - \beta(m)\right) & m \text{ odd}, \\[2ex] \dfrac{1}{m}\left(\zeta(m)\left(1 - \dfrac{1}{2^m}\right) - 1\right) & m \text{ even}, \end{cases} \tag{3.29}$$

Proof. Similar to the previous theorem's proof, we integrate $\sec(xy)$ two different ways. First, integrating regularly and using properties of logarithm, we have

$$\int_0^{\frac{\pi}{2}} \sec(xy)\, dx = \frac{1}{y}\log\left(1 + \sin\left(\frac{\pi y}{2}\right)\right) - \frac{1}{y}\log\left(\cos\left(\frac{\pi y}{2}\right)\right)$$
$$= \frac{1}{y}\frac{d}{d\alpha}\left(-2\,\mathrm{Cl}_2\left(\frac{\pi}{2} + \alpha\right) - \alpha\log 2\right) - \frac{1}{y}\log\left(\prod_{n=1}^{\infty}\left(1 - \left(\frac{y}{2n-1}\right)^2\right)\right),$$

where $\alpha = \dfrac{\pi y}{2}$ and we have used the product formula $\cos(\beta) = \prod_{k=1}^{\infty}\left(1 - \left(\frac{2\beta}{\pi(2k-1)}\right)^2\right)$. Using equation (8) in the first term and applying properties of logarithm and its power series to the second term, the integral becomes

$$\int_0^{\frac{\pi}{2}} \sec(xy)\, dx = \frac{1}{y}\left(2\log\left(\frac{\pi}{2}\right) + 2\log(1 + y) - \log 2\right)$$

$$-\frac{2}{y}\sum_{n=1}^{\infty}\frac{\zeta(2n)(\frac{\pi}{2})^{2n}(1+y)^{2n}}{n(2\pi)^{2n}}+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{y^{2k}}{k(2n-1)^{2k}},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2k}}=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}-\sum_{n=1}^{\infty}\frac{1}{(2n)^{2k}}=\Big(1-\frac{1}{4^k}\Big)\zeta(2k).$$

Alternatively, we can apply Fubini's theorem once again and integrate the power series of the secant function term by term. Doing so will give us the following:

$$\int_0^{\frac{\pi}{2}}\sec(xy)\,dx=\int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(-1)^nE_{2n}}{(2n)!}(xy)^{2n}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^nE_{2n}(\frac{\pi}{2})^{2n+1}}{(2n+1)(2n)!}y^{2n}.$$

Setting the two results equal to each other and simplifying more, we see

$$\frac{2}{y}\log\Big(\frac{\pi}{2\sqrt{2}}\Big)+2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}y^{k-1}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n16^n}\sum_{k=0}^{2n}\binom{2n}{k}y^{k-1}$$
$$+\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k}\Big(1-\frac{1}{4^k}\Big)y^{2k-1}=\sum_{k=0}^{\infty}\frac{(-1)^kE_{2k}(\frac{\pi}{2})^{2k+1}}{(2k+1)!}y^{2k}.$$

Now we can group coefficients. For the even powers of y (i.e., $2j$ for $j=1,2,3,...$), we have

$$2\frac{(-1)^{2j+2}}{2j+1}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n16^n}\binom{2n}{2j+1}=\frac{(-1)^jE_{2j}\pi^{2j+1}}{(2j+1)!2^{2j+1}}=\frac{2\beta(2j+1)}{2j+1}.$$

Setting $m=2j+1$ and rearranging, we achieve the first result of the theorem. For the odd powers of y (i.e., $2j-1$ for $j=1,2,3,...$), we find

$$2\frac{(-1)^{2j}}{2j}-2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{n16^n}\binom{2n}{2j}+\frac{\zeta(2j)}{j}\Big(1-\frac{1}{4^j}\Big)=0,$$

and rearranging this gives the second part of the theorem. □

Corollary 3.2.8. *We have the following representations*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n16^n} = \log\left(\frac{\pi}{2\sqrt{2}}\right), \tag{3.30}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{16^n} = \frac{4-\pi}{8}. \tag{3.31}$$

Proof. Formula (30) follows immediately by setting the coefficients of y^{-1} equal to each other on both sides. For (31), we set the constant terms on both sides equal to find

$$2 - 4\sum_{n=1}^{\infty} \frac{\zeta(2n)}{16^n} = \frac{\pi}{2}.$$

From here, (31) follows immediately.

Remark. In the previous theorem, if we used the Clausen identity for $\log(\cos(x))$ given in the introduction rather than the cosine product formula, we obtain the following relation:

$$\log\left(\frac{\pi}{4}\right) - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n} \sum_{k=0}^{2n} \binom{2n}{k} y^k \left(\frac{2}{4^n} - (-1)^k\right) + \sum_{k=1}^{\infty} \frac{y^k}{k}(1 - 2(-1)^k)$$
$$= \sum_{k=0}^{\infty} \frac{2\beta(2k+1)}{2k+1} y^{2k+1}.$$

Gathering the coefficients of y^0, we find

$$\log\left(\frac{\pi}{4}\right) - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\left(\frac{2}{4^n} - 1\right) = 0.$$

Rearranging and using (30), we can see that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n2^{2n}} = \frac{1}{2}\log\left(\frac{\pi}{2}\right), \tag{3.32}$$

which is a very nice result. This series appears by setting the coefficients of y^0 equal to each other in the proof of Theorem 2.3; however, the series will cancel and you arrive at a truth statement. Here, we are able to recover that series.

Corollary 3.2.9. *We have*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)}{16^n} = \frac{\pi^2}{16} - \frac{1}{2}, \tag{3.33}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)(2n-1)(2n-2)}{16^n} = 1 - \frac{\pi^3}{96}, \tag{3.34}$$

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n}{16^n} = \frac{\pi}{16}\left(\frac{\pi}{2} - 1\right), \tag{3.35}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)n^2}{16^n} = \frac{\pi}{32}\left(\frac{3\pi}{2} - \frac{\pi^2}{4} - 1\right). \tag{3.36}$$

Proof. Define G$(k) := \sum_{n=1}^{\infty} \frac{\zeta(2n)n^k}{16^n}$. Letting $m = 2, 3$ in (29) gives us formulas (33) and (34), respectively. Using (31), we can rewrite (33) as $2G(1) - G(0) = \frac{\pi^2}{16} - G(0) - \frac{\pi}{8}$ and from this, (35) follows immediately. Lastly, for (36), expanding (34) using (31), we have $4G(2) - 6G(1) + 2G(0) = \frac{\pi}{4} + 2G(0) - \frac{\pi^3}{96}$. Using (35), we achieve the desired result. □

Corollary 3.2.10. *We have the following series*

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\left(1 - \frac{1}{4^n}\right)\binom{2n}{2k} = \frac{\zeta(2k)}{2k}\left(1 - \frac{1}{4^k}\right), \tag{3.37}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)}{n4^n}\left(1 - \frac{1}{4^n}\right)\binom{2n}{2k+1} = \frac{\beta(2k+1)}{2k+1}. \tag{3.38}$$

Proof. Denote (29.1) and (29.2) as well as (21.1) and (21.2) as the formula for m odd and m even, respectively. Letting $m = 2k$ for $k = 1, 2, 3...$ and subtracting (29.2) from (21.2), formula (37) follows immediately. Similarly, letting $m = 2k + 1$ for $k = 0, 1, 2, ...$ and subtracting (29.1) from (21.1), the obtain (38). □

Remark. You can add (37) and (38) together to get $\binom{2n+1}{2k+1}$ inside the series. You can also change k to $k - 1$ in (38) and add (37) and (38) again to give you $\binom{2n+1}{2k}$ in the series.

4.0 MULTIPLE ZETA AND SPECIAL HURWITZ ZETA FUNCTIONS AND THEIR SPECIAL VALUES

4.1 INTRODUCTION AND PRELIMINARIES

4.1.1 Gauss hypergeometric function

The generalized hypergeometric function is defined by

$$_pF_q(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdot \dots (a_p)_n}{(b_1)_n \cdot \dots (b_q)_n} \cdot \frac{x^n}{n!}, |x| < 1,$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol.

