ENSEMBLE TIME-STEPPING ALGORITHMS FOR NATURAL CONVECTION

by

Joseph Fiordilino

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This dissertation was presented

by

Joseph Fiordilino

It was defended on

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and approved by

Dr. William Layton, Mathematics, University of Pittsburgh

Dr. Michael Neilan, Mathematics, University of Pittsburgh

Dr. Catalin Trenchea, Mathematics, University of Pittsburgh

Dr. Mehrdad Massoudi, Biomedical Engineering, Carnegie Mellon University

Dissertation Director: Dr. William Layton, Mathematics, University of Pittsburgh

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Joseph Fiordilino, PhD

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Predictability of fluid flow via natural convection is a fundamental issue with implications for, e.g., weather predictions including global climate change assessment and nuclear reactor cooling. In this work, we study numerical methods for natural convection and utilize them to study predictability. Eight new algorithms are devised which are far more efficient than existing ones for ensemble calculations. They allow for either increased ensemble sizes or denser meshes on current computing systems.

The artificial compressibility ensemble (ACE) family produce accurate velocity and temperature approximations and are fastest. The speed of second-order ACE degrades as $\epsilon \to 0$ or $\Delta t \to 0$ due to the iterative solver. However, first-order ACE has a uniform solve time since $\gamma = \mathcal{O}(1)$. The ensemble backward differentiation formula (eBDF) family are most accurate and reliable. The penalty ensemble algorithm (PEA) family are strongly affected by the timestep and are least accurate. In particular, $\gamma = \mathcal{O}(1/\Delta t^2)$ for second-order PEA leads to solver breakdown. We also propose an ACE turbulence (ACE-T) family of methods for turbulence modeling which are both fast and accurate.

A complete numerical analysis is performed which establishes full-reliability. The analysis involves techniques that are novel and results that subsume, elucidate, and expand previous results in closely related fields, e.g., iso-thermal fluid flow. Numerical tests show predicted accuracy is consistent with theory.

Predictability is a highly complex and problem-dependent phenomenon. Predictability studies are performed utilizing the new second-order ACE algorithm. We perform a numerical test where the flow reaches a steady state. It is found that increasing the size of the domain increases predictability. Also, spatial averages increase predictability with increasing filter radius. We also study a problem with a manufactured solution. Sufficiently large rotations increase the predictability of a flow. Further, spatial averages decrease predictability with increasing filter radius.

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PREFACE

The most important choice a person can make for doctoral study is their adviser. After all, it is an apprenticeship. Professor William Layton was instrumental in my success. He is an immensely knowledgeable and skillful researcher. Further, he went out of his way to make himself available to answer questions and provide both career and life advice. It's incredibly rare to find an adviser with a genuine interest in the current and future well-being of their students. Thank you.

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Lastly, I would like to thank my wife Pamela and daughter Angela. Your love and support mean the world to me. Thank you for being there by my side. I could not have done this without you. Our time in Pittsburgh has been wonderful and filled with many great memories we will take with us on the next journey.

1.0 INTRODUCTION

There cannot be a greater mistake than that of looking superciliously upon practical applications of science. The life and soul of science is its practical application...

Sir William Thomson [82]

This thesis is concerned with numerical methods, and their properties, for the study of natural convection. Natural convection is ubiquitous in nature. This phenomenon plays a fundamental role in our planet's atmosphere and oceans and in many technologies, e.g., cooling of nuclear reactor and electronic systems; see, e.g., [6, 47, 101, 109] and references therein. Natural convection can be described mathematically by the Boussinesq equations under a variety of circumstances.

The Boussinesq equations are derived from physical conservation laws [14, 116]; that is, conservation of momentum, mass, and internal energy. Thus, they share the same structure as the Navier-Stokes equations (NSE) and, therefore, many of the difficulties and phenomenon, such as turbulence, are present. The resulting nonlinear set of equations exhibit chaos and the question of predictability arises. Thus, numerical solution of the Boussinesq equations with slightly different initial conditions can exhibit exponential separation of nearby trajectories.

With this in mind, practical computing demands several runs of a code. In particular, for many problems of interest, a single realization solve is not sufficient owing to fundamental uncertainty in initial conditions, forcings, parameters, and etc. [89]. Consequently, ensemble calculations are critical. Briefly, ensemble calculations amount to J solves of a set of equations with slightly perturbed initial data. Averaging these solutions produces the ensemble averaged solution, which tends to perform better as a prediction than any of the realizations; see, e.g., Chapter 6.5 of [80] or [4, 44, 81] and references therein.

It is evident that the ensemble size and mesh density are competing factors; increases in either yield better approximations. Consequently, the development of algorithms which can reduce storage requirements or decrease turnaround time are well justified. Currently, the practice is to solve a single set of equations with slightly perturbed data. Ultimately, this reduces to J linear algebra problems, $1 \leq j \leq J$: $A_j x_j = b_j$; that is, J linear solves with J different coefficient matrices. If the J coefficient matrices could be somehow reduced, this would constitute a major storage decrease. Further, if only a *single* coefficient matrix were needed, efficient block solvers would be applicable and an additional, dramatic reduction in turnaround time would be possible.

In view of the above, several algorithms are proposed and studied in this thesis. We analyze longtime stability of some commonly used algorithms, e.g., the first- and second-order backward differentiation formula (BDF) family. New ensemble algorithms are proposed and a complete numerical analysis is performed. Consequently, full reliability of each algorithm is established. Thus, scientists and engineers can have confidence in our algorithms' capabilities and limitations.

In Chapter 2, we introduce mathematical tools and notation that will be instrumental in establishing rigorous results. For example, the discrete inf-sup condition (2.23), also known as the discrete Ladyzhenskaya-Babŭska-Brezzi condition (LBBh), plays a pivotal role throughout. When satisfied, this condition ensures well-posedness, stability of approximate solutions, and is necessary for convergence to the true solutions of the continuous problems. We will use this condition to establish *full-reliability* of all algorithms.

We confront the issue of long-time stability of solutions to the Boussinesq equations in Chapter 3. Under mild conditions, the temperature is uniformly bounded in time, however, common numerical methods, e.g., the BDF family, have not yet been proven to exhibit this behavior. The main difficulty is that the temperature does not satisfy homogeneous Dirichlet boundary conditions. Therefore, the convective and buoyancy terms couple the velocity and temperature in an essential way that is difficult to treat with current mathematical tools.

We introduce a new interpolant, the discrete Hopf interpolant (Theorem 3), to confront this issue. Using it, we are able to show that the velocity and temperature approximations can exhibit, at most, sub-linear growth in the final simulation time t^* provided the finite element mesh satisfies a mesh condition. In particular, provided that the first mesh-line of the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the hot wall, $\|u_h^{n+1}\| + \|T_h^{n+1}\| \leq C\sqrt{t^*}$.

Chapter 4 forms the central component of this thesis. We confront the issue of efficiency when simulating multiple realizations of the Boussinesq equations with slightly different initial data. Ensemble calculations have proven essential in, e.g., weather forecasting [80] and ocean modeling [89]. Unfortunately, ensemble calculations require J sequential, fine mesh solves or J parallel, coarse mesh solves of a given code. We introduce several ensemble algorithms, which offer potentially dramatic speed ups and reduce storage requirements. We verify and validate these algorithms with numerical experiments.

In Chapter 5, we utilize our flagship algorithm, second-order **ACE** from Chapter 4, to illustrate the use of ensembles. In particular, we study the predictability phenomenon. To this end, we introduce quantities which are useful for quantifying predictability horizons: average effective Lyapunov exponents, predictability horizons, and variance. Predictability horizons of solutions and their spatial averages are calculated for test problems.

We end with conclusions and open questions in Chapter 6. The aim of this thesis is to be more than a collection of results. The intent is to improve understanding on each of the topics confronted in Chapters 3 - 5 and invite graduate students and researchers to tackle interesting open problems. Consequently, we collect, state, and elaborate on many of the open problems that have arisen along our journey. We hope that this thesis will be accessible and provide sufficient insight to improve and extend results and develop understanding.

2.0 MATHEMATICAL PRELIMINARIES

> Young man, in mathematics you don't understand things. You just get used to them.
>
> John von Neumann [144]

In this section, we introduce notation and necessary preliminaries. $H^s(\Omega)$ denotes the Hilbert space of $L^2(\Omega)$ functions with distributional derivatives of order $s \geq 0$ in $L^2(\Omega)$. The corresponding norms and seminorms are $\|\cdot\|_s$ and $|\cdot|_s$. In the special case $s = 0$, $H^0(\Omega) = L^2(\Omega)$ and the associated inner product and induced norm are $(\cdot,\cdot)$ and $\|\cdot\|$. The $L^p(\Omega)$ norm is denoted $\|\cdot\|_{L^p}$ and $\|\cdot\|_{L^p(\Omega_0)}$ for $\Omega_0 \subset \Omega$, $1 \leq p \leq \infty$.

Define the Hilbert spaces,

$$X := H_0^1(\Omega)^d = \{v \in H^1(\Omega)^d : v = 0 \textit{ on } \partial\Omega\},\ Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : (1,q) = 0\},$$
$$W := H^1(\Omega),\ W_{\Gamma_D} := \{S \in W : S = 0 \textit{ on } \Gamma_D\},\ V := \{v \in X : (q, \nabla\cdot v) = 0\ \forall\, q \in Q\}.$$

The dual norm $\|\cdot\|_{-1}$ is understood to correspond to either X or W_{Γ_D}. Further, we utilize the fractional order Hilbert space on the nonhomogeneous Dirichlet boundary $H^{1/2}(\Gamma_{D_1})$ with corresponding norm

$$\|\chi\|_{1/2,\Gamma_{D_1}} := \Big(\int_{\Gamma_{D_1}} |\chi(s)|^2 ds + \int_{\Gamma_{D_1}}\int_{\Gamma_{D_1}} \frac{|\chi(s)-\chi(s')|^2}{|s-s'|^d} ds ds'\Big)^{1/2}.$$

An extension operator of the nonhomogeneous Dirichlet data will be useful.

Theorem 1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\chi \in H^{1/2}(\Gamma_{D_1})$. Then, there exists an extension operator $\tau : H^{1/2}(\Gamma_{D_1}) \to W$ and $C_{tr} > 0$ such that $\tau|_{\Gamma_{D_1}} = \chi$ and*

$$\|\tau\|_1 \leq C_{tr}\|\chi\|_{1/2,\Gamma_{D_1}}. \tag{2.1}$$

Proof. See Lemma 3.2 on p. 1832 of [23]. $\square$

For natural convection within a unit square or cubic enclosure with a pair of differentially heated vertical walls, the linear conduction profile $\tau(x) = 1 - x_1$, where x_1 denotes the spatial coordinate in the horizontal direction, is such an extension satisfying: $\|\tau\|_1 \leq \frac{2\sqrt{3}}{3}$.

The explicitly skew-symmetric trilinear forms are denoted:

$$
\begin{aligned}
b(u, v, w) &= \frac{1}{2}(u \cdot \nabla v, w) - \frac{1}{2}(u \cdot \nabla w, v) \quad \forall u, v, w \in X, \\
b^*(u, T, S) &= \frac{1}{2}(u \cdot \nabla T, S) - \frac{1}{2}(u \cdot \nabla S, T) \quad \forall u \in X,\ \forall T, S \in W.
\end{aligned}
$$

They enjoy the following continuity results and properties.

Lemma 1. *There are constants C_1, C_2, C_3, C_4, C_5, and C_6 such that for all u,v,$w \in X$ and T,$S \in W$, $b(u, v, w)$ and $b^*(u, T, S)$ satisfy*

$$
\begin{aligned}
b(u, v, w) &= (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)v, w), \\
b(u, v, w) &\leq C_1 \|\nabla u\| \|\nabla v\| \|\nabla w\|, \\
b(u, v, w) &\leq C_2 \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \|\nabla w\|, \\
b(u, v, w) &\leq C_3 \|\nabla u\| \|\nabla v\| \sqrt{\|w\| \|\nabla w\|}, \\
b^*(u, T, S) &= (u \cdot \nabla T, S) + \frac{1}{2}((\nabla \cdot u)T, S), \\
b^*(u, T, S) &\leq C_4 \|\nabla u\| \|\nabla T\| \|\nabla S\|, \\
b^*(u, T, S) &\leq C_5 \sqrt{\|u\| \|\nabla u\|} \|\nabla T\| \|\nabla S\|, \\
b^*(u, T, S) &\leq C_6 \|\nabla u\| \|\nabla T\| \sqrt{\|S\| \|\nabla S\|}.
\end{aligned}
$$

Proof. See Lemma 2.1 on p. 12 of [132]. □

It is interesting to note that skew-symmetry of the above trilinear forms requires $u \cdot n = 0$; that is, $T \cdot n \neq 0$ is permitted. Also, there are several equivalent forms of the nonlinearities in the continuous setting which differ in the discrete setting [76]. Consideration of these alternative forms in Chapters 3 and 4 is an interesting open problem.

The Poincaré-Friedrichs inequality will be useful: $\forall \chi \in X$ or W_{Γ_D}, there exists $C_P > 0$ such that

$$
\|\chi\| \leq C_P \|\nabla \chi\|.
$$

The constant C_P depends solely on the domain [76]. In particular, the domain and its homogeneous Dirichlet boundary. Consequently, C_P will generally differ for functions in X or W_{Γ_D}. We do not explicitly differentiate here. Also, the following Sobolev embedding inequality is useful [45]: for $\chi \in X$ or W_{Γ_D},

$$\|\chi\|_{L^4} \leq C_S \|\nabla \chi\|, \tag{2.2}$$

where C_S depends on the domain. These are consequences of the Ladyzhenkaya inequalities.

The differential filter [48] will prove useful for studying predictability in Chapter 5: Given χ, find $\overline{\chi}$ satisfying

$$-\delta^2 \Delta \overline{\chi} + \overline{\chi} = \chi \ \ in\ \Omega, \tag{2.3}$$

$$\overline{\chi} = \chi \ \ on\ \partial\Omega. \tag{2.4}$$

The boundary conditions are a delicate issue. In particular, if $\chi = 0$ on $\partial\Omega$ then $\overline{\chi} = \Delta\overline{\chi} = 0$ on $\partial\Omega$. Thus, $\overline{\chi}$ is a linear function within the boundary layer, which may not reflect the correct behavior [76].

The weak formulation is: Given $\chi \in L^2(\Omega)^d$ or $L^2(\Omega)$, find $\overline{\chi} \in X$ or W satisfying

$$\delta^2(\nabla\overline{\chi}, \nabla v) + (\overline{\chi}, v) = (\chi, v), \ \ \forall v \in X \ or\ W_{\Gamma_D}. \tag{2.5}$$

When solving for $\overline{\chi} \in W$, we require $\overline{\chi} = \Psi + \tau$; $\Psi \in W_{\Gamma_D}$ is the auxiliary solution and $\tau \in W$ is an interpolant satisfying the nonhomogeneous Dirichlet boundary condition. If $\chi \in X$ or $H^1_0(\Omega)$, the filtered solution $\overline{\chi}$ satisfies the following.

Lemma 2. *Let $\Omega \subset \mathbb{R}^d$ be a convex polyhedron and $\chi \in X$ or $H^1_0(\Omega)$. Then, the solution $\overline{\chi}$ to the problem (2.5) satisfies the following:*
If $\chi \in L^2(\Omega)^d$ or $L^2(\Omega)$, then

$$\delta^2\|\nabla\overline{\chi}\|^2 + \|\overline{\chi}\|^2 \leq \|\chi\|^2.$$

Further, if $\nabla\chi \in L^2(\Omega)^d$ or $L^2(\Omega)$, then

$$\delta^2\|\nabla(\overline{\chi} - \chi)\|^2 + \|\overline{\chi} - \chi\|^2 \leq \delta^2\|\nabla\chi\|^2.$$

Moreover, if $\Delta\chi \in L^2(\Omega)^d$ or $L^2(\Omega)$, then

$$\delta^2\|\nabla(\overline{\chi} - \chi)\|^2 + \|\overline{\chi} - \chi\|^2 \leq \delta^4\|\Delta\chi\|^2.$$

Proof. See Lemma 4.0.3 and 4.0.5 on p. 7 of [98]. □

Let N be a positive integer and set both $\Delta t = \frac{t^*}{N}$ and $t^n = n\Delta t$ for $0 \le n \le N$. Then, $[0, t^*] = \bigcup_{n=0}^{N-1} [t^n, t^{n+1}]$ is a partition of the time interval. We define the discrete time-derivatives and associated extrapolations,

$$\partial^i_{\Delta t}(v^n) := \begin{cases} \frac{v^n - v^{n-1}}{\Delta t} & i = 1, \\ \frac{3v^n - 4v^{n-1} + v^{n-2}}{2\Delta t} & i = 2. \end{cases} \quad (2.6) \qquad \mathscr{E}^i(v^n) := \begin{cases} v^{n-1} & i = 1, \\ 2v^{n-1} - v^{n-2} & i = 2. \end{cases} \quad (2.7)$$

Using the polarization identity and the elementary identity $2(3a - 4b + c)a = a^2 + (2a - b)^2 - b^2 + (2b - c)^2$, respectively, the following relations hold:

$$(\partial^1_{\Delta t}(v^n), \Delta t v^n) = \frac{1}{2}\Big(\|v^n\|^2 - \|v^{n-1}\|^2 + \|v^n - v^{n-1}\|^2\Big), \quad (2.8)$$

$$(\partial^2_{\Delta t}(v^n), \Delta t v^n) = \frac{1}{4}\Big(\|v^n\|^2 + \|2v^n - v^{n-1}\|^2 - \|v^{n-1}\|^2 - \|2v^{n-1} - v^{n-2}\|^2 + \|v^n - 2v^{n-1} + v^{n-2}\|^2\Big). \quad (2.9)$$

A discrete Gronwall inequality will play an important role in the stability and error analysis.

Lemma 3. *(Discrete Gronwall Lemma). Let Δt, H, a_n, b_n, c_n, and d_n be finite nonnegative numbers for $n \ge 0$ such that for $N \ge 1$*

$$a_N + \Delta t \sum_0^N b_n \le \Delta t \sum_0^{N-1} d_n a_n + \Delta t \sum_0^N c_n + H,$$

then for all $\Delta t > 0$ and $N \ge 1$

$$a_N + \Delta t \sum_0^N b_n \le exp\big(\Delta t \sum_0^{N-1} d_n\big)\big(\Delta t \sum_0^N c_n + H\big).$$

Proof. See Lemma 5.1 on p. 369 of [64]. □

We will also utilize the following norms in the error analysis: $\forall \ -1 \leq k < \infty$,

$$|||v|||_{\infty,k} := \max_{0\leq n\leq N} \|v^n\|_k, \ |||v|||_{p,k} := \big(\Delta t \sum_{n=0}^{N} \|v^n\|_k^p\big)^{1/p}.$$

The following form of the Boussinesq equations is considered. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, Lipschitz domain. Given $u(x,0) = u^0(x)$ and $T(x,0) = T^0(x)$, find $u(x,t) : \Omega \times (0,t^*] \to \mathbb{R}^d$, $p(x,t) : \Omega \times (0,t^*] \to \mathbb{R}$, and $T(x,t) : \Omega \times (0,t^*] \to \mathbb{R}$ satisfying

$$u_t + u \cdot \nabla u - \nu \Delta u + \Lambda \times u + \nabla p = \beta g T + f_1 \ \ in \ \Omega, \tag{2.10}$$

$$\nabla \cdot u = 0 \ \ in \ \Omega, \tag{2.11}$$

$$T_t + u \cdot \nabla T - \kappa \Delta T = f_2 \ \ in \ \Omega, \tag{2.12}$$

$$u = 0 \ \ on \ \partial\Omega, \quad T = 1 \ \ on \ \Gamma_{D_1}, \quad T = 0 \ \ on \ \Gamma_{D_2}, \quad n \cdot \nabla T = 0 \ \ on \ \Gamma_N. \tag{2.13}$$

In the above, $\beta \leftarrow (T_H - T_C)\beta$ and $p \leftarrow \frac{1}{\rho}p$; see Appendix A. The weak formulation of the Boussinesq equations (2.10) - (2.13) is: Find $u : [0,t^*] \to X$, $p : [0,t^*] \to Q$, $T : [0,t^*] \to W$ for a.e. $t \in (0,t^*]$ satisfying for $j = 1, ..., J$:

$$\begin{aligned}(u_t, v) + b(u,u,v) + \nu(\nabla u, \nabla v) + (\Lambda \times u, v) - (p, \nabla \cdot v) = (\beta g T, v) \\ + (f_1, v) \ \ \forall v \in X,\end{aligned} \tag{2.14}$$

$$(\nabla \cdot u, q) = 0 \ \ \forall q \in Q, \tag{2.15}$$

$$(T_t, S) + b^*(u,T,S) + \kappa(\nabla T, \nabla S) = (f_2, S) \ \ \forall S \in W_{\Gamma_D}. \tag{2.16}$$

For turbulent simulations, we consider the following system: Find $u : [0,t^*] \to X$, $p : [0,t^*] \to Q$, $T : [0,t^*] \to W$ for a.e. $t \in (0,t^*]$ satisfying for $j = 1, ..., J$:

$$\begin{aligned}(u_t, v) + b(u,u,v) + \big((\nu + \nu_{turb})D(u), \nabla v\big) + (\Lambda \times u, v) - (p, \nabla \cdot v) \\ = (\beta g T, v) + (f_1, v) \ \ \forall v \in X,\end{aligned} \tag{2.17}$$

$$(\nabla \cdot u, q) = 0 \ \ \forall q \in Q, \tag{2.18}$$

$$(T_t, S) + b^*(u,T,S) + \big((\kappa + \frac{\nu_{turb}}{\sigma_{turb}})\nabla T, \nabla S\big) = (f_2, S) \ \ \forall S \in W_{\Gamma_D}, \tag{2.19}$$

where $D(u) := \frac{1}{2}\big(\nabla u + \nabla u^T\big)$ is the symmetric part of the deformation tensor, ν_{turb} is the eddy viscosity, and σ_{turb} is the turbulent Prandtl number. The solution quantities u, p, and T are understood to correspond to the mean solution quantities. In the above, the Reynolds stress tensor and turbulent heat flux vector were modeled via $R(u,u) = \nu_{turb} D(u)$ and $H(u,T) = \frac{\nu_{turb}}{\sigma_{turb}} \nabla T$ under the eddy viscosity hypothesis, Boussinesq assumption, and gradient-diffusion hypothesis. Owing to Korn's inequality, the following relationships hold [76]:

$$
\begin{aligned}
(\nabla u, \nabla v) = 2(D(u), \nabla v) = 2(\nabla u, D(v)) \;\; \forall\, u,\; v \in V,\\
\frac{\sqrt{2}}{2}\|\nabla u\| \leq \|D(u)\| \leq \|\nabla u\| \;\; \forall\, u,\; v \in X.
\end{aligned}
$$

As a consequence, results proven with $D(u)$ replaced with ∇u imply results for the former.

2.1 FINITE ELEMENT PRELIMINARIES

Let $\{\mathcal{T}_h\}_{0<h<1}$ be a family of quasi-uniform meshes, unless specified otherwise, with maximum element length $h = \max\limits_{K \in \mathcal{T}_h} h_K$. The geometric interpolation of Ω is defined as $\Omega_h = \bigcup_{K\in\mathcal{T}_h} K$. We will assume Ω to be a convex polyhedron for simplicity; curved boundaries can be dealt with in the usual way, e.g., isoparametric elements [31]. Consequently, we have $\Omega = \Omega_h$. Let $X_h \subset X$, $Q_h \subset Q$, $\hat{W}_h = (W_h, W_{\Gamma_D,h}) \subset (W, W_{\Gamma_D}) = \hat{W}$ be conforming finite element spaces defined as

$$
\begin{aligned}
X_h &:= \{v_h \in C^0(\overline{\Omega}_h)^d : \forall\, K \in \mathcal{T}_h,\; v_h|_K \in \mathbb{P}_j(K)^d \;\} \cap X,\\
Q_h &:= \{q_h \in C^0(\overline{\Omega}_h) : \forall\, K \in \mathcal{T}_h,\; q_h|_K \in \mathbb{P}_l(K) \;\} \cap Q,\\
W_h &:= \{S_h \in C^0(\overline{\Omega}_h) : \forall\, K \in \mathcal{T}_h,\; S_h|_K \in \mathbb{P}_j(K) \;\} \cap W,\\
W_{\Gamma_D,h} &:= \{S_h \in C^0(\overline{\Omega}_h) : \forall\, K \in \mathcal{T}_h,\; S_h|_K \in \mathbb{P}_j(K) \;\} \cap W_{\Gamma_D}.
\end{aligned}
$$

The spaces above satisfy the following approximation properties: $\forall 1 \leq j, l \leq k, m$,

$$\inf_{v_h \in X_h} \Big\{ \|u - v_h\| + h\|\nabla(u - v_h)\| \Big\} \leq Ch^{k+1}|u|_{k+1} \qquad u \in X \cap H^{k+1}(\Omega)^d, \tag{2.20}$$

$$\inf_{q_h \in Q_h} \|p - q_h\| \leq Ch^m |p|_m \qquad p \in Q \cap H^m(\Omega), \tag{2.21}$$

$$\inf_{S_h \in \hat{W}_h} \Big\{ \|T - S_h\| + h\|\nabla(T - S_h)\| \Big\} \leq Ch^{k+1}|T|_{k+1} \qquad T \in \hat{W} \cap H^{k+1}(\Omega). \tag{2.22}$$

Furthermore, we consider those spaces for which the discrete inf-sup condition is satisfied,

$$\inf_{q_h \in Q_h} \sup_{v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\| \|\nabla v_h\|} \geq \alpha > 0, \tag{2.23}$$

where α is independent of h. Examples include the MINI-element, Taylor-Hood, and non-conforming Crouzeix-Raviart elements [76]. The space of discretely divergence free functions is defined by

$$V_h := \{v_h \in X_h : (q_h, \nabla \cdot v_h) = 0, \forall q_h \in Q_h\}.$$

The discrete inf-sup condition (2.23) plays an important role in the stability and error analysis. In fact, it implies that we may approximate functions in V well by functions in V_h.

Lemma 4. *Suppose the discrete inf-sup condition (2.23) holds, then for any $v \in V$*

$$\inf_{v_h \in V_h} \|\nabla(v - v_h)\| \leq C(\alpha) \inf_{v_h \in X_h} \|\nabla(v - v_h)\|.$$

Proof. See Theorem 1.1 on p. 59 of [50]. □

The spaces X_h^* and V_h^*, dual to X_h and V_h, are endowed with the following dual norms

$$\|w\|_{X_h^*} := \sup_{v_h \in X_h} \frac{(w, v_h)}{\|\nabla v_h\|}, \quad \|w\|_{V_h^*} := \sup_{v_h \in V_h} \frac{(w, v_h)}{\|\nabla v_h\|}.$$

Further, these norms are equivalent for functions in V_h.

Lemma 5. *Let $w \in V_h$, then*

$$C_* \|w\|_{X_h^*} \leq \|w\|_{V_h^*} \leq \|w\|_{X_h^*}.$$

Proof. See Lemma 1 on p. 243 of [46]. □

The following local and global inverse estimate holds [31]: $\forall \chi \in X_h$ or W_h,

$$\|\nabla \chi\|_{L^2(K)} \leq C_K h_K^{-1} \|\chi\|_{L^2(K)},$$

$$\|\nabla \chi\| \leq C_{inv} h^{-1} \|\chi\|,$$

where C_{inv} depends on the minimum angle in the triangulation.

The Stokes projection will be useful in the error analysis. Let $I_h^{Stokes} : X \times Q \to X_h \times Q_h$ via $I_h^{Stokes}(u,p) = (U,P)$ satisfy the following discrete Stokes problem:

$$Pr(\nabla(U-u), \nabla v_h) - (P-p, \nabla \cdot v_h) = 0 \; \forall \; v_h \in X_h, \tag{2.24}$$

$$(\nabla \cdot (U-u), q_h) = 0 \; \forall \; q_h \in Q_h. \tag{2.25}$$

The following result holds.

Lemma 6. *Assume the approximation properties (2.20) - (2.21) and associated regularity hold. Then, there exists $C > 0$ such that*

$$h^{-1}\|u-U\| + \|\nabla(u-U)\| + \|p-P\| \leq C(\alpha, Pr, \Omega)\Big\{ \inf_{v_h \in X_h} \|\nabla(u-v_h)\| + \inf_{q_h \in Q_h} \|p-q_h\| \Big\}.$$

Proof. Follows from Theorem 13 on p. 62 of [85] and the Aubin-Nitsche trick. □

We will also need the existence of an interpolant of the extension operator, τ, with optimal approximation properties.

Theorem 2. *Let $\tau : H^{1/2}(\Gamma_{D_1}) \to W$ be an extension operator satisfying Theorem 1. Moreover, suppose $\tau \in W \cap H^{k+1}(\Omega)$. Then, there exists an interpolant $I_h\tau \in W_h$ such that*

$$\|\nabla I_h \tau\| \leq \widetilde{C_I} \|\tau\|_1,$$

$$\|\tau - I_h\tau\| + h\|\nabla(\tau - I_h\tau)\| \leq C h^{k+1} |\tau|_{k+1}.$$

Moreover, if $\tau|_{\Gamma_{D_2}} = 0$, then

$$\|I_h\tau\|_1 \leq C_I \|\tau\|_1.$$

Proof. See Lemma 3.2 on p. 1838 of [23] for the first. The second is a consequence of the Poincaré-Friedrichs inequality. □

In the discrete setting, the explicitly skew-symmetric trilinear forms satisfy an additional estimate.

Lemma 7. *Suppose $u,v,w \in X_h$ and $T,S \in W_{\Gamma_D,h}$. Then, there exists $C_\star$, $C_{\star\star} > 0$ such that $b(u,v,w)$ and $b^*(u,T,S)$ satisfy, for $d = 3$,*

$$\sup_{u,v,w \in X_h} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|w\|} \leq C_\star h^{-1/2},$$
$$\sup_{\substack{u \in X_h \\ T,S \in W_h}} \frac{b^*(u,T,S)}{\|\nabla u\|\|\nabla T\|\|S\|} \leq C_{\star\star} h^{-1/2}.$$

Furthermore, for $d = 2$,

$$\sup_{u,v,w \in X_h} \frac{b(u,v,w)}{\|\nabla u\|\|\nabla v\|\|w\|} \leq C_\star (1 + |\ln(h)|)^{1/2},$$
$$\sup_{\substack{u \in X_h \\ T,S \in W_h}} \frac{b^*(u,T,S)}{\|\nabla u\|\|\nabla T\|\|S\|} \leq C_{\star\star} (1 + |\ln(h)|)^{1/2}.$$

Proof. The first set follow from Lemma 1 and the inverse inequality. In 2d, the following inverse estimate holds [12]: $\|\chi\|_{L^\infty} \leq C(1 + |\ln(h)|)^{1/2}\|\nabla \chi\|$. Consequently,

$$\begin{aligned}
b(u,v,w) &= (u \cdot \nabla v, w) + \frac{1}{2}((\nabla \cdot u)v, w) \\
&\leq \|u\|_{L^\infty}\|\nabla v\|\|w\| + \frac{1}{2}\|\nabla \cdot u\|\|v\|_{L^\infty}\|w\| \\
&\leq C(1 + |\ln(h)|)^{1/2}\|\nabla u\|\|\nabla v\|\|w\| + \frac{C\sqrt{d}(1 + |\ln(h)|)^{1/2}}{2}\|\nabla u\|\|\nabla v\|\|w\| \\
&\leq C(1 + |\ln(h)|)^{1/2}\|\nabla u\|\|\nabla v\|\|w\|,
\end{aligned}$$

and the result follows. Follow analogously for b^*. □

The discrete differential filter is: Given $\chi_h \in L^2(\Omega)^d$ or $L^2(\Omega)$, find $\overline{\chi_h} \in X_h$ or W_h satisfying

$$\delta^2(\nabla \overline{\chi_h}, \nabla v_h) + (\overline{\chi_h}, v_h) = (\chi_h, v_h), \quad \forall v_h \in X_h \text{ or } W_{\Gamma_D,h}. \tag{2.26}$$

When solving for $\overline{\chi_h} \in W_h$, we require $\overline{\chi_h} = \Psi_h + I_h\tau$; again, $\Psi_h \in W_{\Gamma_D,h}$ is the auxiliary solution and $I_h\tau \in W_h$ is an interpolant satisfying the nonhomogeneous Dirichlet boundary condition. For $\chi_h \in X_h$, $d = 1,2,3$, the discrete differential filter satisfies the following.

Lemma 8. *Let $\overline{\chi}$ be the continuous differential filter satisfying (2.5). Suppose the approximation properties (2.20) or (2.22) hold. Then, the discrete problem (2.26) is well-posed and satisfies the following:*

$$\delta\|\nabla(\overline{\chi}-\overline{\chi_h})\|+\|\overline{\chi}-\overline{\chi_h}\|\leq Ch^k(\delta+h)|\overline{\chi}|_{k+1}.$$

Proof. See Lemma 4.0.6 and Theorem 4.0.1 on p. 7-8 of [98]. $\square$

3.0 STABILITY OF FINITE ELEMENT METHODS FOR THE BOUSSINESQ EQUATIONS

I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely in your thoughts advanced to the state of Science, whatever the matter may be.

Sir William Thomson [82]

The temperature in natural convection problems is uniformly bounded in time ($\|T(t)\| \leq C < \infty$) under mild data assumptions [42, 96]. However, when this often analyzed problem is approximated by standard FEM, all available stability bounds, e.g., [125, 126, 142], for the temperature exhibit exponential growth in time unless the heat transfer through the solid container is included in the model, e.g., [10]. Moreover, even in the stationary case, stability estimates can yield extremely restrictive mesh conditions, e.g., $h = \mathcal{O}(Ra^{-30/(6-d)})$ [23].

In this chapter, we prove that, without the aforementioned restrictions, the temperature approximation is bounded sub-linearly in terms of the simulation time t^* provided that the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the heated wall; that is, $\|T_h^n\| \leq C\sqrt{t^*}$. In practice, numerical simulations are carried out on a graded mesh [21, 67, 99, 103] due to the interaction between the boundary layer, which is $\mathcal{O}(Ra^{-1/4})$ in the laminar regime [49], and the core flow. In particular, practitioners place several mesh points within the boundary layer, which envelops the internal core flow. Although our condition is more restrictive, this may be due to a gap in the analysis and, none-the-less, it is indicative of the value of graded meshes for stability as well as accuracy.

A major accomplishment in this chapter is the construction of a new discrete extension operator, the discrete Hopf extension. In Hopf [65], a "background flow" was utilized to study stationary solutions of the NSE. Since then, the background flow method has been utilized and developed in many subsequent works. In Doering and Constantin [27], it was employed to derive an upper bound on the time averaged turbulent energy dissipation rate for shear driven flow. In this work, the background flow takes the following form

$$\tau(x) = \begin{cases} \frac{U}{2\delta}(2\delta - x_\alpha) & 0 \leq x_\alpha \leq \delta, \\ \frac{U}{2} & \delta \leq x_\alpha \leq 1-\delta, \\ \frac{U}{2\delta}(L - x_\alpha) & L-\delta \leq x_\alpha \leq L, \end{cases}$$

where U is the speed of the driven boundary, δ is the so-called boundary layer, α is in the direction orthogonal to the driven wall, and L is the distance between two parallel plates. From the viewpoint of the analyst, δ acts as an additional free parameter. In particular, choosing $\delta = \mathcal{O}(Re^{-1})$, the authors derive the estimate $\varepsilon \leq \frac{1}{8\sqrt{2}}\frac{U^3}{L}$.

Since [27], the method has been developed further and applied in other settings, including natural convection [28]. Wang [139] later derived $\varepsilon \leq C\frac{U^3}{L}$, where L is the diameter of the domain, for bounded, smooth domains in $\mathbb{R}^n$. John, Layton, and Manica [77] treated the discrete setting, deriving the same dissipation rate scaling under the mesh condition $h = \mathcal{O}(Re^{-1})$. Recently, Pakzad [111] utilized the method to analyze energy dissipation rates of the Smagorinsky model with van Driest damping.