An application of the well-known ratio test shows that:

(i) If $p \le q$, the series converges for all finite x

(ii) If $p = q + 1$, the series converges for $x < 1$ and diverges for $|x| > 1$

(iii) Of $p > q + 1$, the series diverges for $x \neq 0$

(iv) If $p = q + 1$, then the series converges absolutely on the circle $|x| = 1$ if

$$\mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0.$$

Some examples of elementary functions which can be interpreted as hypergeometric functions are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = {}_0F_0(-, -; x),$$

$$\log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = z \cdot {}_2F_1(1, 1; 2; -x),$$

$$\arcsin x = x \cdot {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right)$$

$$\frac{1}{1-x} = {}_1F_0(1, -; x).$$

Now, let's turn our attention to the case when $p = q+1 = 2$. In this case, we have the celebrated *Gauss hypergeometric function* given by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!}, |x| < 1,$$

where $(q)_n = q(q+1) \cdot \ldots \cdot (q+n-1)$ is the Pochhammer symbol. From previous considerations, this series converges absolutely when $|x| < 1$ and diverges for $|x| > 1$. The convergence of the Gauss hypergeometric function in the case when $|x| = 1$ is a little bit more delicate, but thanks to Raabe criterion, the series converges when $\mathrm{Re}(a + b - c) < 0$.

For the hyergeometric function ${}_2F_1$ we have the following integral representation due to Euler,

Theorem 4.1.1 (Euler)**.** *For* $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$*, we have*

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt,$$

for all x *in the complex plane cut along the real axis from* 1 *to* ∞*. Here it is understood that* $\arg t = \arg(1-t) = 0$ *and* $(1-xt)^{-a}$ *has its principal value.*

One of the important properties of the function is what happens at $x = 1$. The next result can be proven using Euler's integral representation theorem from above,

Theorem 4.1.2 (Gauss)**.** *If* $\mathrm{Re}(a+b-c)<0$*, then*

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

As an immediate consequence of this result we have the following

Corollary 4.1.3 (Chu-Vandermonde)**.** *We have*

$$ {}_2F_1(-n,b;c;1)=\frac{(c-b)_n}{(c)_n}. $$

The above corollary written in the form, $\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n}$, will be crucial in the proof of the next result which is essential in deriving the formulas for the integer powers of arcsin. We have

Proposition 4.1.4. *(see [37, 57]) Let x and y two indeterminates with* $|y|<1$*. There hold two infinite series identities*

$$ {}_2F_1\left(\frac{x}{2},-\frac{x}{2};\frac{1}{2};y^2\right)=\cos(x\arcsin y), $$

and

$$ {}_2F_1\left(\frac{1+x}{2},\frac{1-x}{2};\frac{3}{2};y^2\right)=\frac{\sin(x\arcsin y)}{xy}. $$

For a proof of this result we recommend [37]. Also, for many other properties of the Gauss hypergeometric and other related function, the reader can consult [5, 70].

4.1.2 Multiple zeta and Hurwitz zeta functions

In analogy with the Riemann zeta function, we define the multiple zeta functions (Euler-Riemann-Zagier functions),

$$ \zeta(s_1,s_2,\ldots,s_r)=\sum_{1\le n_1<n_2<\ldots<n_r}\frac{1}{n_1^{s_1}n_2^{s_2}\ldots n_r^{s_r}},\ \mathrm{Re}(s_r)>1,\ \sum_{j=1}^{r}\mathrm{Re}(s_j)>r, $$

for all $j = 1, 2, \ldots, r$.

Using the following integral representation for the multiple zeta functions,

$$\zeta(s_1, s_2, \ldots, s_r) = \frac{1}{\Gamma(s_1)\Gamma(s_2)\ldots\Gamma(s_r)} \int_0^{\infty} \int_0^{\infty} \ldots \int_0^{\infty} \frac{{t_1}^{s_1-1}{t_2}^{s_2-1}\ldots t_r^{s_r-1}}{\prod_{j=1}^{r}(\exp(t_j + \ldots + t_r) - 1)},$$

J. Zhao [76, 77] was able to provide the analytic continuation for $\zeta(s_1, s_2, \ldots, s_r)$,

Theorem 4.1.5. *The multiple zeta functions $\zeta(s_1, s_2, \ldots, s_r)$ have an analytic continuation to the whole $\mathbb{C}^r$ with possible simple poles at $s_r = 1$ and $s_r(j) := s_j + \ldots + s_r = r - j + 2 - l$, $j, l \in \mathbb{N}, 1 \leq j \leq r$.*

Regarding multiple Hurwitz zeta values, we have the following result [62],

Theorem 4.1.6. *The multiple Hurwitz zeta functions,*

$$\zeta(s_1, \ldots, s_r; x_1, \ldots, x_r),$$

extends to a meromorphic function in $\mathbb{C}^r$ with possible singularities on $s_1 = 1, s_1 + s_2 + \ldots + s_j \in \mathbb{Z}_{\leq j}$, $j = 2, 3, \ldots, r$. If the x_i are all rational, and $x_2 - x_1 \neq 0$ or $1/2$, then the above set coincides with the complete set of singularities. If $x_2 - x_1 = 1/2$, then $s_1 = 1, s_1 + s_2 = 2, 0, -2, -4, -6, \ldots$ and for $3 \leq j \leq r, s_1 + s_2 + \ldots + s_j \in \mathbb{Z}_{\leq j}$ forms the complete set of singularities. If $x_2 - x_1 = 0$, then $s_1 = 1$, $s_1 + s_2 = 2, 1, 0, -2, -4, -6, \ldots$ and for $3 \leq j \leq r$, $s_1 + s_2 + \ldots s_r \in \mathbb{Z}_{\leq j}$ forms the complete set of singularities.

4.1.3 Multiple zeta values

Multiple zeta values were introduced independently by M. Hoffman [50] and D. Zagier [73] in 1992. They are the numbers defined by the following convergent series

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{0<n_1<n_2<\ldots<n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_r^{k_r}},$$

where $k_1, k_2, \ldots, k_r$ are positive integers with $k_r > 1$.

Alternatively, the multiple zeta values can also be rewritten as nested sums:

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_r=1}^{\infty} \frac{1}{n_r^{k_r}} \sum_{n_{r-1}=1}^{n_r-1} \frac{1}{n_{r-1}^{k_{r-1}}} \cdots \sum_{n_2=1}^{n_3-1} \frac{1}{n_2^{k_2}} \sum_{n_1=1}^{n_2-1} \frac{1}{n_1^{k_1}}.$$

The main goal of this theory is to understand all linear relations over $\mathbb{Q}$ among the MZVs of a given weight. Although they look simple, it seems that these numbers have deep connections with Galois representation theory or they even appear in calculating Feynman integrals from quantum field theory. We call the above series a multiple zeta of depth r and weight k, where $k = k_1 + k_2 + \ldots + k_r$. Obviously, $0 < r < k$ and there are $\binom{k-2}{r-1}$ multiple zeta values of given weight k and depth r. For example, there are 2^{13} such numbers in weight 15, but they form a vector space over $\mathbb{Q}$ of dimension at most 28. Multiple zeta values satisfy many relations.

The simplest relation is the product of two single zeta values,

$$\zeta(k_1)\zeta(k_2) = \sum_{1\leq n_1} \frac{1}{n_1^{k_1}} \sum_{1\leq n_2} \frac{1}{n_2^{k_2}} = \sum_{1\leq n_1, n_2} \frac{1}{n_1^{k_1} n_2^{k_2}} = \left(\sum_{1\leq n_1<n_2} + \sum_{1\leq n_1>n_2} + \sum_{1\leq n_1=n_2} \right) \frac{1}{n_1^{k_1} n_2^{k_2}}$$

which gives us the simplest example of what we call *stuffle* product,

$$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).$$

Similarly, for triple zeta values, we have the following identity,

$$\zeta(k_1, k_2)\zeta(k_3) = \zeta(k_1, k_2, k_3) + \zeta(k_3, k_1, k_2) + \zeta(k_1, k_3, k_2) + \zeta(k_1 + k_3, k_2) + \zeta(k_1, k_2 + k_3).$$

These two examples of stuffle product is the first indication that $\mathbb{Q}$-vector space $\mathcal{Z}$ has the structure of an algebra. Basically, one can say that every polynomial relation between Riemann zeta values $\zeta(k)$ gives rise to a linear relation between multiple zeta values.

Generally speaking, if $k = (k_1, k_2, \ldots, k_r)$ and $k' = (k'_1, k'_2, \ldots, k'_r)$ are two admissible multi-indices then

$$\zeta(k)\zeta(k') = \sum_{k''} \text{stuffle}(k, k', k'')\zeta(k'').$$

For example, if we want to compute the product

$$\zeta(k_1)\zeta(k_2, \ldots, k_r) = \sum_{1 \le n_1, n_2 < \ldots < n_r} \frac{1}{n_1} \cdots \frac{1}{n_r},$$

this gives us

$$\zeta(k_1)\zeta(k_2, \ldots, k_r) = \zeta(k_1, k_2, \ldots, k_r) + \zeta(k_2, k_1, k_3, \ldots, k_r) + \ldots + \zeta(k_1 + k_2, k_3, \ldots, k_r) +$$
$$+ \zeta(k_2, k_1 + k_3, \ldots, k_r) + \zeta(k_2, k_3, \ldots, k_1 + k_r).$$

For $M = \mathbb{C} - \{0, 1\}$ let us consider the 1-forms $\omega_0 = \dfrac{dz}{z}$ and $\omega_1 = \dfrac{dz}{1 - z}$. The iterated integral

$$\int_0^1 \omega_{e_1} \ldots \omega_{e_r}, e_j \in \{0, 1\}$$

converges if and only if $e_1 = 1$ and $e_0 = 0$. In this sense, M. Kontsevich observed that all multiple zeta values are periods,

$$\zeta(k_1, k_2, \ldots, k_r) = \int_0^1 \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{k_1 - 1} \cdot \ldots \cdot \omega_1 \underbrace{\omega_0 \ldots \omega_0}_{k_r - 1}.$$

The simplest example is the following integral representation of $\zeta(2) = \displaystyle\int_{0 < t_1 < t_2 < 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2}$. If we multiply the integral representation of two single zeta values, we have

$$\zeta(2)\zeta(2) = \left(\int_{0<t_1<t_2<1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\right)\left(\int_{0<s_1<s_2<1} \frac{ds_1}{1-s_1}\frac{ds_2}{s_2}\right) = \int_{0<t_1<t_2<1\ 0<s_1<s_2<1} \frac{dt_1 dt_2 ds_1 ds_2}{(1-t_1)t_2(1-s_1)s_2}$$

which gives us

$$\zeta(2)\zeta(2) = 4\int_{0<r_1<r_2<r_3<r_4<1} \frac{dr_1}{1-r_1}\frac{dr_2}{1-r_2}\frac{dr_3}{r_3}\frac{dr_4}{r_4} + 2\int_{0<r_1<r_2<r_3<r_4<1} \frac{dr_1}{1-r_1}\frac{dr_2}{r_2}\frac{dr_3}{1-r_3}\frac{dr_4}{r_4}$$

which is finally equivalent to $\zeta(2)^2 = 4\zeta(1,3) + 2\zeta(2,2)$.

These two relations are not enough! Let us denote the stuffle product by $*$ and the shuffle product by $\sqcup$. The sums $\zeta(k_1, \ldots, k_{r-1}, 1)$ are infinite! But it turns out that the formal difference,

$$\zeta(1) * \zeta(k_1, \ldots, k_r) - \zeta(1) \sqcup \zeta(k_1, \ldots, k_r)$$

is finite: all divergent multiple zeta values drop out. This gives us a third linear relation called the *regularization* relation. For example, $\zeta(1)*\zeta(2)-\zeta(1)\sqcup\zeta(2) = 0$, and Euler's relation, $\zeta(1,2) = \zeta(3)$ follows.

A natural question to ask is what kind of $\mathbb{Q}$-linear relations do we have in low weight?

- weight 2: $\zeta(2)$
- weight 3: $\zeta(1,2) = \zeta(3)$
- weight 4: $\zeta(4)$, $\zeta(1,3)$, $\zeta(2,2)$, $\zeta(1,1,2)$

From the regularization procedure, we have $\zeta(1,3) + \zeta(4) = 2\zeta(1,3) + \zeta(2,2)$ and $2\zeta(1,1,2) + \zeta(2,2) + \zeta(1,4) = 3\zeta(1,1,2)$. In particular, this shows that every MZV in weight 4 is a rational multiple of $\zeta(2)^2$.

The theory of the multiple zeta values has been developed intensively over the past two decades in the papers [14, 50, 63, 74, 75].

4.1.4 The evaluation of some classical multiple zeta values

Now, we turn to some important evaluations of multiple zeta values. Since $\zeta(2) = \frac{\pi^2}{6}$ and from the sttufle relations, $\zeta(2,2) = \frac{\pi^4}{120}$, one can ask what is the exact value for the multiple zeta values $\zeta(\underbrace{2,2,\ldots,2}_{n})$? This was answered by Hoffman [50] and Zagier [73] in 1992.

Theorem 4.1.7. *The following equality holds true*

$$\zeta(\underbrace{2,2,\ldots,2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}.$$

The proof given by Hoffman [50] relies on combinatorial methods. Zagier's argument is simpler and it uses generating functions and Euler's formula for the infinite sine product. Another important reslt was to extend Euler's remarkable equality for double zeta values, $\zeta(1,2) = \zeta(3)$. Indeed, this is due to Hoffman [50],

Theorem 4.1.8. *The following inequality is valid*

$$\zeta(\underbrace{1,2,\ldots,1,2}_{n}) = \zeta(\underbrace{3,3,\ldots,3}_{n})$$

Another important evaluation is whether one can find a formula for $\zeta(\underbrace{1,1,\ldots,1}_{n},m)$, with $m \geq 2$ integer. This was answered by Drinfeld [44] in the following

Theorem 4.1.9. *The following duality holds true*

$$\zeta(\underbrace{1,1,\ldots,1}_{m},n+2) = \zeta(\underbrace{1,1,\ldots,1}_{n},m+2).$$

In fact, the above result follows easily if we take into account that the generating function of $\zeta(\underbrace{1,1,\ldots,1}_{m},n+2)$ is given by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\zeta(\underbrace{1,1,\ldots,1}_{m},n+2)x^{m+1}y^{n+1} = 1 - \frac{\Gamma(1-x)\Gamma(1-y)}{\Gamma(1-x-y)} = 1 - \exp\left(\sum_{k=2}^{\infty}(x^k + y^k - (x+y)^k)\frac{\zeta(k)}{k}\right)$$

The next result was a conjecture of Zagier which was solved by Broadhurst and it stems from the shuffle identity $\zeta(1,3) = \frac{1}{4}\zeta(4)$.

Theorem 4.1.10. *The following equality holds true*

$$\zeta(\underbrace{1,3,\dots,1,3}_{n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

This was settled by Borwein, Bradley, Broadhurst and Lisonek [15]. Their proof was based on the identity

$$1+\sum_{n=1}^{\infty}\zeta(\underbrace{1,3,\dots,1,3}_{n})t^{4n} = \frac{\cosh(\pi t)-\cos(\pi t)}{\pi^2 t^2}.$$

4.2 HOFFMAN'S CONJECTURE AND ZAGIER'S FORMULA FOR HOFFMAN ELEMENTS

Let us denote by $\mathcal{Z}$ the $\mathbb{Q}$-vector space spanned by all multiple zeta values. It is not hard to see that $\mathcal{Z}$ has the structure of an algebra.

In 2011, F. Brown [24] proved the following result

Theorem 4.2.1. *The periods of mixed Tate motives over $\mathbb{Z}$ lie in $\mathcal{Z}\left[\frac{1}{2\pi i}\right]$.*

It was known that multiple zeta values are examples of periods of mixed Tate motives over $\mathbb{Z}$ from previous work of Deligne and Goncharov [43], but it turns out that also the converse is true in some sense.

A natural question to ask is what is a basis for $\mathcal{Z}$ as a $\mathbb{Q}$-vector space? The natural thing to try at first is to add a basis for a general family of even zeta values, add odd zeta values, continue with multiple zeta values in low depth (2, 3) with odd arguments and so on. Unfortunately, this approach fails because of linear dependence relations over $\mathbb{Q}$ such as

$$28\zeta(3,9)+150\zeta(5,7)+168\zeta(7,5) = \frac{5197}{691}\zeta(12).$$

This is one of the many identities discovered by Gangl, Kaneko and Zagier [48] which are related to periods of polynomials for cusp formas in $PSL(2, \mathbb{Z})$. We can circumvent this problem with the following "high depth" result which was conjectured by M. Hoffman [51] in 1997 and settled by F. Brown [24] in 2012.