The outline of this chapter is as follows. In Section 3.1, we present a new interpolant which will play a pivotal role in the stability analyses. In Sections 3.2 and 3.3, we present and analyze the stability of four numerical schemes: first- and second-order BDF and linearly implicit variants. In particular, it is shown that the velocity and temperature approximations can grow at most sub-linearly in time provided that the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the nonhomogeneous Dirichlet boundary. Moreover, the pressure approximation can grow at most linearly. We end with conclusions in Section 3.4.

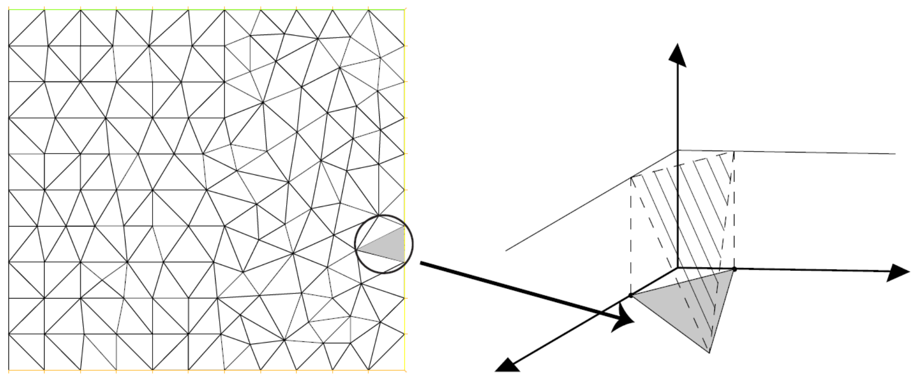

Figure 3.1: The discrete Hopf interpolant on a FE mesh.

3.1 CONSTRUCTION OF THE DISCRETE HOPF EXTENSION

The mesh condition $h = \mathcal{O}(Ra^{-30/(6-d)})$ from [23] arises from the use of the Scott-Zhang interpolant of degree j. To improve upon this condition, we develop a special interpolant for the upcoming analysis. We construct it as follows; see Figure 3.1.

Algorithm: Construction of the discrete Hopf extension

Step one: Consider those mesh elements K such that $K \cap \Gamma_{D_1} \neq \emptyset$. Enumerate these mesh elements from 1 to l'.

Step two: $\forall\, 1 \leq l \leq l'$, let $\{\phi_k^l\}_{k=1}^{d+1}$ be the usual piecewise linear hat functions with *supp* $\phi_k^l \subset K_l$.

Step three: Fix l, select those ϕ_k^l such that $\phi_k^l(x) = 1$ for $x \in K_l \cap \Gamma_{D_1}$.

Step four: Define ψ_i such that $\{\psi^i\}_{i=1}^{i'} = \{\phi_k^l\}_{k,l=1}^{k',l'}$.

Step five: Define $\tau = \sum_{i=1}^{i'} \tilde{T}^i \psi^i$ where $-\infty < \tilde{T}_{min} \leq \tilde{T}^i \leq \tilde{T}_{max} < \infty$ are arbitrary constants.

Then,

Theorem 3. *Suppose* $\tilde{T} : \Gamma_{D_1} \to \mathbb{R}$ *is a piecewise linear function defined on* Γ_{D_1}. *The* *discrete Hopf extension $\tau : \Omega \to \mathbb{R}$ satisfies*

$$\tau(x) = \tilde{T} \text{ on } \Gamma_{D_1},$$
$$\tau(x) = 0 \text{ on } \Omega - \cup_{l=1}^{l'} K_l.$$

Moreover, let $\delta = \max_{1\le l\le l'} h_l$. Then, the following estimate holds: $\forall\sigma > 0$, $\forall(\chi_1, \chi_2) \in (X_h, W_{\Gamma_D,h})$

$$|b^*(\chi_1, \tau, \chi_2)| \le C\delta\Big(\sigma^{-1}\|\nabla\chi_1\|^2 + \sigma\|\nabla\chi_2\|^2\Big). \tag{3.1}$$

Proof. The properties are a consequence of the construction. For the estimate (3.1), it suffices to consider $|b^*(\chi_1, \tilde{T}^i\psi^i, \chi_2)|$ where $\tilde{T}^i = \tilde{T}(x_i)$ is the corresponding nodal value of $\tilde{T}$. For each ψ^i there is a corresponding mesh element K_l such that *supp* $\psi^i \subset K_l$. Let $\hat{K} \subset \mathbb{R}^d$ be the reference element and $F_{K_l} : \hat{K} \to K_l$ the associated affine transformation given by $x = F_{K_l}\hat{x} = B_{K_l}\hat{x} + b_{K_l}$. We will utilize the operator norm $\|\cdot\|_{op}$ and the Euclidean norm $|\cdot|_{\ell_2}$ below.

Consider $\frac{1}{2}|(\chi_1 \cdot \nabla\tilde{T}^i\psi^i, \chi_2)|$, the estimate for $\frac{1}{2}|(\chi_1 \cdot \nabla\chi_2, \tilde{T}^i\psi^i)|$ follows analogously. Transform to the reference element, use standard FEM estimates, the Cauchy-Schwarz inequality, and equivalence of norms. Then,

$$\begin{aligned}
\frac{1}{2}|(\chi_1 \cdot \nabla\tilde{T}^i\psi^i, \chi_2)| &= \frac{|\tilde{T}^i||det(B_{K_l})|}{2}\Big|\int_{\hat{K}} \hat{\chi_1} \cdot B_{K_l}^{-T}\hat{\nabla}\hat{\psi^i}\hat{\chi_2}d\hat{x}\Big| \\
&\le \frac{|\tilde{T}^i||det(B_{K_l})|}{2}\|B_{K_l}^{-T}\|_{op}|\hat{\nabla}\hat{\psi^i}|_{\ell_2}\int_{\hat{K}} |\hat{\chi_1}|_{\ell_2}|\hat{\chi_2}|d\hat{x} \\
&\le Ch_l^{d-1}\|\hat{\chi_1}\|_{L^2(\hat{K})}\|\hat{\chi_2}\|_{L^2(\hat{K})} \\
&\le Ch_l^{d-1}\|\hat{\nabla}\hat{\chi_1}\|_{L^2(\hat{K})}\|\hat{\nabla}\hat{\chi_2}\|_{L^2(\hat{K})}.
\end{aligned} \tag{3.2}$$

Consider $\|\hat{\nabla}\hat{\chi_2}\|_{L^2(\hat{K})}$ and $\|\hat{\nabla}\hat{\chi_1}\|_{L^2(\hat{K})}$. Transforming back to the mesh element and using standard FEM estimates yields

$$\begin{aligned}
\|\hat{\nabla}\hat{\chi_2}\|^2_{L^2(\hat{K})} &= |det(B_{K_l}^{-1})|\int_{K_l} B_{K_l}^T\nabla\chi_2 \cdot B_{K_l}^T\nabla\chi_2 dx \\
&\le |det(B_{K_l}^{-1})|\|B_{K_l}^T\|^2_{op}\|\nabla\chi_2\|^2_{L^2(K_l)} \\
&\le Ch_l^{2-d}\|\nabla\chi_2\|^2_{L^2(K_l)}
\end{aligned}$$

$$\leq Ch_l^{2-d}\|\nabla\chi_2\|^2, \tag{3.3}$$

$$\|\hat{\nabla}\hat{\chi}_1\|^2_{L^2(\hat{K})} \leq Ch_l^{2-d}\|\nabla\chi_1\|^2. \tag{3.4}$$

Use (3.3) and (3.4) in (3.2) and Young's inequality. This yields

$$\frac{1}{2}|(\chi_1 \cdot \nabla\tilde{T}^i\psi^i, \chi_2)| \leq Ch_l\Big(\sigma\|\nabla\chi_1\|^2 + \sigma^{-1}\|\nabla\chi_2\|^2\Big).$$

Summing from $i = 1$ to $i = i'$ and taking the maximum h_l yields the result. □

The equivalence of norms argument in (3.2) is subtle. Consider K and let p be a polynomial of fixed degree satisfying $p \in C^0(\overline{K})$ and $p(x) = 0$ for some $x \in \partial K$. If $\|\nabla p\|_{L^2(K)} = 0$, then $p = C \in \mathbb{R}$. Further, since p is continuous on K, $C = 0$. Thus, $\|\nabla\cdot\|_{L^2(K)}$ and $\|\cdot\|_{L^2(K)}$ are equivalent norms for such functions.

If we allow the interpolant to be constructed with the basis elements of W_h, we can reconstruct any function $S_h \in W_h$ exactly on the boundary Γ_{D_1} with the same properties. For square and cubic domains we can define such an interpolant explicitly, e.g.,

$$\tau(x) = \begin{cases} \frac{1}{\delta}(\delta - x_\alpha) & 0 \leq x_\alpha \leq \delta, \\ 0 & \delta \leq x_\alpha \leq 1, \end{cases}$$

where α is in the direction orthogonal to the differentially heated walls or in the direction of gravity for the differentially heated vertical wall problem and Rayleigh-Bénard problem, respectively.

3.2 NUMERICAL SCHEMES

In this section, we consider the following popular temporal discretizations: BDF1, linearly implicit BDF1, BDF2, and linearly implicit BDF2; see, e.g., [3,69] regarding linearly implicit variants of some common time-stepping schemes. Denote the fully discrete solutions by u_h^n, p_h^n, and T_h^n at time levels $t^n = n\Delta t$, $n = 0, 1, ..., N$, and $t^* = N\Delta t$. Recall, the first- and second-order extrapolations are defined via $\mathscr{E}^1(v^{n+1}) = v^n$ and $\mathscr{E}^2(v^{n+1}) = 2v^n - v^{n-1}$. All algorithms require $f_1^{n+1}, f_2^{n+1}, Pr, Ra$, and Ta to be provided. Moreover, both $\{u_h^k\}_{k=n+1-i}^n$ and $\{T_h^k\}_{k=n+1-i}^n$ must be prescribed for $i = 1, 2$.

BDFi: Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$\begin{aligned}(\partial^i_{\Delta t}(u_h^{n+1}), v_h) + b(u_h^{n+1}, u_h^{n+1}, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) + PrTa^{1/2}(e_\Lambda \times u_h^{n+1}, v_h)\\ - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\xi T_h^{n+1}, v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,\end{aligned} \quad (3.5)$$

$$(\nabla \cdot u_h^{n+1}, q_h) = 0 \ \ \forall q_h \in Q_h, \quad (3.6)$$

$$(\partial^i_{\Delta t}(T_h^{n+1}), S_h) + b^*(u_h^{n+1}, T_h^{n+1}, S_h) + (\nabla T_h^{n+1}, \nabla S_h) = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{h,\Gamma_D}, \quad (3.7)$$

linearly implicit BDFi: Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$\begin{aligned}(\partial^i_{\Delta t}(u_h^{n+1}), v_h) + b(\mathscr{E}^i(u_h^{n+1}), u_h^{n+1}, v_h) + Pr(\nabla u_h^{n+1}, \nabla v_h) + PrTa^{1/2}(e_\Lambda \times u_h^{n+1}, v_h)\\ - (p_h^{n+1}, \nabla \cdot v_h) = PrRa(\xi \mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,\end{aligned} \quad (3.8)$$

$$(\nabla \cdot u_h^{n+1}, q_h) = 0 \ \ \forall q_h \in Q_h, \quad (3.9)$$

$$\begin{aligned}(\partial^i_{\Delta t}(T_h^{n+1}), S_h) + b^*(\mathscr{E}^i(u_h^{n+1}), T_h^{n+1}, S_h) + (\nabla T_h^{n+1}, \nabla S_h)\\ = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{h,\Gamma_D}.\end{aligned} \quad (3.10)$$

It is not necessary to introduce an extrapolation for the buoyancy term $PrRa(\xi T_h, v_h)$ in the linearly implicit schemes. However, a speed advantage is gained since the velocity and temperature solves become uncoupled. These algorithms will be modified for use in ensemble calculations in Chapter 4; consequently, results in this chapter will remain valid for the presented ensemble algorithms.

3.3 STABILITY ANALYSIS

We present stability results for the aforementioned algorithms provided the first meshline in the finite element mesh is within $\mathcal{O}(Ra^{-1})$ of the heated wall. We begin with the **BDFi** schemes, first proving stability of the velocity and temperature approximations. As a corollary, the pressure approximation is proven stable. We then follow analogously for the **linearly implicit BDFi** schemes.

Theorem 4. *Consider* ***BDF1*** *or* ***BDF2****. Suppose $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ and $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. If $\delta = \mathcal{O}(Ra^{-1})$, then there exist $C > 0$, independent of t^*, such that*
BDF1:

$$\frac{1}{2}\|T_h^N\|^2 + \|u_h^N\|^2 + \sum_{n=0}^{N-1}\|T_h^{n+1} - T_h^n\|^2 + \sum_{n=0}^{N-1}\|u_h^{n+1} - u_h^n\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla T_h^{n+1}\|^2 + \frac{Pr\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\|^2 \leq Ct^*,$$

BDF2:

$$\frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \sum_{n=1}^{N-1}\|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2 + \sum_{n=1}^{N-1}\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\|\nabla T_h^{n+1}\|^2 + Pr\Delta t\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2 \leq Ct^*,$$

Proof. Our strategy is to first estimate an auxiliary temperature approximation in terms of the velocity approximation and data. We then bound the velocity approximation in terms of data yielding stability of both approximations. Denote the auxiliary temperature approximation $\theta_h^{n+1} = T_h^{n+1} - \tau$. Consider the **BDFi** family, $i = 1$ or 2. Let $S_h = \Delta t\theta_h^{n+1} \in W_{\Gamma_D,h}$ in equation (3.7), rewrite all quantities in terms of θ_h^k, $k = n,\ n+1$, and rearrange. Since $(\nabla\tau, \nabla\theta_h^{n+1}) = -(\Delta\tau, \theta_h^{n+1}) + \int_{\partial\Omega}(\nabla\tau\cdot n)\theta_h^{n+1}dx = 0$, we have

$$(\partial_{\Delta t}^i(\theta_h^{n+1}), \Delta t\theta_h^{n+1}) + \Delta t\|\theta_h^{n+1}\|^2 = -\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1}) + \Delta t(f_2^{n+1}, \theta_h^{n+1}) \quad (3.11)$$

Consider $-\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1})$. Use skew-symmetry and apply Theorem 3. Then,

$$\begin{aligned} -\Delta t b^*(u_h^{n+1}, \theta_h^{n+1} + \tau, \theta_h^{n+1}) &= -\Delta t b^*(u_h^{n+1}, \tau, \theta_h^{n+1}) \\ &\le C\Delta t\delta\big(\sigma_1^{-1}\|\nabla u_h^{n+1}\|^2 + \sigma_1\|\nabla\theta_h^{n+1}\|^2\big). \end{aligned} \quad (3.12)$$

Use the Cauchy-Schwarz-Young inequality on $\Delta t(f_2^{n+1}, \theta_h^{n+1})$,

$$\Delta t(f_2^{n+1}, \theta_h^{n+1}) \le \frac{\Delta t}{2\sigma_2}\|f_2^{n+1}\|_{-1}^2 + \frac{\sigma_2\Delta t}{2}\|\nabla\theta_h^{n+1}\|^2. \quad (3.13)$$

Using (3.12) and (3.13) in (3.11) and rearranging leads to

$$(\partial_{\Delta t}^i(\theta_h^{n+1}), \Delta t\theta_h^{n+1}) + \frac{3\Delta t}{4}\big(1 - \frac{4C\delta\sigma_1}{3} - \frac{2\sigma_2}{3}\big)\|\nabla\theta_h^{n+1}\|^2 \le C\Delta t\delta\epsilon_1^{-1}\|\nabla u_h^{n+1}\|^2 + \frac{\Delta t}{2\sigma_2}\|f_2^{n+1}\|_{-1}^2.$$

Letting $C\delta\sigma_1 = \sigma_2 = 1/2$ yields

$$(\partial_{\Delta t}^i(\theta_h^{n+1}), \Delta t\theta_h^{n+1}) + \frac{\Delta t}{4}\|\nabla\theta_h^{n+1}\|^2 \le 2C^2\Delta t\,\delta^2\,\|\nabla u_h^{n+1}\|^2 + \Delta t\|f_2^{n+1}\|_{-1}^2.$$

Sum from $n = i - 1$ to $n = N - 1$ and put all data on the right hand side. This yields the following bounds on the auxiliary temperature approximation in terms of the velocity approximation and data, for $i = 1$:

$$\begin{aligned} \frac{1}{2}\|\theta_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|\theta_h^{n+1} - \theta_h^n\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla\theta_h^{n+1}\|^2 &\le 2C^2\delta^2\Delta t\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\|^2 \\ &\quad + \Delta t\sum_{n=0}^{N-1}\|f_2^{n+1}\|_{-1}^2 + \frac{1}{2}\|\theta_h^0\|^2, \end{aligned} \quad (3.14)$$

and for $i = 2$:

$$\begin{aligned} &\frac{1}{4}\Big(\|\theta_h^N\|^2 + \|2\theta_h^N - \theta_h^{N-1}\|^2\Big) + \frac{1}{4}\sum_{n=1}^{N-1}\|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2 + \frac{\Delta t}{4}\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2 \\ &\le 2C^2\delta^2\Delta t\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2 + \Delta t\sum_{n=1}^{N-1}\|f_2^{n+1}\|_{-1}^2 + \frac{1}{4}\Big(\|\theta_h^0\|^2 + \|2\theta_h^1 - \theta_h^0\|^2\Big). \end{aligned} \quad (3.15)$$

Next, let $v_h = \Delta t u_h^{n+1} \in V_h$ in (3.5) and rearrange terms. Then,

$$(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + Pr\Delta t\|\nabla u_h^{n+1}\|^2 = \Delta t Pr Ra(\xi(\theta_h^{n+1}+\tau), u_h^{n+1}) \\ + \Delta t(f_1^{n+1}, u_h^{n+1}). \quad (3.16)$$

Use the Cauchy-Schwarz, Poincaré-Friedrichs, and Young's inequalities on $\Delta t(f_1^{n+1}, u_h^{n+1})$ and $\Delta t Pr Ra(\xi(\theta_h^{n+1}+\tau), u_h^{n+1})$ and note that $\|\xi\|_{L^\infty(\Omega)} = 1$,

$$\Delta t Pr Ra(\xi\theta_h^{n+1}, u_h^{n+1}) \le \frac{Pr Ra^2 C_P^4 \Delta t}{\|}\nabla\theta_h^{n+1}\|^2 + \frac{Pr\Delta t}{4}\|\nabla u_h^{n+1}\|^2, \quad (3.17)$$

$$\Delta t Pr Ra(\xi\tau, u_h^{n+1}) \le \frac{Pr Ra^2\Delta t}{2\sigma_3}\|\tau\|_{-1}^2 + \frac{Pr\sigma_3\Delta t}{2}\|\nabla u_h^{n+1}\|^2, \quad (3.18)$$

$$\Delta t(f_1^{n+1}, u_h^{n+1}) \le \frac{\Delta t}{\sigma_4}\|f_1^{n+1}\|_{-1}^2 + \frac{\sigma_4\Delta t}{2}\|\nabla u_h^{n+1}\|^2. \quad (3.19)$$

Using (3.17), (3.18), and (3.19) in (3.16) leads to

$$(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + \frac{3Pr\Delta t}{4}\big(1 - \frac{2\sigma_3}{3} - \frac{2\sigma_4}{3Pr}\big)\|\nabla u_h^{n+1}\|^2 \le Pr Ra^2 C_P^4\Delta t\|\nabla\theta_h^{n+1}\|^2 \\ + \frac{Pr Ra^2\Delta t}{2\sigma_3}\|\tau\|_{-1}^2 + \frac{\Delta t}{\sigma_4}\|f_1^{n+1}\|_{-1}^2.$$

Let $Pr\sigma_3 = \sigma_4 = Pr/4$. Then,

$$(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + \frac{Pr\Delta t}{2}\|\nabla u_h^{n+1}\| \le Pr Ra^2 C_P^4\Delta t\|\nabla\theta_h^{n+1}\|^2 \\ + 2Pr Ra^2\Delta t\|\tau\|_{-1}^2 + \frac{2\Delta t}{Pr}\|f^{n+1}\|_{-1}^2.$$

Using the identity (2.8), summing from $n = i-1$ to $n = N-1$, and putting all data on right-hand side yields, for $i = 1$:

$$\frac{1}{2}\|u_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+1} - u_h^n\|^2 + \frac{Pr\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\|^2 \le Pr Ra^2 C_P^4\Delta t\sum_{n=0}^{N-1}\|\nabla\theta_h^{n+1}\|^2 \\ + \frac{2\Delta t}{Pr}\sum_{n=0}^{N-1}\Big(Pr^2 Ra^2\|\tau\|_{-1}^2 + \|f_1^{n+1}\|_{-1}^2\Big) + \frac{1}{2}\|u_h^0\|^2, \quad (3.20)$$

and for $i = 2$:

$$\frac{1}{4}\Big(\|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2\Big) + \frac{1}{4}\sum_{n=1}^{N-1}\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \frac{Pr\Delta t}{2}\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2$$

$$
\leq PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2+\frac{2\Delta t}{Pr}\sum_{n=1}^{N-1}\Big(Pr^2Ra^2\|\tau\|_{-1}^2+\|f^{n+1}\|_{-1}^2\Big)
+\frac{1}{4}\Big(\|u_h^0\|^2+\|2u_h^1-u_h^0\|^2\Big). \quad (3.21)
$$

Now, from equation (3.14), we have

$$
PrRa^2C_P^4\Delta t\sum_{n=0}^{N-1}\|\nabla\theta_h^{n+1}\|^2\leq 8C^2C_P^4PrRa^2\delta^2\Delta t\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\|^2
+4PrRa^2C_P^4\Delta t\sum_{n=0}^{N-1}\|f_2^{n+1}\|_{-1}^2+2PrRa^2C_P^4\|\theta_h^0\|^2. \quad (3.22)
$$

Using the above in (3.20) with $\delta=\frac{1}{4\sqrt{2}CC_P^2}Ra^{-1}$ leads to

$$
\frac{1}{2}\|u_h^N\|^2+\frac{1}{2}\sum_{n=0}^{N-1}\|u_h^{n+1}-u_h^n\|^2+\frac{Pr\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\|^2
\leq 4PrRa^2C_P^4\Delta t\sum_{n=0}^{N-1}\|f_2^{n+1}\|_{-1}^2+2PrRa^2C_P^4\|\theta_h^0\|^2
+\frac{2\Delta t}{Pr}\sum_{n=0}^{N-1}\Big(Pr^2Ra^2\|\tau\|_{-1}^2+\|f_1^{n+1}\|_{-1}^2\Big)+\frac{1}{2}\|u_h^0\|^2. \quad (3.23)
$$

Similarly, from equation (3.15), we have

$$
PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2\leq 8C^2C_P^4PrRa^2\delta^2\Delta t\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2
+4PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|f_2^{n+1}\|_{-1}^2+PrRa^2C_P^4\Big(\|\theta_h^0\|^2+\|2\theta_h^1-\theta_h^0\|^2\Big). \quad (3.24)
$$

Using the above in (3.15) and the same choice of δ yields

$$
\frac{1}{4}\Big(\|u_h^N\|^2+\|2u_h^N-u_h^{N-1}\|^2\Big)+\frac{1}{4}\sum_{n=1}^{N-1}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\frac{Pr\Delta t}{4}\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2
\leq 4PrRa^2C_P^4\Delta t\sum_{n=0}^{N-1}\|f_2^{n+1}\|_{-1}^2+PrRa^2C_P^4\Big(\|\theta_h^0\|^2+\|2\theta_h^1-\theta_h^0\|^2\Big)
+\frac{2\Delta t}{Pr}\sum_{n=1}^{N-1}\Big(Pr^2Ra^2\|\tau\|_{-1}^2+\|f^{n+1}\|_{-1}^2\Big)+\frac{1}{4}\Big(\|u_h^0\|^2+\|2u_h^1-u_h^0\|^2\Big). \quad (3.25)
$$

Thus, the velocity approximation is bounded above by data and therefore the temperature approximation as well; that is, both the velocity and temperature approximations are stable. Adding (3.14) and (3.23), multiplying by 2, and using the identity $T_h^n = \theta_h^n + \tau$ together with the triangle inequality yields the first result. The second follows similarly. $\square$

As a corollary, the pressure approximation is stable, allowing for linear growth with respect to t^* in $L^1(0, t^*; L^2(\Omega))$.

Corollary 1. *Suppose the hypotheses of Theorem 4 hold. Then,*

$$\alpha \Delta t \sum_{n=i-1}^{N-1} \|p_h^{n+1}\| \leq C t^*.$$

Proof. Consider (3.5), isolate $(\partial_{\Delta t}^i(u_h^{n+1}), v_h)$, and let $v_h \in V_h$. Then,

$$\begin{aligned}(\partial_{\Delta t}^i(u_h^{n+1}), v_h) = -b(u_h^{n+1}, u_h^{n+1}, v_h) - Pr(\nabla u_h^{n+1}, \nabla v_h) - PrTa^{1/2}(e_\Lambda \times u_h^{n+1}, v_h)& \\ + PrRa(\xi T_h^{n+1}, v_h) + (f_1^{n+1}, v_h).& \quad (3.26)\end{aligned}$$

Applying Lemma 1 to the skew-symmetric trilinear term and the Cauchy-Schwarz and Poincaré-Friedrichs inequalities to the remaining terms yields

$$-b(u_h^{n+1}, u_h^{n+1}, v_h) \leq C_1 \|\nabla u_h^{n+1}\| \|\nabla u_h^{n+1}\| \|\nabla v_h\|, \quad (3.27)$$

$$-Pr(\nabla u_h^{n+1}, \nabla v_h) \leq Pr \|\nabla u_h^{n+1}\| \|\nabla v_h\|, \quad (3.28)$$

$$-PrTa^{1/2}(e_\Lambda \times u_h^{n+1}, v_h) \leq PrTa^{1/2} C_P^2 \|\nabla u_h^{n+1}\| \|\nabla v_h\|, \quad (3.29)$$

$$PrRa(\xi T_h^{n+1}, v_h) \leq PrRaC_P^2 \|\nabla T_h^{n+1}\| \|\nabla v_h\|, \quad (3.30)$$

$$(f_1^{n+1}, v_h) \leq \|f_1^{n+1}\|_{-1} \|\nabla v_h\|. \quad (3.31)$$

Apply the above estimates in (3.26), divide by the common factor $\|\nabla v_h\|$ on both sides, and take the supremum over all $0 \neq v_h \in V_h$. Then,

$$\begin{aligned}\|\partial_{\Delta t}^i(u_h^{n+1})\|_{V_h^*} \leq \Big(C_1 \|\nabla u_h^{n+1}\| + Pr + PrTa^{1/2} C_P^2\Big) \|\nabla u_h^{n+1}\|& \\ + PrRaC_P^2 \|\nabla T_h^{n+1}\| + \|f_1^{n+1}\|_{-1}.& \quad (3.32)\end{aligned}$$

By Lemma 5,

$$\|\partial^i_{\Delta t}(u_h^{n+1})\|_{X_h^*} \leq C_*\Bigg(\Big(C_1\|\nabla u_h^{n+1}\| + Pr + PrTa^{1/2}C_P^2\Big)\|\nabla u_h^{n+1}\| + PrRaC_P^2\|\nabla T_h^{n+1}\| + \|f_1^{n+1}\|_{-1}\Bigg). \quad (3.33)$$

Reconsider equation (3.5). Isolate the pressure term, apply (3.27) - (3.31) on the right-hand side terms. Then,

$$(p_h^{n+1}, \nabla\cdot v_h) \leq (\partial^i_{\Delta t}(u_h^{n+1}), v_h) + \Bigg(\Big(C_1\|\nabla u_h^{n+1}\| + Pr + PrTa^{1/2}C_P^2\Big)\|\nabla u_h^{n+1}\| + PrRaC_P^2\|\nabla T_h^{n+1}\| + \|f_1^{n+1}\|_{-1}\Bigg)\|\nabla v_h\|. \quad (3.34)$$

Divide by $\|\nabla v_h\|$ and take the supremum over all $0\neq v_h \in X_h$. Then,

$$\sup_{0\neq v_h\in X_h}\frac{(p_h^{n+1}, \nabla\cdot v_h)}{\|\nabla v_h\|} \leq \big(1+C_*\big)\Bigg(\Big(C_1\|\nabla u_h^{n+1}\| + Pr + PrTa^{1/2}C_P^2\Big)\|\nabla u_h^{n+1}\| + PrRaC_P^2\|\nabla T_h^{n+1}\| + \|f_1^{n+1}\|_{-1}\Bigg). \quad (3.35)$$

Use the inf-sup condition (2.23),

$$\alpha\|p_h^{n+1}\| \leq \big(1+C_*\big)\Bigg(\Big(C_1\|\nabla u_h^{n+1}\| + Pr + PrTa^{1/2}C_P^2\Big)\|\nabla u_h^{n+1}\| + PrRaC_P^2\|\nabla T_h^{n+1}\| + \|f_1^{n+1}\|_{-1}\Bigg). \quad (3.36)$$

Multiplying by Δt, summing from $n = i-1$ to $n = N-1$, applying the Cauchy-Schwarz inequality to all but the first term on the right-hand side yields

$$\alpha\Delta t\sum_{n=i-1}^{N-1}\|p_h^{n+1}\| \leq \big(1+C_*\big)\Bigg(C_1\Delta t\sum_{n=i-1}^{N-1}\|\nabla u_h^{n+1}\|^2 + \big(Pr + PrTa^{1/2}C_P^2\big)\sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla u_h^{n+1}\|^2\Big)^{1/2} + PrRaC_P^2\sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla T_h^{n+1}\|^2\Big)^{1/2} + \sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|f_1^{n+1}\|_{-1}^2\Big)^{1/2}\Bigg). \quad (3.37)$$

Consequently, stability of the pressure approximation follows, built upon the stability of the temperature and velocity approximations. □

We now prove analagous results for the linearly implicit schemes.

Theorem 5. *Consider **linearly implicit BDF1** or **linearly implicit BDF2**. Suppose $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ and $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. If $\delta = \mathcal{O}(Ra^{-1})$, then there exist $C>0$, independent of t^*, such that*

linearly implicit BDF1:

$$\frac{1}{2}\|T_h^N\|^2 + \|u_h^N\|^2 + \sum_{n=0}^{N-1}\|T_h^{n+1}-T_h^n\|^2 + \sum_{n=0}^{N-1}\|u_h^{n+1}-u_h^n\|^2 + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\|\nabla T_h^{n+1}\|^2 + \frac{Pr\Delta t}{2}\sum_{n=0}^{N-1}\|\nabla u_h^{n+1}\| + \frac{Pr\Delta t}{2}\|\nabla u_h^N\|^2 \leq Ct^*.$$

linearly implicit BDF2:

$$\frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2 + \|u_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \sum_{n=1}^{N-1}\|T_h^{n+1}-2T_h^n+T_h^{n-1}\|^2 + \sum_{n=1}^{N-1}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2 + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\|\nabla T_h^{n+1}\|^2 + \frac{Pr\Delta t}{2}\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2 + \frac{Pr\Delta t}{4}\Big(2\|\nabla u_h^N\|^2 + \|\nabla u_h^{N-1}\|^2\Big) \leq Ct^*.$$

Proof. We follow the general strategy in Theorem 4. Let $v_h = \Delta t u_h^{n+1} \in V_h$ in (3.8) and $S_h = \Delta t\theta_h^{n+1} \in W_{\Gamma_D,h}$ in equation (3.10). Rewrite all quantities in terms of θ_h^k, $k = n,\ n+1$, and rearrange. Then,

$$(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + Pr\Delta t\|\nabla u_h^{n+1}\|^2 = PrRa\Delta t(\xi(\mathscr{E}^i(\theta_h^{n+1})+\tau), u_h^{n+1}) + \Delta t(f_1^{n+1}, u_h^{n+1}), \quad (3.38)$$

and

$$(\partial_{\Delta t}^i(\theta_h^{n+1}), \Delta t\theta_h^{n+1}) + \Delta t\|\nabla\theta_h^{n+1}\|^2 = -\Delta t b^*(\mathscr{E}^i(u_h^{n+1}), \tau, \theta_h^{n+1}) + \Delta t(f_2^{n+1}, \theta_h^{n+1}). \quad (3.39)$$

We present the analysis only for the case $i=2$ (**linearly implicit BDF2**). Consider $-\Delta t b^*(\mathscr{E}^2(u_h^{n+1}), \tau, \theta_h^{n+1})$. We have that $-\Delta t b^*(\mathscr{E}^2(u_h^{n+1}), \tau, \theta_h^{n+1}) = -\Delta t b^*(2u_h^n - u_h^{n-1}, \tau, \theta_h^{n+1})$ $= -2\Delta t b^*(u_h^n, \tau, \theta_h^{n+1}) + \Delta t b^*(u_h^{n-1}, \tau, \theta_h^{n+1})$. Thus, using Theorem 2 yields

$$-2\Delta t b^*(u_h^n, \tau, \theta_h^{n+1}) \leq C\,\delta\,\Delta t\Big(4\sigma_5^{-1}\|\nabla u_h^n\|^2 + \sigma_5\|\nabla\theta_h^{n+1}\|^2\Big), \quad (3.40)$$

$$\Delta t b^*(u_h^{n-1},\tau,\theta_h^{n+1}) \le C\,\delta\,\Delta t\Big(\sigma_6^{-1}\|\nabla u_h^{n-1}\|^2 + \sigma_6\|\nabla\theta_h^{n+1}\|^2\Big). \tag{3.41}$$

Moreover, use the Cauchy-Schwarz-Young and Poincare-Friedrichs inequalities on $PrRa\Delta t(\xi\mathscr{E}^2(\theta_h^{n+1}),u_h^{n+1})$,

$$2PrRa\Delta t(\xi\theta_h^n,u_h^{n+1}) \le \frac{2Pr^2Ra^2C_P^4\Delta t}{\sigma_7}\|\nabla\theta_h^n\|^2 + \frac{\Delta t\sigma_7}{2}\|\nabla u_h^{n+1}\|^2, \tag{3.42}$$

$$-PrRa\Delta t(\xi\theta_h^{n-1},u_h^{n+1}) \le \frac{Pr^2Ra^2C_P^4\Delta t}{2\sigma_8}\|\nabla\theta_h^{n-1}\|^2 + \frac{\Delta t\sigma_8}{2}\|\nabla u_h^{n+1}\|^2. \tag{3.43}$$

Use estimates (3.13), (3.40), and (3.41) in equation (3.39). Let $\sigma_5=\sigma_6=\frac{1}{4C\delta}$ and $\sigma_2=1/4$, using the identity (2.9), sum from $n=1$ to $n=N-1$, and rearrange. Then,

$$\begin{aligned}
&\frac14\|\theta_h^N\|^2 + \frac14\|2\theta_h^N-\theta_h^{N-1}\|^2 + \frac14\sum_{n=1}^{N-1}\|\theta_h^{n+1}-2\theta_h^n+\theta_h^{n-1}\|^2 + \frac{\Delta t}{4}\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2\\
&\quad\le 16C^2\Delta t\,\delta^2\sum_{n=1}^{N-1}\|\nabla u_h^n\|^2 + 4C^2\Delta t\,\delta^2\sum_{n=1}^{N-1}\|\nabla u_h^{n-1}\|^2 + 2\Delta t\sum_{n=1}^{N-1}\|f_2^{n+1}\|_{-1}^2\\
&\qquad + \frac14\|\theta_h^0\|^2 + \frac14\|\theta_h^1-\theta_h^0\|^2.
\end{aligned} \tag{3.44}$$

Using (3.18), (3.19), (3.42), and (3.43) in (3.39) leads to

$$\begin{aligned}
(\partial_{\Delta t}^i(u_h^{n+1}),\Delta t u_h^{n+1}) + Pr\Delta t\big(1-\frac{Pr\sigma_3\Delta t}{2}-\frac{\sigma_4+\sigma_7+\sigma_8}{2Pr}\big)\|\nabla u_h^{n+1}\|^2 &\le \frac{2Pr^2Ra^2C_P^4\Delta t}{\sigma_7}\|\nabla\theta_h^n\|^2\\
&+\frac{Pr^2Ra^2C_P^4\Delta t}{2\sigma_8}\|\nabla\theta_h^{n-1}\|^2 + \frac{\Delta t}{2\epsilon_4}\|\tau\|_{-1}^2 + \frac{\Delta t}{2\epsilon_5}\|f_1^{n+1}\|_{-1}^2.
\end{aligned}$$