Theorem 4.2.2. *Every multiple zeta value of weight k can be expressed as a $\mathbb{Q}$-linear combination of multiple zeta values of the same weight involving 2's and 3's.*

In other words, every multiple zeta value of weight k is a $\mathbb{Q}$-linear combination of $\zeta(k_1, k_2, \ldots, k_r)$, where $k_i \in \{2, 3\}$ and $\sum_{i=1}^{r} k_i = k$. The arguments used by Brown in proving the above theorem are purely motivic and it used motivic multiple zeta values. In fact, Brown showed that the multiple zeta values involving Hoffman elements,

$$H(r, s) = \zeta(\underbrace{2, 2, \ldots, 2}_{r}, 3, \underbrace{2, 2, \ldots, 2}_{s})$$

can be expressed as a $\mathbb{Q}$-linear combination of products $\pi^{2m}\zeta(2n+1)$, with $m + n = r + s + 1$.

The next result gives an explicit formula for $H(r, s)$ which are part of Hoffman's conjectural basis. This result confirms the 2-adic properties of the coefficients required for Brown's proof. The gap was filled by D. Zagier [75] in 2012 who proved the following

Theorem 4.2.3. *For all integers $r, s \geq 0$, we have*

$$H(r, s) = 2\sum_{k=1}^{r+s+1} (-1)^k \left[\binom{2k}{2r+2} - \left(1 - \frac{1}{2^{2k}}\right)\binom{2k}{2s+1}\right] \zeta(2k+1)\zeta(\underbrace{2, 2, \ldots, 2}_{r+s+1-k}).$$

The above formula is proven indirectly by computing the generating functions of both sides in closed form and then showing that they are both entire functions of exponential growth which agree at sufficiently many points to force their equality. Specifically, Zagier considered the following two generating functions:

$$F(x,y) = \sum_{r,s\geq 0} (-1)^{r+s+1} H(r,s) x^{2r+2} y^{2s+1},$$

and

$$\hat{F}(x,y) = \sum_{r,s\geq 0} (-1)^{r+s+1} \hat{H}(r,s) x^{2r+2} y^{2s+1},$$

where $H(r,s)$ is given by the multiple zeta values involving 2's and 3's and $\hat{H}(r,s)$ is the $\mathbb{Q}$-linear finite combinations of powers of π and odd zeta values.

Step 1. $F(x,y)$ is the product of

$$\frac{1}{\pi}\sin(\pi y)\cdot {}_3F_2'(x,-x,0;1+y,1-y;1)_{|z=0},$$

where ${}_3F_2$ is the hypergeometric function.

Step 2. $\hat{F}(x,y)$ is a linear combination of 14 functions of the form

$$\Psi\left(1+\frac{u}{2}\right)\cdot\frac{\sin(\pi v)}{2\pi},$$

with $u \in \{\pm x \pm y, \pm 2x \pm 2y, \pm 2y\}$ and $v \in \{x,y\}$, where $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

Step 3. The functions F and $\hat{F}$ are both entire on $\mathbb{C}\times\mathbb{C}$ and they are both bounded by a constant multiple of $e^{\pi X}\log X$, when $X = \max\{|x|,|y|\}$ tends to infinity and also by a constant multiple of $e^{\pi|\tau(y)|}$, when $|y|$ tends to infinity while $x \in \mathbb{C}$ is fixed.

Step 4. The diagonal argument: for $z \in \mathbb{C}$, we have

$$F(z,z) = \hat{F}(z,z) = -\frac{\sin(\pi z)}{z}A(z),$$

where $A(z) = \sum_{n=1}^{\infty}\frac{z^2}{n(n^2-z^2)}$ is a meromorphic function in the whole complex plane, with simple poles at $z \in \mathbb{Z} - \{0\}$.

In this case we have some explicit formulas!

Step 5. We have

$$F(n, y) = \hat{F}(n, y), n \in \mathbb{N}, y \in \mathbb{C}.$$

Step 6. We have

$$F(x, k) = \hat{F}(x, k), k \in \mathbb{N}, x \in \mathbb{C}.$$

Step 7. The final ingredient is the following famous,

Lemma 4.2.4. *An entire function $f : \mathbb{C} \to \mathbb{C}$ that vanishes at all integers and satisfies*

$$f(z) = O(e^{\pi|\tau(z)|}),$$

is a constant multiple of $\sin(\pi z)$.

Now, the only thing left to do is to to fix x and apply the lemma to the function $f(y) = F(x, y) - \hat{F}(x, y)$ which is an entire function of order 1 and vanishes at all integers.

Denote $\mathcal{Z}_k$ the $\mathbb{Q}$-vector space spanned by all multiple zeta values of weight k. In [73], D. Zagier conjectured that

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k,$$

where the number d_k is determined by the Fibonacci-like sequence

$$d_2 = d_3 = d_4 = 1, d_k = d_{k-2} + d_{k-3}, k \geq 5,$$

or equivalently d_k is the coefficient from the power series $\sum_{k \geq 0} d_k t^k = \dfrac{1}{1 - t^2 - t^3}$.

Brown's result (Theorem 4.2.2) implies that $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$. Also, let us remark the fact that Zagier's dimensional conjecture implies the transcendence conjecture. For example, we have

$$\zeta(1,1,2) = \zeta(4), \zeta(1,1,1,2) = \zeta(5),$$
$$\zeta(1,1,3) = \zeta(1,4) = 2\zeta(5) - \zeta(2)\zeta(3),$$
$$\zeta(1,1,2) = \zeta(3,2) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3),$$
$$\zeta(1,2,2) = \zeta(2,3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5).$$

This shows that $d_5 \leq 2$. We want $d_5 = 2$ for the Zagier's conjecture to hold true! This is equivalent to $\frac{\zeta(2)\zeta(3)}{\zeta(5)} \notin \mathbb{Q}$.

Last but not least, there is another unsolved problem involving Hoffman elements,

Conjecture 4.2.5 (Hoffman)**.** *The following identity holds true:*

$$\zeta(2,1,\underbrace{2,2,\ldots,2}_{r},3) = \zeta(\underbrace{2,2,\ldots,2}_{r+3}) + 2\zeta(\underbrace{2,2,\ldots,2}_{r},3,3).$$

5.0 ANOTHER LOOK AT THE EVALUATION OF SOME MULTIPLE ZETA AND SPECIAL HURWITZ ZETA VALUES

5.1 INTRODUCTION AND PRELIMINARIES

In [30], using the power series expansion of $\arcsin x$ near 0, Boo Rim Choe derived a proof for Euler's famous formula

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

in the form

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the function $g : (-1, 1) \to \mathbb{R}$, $g(x) = (\arcsin x)^2$ satisfies the initial value problem $(1 - x^2)y'' - xy' - 2 = 0$, $y(0) = y'(0) = 0$, by looking for a power series solutions, one has

$$\arcsin^2 x = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n^2 \binom{2n}{n}} x^{2n}, |x| \leq 1. \tag{5.1}$$

As it has been already highlighted, the above formula plays an important role in evaluating series involving the central binomial coefficient or representations of $\zeta(3)$ such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} = \frac{\pi^2}{18}, \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}} = \frac{2}{5}\zeta(3), \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}} = \frac{17}{3456}\pi^4$$

and

$$\zeta(3) = -2\pi^2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2)(2n+3)2^{2n}}.$$

The history and details for the derivation of (1) are presented in Chapter 1, but the reader can find it also in [20]. It is interesting to mention that if we use the same technique as in [30] but starting from (5.1) we deduce exactly $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. We omit the details here since Chapter 1 covers this fact. Using these ideas, Ewell [46] found new series representations for π and $\zeta(3)$. Another important aspect is the fact that if we integrate (1) we have

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{n^3\binom{2n}{n}} x^{2n} = 4\int_0^y \frac{\arcsin^2(x)}{x} dx =$$

$$= 2x^2 \text{ Hypergeometric}_4 F_3 \left[\{1,1,1,1\}, \left\{\frac{3}{2}, 2, 2\right\}, x^2\right].$$