Let $2Pr\sigma_3 = 2\sigma_4=\sigma_7=\sigma_8=Pr/2$. Then,

$$\begin{aligned}
(\partial_{\Delta t}^i(u_h^{n+1}),\Delta t u_h^{n+1}) + \frac{Pr\Delta t}{4}\|\nabla u_h^{n+1}\|^2 &\le 4PrRa^2C_P^4\Delta t\|\nabla\theta_h^n\|^2 + Pr^2Ra^2C_P^4\Delta t\|\nabla\theta_h^{n-1}\|^2\\
&+\frac{2\Delta t}{Pr}\|\tau\|_{-1}^2 + \frac{2\Delta t}{Pr}\|f^{n+1}\|_{-1}^2.
\end{aligned}$$

Using the identity (2.9), summing from $n=1$ to $n=N-1$, and rearranging yields

$$\begin{aligned}
&\frac14\|u_h^N\|^2 + \frac14\|2u_h^N-u_h^{N-1}\|^2 + \frac14\sum_{n=1}^{N-1}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2 + \frac{Pr\Delta t}{4}\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2\\
&\le PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\Big(4\|\nabla\theta_h^n\|^2+\|\nabla\theta_h^{n-1}\|^2\Big)+\frac{2\Delta t}{Pr}\sum_{n=1}^{N-1}\Big(\|\tau\|_{-1}^2+\|f^{n+1}\|_{-1}^2\Big)+\frac14\|u_h^0\|^2+\frac14\|2u_h^1-u_h^0\|^2
\end{aligned}$$

$$\leq 5PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2+\frac{2\Delta t}{Pr}\sum_{n=1}^{N-1}\Big(\|\tau\|_{-1}^2+\|f^{n+1}\|_{-1}^2\Big)+\frac{1}{4}\|u_h^0\|^2+\frac{1}{4}\|2u_h^1-u_h^0\|^2$$
$$+PrRa^2C_P^4\Delta t\Big(5\|\nabla\theta_h^1\|^2+\|\nabla\theta_h^0\|^2\Big). \quad (3.45)$$

Now, from equation (3.44), we have

$$5PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|\nabla\theta_h^{n+1}\|^2\leq 80C^2PrRa^2C_P^4\delta^2\Delta t\sum_{n=1}^{N-1}\Big(4\|\nabla u_h^n\|^2+\|\nabla u_h^{n-1}\|^2\Big)$$
$$+40PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|f_2^{n+1}\|_{-1}^2+5PrRa^2C_P^4\Big(\|\theta_h^0\|^2+\|2\theta_h^1-\theta_h^0\|^2\Big). \quad (3.46)$$

Add and subtract $\frac{Pr\Delta t}{8}\sum_{n=1}^{N-1}\|\nabla u_h^n\|^2$ and $\frac{Pr\Delta t}{16}\sum_{n=1}^{N-1}\|\nabla u_h^{n-1}\|^2$ in (3.45) and use the above estimate with $\delta=\frac{1}{32\sqrt{5}C_P^2}Ra^{-1}$. Then,

$$\frac{1}{4}\|u_h^N\|^2+\frac{1}{4}\|2u_h^N-u_h^{N-1}\|^2+\frac{1}{4}\sum_{n=1}^{N-1}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\frac{Pr\Delta t}{8}\sum_{n=1}^{N-1}\|\nabla u_h^{n+1}\|^2+\frac{Pr\Delta t}{8}\|\nabla u_h^N\|^2$$
$$+\frac{Pr\Delta t}{16}\|\nabla u_h^{N-1}\|^2\leq 40PrRa^2C_P^4\Delta t\sum_{n=1}^{N-1}\|f_2^{n+1}\|_{-1}^2+5PrRa^2C_P^4\Big(\|\theta_h^1\|^2+\|2\theta_h^1-\theta_h^0\|^2\Big)$$
$$+\frac{2\Delta t}{Pr}\sum_{n=0}^{N-1}\Big(\|\tau\|_{-1}^2+\|f^{n+1}\|_{-1}^2\Big)+\frac{1}{4}\|u_h^1\|^2+\frac{1}{4}\|2u_h^1-u_h^0\|^2$$
$$+PrRa^2C_P^4\Delta t\Big(5\|\nabla\theta_h^1\|^2+\|\nabla\theta_h^0\|^2\Big)+\frac{Pr\Delta t}{16}\Big(2\|\nabla u_h^1\|^2+\|\nabla u_h^0\|^2\Big). \quad (3.47)$$

The result follows. □

In similar fashion, we can now prove that pressure approximations of **linear implicit BDFi** can grow at most linearly in $L^1(0,t^*;L^2(\Omega))$.

Corollary 2. *Suppose the hypotheses of Theorem 5 hold. Then,*

$$\alpha\Delta t\sum_{n=i-1}^{N-1}\|p_h^{n+1}\|\leq Ct^*.$$

Proof. Consider (3.8), isolate $(\partial^i_{\Delta t}(u_h^{n+1}), v_h)$, and let $v_h \in V_h$. Then,

$$(\partial^i_{\Delta t}(u_h^{n+1}), v_h) = -b(\mathscr{E}^i(u_h^{n+1}), u_h^{n+1}, v_h) - Pr(\nabla u_h^{n+1}, \nabla v_h) - PrTa^{1/2}(e_\Lambda \times u_h^{n+1}, v_h) + PrRa(\xi \mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h). \quad (3.48)$$

Applying Lemma 1 to the skew-symmetric trilinear term and the Cauchy-Schwarz and Poincaré-Friedrichs inequalities to the remaining terms yields

$$-b(\mathscr{E}^i(u_h^{n+1}), u_h^{n+1}, v_h) \leq C_1 \|\nabla \mathscr{E}^i(u_h^{n+1})\| \|\nabla u_h^{n+1}\| \|\nabla v_h\|, \quad (3.49)$$

$$PrRa(\xi \mathscr{E}^i(T_h^{n+1}), v_h) \leq PrRaC_P^2 \|\nabla \mathscr{E}^i(T_h^{n+1})\| \|\nabla v_h\|. \quad (3.50)$$

Apply the above estimates in (3.48), divide by the common factor $\|\nabla v_h\|$ on both sides, take the supremum over all $0 \neq v_h \in V_h$, and apply Lemma 5. Then,

$$\|\partial^i_{\Delta t}(u_h^{n+1})\|_{X_h^*} \leq C_* \Bigg(\Big(C_1 \|\nabla \mathscr{E}^i(u_h^{n+1})\| + Pr + PrTa^{1/2} C_P^2 \Big) \|\nabla u_h^{n+1}\| + PrRaC_P^2 \|\nabla \mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1} \Bigg). \quad (3.51)$$

Reconsider equation (3.5). Isolate the pressure term, apply (3.28), (3.29), (3.31), (3.49), and (3.50) on the right-hand side terms, divide by $\|\nabla v_h\|$, take the supremum over all $0 \neq v_h \in X_h$. The inf-sup condition (2.23) then yields

$$\alpha \|p_h^{n+1}\| \leq \big(1 + C_*\big) \Bigg(\Big(C_1 \|\nabla \mathscr{E}^i(u_h^{n+1})\| + Pr + PrTa^{1/2} C_P^2 \Big) \|\nabla u_h^{n+1}\| + PrRaC_P^2 \|\nabla \mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1} \Bigg). \quad (3.52)$$

Multiplying by Δt, and summing from $n = i-1$ to $n = N-1$, yields

$$\alpha \Delta t \sum_{n=i-1}^{N-1} \|p_h^{n+1}\| \leq \big(1 + C_*\big) \Bigg(C_1 \Delta t \sum_{n=i-1}^{N-1} \|\nabla \mathscr{E}^i(u_h^{n+1})\| \|\nabla u_h^{n+1}\| + \big(Pr + PrTa^{1/2} C_P^2\big) \Delta t \sum_{n=i-1}^{N-1} \|\nabla u_h^{n+1}\| + PrRaC_P^2 \Delta t \sum_{n=i-1}^{N-1} \|\nabla \mathscr{E}^i(T_h^{n+1})\| + \Delta t \sum_{n=i-1}^{N-1} \|f_1^{n+1}\|_{-1} \Bigg). \quad (3.53)$$

Applying the Cauchy-Schwarz inequality,

$$\begin{aligned}\alpha\Delta t\sum_{n=i-1}^{N-1}\|p_h^{n+1}\| \leq \big(1+C_*\big)\Bigg[\Bigg(C_1\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla\mathscr{E}^i(u_h^{n+1})\|^2\Big)^{1/2} \\ + Pr + PrTa^{1/2}C_P^2\Bigg)\sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla u_h^{n+1}\|^2\Big)^{1/2} \\ + PrRaC_P^2\sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla\mathscr{E}^i(T_h^{n+1})\|^2\Big)^{1/2} + \sqrt{t^*}\Big(\Delta t\sum_{n=i-1}^{N-1}\|f_1^{n+1}\|_{-1}^2\Big)^{1/2}\Bigg]. \quad (3.54)\end{aligned}$$

As needed. □

The differences between the estimates appearing in Corollaries 1 and 2 are the arbitrary constant and the requirements on the mesh, which are given in the corresponding theorems.

The mesh conditions appearing for the linearly implicit schemes, Theorem 5 and Corollary 2, are more restrictive than for the fully implicit schemes, Theorems 4 and Corollary 1; that is, the proportionality constants are relatively smaller. This is interesting since typical analyses of the fully implicit schemes require a discrete Gronwall inequality which imposes a crippling timestep condition: $\Delta t = \mathcal{O}(Ra^{-2})$. Whereas the presented linearly implicit schemes have no such condition, utilizing an alternative Gronwall inequality, Lemma 3.

3.4 CONCLUSION

The coupling terms $b^*(\mathscr{E}^i(u_h), T_h^{n+1}, S_h)$ and $PrRa(\xi\mathscr{E}^i(T_h), v_h)$ that arise in stability analyses of FEM discretizations of natural convection problems with sidewall heating are the major source of difficulty. The former term forces the stability of the temperature approximation to be dependent on the velocity approximation and vice versa for the latter term. Standard techniques fail to overcome this imposition, in the absence of a discrete Gronwall inequality.

A new discrete Hopf interpolant was introduced to overcome this issue. Fully discrete stability estimates were proven which improve upon previous estimates. In particular, it was shown that provided that the first mesh line in the finite element mesh is within $\mathcal{O}(Ra^{-1})$

of the nonhomogeneous Dirichlet boundary, the velocity and temperature approximations are stable allowing for sub-linear growth in t^*. Further, the pressure approximation is stable allowing for linear growth.

A uniform in time stability estimate was not able to be achieved due to the term $PrRa(\xi\tau, v_h)$, which arises when an interpolant of the boundary is introduced. We conjecture that the results proven herein may be improved, owing to a gap in the analysis. In particular, it appears likely that the mesh condition can be improved to $\delta = \mathcal{O}(Ra^{-1/a})$, for $a > 1$.

4.0 ENSEMBLE ALGORITHMS FOR THE BOUSSINESQ EQUATIONS WITH UNCERTAIN DATA

At the limits of what is computable, it is impressive for a new computer to halve the turnaround time... but a new algorithm can reduce the exponent!

William Layton, paraphrased discussion on the curse of dimensionality.

In physical applications, initial conditions, forcings, and parameters are never known exactly. In particular, any measurement device, such as a radiosonde, will specify a value up to a prescribed tolerance. Limitations imposed on dynamical systems due to these uncertainties has been discussed and exhibited in the works of Charney [15], Philips [112], Thompson [134], and Lorenz [92], among others. Essentially, uncertainty in these quantities can render a computer code into an expensive random number generator.

Ensemble calculations improve the quality of a prediction given inherent uncertainties in a choice of model, the initial conditions, parameters, domain, and etc. The historical roots of ensemble forecasting are discussed by Lewis [91]. Applications of ensemble usage include, e.g., weather prediction [80, 102, 128, 129], ocean dynamics [89], turbulence modeling [72], magnetohydrodynamics (MHD) [105], and 3D printing [37,121]. The ensemble average is the most likely distribution and the variance gives an estimate of prediction reliability. Moreover, the predictability horizon and the average effective Lyapunov exponent can be estimated and used to quantify how predictable a flow is and, therefore, the potential reliability of the numerical approximation.

Typically, these calculations involve the numerical solution of J sequential, fine mesh runs or J parallel, coarse mesh runs of a given code for the governing equations of a physical phenomenon with slightly varying initial conditions or parameter values. Evidently, there is a substantial increase in computational resources over single realization solves, which severely limits the ensemble size. This increase begs the question: Can ensemble size be increased without decreasing mesh density (and vice versa) on a fully utilized computer system?

Early work on improving the efficiency of ensemble algorithms for fluid flow problems was performed by Jiang and Layton [71]. In a sequence of papers [71,72], they develop ensemble algorithms for the Navier-Stokes system of equations (NSE) subject to uncertain initial conditions and body forces. As it will be instructive, we present the NSE here. Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, Lipschitz domain. Given $u(x,0;\omega_j) = u^0(x;\omega_j)$ for $j = 1, 2, ..., J$, let $u(x,t;\omega_j)$ and $p(x,t;\omega_j)$ satisfy

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f \;\; in \; \Omega, \tag{4.1}$$

$$\nabla \cdot u = 0 \;\; in \; \Omega, \tag{4.2}$$

$$u = 0 \;\; on \; \partial\Omega, \tag{4.3}$$

where ν is the viscosity and f is a body force. Applying a BDF1 discretization in time and standard FEM discretization in space for the above system, we arrive at the following block linear system for each ensemble member j:

$$\begin{bmatrix} \frac{1}{\Delta t} M_u + \nu S_u + N_u(u^n) & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_j^{n+1} \\ p_j^{n+1} \end{bmatrix} = \begin{bmatrix} (f^{n+1} + \frac{1}{\Delta t} M_u) u_j^n \\ 0 \end{bmatrix}, \tag{4.4}$$

where M_u is the mass matrix, S_u is the diffusion matrix, $N_u(u^n)$ is the convection matrix, and B is the continuity matrix. The above is equivalent to solving the J linear systems:

$$A_j x_j = b_j,$$

with coefficient matrices A_j, solution vectors x_j, and right-hand sides b_j.

The convection matrix $N_u(u^n)$ is the only matrix dependent on the ensemble member j. Jiang and Layton noticed that if this matrix can be modified so that it is independent of j, via a consistent modification of the convective term $u \cdot \nabla u$, then the above block linear system will be equivalent to the following: Let A be the resulting coefficient matrix (independent of j). Then, the following set of J linear systems must be solved at each timestep:

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 | x_2 | ... | x_J \end{bmatrix} = \begin{bmatrix} b_1 | b_2 | ... | b_J \end{bmatrix}. \tag{4.5}$$

The choice they made was:

$$u^{n+1} \cdot \nabla u^{n+1} \leftarrow \langle u \rangle^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n,$$

where $\langle u \rangle^n = \frac{1}{J} \sum_{j=1}^{J} u(x, t^n; \omega_j)$ and $u'^n = u(x, t^n; \omega_j) - \langle u \rangle^n$ are the ensemble average and fluctuation. Using this splitting, they are able to prove stability and optimal-order convergence provided the following CFL-type condition holds:

$$\frac{\Delta t}{\nu h} \|\nabla u'^n_h\|^2 \leq C. \tag{4.6}$$

The system (4.5) can be solved with efficient block solvers; for example, block LU factorizations [26], block QMR [43], block GMRES [70], block BiCGSTAB [30], and etc. [57]. Moreover, since only one coefficient matrix is required for computation per timestep, the storage requirement is reduced.

Since this pioneering effort, there has been a rapid progression of developments. In [72], Jiang introduces an eddy viscosity model, utilizing the Kolmogorov-Prandtl relation, into their laminar flow ensemble algorithm. They are able to prove that the new algorithm is stable under a less restrictive timestep condition:

$$\frac{\Delta t}{\nu} \|\nabla \cdot u'^n_h\|^2_{L^4} \leq C. \tag{4.7}$$

In particular, $-\nabla \cdot R(u, u)$, where $R(u, u)$ is the Reynolds stress, is replaced with $-\nabla \cdot (2\nu_{turb} D(u))$ with

$$\nu_{turb}(l, k') = Cl\sqrt{k'}, \tag{4.8}$$

where $D(u)$ is the symmetric part of the deformation tensor, C is an arbitrary constant, $l = \Delta t|u'|$ is the mixing length, and $k'(x, t) = \frac{1}{2}|u'|^2(x, t) = \frac{1}{2} \sum_{j=1}^{J} |u'|^2(x, t; \omega_j)$ is the kinetic energy associated with velocity fluctuations.

Interestingly, the turbulent viscosity is directly parametrized by fluctuations of ensemble members. This results in a dramatic decrease in complexity over alternative, widely used turbulence models, e.g., the $k - \varepsilon$ model. The typical system [118] that must be solved is as follows: Find $(u, p, k_1, k_2, ..., k_R)$ satisfying

$$u_t + u \cdot \nabla u - \nabla \cdot \big(\nu_{turb}(k_1, ..., k_R) D(u)\big) + \nabla p = f, \tag{4.9}$$

$$\nabla \cdot u = 0, \tag{4.10}$$

$$k_{r,t} + u \cdot \nabla k_r - \nabla \cdot \left(\mu_{turb,r}(k_1, ..., k_R)\nabla k_r\right) - \eta_r(k_1, ..., k_R)|D(u)|^2 + G_r(k_1, ..., k_R) = 0, \tag{4.11}$$

where $\{k_r\}_{r=1}^{R}$ are turbulence statistics, ν_{turb} is the eddy viscosity function, $\mu_{turb,r}$ are eddy diffusion statistics associated with k_r, η_r and G_r are rational functions of k_r. Appropriate boundary conditions must be prescribed for the additional R equations.

It is expected that as R increases, accuracy improves [118]; intuitively, this makes sense since we introduces additional parameters that are data-fitted. The cases $R = 1$ and $R = 2$ are associated with the TKE and $k - \epsilon$ models. We see that, compared with using the turbulence model proposed by Jiang [71], the above model requires R extra solves and many additional parameter and function determinations. In the case of ensembles, the increased complexity is obviously compounded. Naturally, complexity increases further for non-isothermal fluid flow [1, 7, 19, 61, 67].

Returning to the historical progression of ensemble algorithms, Jiang, Kaya, and Layton [73] later develop a new ensemble eddy viscosity model inspired by Leray regularization [87,88] and utilizing the eddy viscosity model (4.8). Interestingly, the method is proven to be unconditionally stable and, as $t^* \to \infty$, the solution approaches statistical equilibrium and its variance approaches zero. Mohebujjaman and Rebholz [105] introduce a first-order ensemble timestepping algorithm also including the above eddy viscosity model for the Elsässer variable formulation of equations for MHD. They present stability and convergence results for their algorithm.

Further, Takhirov, Neda, and Waters [131] introduce time relaxation and study the effects of grad-div stabilization. Noticeably, they found that grad-div stabilization increases stability. Khankan [83] developed a first-order turbulence model for natural convection based on (4.8) and under a similar condition for stability. More recently, Gunzburger, Jiang, and Wang [54] considered ensemble dependent constant viscosity. In this work, they decompose the viscosity into its ensemble average and fluctuating components and use the following IMEX discretization:

$$\langle \nu \rangle \Delta u^{n+1} + \nu' \Delta u^n. \tag{4.12}$$

Under this splitting, the resulting algorithm is stable under condition 4.6 and

$$\left| \frac{\nu'}{\langle \nu \rangle} \right| \leq C. \tag{4.13}$$

They later developed their idea into a second-order time accurate method [55] under the same conditions.

Although each of these works represents a significant advance, there is a need for more efficient algorithms due to ensemble size and resolution demands. New methodologies must be applied to reach further. One possible entry point is the saddle point structure. Operator splitting [51, 103, 143], artificial compressibility [20, 25, 53, 119, 122, 123, 133], and projection methods [52, 113], among others, address this. Artificial compressibility, in particular, decouples the velocity and pressure solves, decreasing storage, complexity, and turnaround time.

Additionally, the Boussinesq equations subject to the Coriolis force have been neglected. This system of equations forms the backbone of all models to numerically simulate the atmosphere and ocean. Further, it is rich in complex features, depending on the Ra number, domain, and boundary conditions, e.g., boundary layers, centro-symmetry [49], bifurcation [96, 141], and turbulence; see [13, 42, 79, 106, 114, 116] and references therein for more details (derivation/stability/existence/uniqueness). Therefore, this is the vital next step in the development of ensemble algorithms. In particular, there is a need for efficient ensemble algorithms for the Boussinesq equations subject to the Coriolis force, including turbulence models, with uncertain data.

Recall, the Boussinesq equations are given by: Suppose we are given, for $j = 1, 2, ..., J$,

$$\text{initial conditions: } u(x, 0; \omega_j) = u^0(x; \omega_j) \text{ and } T(x, 0; \omega_j) = T^0(x; \omega_j),$$
$$\text{parameters: } \nu(\omega_j),\ \beta(\omega_j),\ \kappa(\omega_j), \text{ and } \Lambda(\omega_j),$$
$$\text{forcings: } f_1(x, t; \omega_j) \text{ and } f_2(x, t; \omega_j).$$

Then, find $u(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}^d$, $p(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}$, and $T(x, t; \omega_j) : \Omega \times (0, t^*] \to \mathbb{R}$ satisfying

$$u_t + u \cdot \nabla u - \nu \Delta u + \Lambda \times u + \nabla p = \beta g T + f_1 \ \ in\ \Omega, \tag{4.14}$$
$$\nabla \cdot u = 0 \ \ in\ \Omega, \tag{4.15}$$

$$T_t + u \cdot \nabla T - \kappa \Delta T = f_2 \ \ in \ \Omega, \tag{4.16}$$

$$u = 0 \ \ on \ \partial\Omega, \quad T = 1 \ \ on \ \Gamma_{D_1}, \quad T = 0 \ \ on \ \Gamma_{D_2}, \quad n \cdot \nabla T = 0 \ \ on \ \Gamma_N. \tag{4.17}$$

In the above, $\beta \leftarrow (T_H - T_C)\beta$ and $p \leftarrow \frac{1}{\rho}p$; see Appendix A. Notice that if we apply typical BDF discretizations to the above, the resulting linear system, after FEM discretization, will be of the form (4.5). In view of this, consider the following approximations:

$$u^{n+1} \cdot \nabla u^{n+1} \approx \langle u \rangle^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n, \tag{4.18}$$

$$-\nu \Delta u^{n+1} \approx -\langle \nu \rangle \Delta u^{n+1} - \nu' \Delta u^n, \tag{4.19}$$

$$-\Lambda \times u^{n+1} \approx -\langle \Lambda \rangle \times u^{n+1} - \Lambda' \times u^n, \tag{4.20}$$

$$u^{n+1} \cdot \nabla T^{n+1} \approx \langle u \rangle^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n, \tag{4.21}$$

$$-\kappa \Delta T^{n+1} \approx -\langle \kappa \rangle \Delta T^{n+1} - \kappa' \Delta T^n, \tag{4.22}$$

$$\beta g T^{n+1} \approx \beta g T^n. \tag{4.23}$$

Using the above in (4.14) and (4.16) yields

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &+ \langle u \rangle^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n - \langle \nu \rangle \Delta u^{n+1} + \nu' \Delta u^n \\ &+ \langle \Lambda \rangle \times u^{n+1} + \Lambda' \times u^n + \nabla p^{n+1} = \beta g T^n + f_1^{n+1}, \end{aligned} \tag{4.24}$$

$$\nabla \cdot u^{n+1} = 0, \tag{4.25}$$

and

$$\frac{T^{n+1} - T^n}{\Delta t} + \langle u \rangle^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n - \langle \kappa \rangle \Delta T^{n+1} + \kappa' \Delta T^n = f_2^{n+1}. \tag{4.26}$$

Now, rearranging and applying a typical FEM discretization in space, e.g., Taylor-Hood, then the resulting set of linear systems must be solved:

$$\begin{bmatrix} \frac{1}{\Delta t} M_u + N_u(\langle u_h \rangle^n) + \langle \nu \rangle S_u + \langle \Lambda \rangle R_u & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u_{h,j}^{n+1} \\ p_{h,j}^{n+1} \end{bmatrix} = \begin{bmatrix} F_{u,j} \\ 0 \end{bmatrix}, \tag{4.27}$$

and

$$\Big[\frac{1}{\Delta t} M_T + N_T(\langle u_h \rangle^n) + \langle \kappa \rangle S_T \Big] T_{h,j}^{n+1} = F_{T,j}, \tag{4.28}$$

where M_u is the mass matrix, $N_u(\langle u_h\rangle^n)$ is the convection matrix associated with convective velocity $\langle u_h\rangle^n$, S_u is the diffusion matrix, R is the rotation matrix, and B is the continuity matrix. Analagous relations hold for the matrices in the temperature system. Evidently, we have an equivalent form of the linear system (4.5) and can take advantage of efficient block solvers.

Now, the method outlined above is useful only if it is stable and accurate. We expect that if the fluctuating quantities, u' or $\nabla u'$, ν', Λ', and κ' are "small", then each of the associated approximations we have made will be accurate and the resulting algorithm will produce good results. The algorithm (4.24) - (4.26) corresponds to **eBDF** (4.40) - (4.42), below; we will prove, that this family of algorithms is nonlinearly, energy stable (Theorem 7) and optimally convergent (Theorems 13 and 14) under certain "smallness" conditions: conditions (4.52) and (4.53).

Following the progression of ideas, we see that if we can break the saddle point structure of (4.27), the resulting algorithm will be less complex. Utilizing, the penalty and artificial compressibility methods, this can be accomplished. The penalty method involves modifying the continuity equation (4.15) via

$$\epsilon p + \nabla \cdot u = 0,$$

where $\epsilon > 0$ is the penalization parameter. Formally, taking the gradient of this equation, multiplying by $\frac{1}{\epsilon}$, and rearranging yields the relationship

$$\nabla p = -\frac{1}{\epsilon}\nabla\nabla \cdot u. \tag{4.29}$$

Using (4.29), we can eliminate the pressure term in (4.24) yielding full velocity-pressure decoupling,

$$\begin{aligned}\frac{u^{n+1}-u^n}{\Delta t} + \langle u\rangle^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n - \langle \nu\rangle \Delta u^{n+1} + \nu' \Delta u^n \\ + \langle \Lambda\rangle \times u^{n+1} + \Lambda' \times u^n - \frac{1}{\epsilon}\nabla\nabla \cdot u^{n+1} = \beta g T^n + f_1^{n+1}\end{aligned} \tag{4.30}$$

and

$$p^{n+1} = -\frac{1}{\epsilon}\nabla \cdot u^{n+1}. \tag{4.31}$$

After a rearrangement, we see that the momentum and continuity equations are replaced with a convection-diffusion equation and algebraic update; it is consistent [122], with order i, provided $\epsilon = \mathcal{O}(\Delta t^i)$. Furthermore, after FEM spatial discretization, the following block linear system for the velocity and temperature and algebraic pressure update must be solved.

Step one:

$$\Big[\frac{1}{\Delta t}M_u + \langle\nu\rangle S_u + N_u(\langle u_h\rangle^n) + \frac{1}{\epsilon}D\Big]u_{h,j}^{n+1} = F_{u,j}, \tag{4.32}$$

$$\Big[\frac{1}{\Delta t}M_T + N_T(\langle u_h\rangle^n) + \langle\kappa\rangle S_T\Big]T_{h,j}^{n+1} = F_{T,j}, \tag{4.33}$$

Step two:

$$p_{h,j}^{n+1} = -\frac{1}{\epsilon}\nabla\cdot u_{h,j}^{n+1}, \tag{4.34}$$

where D is the matrix associated with $-\nabla\nabla\cdot$ operator. Clearly, the velocity solve is now decoupled from the pressure solve. Consequently, the system is fully decoupled. In practice, the second step is a pressure mass matrix solve; that is, the pressure mass matrix is built and resulting system is solved. If, e.g., the non-conforming Crouzeix-Raviart (P1nc-P0) element is used, it is a true algebraic update. We will prove that the corresponding fully-discrete algorithm, **PEA** (4.43) - (4.44), is stable and convergent under similar conditions as **eBDF** and proper choice of ϵ.

An alternative approach is to utilize artificial compressibility. Artificial compressibility methods [133] involve adding a "compressibility" term to the continuity equation:

$$\epsilon p_t + \nabla\cdot u = 0,$$

where $\epsilon > 0$ is the artificial compressibility parameter, related to the Mach number [97]. Approximating this equation with BDF and rearranging yields

$$p^{n+1} = p^n - \frac{\Delta t}{\epsilon}\nabla\cdot u^{n+1}.$$

Consequently, the momentum equation can be rewritten as

$$\frac{u^{n+1}-u^n}{\Delta t} + \langle u\rangle^n\cdot\nabla u^{n+1} + u'^n\cdot\nabla u^n - \langle\nu\rangle\Delta u^{n+1} + \nu'\Delta u^n$$

$$+\langle\Lambda\rangle \times u^{n+1} + \Lambda' \times u^{n} + \nabla p^{n} - \frac{\Delta t}{\epsilon}\nabla\nabla\cdot u^{n+1} = \beta g T^{n} + f_1^{n+1}.$$

Once again, this is consistent [123, 124] provided $\epsilon = \mathcal{O}(\Delta t^{i})$. After, a FEM spatial discretization, we must solve

Step one:

$$\Big[\frac{1}{\Delta t}M_u + \langle\nu\rangle S_u + N_u(\langle u_h\rangle^n) + \frac{\Delta t}{\epsilon}D\Big]u_{h,j}^{n+1} = F_{u,j}, \tag{4.35}$$

$$\Big[\frac{1}{\Delta t}M_T + N_T(\langle u_h\rangle^n) + \langle\kappa\rangle S_T\Big]T_{h,j}^{n+1} = F_{T,j}, \tag{4.36}$$

Step two:

$$p_{h,j}^{n+1} = p_{h,j}^{n} - \frac{\Delta t}{\epsilon}\nabla\cdot u_{h,j}^{n+1}. \tag{4.37}$$

The resulting set of equations have the same structure as the penalty approximation, however, we see that $\frac{\Delta t}{\epsilon}$ replaces $\frac{1}{\epsilon}$ in front the grad-div term; this is a critical difference. The grad-div term has drawn significant attention due to its positive impact on solution quality; see, e.g., [11, 36, 108] and references therein. Unfortunately, the condition number of the matrix γD generally grows without bound as $\gamma \to \infty$ [51]. Consequently, iterative solvers can slow dramatically.

Recall, for penalty and artificial compressibility methods, ϵ is selected to be $\mathcal{O}(\Delta t^{i})$ to ensure convergence. Due to the $\frac{1}{\epsilon} = \frac{C}{\Delta t^{i}} = \gamma$ factor in front of the grad-div matrix D, penalty methods are better suited for producing quick results or initial conditions for artificial compressibility methods. Unfortunately, both results of Theorem 15 and the second-order result of Theorem 16 are sub-optimal with respect to ϵ. Regarding the latter, second-order accuracy is recovered with the choice $\epsilon = \mathcal{O}(\Delta t^{3})$. Numerical experiments suggest that this is improvable. Theoretical justification is left as an open problem.

Earlier, we mentioned that these algorithms should produce accurate results provided, e.g., $\nabla u'$ was "small". This is not an unreasonable demand for laminar flows but for turbulent flows it is. However, requiring $\nabla\cdot u'$ to be "small" would not be unreasonable since $\nabla\cdot u = 0$ for the continuous system. It turns out that **ACE-T** is such an algorithm.

Typically, for turbulent flows, we are not interested in the point-wise solution quantities but the temporal, spatial, or ensemble averages of these quantities. The aversion towards point-wise solution values is both out of necessity and practicality; for large Ra numbers, computers aren't yet powerful enough and engineers are often interested in the averaged quantities. Therefore, turbulence modeling is implemented. There are several important variants, however, they generally involve decomposing the solution variables into mean and fluctuating components and solving the resulting closure problem utilizing the eddy viscosity hypothesis, Boussinseq assumption, and a relationship for the turbulent heat fluxes.

Typical choices for the eddy viscosity ν_{turb} are prescribed via the Prandtl length model, Komolgorov-Prandtl relation, and or Smagorinsky model [127], among others [140]. For the turbulent heat flux, models include gradient-diffusion, algebraic flux, and differential flux models [19]. Herein, we utilize the Kolmogorov-Prandtl relation and gradient-diffusion hypothesis yielding the following models:

$$\nabla \cdot R(u,u) = \nu_{turb}(l,k')\nabla u = C\Delta t k' \nabla u, \tag{4.38}$$

$$\nabla \cdot H(u,T) = \frac{\nu_{turb}(l,k')}{\sigma_{turb}}\nabla T = \frac{C\Delta t k'}{\sigma_{turb}}\nabla T, \tag{4.39}$$

where $k'(x,t) = \frac{1}{2}\sum_{j=1}^{J} |u'|^2(x,t;\omega_j)$ is the kinetic energy associated with velocity fluctuations. In the above, we have replaced $D(u)$ with ∇u. Owing to Korn's inequality, results proven with the latter imply the same for the former; constants may change. Additionally, the Kolmogorov-Prandtl relation exhibits the correct near wall behavior: $l(y) = \mathcal{O}(y)$ as $y \to 0$ [72]. This suggests that our proposed turbulence model does not need additional near-wall damping. Moreover, we see that k' directly parametrizes the kinetic energy fluctuations.

The resulting time-stepping scheme is

$$\frac{u^{n+1}-u^n}{\Delta t} + \langle u\rangle^n \cdot \nabla u^{n+1} + u'^n \cdot \nabla u^n - \nabla \cdot \Big(\big(\langle \nu\rangle + C_\nu \Delta t |u'^n|^2\big)\nabla u\Big) + \nu' \Delta u^n$$
$$+ \langle \Lambda\rangle \times u^{n+1} + \Lambda' \times u^n + \nabla p^n - \frac{\Delta t}{\epsilon}\nabla\nabla \cdot u^{n+1} = \beta g T^n + f_1^{n+1},$$

$$\frac{T^{n+1}-T^n}{\Delta t} + \langle u\rangle^n \cdot \nabla T^{n+1} + u'^n \cdot \nabla T^n - \nabla \cdot \Big(\big(\langle \kappa\rangle + \frac{C_\nu \Delta t}{\sigma_{turb}} |u'^n|^2\big)\nabla T^{n+1}\Big) + \kappa' \Delta T^n = f_2^{n+1},$$

$$p^{n+1} = p^n - \frac{\Delta t}{\epsilon} \nabla \cdot u^{n+1}.$$

Therefore, the fully-discrete system will share a similar structure to **ACE**.

4.1 NUMERICAL SCHEMES

In this section, we will introduce eight efficient algorithms for computing an ensemble of solutions to the Boussinesq system (4.14) - (4.17). **eBDF** (4.40) - (4.42), utilize techniques from Jiang [71] resulting in two linear systems, each involving a shared coefficient matrix, for multiple right-hand sides at each timestep. **PEA** (4.43) - (4.44), utilize the penalty method to decouple the velocity and pressure solution variables. A significant speed ($\simeq$ 2.5 to 22.5) up is seen for first-order **PEA** over **eBDF**. Second-order **PEA** performs poorly on timing due to solver breakdown owing to the $\mathcal{O}(1/\Delta t^2)$ factor in front of the grad-div matrix.

ACE (4.46) - (4.47) incorporates artificial compression for the same purpose as **PEA**: decoupling the velocity-pressure solve. The same speed ups are seen over **eBDF** as with first-order **PEA**. Lastly, we develop **ACE-T** (4.49) - (4.50) for turbulent flows. We employ the eddy viscosity model (4.38), utilizing the Kolmogorov-Prandtl relation, and a generalized gradient-diffusion model.