This last integral is a "higher transcendent" and it is related top Clausen's integral and the trigamma function. In other words, for $y = \frac{1}{2}$ we have

$$\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}} = 4\int_0^{\frac{1}{2}} \frac{\arcsin^2(x)}{x} dx = -2\int_0^{\frac{\pi}{3}} z \log\left(2\sin\frac{z}{2}\right) dz =$$

$$= -\frac{\zeta(3)}{3} - \frac{\pi\sqrt{3}}{72}\left(\Psi\left(\frac{1}{3}\right) - \Psi\left(\frac{2}{3}\right)\right),$$

where $\Psi(z)$ is the trigamma function. Also, for $y = 1$ we have the special value

$$\int_0^1 \frac{\arcsin^2 x}{x} dx = \frac{\pi^2 \log 2}{4} - \frac{7}{8}\zeta(3).$$

Recall that for $k_1, k_2, \ldots, k_{r-1} \geq 1$, and $k_r \geq 2$, we define the multiple zeta values by

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{1 \leq n_1 < n_2 < \ldots < n_r} \frac{1}{n_1^{k_1} n_2^{k_2} \ldots n_r^{k_r}},$$

where we fix the weight $k = k_1 + k_2 + \ldots + k_r$ and the depth (length) r. Similarly, we also define

the multiple Hurwitz zeta values and multiple t-values,

$$\zeta(k_1, k_2, \ldots, k_r; a_1, a_2, \ldots, a_r) = \sum_{1 \le n_1 < n_2 < \ldots < n_r} \frac{1}{(n_1 + a_1)^{k_1}(n_2 + a_2)^{k_2} \ldots (n_r + a_r)^{k_r}}$$

and

$$t(k_1, k_2, \ldots, k_r) = 2^{-(k_1+k_2+\ldots+k_r)} \zeta(k_1, k_2, \ldots, k_r; -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}) =$$

$$= \sum_{1 \le n_1 < n_2 < \ldots < n_r} \frac{1}{(2n_1 - 1)^{k_1}(2n_2 - 1)^{k_2} \ldots (2n_r - 1)^{k_r}}.$$

Also, let us recall the Gauss hypergeometric function which is defined for $|x| < 1$ by the power series

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!},$$

where $(q)_n = q(q+1) \cdot \ldots \cdot (q+n-1)$ is the Pochhammer symbol. As we have already proved in the previous chapter, the even and odd integer powers from the Taylor series of arcsin can be obtained by comparing the coefficients like powers of λ in the formulas:

$$\cos(\lambda \arcsin(x)) = {}_2F_1\left(\frac{\lambda}{2}, -\frac{\lambda}{2}; \frac{1}{2}; x^2\right),$$

$$\sin(\lambda \arcsin(x)) = \lambda x \cdot {}_2F_1\left(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}; \frac{1}{2}; x^2\right).$$

This implies the following Taylor series for the integer powers of arcsin function:

$$\frac{\arcsin^{2r}(x)}{(2r)!} = \frac{1}{4^r} \sum_{n=1}^{\infty} \frac{4^n}{n^2\binom{2n}{n}} \cdot x^{2n} \cdot \sum_{1 \le n_1 < n_2 < \ldots n_{r-1} < n} \frac{1}{n_1^2 n_2^2 \ldots n_{r-1}^2} \tag{5.2}$$

and

$$\frac{\arcsin^{2r+1}(x)}{(2r+1)!} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)4^n} \cdot x^{2n+1} \cdot \sum_{0\le n_1<n_2<\ldots<n_r<n} \frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}. \tag{5.3}$$

Also, using Taylor series expansion of $\exp(a \arcsin x)$ and iterated integrals, Borwein and Chamberland [20] derived the same formulae in the following form:

$$\arcsin^{2r} x = (2r)! \sum_{n=1}^{\infty} \frac{H_r(n)4^n}{n^2\binom{2n}{n}} x^{2n}, |x| \le 1 \tag{5.4}$$

where $H_1(n) = \dfrac{1}{4}$ and $H_{r+1}(n) := \dfrac{1}{4} \displaystyle\sum_{n_1=1}^{n-1} \frac{1}{(2n_1)^2} \sum_{n_2=1}^{n_1-1} \frac{1}{(2n_2)^2} \cdots \sum_{n_r=1}^{n_{r-1}-1} \frac{1}{(2n_r)^2}$.

and

$$\arcsin^{2r+1} x = (2r+1)! \sum_{n=0}^{\infty} \frac{G_r(n)\binom{2n}{n}}{(2n+1)4^n} x^{2n+1}, |x| \le 1 \tag{5.5}$$

where $G_0(n) = 1$ and $G_r(n) = \displaystyle\sum_{n_1=0}^{n-1} \frac{1}{(2n_1+1)^2} \sum_{n_2=0}^{n_1-1} \frac{1}{(2n_2+1)^2} \cdots \sum_{n_r=0}^{n_{r-1}-1} \frac{1}{(2n_r+1)^2}$

In 2009, Chu and Zheng [37] gave explicit closed form for integer powers of arcsin in terms of elementary symmetric functions. In other words, let

$$e_N(x_1, x_2, \ldots, x_N) = \prod_{i=1}^{N}(1 + tx_i) = \sum_{N} e_N(x_1, x_2, \ldots, x_N) \cdot t^m.$$

By methods which rely on two hypergeometric summation formulae, they reproved the above formulas in the following form:

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}(x/2)^{2n}}{2n+1} e_N\left(1, \frac{1}{3^2}, \ldots, \frac{1}{(2n-1)^2}\right) = \frac{(\arcsin x)^{2N+1}}{(2N+1)!x}$$

and

$$\sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2\binom{2n}{n}} e_N\left(1, \frac{1}{2^2}, \ldots, \frac{1}{(n-1)^2}\right) = \frac{(2\arcsin x)^{2N+2}}{(2N+2)!}$$

Therefore, this implies the following for integer power of arcsin(x) near $x = 0$:

$$\arcsin^3 x = 6\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{(2m+1)^2}\right)\frac{\binom{2n}{n}}{2^{2n}(2n+1)}x^{2n+1}, \tag{5.6}$$

$$\arcsin^4 x = \frac{3}{2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m^2}\right)\frac{1}{2^{2n}n^2\binom{2n}{n}}x^{2n}, \tag{5.7}$$

$$\arcsin^5 x = 120\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-1}\frac{1}{(2m+1)^2}\sum_{p=0}^{m-1}\frac{1}{(2p+1)^2}\right)\frac{\binom{2n}{n}}{(2n+1)2^{2n}}x^{2n+1}, \tag{5.8}$$

$$\arcsin^6 x = \frac{45}{4}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\frac{1}{m^2}\sum_{p=1}^{m-1}\frac{1}{p^2}\right)\frac{4^n}{\binom{2n}{n}n^2}x^{2n}. \tag{5.9}$$

5.2 MAIN RESULTS

In this section we give a very different approach in evaluating multiple zeta and special Hurwitz zeta values for the Hoffman elements. Some of the ideas were presented for evaluating single zeta values in Chapter 1. In what follows a present new proofs for evaluations of $\zeta(\underbrace{2,2,\ldots,2}_{r})$ and $t(\underbrace{2,2,\ldots,2}_{r})$.

Theorem 5.2.1. *We have the following evaluation*

$$\zeta(\underbrace{2,2,\ldots,2}_{r}) = \frac{\pi^{2r}}{(2r+1)!},$$

Proof. Substituting $x = \sin t$ in formula (5.2) we have

$$\frac{t^{2r}}{(2r)!} = \frac{1}{4^r}\sum_{n=1}^{\infty}\frac{4^n}{n^2\binom{2n}{n}}\sin^{2n} t\sum_{1\le n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2 n_2^2\ldots n_{r-1}^2}$$

Integrating from 0 to $\frac{\pi}{2}$, and using Wallis' formula, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n} t dt = \frac{\pi\binom{2n}{n}}{2^{2n+1}}$, we have

$$\frac{\pi^{2r}}{(2r+1)!} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{1 \le n_1 < n_2 < \ldots n_{r-1} < n} \frac{1}{n_1^2 n_2^2 \ldots n_{r-1}^2} = \zeta(\underbrace{2, 2, \ldots, 2}_{r})$$

and we are done.□

Theorem 5.2.2. *We have the following evaluation*

$$t(\underbrace{2, 2, \ldots, 2}_{r}) = \frac{\pi^{2r}}{2^{2r}(2r)!}.$$

Proof. Similarly, substitute $x = \sin t$ in formula (5.3) and we have

$$\frac{t^{2r+1}}{(2r+1)!} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{(2n+1)4^n} \sin^{2n+1} t \sum_{0 \le n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

Again, integrating from 0 to $\frac{\pi}{2}$ and using Wallis' formula in the form, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n+1} t dt = \frac{1}{2n+1} \cdot \frac{2^{2n}}{\binom{2n}{n}}$, $n \ge 0$, we have

$$\frac{\pi^{2r+2}}{2^{2r+2}(2r+1)!(2r+2)} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \sum_{0 \le n_1 < n_2 < \ldots < n_r < n} \frac{1}{\prod_{i=1}^{r}(2n_i+1)^2} = t(\underbrace{2, 2, \ldots, 2}_{r+1}),$$

and the conclusion follows immediately.□

Next, we evaluate $\zeta(\underbrace{2, 2, \ldots, 2}_{r}, 3)$ and $t(\underbrace{2, 2, \ldots, 2}_{r}, 3)$ in terms of rational zeta series involving $\zeta(2n)$.