Denote the fully discrete solutions by u_h^n, p_h^n, and T_h^n at time levels $t^n = n\Delta t$, $n = 0, 1, ..., N$, and $t^* = N\Delta t$. Recall, the first- and second-order extrapolations are defined via $\mathscr{E}^1(v^{n+1}) = v^n$ and $\mathscr{E}^2(v^{n+1}) = 2v^n - v^{n-1}$. Consequently,

$$\begin{aligned}
\mathscr{E}^1(\langle v\rangle^{n+1}) &= \langle v\rangle^n, \\
\mathscr{E}^1(v'^{n+1}) &= v'^n, \\
\mathscr{E}^2(\langle v\rangle^{n+1}) &= 2\langle v\rangle^n - 2\langle v\rangle^{n-1} = \frac{1}{J}\sum_{j=1}^{J} 2v^n - v^{n-1}, \\
\mathscr{E}^2(v'^{n+1}) &= 2v'^n - v'^{n-1} = (2v^n - v^{n-1}) - \frac{1}{J}\sum_{j=1}^{J} 2v^n - v^{n-1} = \mathscr{E}^2(v^{n+1}) - \mathscr{E}^2(\langle v\rangle^{n+1}).
\end{aligned}$$

For the algorithms below, it will be understood that $f_1^{n+1}, f_2^{n+1}, \nu, \kappa, \Lambda,$ and β must be provided. Further, both $\{u_h^k\}_{k=n+1-i}^n$ and $\{T_h^k\}_{k=n+1-i}^n$ must be prescribed for $i = 1, 2$; for

ACE and **ACE-T**, p_h^n must also be prescribed in the first step.

Algorithm (eBDF): Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$
\begin{aligned}
&(\partial_{\Delta t}^i(u_h^{n+1}), v_h) + b(\mathscr{E}^i(\langle u_h \rangle^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h) \\
&\quad + \langle \nu \rangle (\nabla u_h^{n+1}, \nabla v_h) + \nu' (\nabla \mathscr{E}^i(u_h^{n+1}), \nabla v_h) + (\langle \Lambda \rangle \times u_h^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) \\
&\quad - (p_h^{n+1}, \nabla \cdot v_h) = (\beta g \mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,
\end{aligned} \tag{4.40}
$$

$$
(\nabla \cdot u_h^{n+1}, q_h) = 0 \ \ \forall q_h \in Q_h, \tag{4.41}
$$

$$
\begin{aligned}
&(\partial_{\Delta t}^i(T_h^{n+1}), S_h) + b^*(\mathscr{E}^i(\langle u_h \rangle^{n+1}), T_h^{n+1}, S_h) + b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(T_h^{n+1}), S_h) \\
&\quad + \langle \kappa \rangle (\nabla T_h^{n+1}, \nabla S_h) + \kappa' (\nabla \mathscr{E}^i(T_h^{n+1}), \nabla S_h) = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{\Gamma_D,h}.
\end{aligned} \tag{4.42}
$$

Algorithm (PEA): <u>Step 1.</u> Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$
\begin{aligned}
&(\partial_{\Delta t}^i(u_h^{n+1}), v_h) + b(\mathscr{E}^i(\langle u_h \rangle^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h) \\
&\quad + \langle \nu \rangle (\nabla u_h^{n+1}, \nabla v_h) + \nu' (\nabla \mathscr{E}^i(u_h^{n+1}), \nabla v_h) + (\langle \Lambda \rangle \times u_h^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) \\
&\quad + \frac{1}{\epsilon}(\nabla \cdot u_h^{n+1}, \nabla \cdot v_h) = (\beta g \mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,
\end{aligned} \tag{4.43}
$$

$$
\begin{aligned}
&(\partial_{\Delta t}^i(T_h^{n+1}), S_h) + b^*(\mathscr{E}^i(\langle u_h \rangle^{n+1}), T_h^{n+1}, S_h) + b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(T_h^{n+1}), S_h) \\
&\quad + \langle \kappa \rangle (\nabla T_h^{n+1}, \nabla S_h) + \kappa' (\nabla \mathscr{E}^i(T_h^{n+1}), \nabla S_h) = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{\Gamma_D,h}.
\end{aligned} \tag{4.44}
$$

<u>Step 2.</u> Given $u_h^{n+1} \in X_h$, find $p_h^{n+1} \in Q_h$ satisfying

$$
p_h^{n+1} = -\frac{1}{\epsilon} \nabla \cdot u_h^{n+1}. \tag{4.45}
$$

Algorithm (ACE): <u>Step 1.</u> Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$
\begin{aligned}
&(\partial^i_{\Delta t}(u_h^{n+1}), v_h) + b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h)\\
&\quad + \langle\nu\rangle(\nabla u_h^{n+1}, \nabla v_h) + \nu'(\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h) + (\langle\Lambda\rangle \times u_h^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h)\\
&\quad - (p_h^n, \nabla\cdot v_h) + \frac{\Delta t}{\epsilon}(\nabla\cdot u_h^{n+1}, \nabla\cdot v_h) = (\beta g\mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,
\end{aligned}
\tag{4.46}
$$

$$
\begin{aligned}
&(\partial^i_{\Delta t}(T_h^{n+1}), S_h) + b^*(\mathscr{E}^i(\langle u_h\rangle^{n+1}), T_h^{n+1}, S_h) + b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(T_h^{n+1}), S_h)\\
&\quad + \langle\kappa\rangle(\nabla T_h^{n+1}, \nabla S_h) + \kappa'(\nabla\mathscr{E}^i(T_h^{n+1}), \nabla S_h) = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{\Gamma_D,h}.
\end{aligned}
\tag{4.47}
$$

<u>Step 2.</u> Given $(u_h^{n+1}, p_h^n) \in (X_h, Q_h)$, find $p_h^{n+1} \in Q_h$ satisfying

$$
p_h^{n+1} = p_h^n - \frac{\Delta t}{\epsilon}\nabla\cdot u_h^{n+1}. \tag{4.48}
$$

Algorithm (ACE-T): <u>Step 1.</u> Find $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \in (X_h, Q_h, W_h)$ satisfying, for every $n = i-1, i, ..., N-1$,

$$
\begin{aligned}
&(\partial^i_{\Delta t}(u_h^{n+1}), v_h) + b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h)\\
&\quad + \langle\nu\rangle(\nabla u_h^{n+1}, \nabla v_h) + \nu'(\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h) + (\nu_{turb} D(u_h^{n+1}), D(v_h))\\
&\quad + (\langle\Lambda\rangle \times u_h^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) - (p_h^n, \nabla\cdot v_h) + \frac{\Delta t}{\epsilon}(\nabla\cdot u_h^{n+1}, \nabla\cdot v_h)\\
&\quad = (\beta g\mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \ \ \forall v_h \in X_h,
\end{aligned}
\tag{4.49}
$$

$$
\begin{aligned}
&(\partial^i_{\Delta t}(T_h^{n+1}), S_h) + b^*(\mathscr{E}^i(\langle u_h\rangle^{n+1}), T_h^{n+1}, S_h) + b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(T_h^{n+1}), S_h)\\
&\quad + \langle\kappa\rangle(\nabla T_h^{n+1}, \nabla S_h) + \kappa'(\nabla\mathscr{E}^i(T_h^{n+1}), \nabla S_h) + (\frac{\nu_{turb}}{\sigma_{turb}}\nabla T_h^{n+1}, \nabla S_h)\\
&\quad = (f_2^{n+1}, S_h) \ \ \forall S_h \in W_{\Gamma_D,h}.
\end{aligned}
\tag{4.50}
$$

<u>Step 2.</u> Given $(u_h^{n+1}, p_h^n) \in (X_h, Q_h)$, find $p_h^{n+1} \in Q_h$ satisfying

$$
p_h^{n+1} = p_h^n - \frac{\Delta t}{\epsilon}\nabla\cdot u_h^{n+1}. \tag{4.51}
$$

The second-order **eBDF** is similar to a BDF2-AB2 method used in [86] to uncouple a pair of evolution equations with exactly skew-symmetric coupling. As it will be instructive, we state and prove that solutions exist uniquely for each of the above algorithms. The results are collected into one theorem.

Theorem 6. *Consider each of the above algorithms. Suppose* $f_1^{n+1} \in H^{-1}(\Omega)^d$*,* $f_2^{n+1} \in H^{-1}(\Omega)$*,* $u_h^n \in X_h$*,* $T_h^n \in W_h$*, and* $p_h^n \in Q_h$ *(when required). Then, there exists unique solutions* $u_h^{n+1} \in X_h$*,* $T_h^{n+1} \in W_h$*, and* $p_h^{n+1} \in Q_h$*.*

Proof. See Appendix C. □

It is interesting to note that if $\langle u \rangle^n$ is replaced with the weighted arithmetic mean, e.g., $\langle u \rangle_w^n = \sum_{j=1}^J w_j u(x, t^n; \omega_j)$ such that $\sum_{j=1}^J w_j = 1$, all results proven below will hold. It would be interesting to utilize the arithmetic mean and associated fluctuation in the above algorithms whereby an additional calculation is made,

$$\max_{1 \leq j \leq J} \min_{w \in B^J(0,1)} \|\nabla \mathscr{E}_w^i(u'^{n+1}_h)\|,$$

where $B^J(0,1)$ is the J-dimensional unit ball. This optimization problem could lead to increased stability.

4.2 STABILITY ANALYSIS

In this section, we prove conditional, nonlinear, energy stability of solutions for each of the proposed algorithms. Our analysis is general, encompassing both the 2d and 3d cases. Restricting to 2d [71], condition (4.52) can be relaxed. Sufficient conditions for stability are as follows:

$$\frac{\Delta t}{h} \max_{1 \leq j \leq J} \|\nabla \mathscr{E}^i(u'^{n+1}_h)\|^2 \leq C_\dagger \min\{\langle \nu \rangle, \langle \kappa \rangle\}, \tag{4.52}$$

$$\max\left\{ \max_{1 \leq j \leq J} \left| \frac{\nu'}{\langle \nu \rangle} \right|^2, \max_{1 \leq j \leq J} \left| \frac{\kappa'}{\langle \kappa \rangle} \right|^2 \right\} \leq C_{\dagger\dagger}, \tag{4.53}$$

where $C_{\dagger\dagger} < \frac{1}{4}, \frac{1}{36}$ for first- and second-order methods, respectively. Typically, $C_\dagger$ is determined with pre-computations. Dimensional analysis indicates that $[C_\dagger] = L^{3-d}$, where L is a typical length scale. For **ACE-T**, condition (4.52) is improvable:

$$\Delta t \max_{1 \leq j \leq J} \|\nabla \cdot \mathscr{E}^i(u'^{n+1}_h)\|_{L^4}^2 \leq C_\dagger \min\{\langle \nu \rangle, \langle \kappa \rangle\}, \tag{4.54}$$

provided $C_\nu \geq C_{\dagger\dagger\dagger}$ and $\frac{C_\nu}{\sigma_{turb}} \geq C_{\dagger\dagger\dagger\dagger}$. Here, $[C_\dagger] = L^{d-2}$.

If the viscosity and thermal conductivity are not ensemble dependent, e.g., $\nu(\omega_j) = \langle \nu \rangle = \nu$ and $\kappa(\omega_j) = \langle \kappa \rangle = \kappa$ for all $1 \leq j \leq J$, condition (4.53) is automatically satisfied. Further, the following alternative non-dimensional forms of condition (4.52) and (4.54) are:

$$\frac{\Delta t}{h} \max_{1\leq j\leq J} \|\nabla \mathscr{E}^i(u'^{n+1}_h)\|^2 \leq C_\dagger Pr,$$
$$\Delta t \max_{1\leq j\leq J} \|\nabla \cdot \mathscr{E}^i(u'^{n+1}_h)\|^2_{L^4} \leq C_\dagger Pr,$$

recalling that $Pr = \frac{\nu}{\kappa}$ is the Prandtl number. Also, if $J = 1$, we see that all conditions are automatically satisfied and the resulting algorithms are unconditionally, nonlinearly, energy stable. In fact, when $J = 1$, we recover the standard linearly implicit BDF, penalty, and artificial compressibility methods.

Theorem 7. *Consider **eBDF** (4.40) - (4.42). Suppose $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ and $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. If conditions (4.52) and (4.53) hold, then there exists $C_\#$, $C_\triangle > 0$ such that,*

$$\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1} - u_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2\Big) \\
&\qquad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big) \\
&+ \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big) \leq \exp(C_\# t^*)\Bigg(C_I^2\Big(\frac{16|\beta g|^2 C_P^2}{\nu_{min}} + \frac{12\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2 + \\
&\qquad + 4\Delta t\sum_{n=0}^{N-1}\Big(\frac{4}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{3}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big) + \|u_h^0\|^2 + 2\|T_h^0\|^2 \\
&\qquad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg) \\
&\qquad + \Big(1 + 2\exp(C_\# t^*) + \frac{\langle\kappa\rangle t^*}{4} + \frac{\big(1 + 2\exp(C_\# t^*)\big)\langle\kappa\rangle\Delta t}{2}\Big)\|\tau\|_1^2
\end{aligned}$$

and

$$\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2$$

$$+\frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\|T_h^{n+1}-2T_h^n+T_h^{n-1}\|^2\Big)$$
$$+\frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big)+2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big)$$
$$+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big)\leq\exp(C_\triangle t^*)\Bigg(32C_I^2\Big(\frac{|\beta g|^2C_P^2}{\nu_{min}}+\frac{\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2$$
$$+32\Delta t\sum_{n=1}^{N-1}\Big(\frac{1}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2+\frac{1}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big)+\|u_h^1\|^2+2\|T_h^1\|^2+\|2u_h^1-u_h^0\|^2+2\|2T_h^1-T_h^0\|^2$$
$$+2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^1\|^2+2\|\langle\kappa\rangle^{1/2}\nabla T_h^1\|^2\Big)+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2+2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg)$$
$$+\Big(2+4\exp(C_\triangle t^*)+\frac{\langle\kappa\rangle t^*}{2}+\big(3+6\exp(C_\triangle t^*)\big)\langle\kappa\rangle\Delta t\Big)\|\tau\|_1^2.$$

Proof. Our strategy is to prove stability of an auxiliary temperature approximation $\theta_h \in W_{\Gamma_D,h}$ given by the relationship $T_h^n = \theta_h^n + I_h\tau$, where $I_h\tau \in W_h$ is an interpolant of τ in the finite element space satisfying $\|I_h\tau\|_1 \leq C_I\|\tau\|_1$. Using the above relationship and the triangle inequality will yield the result. Thus, let $T_h^n = \theta_h^n + I_h\tau$ in equation (4.42). Let $S_h = \Delta t\theta_h^{n+1} \in W_{\Gamma_D,h}$, add $0 = \Delta t b^*(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(\theta_h^{n+1}), \mathscr{E}^i(\theta_h^{n+1}))$, and reorganize. Then,

$$(\partial^i_{\Delta t}(\theta_h^{n+1}),\Delta t\theta_h^{n+1})+\Delta t\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2=-\Delta tb^*(\mathscr{E}^i(u'^{n+1}_h),\mathscr{E}^i(\theta_h^{n+1}),\theta_h^{n+1}-\mathscr{E}^i(\theta_h^{n+1}))$$
$$-\Delta tb^*(\mathscr{E}^i(u_h^{n+1}),I_h\tau,\theta_h^{n+1})-\Delta t(\kappa\nabla I_h\tau,\nabla\theta_h^{n+1})$$
$$-\Delta t(\kappa'\nabla\mathscr{E}^i(\theta_h^{n+1}),\nabla\theta_h^{n+1})+\Delta t(f_2^{n+1},\theta_h^{n+1}). \quad (4.55)$$

Similarly, for the velocity, consider equation (4.40), letting $v_h = \Delta t u_h^{n+1} \in X_h$, adding $0 = \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(u_h^{n+1}), \mathscr{E}^i(u_h^{n+1}))$, and reorganizing yields

$$(\partial^i_{\Delta t}(u_h^{n+1}),\Delta tu_h^{n+1})+\Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2=-\Delta tb(\mathscr{E}^i(u'^{n+1}_h),\mathscr{E}^i(u_h^{n+1}),u_h^{n+1}-\mathscr{E}^i(u_h^{n+1}))$$
$$-\Delta t(\nu'\nabla\mathscr{E}^i(u_h^{n+1}),\nabla u_h^{n+1})-\Delta t(\Lambda'\times\mathscr{E}^i(u_h^{n+1}))$$
$$+\Delta t(\beta g\mathscr{E}^i(T_h^{n+1}),u_h^{n+1})+\Delta t(f_1^{n+1},u_h^{n+1}). \quad (4.56)$$

Note that $(\langle\Lambda\rangle \times u_h^{n+1}, u_h^{n+1}) = 0$ by skew-symmetry. We treat the cases $i = 1$ and $i = 2$ separately; let $i = 1$. Consider (4.55), then the following estimates holds

$$-\Delta tb^*(u'^n_h,\theta_h^n,\theta_h^{n+1}-\theta_h^n)\leq C_{\star\star}\Delta th^{-1/2}\|\langle\kappa\rangle^{-1/2}\nabla u'^n_h\|\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|\|\theta_h^{n+1}-\theta_h^n\| \quad (4.57)$$

$$\leq \frac{2C_{\star\star}^2\Delta t^2}{h}\|\langle\kappa\rangle^{-1/2}\nabla {u'}_h^n\|^2\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2 + \frac{1}{4}\|\theta_h^{n+1}-\theta_h^n\|^2,$$

$$\begin{aligned} -\Delta t b^*(u_h^n, I_h\tau, \theta_h^{n+1}) &\leq \frac{\Delta t}{2}\|u_h^n\cdot\langle\kappa\rangle^{-1/2}\nabla I_h\tau\|\|\langle\kappa\rangle^{1/2}\theta_h^{n+1}\| \\ &+ \frac{\Delta t}{2}\|u_h^n\cdot\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|\|\langle\kappa\rangle^{-1/2}I_h\tau\| \\ &\leq \frac{(1+C_P)^2C_I^2\Delta t}{4\kappa_{min}\sigma_0}\|\tau\|_1^2\|u_h^n\|^2 + \frac{\sigma_0\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2, \end{aligned} \tag{4.58}$$

$$-\Delta t(\kappa\nabla I_h\tau, \nabla\theta_h^{n+1}) \leq \frac{C_I^2\kappa_{max}^2\Delta t}{2\kappa_{min}\sigma_1}\|\tau\|_1^2 + \frac{\sigma_1\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 \tag{4.59}$$

$$\begin{aligned} -\Delta t(\kappa'\nabla\theta_h^n, \nabla\theta_h^{n+1}) &= -\Delta t(\kappa'\frac{\langle\kappa\rangle^{1/2}}{\langle\kappa\rangle^{1/2}}\nabla\theta_h^n, \nabla\theta_h^{n+1}) \\ &\leq \Delta t\|\kappa'\langle\kappa\rangle^{-1/2}\nabla\theta_h^n\|\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\| \\ &= \Delta t\|\frac{\kappa'}{\langle\kappa\rangle}\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\| \\ &\leq \frac{\Delta t}{2}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2 + \frac{\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2, \end{aligned} \tag{4.60}$$

$$\Delta t(f_2^{n+1}, \theta_h^{n+1}) \leq \frac{\Delta t}{2\kappa_{min}\sigma_2}\|f_2^{n+1}\|_{-1}^2 + \frac{\sigma_2\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2. \tag{4.61}$$