Theorem 5.2.3. *We have the following evaluations*

$$H(r-1, 0) := \zeta(\underbrace{2, 2, \ldots, 2}_{r-1}, 3) = -4(2r+1) \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}} \cdot \zeta(\underbrace{2, 2, \ldots, 2}_{r})$$

Proof. We start again with formula (5.2),

$$\frac{\arcsin^{2r}(x)}{(2r)!} = \frac{1}{4^r}\sum_{n=1}^{\infty}\frac{4^n}{n^2\binom{2n}{n}}\cdot x^{2n}\cdot\sum_{1\le n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2n_2^2\ldots n_{r-1}^2}.$$

Dividing by x and integrating from 0 to $\sin t$, we have

$$\int_0^{\sin t}\frac{\arcsin^{2r}(x)}{x}dx = \frac{(2r)!}{4^r}\sum_{n=1}^{\infty}\frac{4^n}{2n^3\binom{2n}{n}}\sin^{2n}t\sum_{n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2n_2^2\ldots n_{r-1}^2}.$$

By the substitution $x = \sin u$ in the integral from the left hand side, we have

$$\int_0^{\sin t}\frac{\arcsin^{2r}(x)}{x}dx = \int_0^t u^{2r}\cot u du = \frac{(2r)!}{4^r}\sum_{n=1}^{\infty}\frac{4^n}{2n^3\binom{2n}{n}}\sin^{2n}t\sum_{n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2n_2^2\ldots n_{r-1}^2}.$$

On the other hand, using $u\cot u = -2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}u^{2n}$, $|u|<\pi$, we derive

$$\int_0^t u^{2r}\cot u du = -2\int_0^t\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}u^{2n+2r-1}du = -2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\cdot\frac{t^{2n+2r}}{(2n+2r)}.$$

Therefore, we obtain

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\cdot\frac{t^{2n+2r}}{(2n+2r)} = \frac{(2r)!}{4^r}\sum_{n=1}^{\infty}\frac{4^n}{2n^3\binom{2n}{n}}\sin^{2n}t\sum_{n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2n_2^2\ldots n_{r-1}^2}.$$

Finally, integrating from 0 to $\frac{\pi}{2}$ and using Wallis' integral formula, $\int_0^{\frac{\pi}{2}}\sin^{2n}tdt = \frac{\pi\binom{2n}{n}}{2^{2n+1}}$, we obtain

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\frac{\pi^{2n+2r+1}}{2^{2n+2r+1}(2n+2r)(2n+2r+1)} = \frac{(2r)!\pi}{4^{r+1}}\sum_{n=1}^{\infty}\frac{1}{n^3}\sum_{n_1<n_2<\ldots n_{r-1}<n}\frac{1}{n_1^2n_2^2\ldots n_{r-1}^2}$$

or equivalently,

$$-4\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}}\cdot\frac{\pi^{2r}}{(2r)!}=\zeta(\underbrace{2,2,\ldots,2}_{r-1},3).$$

Now, using the fact that $\zeta(\underbrace{2,2,\ldots,2}_{r})=\dfrac{\pi^{2r}}{(2r+1)!}$, we obtain the conclusion of the theorem.□

Theorem 5.2.4. *We have*

$$T(r):=t(\underbrace{2,2,\ldots,2}_{r},3)=-4(r+1)\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2r+1)(2n+2r+2)2^{2n}}\cdot t(\underbrace{2,2,\ldots,2}_{r+1}).$$

Proof. We start with formula (5.3),

$$\frac{\arcsin^{2r+1}(x)}{(2r+1)!}=\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)4^n}\cdot x^{2n+1}\cdot\sum_{0\le n_1<n_2<\ldots<n_r<n}\frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

Dividing by x and integrating from 0 to $\sin t$, we have

$$\int_0^{\sin t}\frac{\arcsin^{2r+1}(x)}{x}dx=(2r+1)!\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^2 4^n}\sin^{2n+1}t\sum_{0\le n_1<n_2<\ldots<n_r<n}\frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

By the substitution $x=\sin u$ in the integral, we have

$$\int_0^{t}u^{2r+1}\cot u du=(2r+1)!\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)^2 4^n}\sum_{0\le n_1<n_2<\ldots<n_r<n}\frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

On the other hand, using $u\cot u=-2\sum_{n=0}^{\infty}\dfrac{\zeta(2n)}{\pi^{2n}}u^{2n}$, $|u|<\pi$, we derive

$$\int_0^{t}u^{2r+1}\cot u du=-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\cdot\frac{t^{2n+2r+1}}{2n+2r+1}.$$

Therefore, we obtain

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}}\cdot\frac{t^{2n+2r+1}}{2n+2r+1}=(2r+1)!\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(2n+1)2^{4n}}\sin^{2n+1}t\sum_{0\le n_1<n_2<\ldots<n_r<n}\frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

Finally, integrating from 0 to $\frac{\pi}{2}$ and using Wallis' integral formula in the form, $\displaystyle\int_0^{\frac{\pi}{2}}\sin^{2n+1}t dt=\frac{1}{2n+1}\cdot\frac{2^{2n}}{\binom{2n}{n}}$, $n\ge 0$, we have

$$\frac{\pi^{2r+2}}{(2r+1)!2^{2r+2}}\left(-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+2r+1)(2n+2r+2)2^{2n}}\right)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sum_{0\le n_1<n_2<\ldots<n_r<n}\frac{1}{\prod_{i=1}^{r}(2n_i+1)^2}.$$

Now, using the fact that $t(\underbrace{2,2,\ldots,2}_{r})=\frac{\pi^{2r}}{2^{2r}(2r)!}$ (Theorem 5.2.2) we obtain the desired result. □

The main question is if Theorems 5.2.3 and 5.2.4 will allow us to express the family of multiple zeta and t-values involving Hoffman elements as a $\mathbb{Q}$-linear combinations of powers of π and odd zeta values. Before we go into more detail, let us state the following

Theorem 5.2.5 (Orr, [64])**.** *For $p\in\mathbb{N}$, and $|z|<1$,*

$$\int_0^{\pi z}x^p\cot x dx=(\pi z)^p\sum_{k=0}^{p}\frac{p!(-1)^{[\frac{k+3}{2}]}}{(p-k)!(2\pi z)^k}\mathrm{Cl}_{k+1}(2\pi z)+\delta_{[\frac{p}{2}],\frac{p}{2}}\frac{p!(-1)^{\frac{p}{2}}}{2^p}\zeta(p+1),$$

where Cl_{k+1} is the $(k+1)$-Clausen function.

The above theorem combined with the following elementary properties of Clausen functions from Chapter 1, subsection 2.1.3 give us the missing ingredient

Lemma 5.2.6. *We have the following equality*

$$-2\sum_{n=0}^{\infty}\frac{\zeta(2n)}{(2n+p)2^{2n}}=\log 2+\sum_{k=1}^{[p/2]}\frac{p!(-1)^k(4^k-1)\zeta(2k+1)}{(p-2k)!(2\pi)^{2k}}+\delta_{[p/2],p/2}\frac{p!(-1)^{p/2}\zeta(p+1)}{\pi^p}.$$

Proof of the lemma. (see also [64]) Theorem 5.2.5 for $z = \frac{1}{2}$ give us

$$\int_0^{\frac{\pi}{2}} x^p \cot x dx = \left(\frac{\pi}{2}\right)^p \left(\log 2 + \sum_{k=1}^{[p/2]} \frac{p!(-1)^k(4^k-1)}{(p-2k)!(2\pi)^{2k}} \zeta(2k+1)\right) + \delta_{[\frac{p}{2}],\frac{p}{2}} \frac{p!(-1)^{\frac{p}{2}}\zeta(p+1)}{2^p}.$$

On othe other hand, since $\cot x = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \cdot x^{2n-1}, |x| < \pi$, by integration and Fubini's theorem, we obtain

$$\int_0^{\frac{\pi}{2}} x^p \cot x dx = \int_0^{\frac{\pi}{2}} x^p \left(-2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \cdot x^{2n-1}\right) dx = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} \int_0^{\frac{\pi}{2}} x^{2n+p-1} dx$$

or equivalently,

$$\int_0^{\frac{\pi}{2}} x^p \cot x dx = -2\left(\frac{\pi}{2}\right)^p \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+p)2^{2n}},$$

and the lemma follows immediately.□

The special case $s = 0$ of Zagier's formula reads as the following

Corollary 5.2.7. *We have*

$$H(r-1,0) = 2\sum_{k=1}^{r}(-1)^k \left[\binom{2k}{2r} - \left(1 - \frac{1}{2^{2k}}\right) 2k\right] \zeta(2k+1)\zeta(\underbrace{2,2,\ldots,2}_{r-k}).$$

Proof. By applying Lemma 5.2.6 (for $p = 2r$ and $p = 2r+1$), we have

$$-2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}} = -2\left(\sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r)2^{2n}} - \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r+1)2^{2n}}\right) =$$

$$= \sum_{k=1}^{r} \frac{(2r)!(-1)^k(4^k-1)\zeta(2k+1)}{(2r-2k)!(2\pi)^{2k})} + \delta_{r,r} \frac{r!(-1)^r\zeta(2r+1))}{\pi^{2r}} -$$

$$-\sum_{k=1}^{r} \frac{(2r+1)!(-1)^k(4^k-1)\zeta(2k+1)}{(2(r-k)+1)!(2\pi)^{2k}} =$$

$$= \sum_{k=1}^{r} \left[\frac{(2r)!(-1)^k(4^k-1)}{(2r-2k)!(2\pi)^{2k})} - \frac{(2r+1)!(-1)^k(4^k-1))}{(2(r-k)+1)!(2\pi)^{2k}} + \frac{(2r)!(-1)^k)}{\pi^{2k}}\delta_{r,k} \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \left[\frac{(-1)^k(4^k-1)}{(2\pi)^{2k}}(2k)! \left(\binom{2r}{2k} - \binom{2r+1}{2k} \right) + \delta_{k,r} \frac{(2r)!(-1)^r}{\pi^{2r}} \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \frac{(-1)^k(2k)!}{\pi^{2k}} \left[\left(1 - \frac{1}{4^k}\right) \left(-\binom{2r}{2k-1} \right) + \delta_{r,k} \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \frac{(-1)^k(2k)!}{\pi^{2k}} \binom{2r}{2k-1} \left[\frac{\delta_{r,k}}{\binom{2r}{2k-1}} - \left(1 - \frac{1}{4^k}\right) \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \frac{(-1)^k(2k)!}{\pi^{2k}} \cdot \frac{(2r)!}{(2k-1)!(2r-2k+1)!} \left[\frac{\delta_{r,k}}{\binom{2r}{2k-1}} - \left(1 - \frac{1}{4^k}\right) \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \frac{(-1)^k 2k}{\pi^{2k}} \cdot \frac{(2r)!}{(2(r-k)+1)!} \left[\frac{\delta_{r,k}}{\binom{2r}{2k-1}} - \left(1 - \frac{1}{4^k}\right) \right] \zeta(2k+1) =$$

$$= \sum_{k=1}^{r} \frac{(-1)^k(2k)(2r)!\pi^{2(r-k)}}{\pi^{2k} \cdot \pi^{2(r-k)} \cdot (2(r-k)+1)!} \left[\frac{\delta_{r,k}}{\binom{2r}{2k-1}} - \left(1 - \frac{1}{4^k}\right) \right] \zeta(2k+1) =$$

$$= \frac{1}{(2r+1)} \sum_{k=1}^{r} \frac{(2r+1)!}{\pi^{2r}} \cdot \frac{\pi^{2(r-k)}}{(2(r-k)+1)!} (-1)^k \left[\frac{2k\delta_{r,k}}{\binom{2r}{2k-1}} - \left(1 - \frac{1}{4^k}\right) 2k \right] \zeta(2k+1) =$$

$$= \frac{1}{(2r+1)\zeta(\underbrace{2,2,\ldots,2}_{r})} \sum_{k=1}^{r} (-1)^k \zeta(\underbrace{2,2,\ldots,2}_{r-k}) c_{r-1,0}^{k} \zeta(2k+1),$$

where obvisouly $\frac{2k\delta_{r,k}}{\binom{2r}{2k-1}} = \binom{2k}{2r}$.

By Theorem 5.2.3, we deduce

$$H(r-1,0) := \zeta(\underbrace{2,2,\ldots,2}_{r-1},3) = 2(2r+1) \cdot \left(-2 \sum_{n=0}^{\infty} \frac{\zeta(2n)}{(2n+2r)(2n+2r+1)2^{2n}} \right) \cdot$$
$$\cdot \zeta(\underbrace{2,2,\ldots,2}_{r}),$$

which will give us

$$H(r-1,0) = 2(2r+1)\cdot\frac{1}{(2r+1)\zeta(\underbrace{2,2,\ldots,2}_{r})}\sum_{k=1}^{r}(-1)^k\zeta(\underbrace{2,2,\ldots,2}_{r-k})c_{r-1,0}^{k}\zeta(2k+1)\cdot\zeta(\underbrace{2,2,\ldots,2}_{r}),$$

which finally reduces to

$$H(r-1,0) = 2\sum_{k=1}^{r}(-1)^k c_{r-1,0}^{k}\zeta(\underbrace{2,2,\ldots,2}_{r-k})\zeta(2k+1),$$

which is exactly what we wanted to prove.□

Moreover, following the same computational path as above, we have the following

Corollary 5.2.8.

$$T(r) = 2\sum_{k=1}^{r+1}(-1)^k\frac{1}{4^k(2(r-k+1)+1)}\left[\left(1-\frac{1}{4^k}\right)2k-(2r+1)(2r+2)\binom{2k}{2r+2}\right]t(\underbrace{2,2,\ldots,2}_{r-k+1})\zeta(2k+1).$$

6.0 FUTURE RESEARCH PROJECTS

- My first goal, which is a work in progress at this point [60], is to find a more general formula for multiple zeta values of the following type,

$$H^m(r-1,0) = \zeta(\underbrace{2,2,\ldots,2}_{r-1},m), m \geq 3.$$

I imagine that a Zagier type formula for $H^m(r-1,0)$ will be hopeless, but I suspect that for some particular values of $m = 4, 5$ will provide us some formulas as a $\mathbb{Q}$-linear combinations of powers of π and odd zeta values.

- Perhaps there is a rational zeta series approach for the general Zagier formula for $H(r, s)$. The first stept would be to see what happens in the case when $s = 1$, and from there we can find a pattern for the general case.

- It would be interesting to see whether we can find a Zagier type formula for the multiple t-values, $t(2, \ldots, 2, 3, 2, \ldots, 2)$

- Another breakthrough would be to find an evaluation for the multiple zeta values $\zeta(2, \ldots, 2, 3, 3)$. Perhaps, this could lead to a proof of Hoffman's conjecture.

7.0 APPENDIX

7.1 TAYLOR SERIES EXPANSION FOR THE TANGENT, COTANGENT, AND COSECANT FUNCTIONS

Let us start with the following formula

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} = \frac{t}{e^t - 1}, \quad |t| < 2\pi.$$

On the left-hand side we take only the even powers of t, and we have

$$\sum_{n=0}^{\infty} B_n \cdot \frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n \cdot \frac{(-t)^n}{n!} = \frac{t}{e^t - 1} - \frac{t}{e^{-t} - 1}.$$

This gives us

$$2\sum_{n=0}^{\infty} B_{2n} \cdot \frac{t^{2n}}{(2n)!} = \frac{t(e^{-t} - e^t)}{(e^t - 1)(e^{-t} - 1)} = \frac{t(e^{-t/2} + e^{t/2})(e^{-t/2} - e^{t/2})}{e^{t/2}(e^{t/2} - e^{-t/2})(e^{-t/2} - e^{-t/2})e^{-t/2}},$$

which is equivalent with

$$2\sum_{n=0}^{\infty} B_{2n} \cdot \frac{t^{2n}}{(2n)!} = \frac{t(e^{t/2} + e^{-t/2})}{e^{t/2} - e^{t/2}}.$$

For $t = 2ix$, we have

$$2\sum_{n=0}^{\infty} B_{2n} \cdot \frac{(2ix)^{2n}}{(2n)!} = \frac{2ix(e^{ix}+e^{-ix})}{e^{ix}-e^{-ix}},$$

or

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} x^{2n-1} = \frac{i(e^{ix}+e^{-ix})}{e^{ix}-e^{-ix}}.$$

Recall that $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$ and $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$, and since $\cot x = \dfrac{\cos x}{\sin x}$, it follows that

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} \cdot x^{2n-1}.$$

On the other hand, we have $\cot x = 2\cot(2x) = \cot x - \dfrac{\cot^2 x - 1}{\cot x} = \dfrac{1}{\cot x} = \tan x$, and this implies that

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} \cdot x^{2n-1} - 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n 2^{2n-1} B_{2n}}{(2n)!} \cdot x^{2n-1},$$

which is equivalent with

$$\tan x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n (1-4^n) B_{2n}}{(2n)!} \cdot x^{2n-1}.$$

In a similar way, we can also deduce a formula for the cosecant function. Indeed, using the identity

$$\cot x - \cot(2x) = \cot x - \frac{\cot^2 x - 1}{2\cot x} = \frac{\csc^2 x}{2\cot x} = \frac{1}{2\sin x\cos x} = \csc(2x),$$

and thus

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^n 4^n 2^{2n-1} B_{2n}}{(2n)!} \cdot x^{2n-1} - \sum_{n=0}^{\infty} \frac{(-1)^n 4^n B_{2n}}{(2n)!} \cdot x^{2n-1},$$

and this gives us

$$\csc x = \sum_{n=0}^{\infty} \frac{(-1)^n(2-4^n)B_{2n}}{(2n)!} \cdot x^{2n-1}.$$

7.2 AN INTEGRAL FORMULA INVOLVING APERY AND CATALAN CONSTANTS

In the proof of Theorem 3.2.3, pp. 40 we used the following formula

Lemma 7.2.1.

$$\int_0^{\frac{\pi}{4}} u \log(\sin u) du = \frac{35}{128}\zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32} \log 2.$$

In what follows we shall give an account into the idea of the proof of the above formula. In [35], the authors consider the Barnes G-function which satisfies the following fundamental functional equation:

$$G(1) = 1 \quad \textit{and} \quad G(z+1) = \Gamma(z)G(z), \quad z \in \mathbb{C},$$

where $\Gamma(z)$ is the Euler gamma function.

Also, Barnes generalized the above relations to the case of *multiple gamma functions* denoted by $G_n(z)$ or $\Gamma_n(z)$. Thus, we have

$$\Gamma_n(z) := (G_n(z))^{(-1)^{n-1}}, \quad n \in \mathbb{N},$$

so that

$$G_1(z) = \Gamma_1(z) = \Gamma(z), \quad G_2(z) = \frac{1}{\Gamma_2(z)} = G(z), \quad G_3(z) = \Gamma_3(z),$$

and so on.

In terms of multiple gamma functions $G_n(z)$ of order n, we have the following functional relationships,

$$G_n(1) = 1, \quad G_{n+1}(z) = G_n(z+1) = G_n(z)G_{n+1}(z), \quad z \in \mathbb{C}, \quad n \in \mathbb{N},$$

or equivalently

$$\Gamma_n(1) = 1 \quad \textit{and} \quad \Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)}, \quad z \in \mathbb{C} \quad n \in \mathbb{N}.$$

In [35] the following identity is deduced:

$$\begin{aligned}\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n\pi t)-1}{n^3} = \frac{1}{\pi^2}\int_0^{\pi t} u\log(\sin u)du + \frac{t^2}{2}\log 2+ \\ (t-1)\log G(t+1) - (1+t)G(1-t) - 2\log(\Gamma_3(1+t)\Gamma_3(1-t))\end{aligned} \tag{7.1}$$

Note that the left-hand side can be expressed in terms of Clausen function $\mathrm{Cl}_n(t)$ as follows:

$$\sum_{n=1}^{\infty}\frac{\cos(2n\pi t)-1}{n^3} = \mathrm{Cl}_3(2\pi t) - \zeta(3). \tag{7.2}$$

Combining the two identities, we infer

$$\frac{1}{4\pi^2}[\mathrm{Cl}_3(2\pi t) - \zeta(3)] = \frac{1}{\pi^2}\mathcal{S}(t) + \frac{t^2}{2}\log 2 + (t-1)\log G(t+1) - (1+t)\log G(1-t) - 2\log\mathcal{A}(t)$$

where $\mathcal{S}(t) := \int_0^{\pi t} u\log(\sin u)du$ and $\mathcal{A}(t) = \Gamma_3(1+t)\Gamma_3(1-t)$.

Using evaluations in terms of rational zeta series involving the values $\zeta(2n)$ and $\zeta(2n+1)$ for $\log\mathcal{A}(t)$ and $\log(G(1+t)G(1-t))$ as well as $\mathrm{Cl}_{2n+1}(\pi/2) = \dfrac{1-2^n}{2^{4n+1}}\zeta(2n+1)$, we arrive at the following

$$\int_0^{\frac{\pi}{4}} u\log(\sin u)du = \mathcal{S}(1/4) = \frac{35}{128}\zeta(3) - \frac{\pi G}{8} - \frac{\pi^2}{32}\log 2.$$

7.3 PROOF OF THEOREM 5.2.5

For the sake of completeness, we decided to incorporate Orr's original proof of Theorem 5.2.5 from [64].

Let us consider the following functions,

$$f(z) = \int_0^{\pi z} x^p \cot(x)\, dx,$$

and

$$g(z) = (\pi z)^p \sum_{k=0}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}}{(p-k)!(2\pi z)^k} \operatorname{Cl}_{k+1}(2\pi z) + \delta_{\lfloor \frac{p}{2} \rfloor, \frac{p}{2}} \frac{p!(-1)^{\frac{p}{2}}}{2^p} \zeta(p+1).$$

Clearly, $f'(z) = \pi^{p+1} z^p \cot(\pi z)$. On the other hand, using basic properties of the Clausen function (pp. 24), we have

$$\begin{aligned} g'(z) = \frac{1}{2^p} \sum_{k=0}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}(p-k)(2\pi)^{p-k} z^{p-k-1}}{(p-k)!} \operatorname{Cl}_{k+1}(2\pi z) + \frac{(\pi z)^p 2\pi \cos(\pi z)}{2\sin(\pi z)} \\ + \frac{1}{2^p} \sum_{k=1}^{p} \frac{p!(-1)^{\lfloor \frac{k+3}{2} \rfloor}(2\pi z)^{p-k}}{(p-k)!} \Big\{(-1)^{k+1} 2\pi \operatorname{Cl}_k(2\pi z)\Big\} \\ = \pi^{p+1} z^p \cot(\pi z) + (\pi z)^p \sum_{k=1}^{p} \frac{p! \operatorname{Cl}_k(2\pi z)}{(p-k)! z^{k+1} (2\pi)^{k-1}} \Big((-1)^{\lfloor \frac{k+2}{2} \rfloor} + (-1)^{\lfloor \frac{k+3}{2} \rfloor}(-1)^k\Big). \end{aligned}$$

This implies that $g'(z) = \pi^{p+1} z^p \cot(\pi z) = f'(z)$. On the other hand, $f(0) = 0$, and using again the basic properties of the Clausen function, we have

$$g'(0) = g(0) = \frac{1}{2^p} p!(-1)^{\lfloor \frac{p+3}{2} \rfloor} \operatorname{Cl}_{p+1}(0) + \delta_{\lfloor \frac{p}{2} \rfloor, \frac{p}{2}} \frac{p!(-1)^{\frac{p}{2}}}{2^p} \zeta(p+1) = 0.$$

Now, since $f(0) = g(0) = 0$ and $f'(z) = g'(z)$, it follows that $f(z) = g(z)$ and we are done.

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