Let $\sigma_0 = \sigma_1 = \sigma_2 = \frac{1}{12}$, use the above estimates in equation (4.55), multiply by 2, and rearrange. Then,

$$\begin{aligned} &\|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \frac{1}{2}\|\theta_h^{n+1}-\theta_h^n\|^2 + \frac{\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 + \frac{\Delta t}{2}\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 - \|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\Big) \\ &\quad + \frac{\Delta t}{2}\Big(1 - 2\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2 - \frac{4C_{\star\star}^2\Delta t}{h}\|\langle\kappa\rangle^{-1/2}\nabla {u'}_h^n\|^2\Big)\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2 \\ &\quad \leq \frac{6(1+C_P)^2C_I^2\Delta t}{\kappa_{min}}\|\tau\|_1^2\|u_h^n\|^2 + \frac{12C_I^2\kappa_{max}^2\Delta t}{\kappa_{min}}\|\tau\|_1^2 + \frac{12\Delta t}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2. \end{aligned} \tag{4.62}$$

Since conditions (4.52) and (4.53) hold, the last term on the left-hand side is non-negative and we may drop it yielding

$$\begin{aligned} &\|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2 + \frac{1}{2}\|\theta_h^{n+1}-\theta_h^n\|^2 + \frac{\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 + \frac{\Delta t}{2}\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 - \|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\Big) \\ &\quad \leq \frac{6(1+C_P)^2C_I^2\Delta t}{\kappa_{min}}\|\tau\|_1^2\|u_h^n\|^2 + \frac{12C_I^2\kappa_{max}^2\Delta t}{\kappa_{min}}\|\tau\|_1^2 + \frac{12\Delta t}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2. \end{aligned} \tag{4.63}$$

Now we follow analogously for the velocity equation. Considering equation (4.56), the following estimates hold

$$-\Delta t b({u'}_h^n, u_h^n, u_h^{n+1}-u_h^n) \leq \frac{2C_\star^2\Delta t^2}{h}\|\langle\nu\rangle^{-1/2}\nabla {u'}_h^n\|^2\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2 + \frac{1}{4}\|u_h^{n+1}-u_h^n\|^2, \tag{4.64}$$

$$-\Delta t(\nu'\nabla u_h^n, \nabla u_h^{n+1}) \leq \frac{\Delta t}{2}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2 \|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2 + \frac{\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2, \tag{4.65}$$

$$-\Delta t(\Lambda' \times u_h^n, u_h^{n+1}) \leq \frac{C_P^2\Delta t}{2\nu_{min}\sigma_3}|\Lambda'|^2\|u_h^n\|^2 + \frac{\sigma_3\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2, \tag{4.66}$$

$$\Delta t(\beta g\theta_h^n, u_h^{n+1}) \leq \frac{|\beta g|^2 C_P^2\Delta t}{2\nu_{min}\sigma_4}\|\theta_h^n\|^2 + \frac{\sigma_4\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2, \tag{4.67}$$

$$\Delta t(\beta g I_h\tau, u_h^{n+1}) \leq \frac{|\beta g|^2 C_P^2 C_I^2\Delta t}{2\nu_{min}\sigma_5}\|\tau\|_1^2 + \frac{\sigma_5\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2, \tag{4.68}$$

$$\Delta t(f_1^{n+1}, u_h^{n+1}) \leq \frac{\Delta t}{2\nu_{min}\sigma_6}\|f_1^{n+1}\|_{-1}^2 + \frac{\sigma_6\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2. \tag{4.69}$$

Let $\sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = \frac{1}{16}$, use the above estimates in equation (4.56), multiply by 2, and rearrange. Then,

$$\begin{aligned}
&\|u_h^{n+1}\|^2 - \|u_h^n\|^2 + \frac{1}{2}\|u_h^{n+1} - u_h^n\|^2 + \frac{\Delta t}{4}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 - \|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\Big) \\
&\quad + \frac{\Delta t}{2}\Big(1 - 2\left|\frac{\nu'}{\langle\nu\rangle}\right|^2 - \frac{4C_\star^2\Delta t}{h}\|\langle\nu\rangle^{-1/2}\nabla u_h'^n\|^2\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2 \leq \frac{16C_P^2\Delta t}{\nu_{min}}|\Lambda'|^2\|u_h^n\|^2 \\
&\qquad \frac{16|\beta g|^2 C_P^2\Delta t}{\nu_{min}}\|\theta_h^n\|^2 + \frac{16|\beta g|^2 C_P^2 C_I^2\Delta t}{\nu_{min}}\|\tau\|_1^2 + \frac{16\Delta t}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2.
\end{aligned} \tag{4.70}$$

Conditions (4.52) and (4.53) imply we may drop the last term on the left-hand side. Thus, adding inequality (4.63) to (4.70), summing over n from $n = 0$ to $n = N-1$, and rearranging yields

$$\begin{aligned}
&\|u_h^N\|^2 + \|\theta_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1} - u_h^n\|^2 + \|\theta_h^{n+1} - \theta_h^n\|^2\Big) \\
&\qquad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big) \\
&+ \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^N\|^2\Big) \leq \Big(\frac{16C_P^2}{\nu_{min}}|\Lambda'|^2 + \frac{6(1+C_P)^2 C_I^2}{\kappa_{min}}\|\tau\|_1^2\Big)\Delta t\sum_{n=0}^{N-1}\|u_h^n\|^2 \\
&\qquad + \frac{16|\beta g|^2 C_P^2}{\nu_{min}}\Delta t\sum_{n=0}^{N-1}\|\theta_h^n\|^2 + C_I^2\Big(\frac{16|\beta g|^2 C_P^2}{\nu_{min}} + \frac{12\kappa_{max}^2}{\kappa_{min}}\Big)\Delta t\sum_{n=0}^{N-1}\|\tau\|_1^2 \\
&\qquad + 4\Delta t\sum_{n=0}^{N-1}\Big(\frac{4}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{3}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big) + \|u_h^0\|^2 + \|\theta_h^0\|^2 \\
&\qquad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^0\|^2\Big).
\end{aligned} \tag{4.71}$$

Add $\frac{\Delta t^2}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^N\|^2\Big)$ to the right-hand side and let $C_{\#}=\max\{1, \frac{16C_P^2}{\nu_{min}}|\Lambda'|^2, \frac{6(1+C_P)^2C_I^2}{\kappa_{min}}, \frac{16|\beta g|^2C_P^2}{\nu_{min}}\}$. Applying Lemma 3 yields

$$
\begin{aligned}
&\|u_h^N\|^2+\|\theta_h^N\|^2+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1}-u_h^n\|^2+\|\theta_h^{n+1}-\theta_h^n\|^2\Big)\\
&\qquad\qquad+\frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big)\\
&+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^N\|^2\Big)\le\exp(C_{\#}t^*)\Bigg(C_I^2\Big(\frac{16|\beta g|^2C_P^2}{\nu_{min}}+\frac{12\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2\\
&\qquad+4\Delta t\sum_{n=0}^{N-1}\Big(\frac{4}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2+\frac{3}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big)+\|u_h^0\|^2+\|\theta_h^0\|^2\\
&\qquad\qquad+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^0\|^2\Big)\Bigg).
\end{aligned}
\tag{4.72}
$$

Lastly, using the relationship $T_h^n=\theta_h^n+I_h\tau$ and the triangle inequality yields the first result; that is, the velocity and temperature approximations are stable. Moving to the second-order algorithm, let $i=2$. The following estimates hold

$$
\begin{aligned}
&-\Delta t b^*(\mathscr{E}^2(u'^{n+1}_h),2\theta_h^n-\theta_h^{n-1},\theta_h^{n+1}-2\theta_h^n+\theta_h^{n-1})\\
&\qquad\le\frac{4C_{\star\star}^2\Delta t^2}{h}\|\langle\kappa\rangle^{-1/2}\nabla\mathscr{E}^2(u'^{n+1}_h)\|^2\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n-1}\|^2\Big)\\
&\qquad\qquad+\frac{1}{8}\|\theta_h^{n+1}-2\theta_h^n+\theta_h^{n-1}\|^2,
\end{aligned}
$$

$$
\begin{aligned}
-\Delta t b^*(2u_h^n-u_h^{n-1},I_h\tau,\theta_h^{n+1})&\le\frac{(1+C_P)^2C_I^2\Delta t}{\kappa_{min}\sigma_7}\|\tau\|_1^2\|2u_h^n-u_h^{n-1}\|^2 \qquad (4.73)\\
&\quad+\frac{\sigma_7\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2,\\
-\Delta t(\kappa'\nabla(2\theta_h^n-\theta_h^{n-1}),\nabla\theta_h^{n+1})&\le\Delta t\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\Big(4\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2+\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n-1}\|^2\Big) \qquad (4.74)\\
&\quad+\frac{\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2.
\end{aligned}
$$

Let $4\sigma_1=4\sigma_2=\sigma_7=\frac{1}{4}$, use the above estimates in equation (4.55), multiply by 4, and rearrange. Then,

$$
\begin{aligned}
&\|\theta_h^{n+1}\|^2+\|2\theta_h^{n+1}-\theta_h^n\|^2-\|\theta_h^n\|^2-\|2\theta_h^n-\theta_h^{n-1}\|^2+\frac{1}{2}\|\theta_h^{n+1}-2\theta_h^n-\theta_h^{n-1}\|^2+\frac{\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\\
&\quad+2\Delta t\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2-\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\Big)+\Delta t\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2-\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n-1}\|^2\Big)\\
&\qquad+2\Delta t\Big(1-8\Big|\frac{\kappa'}{\langle\kappa\rangle}\Big|^2-\frac{4C_{\star\star}^2\Delta t}{h}\|\langle\kappa\rangle^{-1/2}\nabla\mathscr{E}^2(u'^{n+1}_h)\|^2\Big)\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\\
&\qquad+\Delta t\Big(1-4\Big|\frac{\kappa'}{\langle\kappa\rangle}\Big|^2-\frac{4C_{\star\star}^2\Delta t}{h}\|\langle\kappa\rangle^{-1/2}\nabla\mathscr{E}^2(u'^{n+1}_h)\|^2\Big)\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n-1}\|^2\\
&\quad\le\frac{16(1+C_P)^2C_I^2\Delta t}{\kappa_{min}}\|\tau\|_1^2\|2u_h^n-u_h^{n-1}\|^2+\frac{32C_I^2\kappa_{max}^2\Delta t}{\kappa_{min}}\|\tau\|_1^2+\frac{32\Delta t}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2. \qquad (4.75)
\end{aligned}
$$

Similarly, for the velocity, consider equation (4.56). Then, the following estimates hold

$$
\begin{aligned}
&-\Delta t b(\mathscr{E}^2(u'^{n+1}_h),2u_h^n-u_h^n,u_h^{n+1}-2u_h^n+u_h^n)\\
&\qquad\le\frac{4C_\star^2\Delta t^2}{h}\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^2(u'^{n+1}_h)\|^2\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2+\|\langle\nu\rangle^{1/2}\nabla u_h^{n-1}\|^2\Big)\\
&\qquad\qquad+\frac{1}{8}\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2, \qquad (4.76)
\end{aligned}
$$

$$
\begin{aligned}
-\Delta t(\nu'\nabla(2u_h^n-u_h^{n-1}),\nabla u_h^{n+1})&\le\Delta t\Big|\frac{\nu'}{\langle\nu\rangle}\Big|^2\Big(4\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2+\|\langle\nu\rangle^{1/2}\nabla u_h^{n-1}\|^2\Big) \qquad (4.77)\\
&\quad+\frac{\Delta t}{4}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2,\\
-\Delta t(\Lambda'\times(2u_h^n-u_h^{n-1}),u_h^{n+1})&\le\frac{C_P^2\Delta t}{\nu_{min}\sigma_8}|\Lambda'|^2\|2u_h^n-u_h^{n-1}\|^2+\frac{\sigma_8\Delta t}{4}\|\langle\nu\rangle^{1/2}u_h^{n+1}\|^2, \qquad (4.78)\\
\Delta t(\beta g(2\theta_h^n-\theta_h^{n-1}),u_h^{n+1})&\le\frac{|\beta g|^2C_P^2\Delta t}{\nu_{min}\sigma_9}\|2\theta_h^n-\theta_h^{n-1}\|^2+\frac{\sigma_9\Delta t}{4}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2. \qquad (4.79)
\end{aligned}
$$

Let $4\sigma_5=4\sigma_6=\sigma_8=\sigma_9=\frac{1}{8}$, use the above estimates in equation (4.56), multiply by 4, and rearrange. Then,

$$
\begin{aligned}
&\|u_h^{n+1}\|^2+\|2u_h^{n+1}-u_h^n\|^2-\|u_h^n\|^2-\|2u_h^n-u_h^{n-1}\|^2+\frac{1}{2}\|u_h^{n+1}-2u_h^n-u_h^{n-1}\|^2+\frac{\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\\
&\quad+2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2-\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\Big)+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2-\|\langle\nu\rangle^{1/2}\nabla u_h^{n-1}\|^2\Big)\\
&\qquad+2\Delta t\Big(1-8\Big|\frac{\nu'}{\langle\nu\rangle}\Big|^2-\frac{4C_\star^2\Delta t}{h}\|\langle\nu\rangle^{-1/2}\nabla u'^n_h\|^2\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\\
&\qquad+\Delta t\Big(1-4\Big|\frac{\nu'}{\langle\nu\rangle}\Big|^2-\frac{4C_\star^2\Delta t}{h}\|\langle\nu\rangle^{-1/2}\nabla u'^n_h\|^2\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n-1}\|^2
\end{aligned}
$$

$$\le \frac{32C_P^2\Delta t}{\nu_{min}}|\Lambda'|^2\|2u_h^n - u_h^{n-1}\|^2 + \frac{32|\beta g|^2 C_P^2\Delta t}{\nu_{min}}\|2\theta_h^n - \theta_h^{n-1}\|^2 + \frac{32|\beta g|^2 C_P^2 C_I^2\Delta t}{\nu_{min}}\|\tau\|_1^2 + \frac{32\Delta t}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2. \quad (4.80)$$

Denote $C_\triangle = \max\{1, \frac{32C_P^2}{\nu_{min}}|\Lambda'|^2, \frac{32|\beta g|^2 C_P^2}{\nu_{min}}, \frac{16(1+C_P)^2 C_I^2}{\kappa_{min}}\|\tau\|_1^2\}$. Then, adding (4.75) to (4.80), using conditions (4.52) and (4.53), summing over n from $n = 1$ to $n = N-1$, and rearranging yields

$$\begin{aligned}
&\|u_h^N\|^2 + \|\theta_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \|2\theta_h^N - \theta_h^{N-1}\|^2 \\
&\quad + \frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \|\theta_h^{n+1} - 2\theta_h^n + \theta_h^{n-1}\|^2\Big) \\
&+ \frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big) + 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^N\|^2\Big) \\
&\quad + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^{N-1}\|^2\Big) \\
&\le C_\triangle\Delta t\sum_{n=1}^{N-1}\Big(\|u_h^n\|^2 + \|2u_h^n - u_h^{n-1}\|^2 + \|\theta_h^n\|^2 + \|2\theta_h^n - \theta_h^{n-1}\|^2\Big) \\
&+ 32C_I^2\Delta t\sum_{n=1}^{N-1}\Big(\frac{|\beta g|^2 C_P^2}{\nu_{min}} + \frac{\kappa_{max}^2}{\kappa_{min}}\Big)\|\tau\|_1^2 + 32\Delta t\sum_{n=1}^{N-1}\Big(\frac{1}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{1}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2 \\
&\quad + \|u_h^1\|^2 + \|\theta_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 + \|2\theta_h^1 - \theta_h^0\|^2 \\
&+ 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^1\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^1\|^2\Big) + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\theta_h^0\|^2\Big)\Big). \quad (4.81)
\end{aligned}$$

Apply Lemma 3, recall the relation $T_h^n = \theta_h^n + I_h\tau$, and apply the triangle inequality. This yields the result. □

As a corollary, stability of the pressure approximation follows.

Corollary 3. *Suppose Theorem 7 holds. Let $i = 1$. Then, the pressure approximation satisfies for all $N \ge 1$,*

$$\begin{aligned}
\alpha\Delta t\sum_{n=0}^{N-1}\|p_h^{n+1}\| \le (1+C_*^{-1})\Bigg(&\frac{C_1}{\nu_{min}}\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\nu\rangle^{1/2}\nabla\langle u_h\rangle^n\|^2\Big)^{1/2}\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)^{1/2} \\
&+ t^{*1/2}\Big(\nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)^{1/2}
\end{aligned}$$

$$+ t^{*1/2}\Big(\frac{C_1 C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\Big)^{1/2}$$
$$+ \frac{C_P^2|\beta g|t^{*1/2}}{\kappa_{min}^{1/2}}\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\kappa\rangle^{1/2}\nabla T_h^n\|^2\Big)^{1/2} + t^{*1/2}\Big(\Delta t\sum_{n=0}^{N-1}\|f_1^{n+1}\|_{-1}^2\Big)^{1/2}\Bigg).$$

Moreover, for $i = 2$,

$$\begin{aligned}
&\alpha\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\| \\
&\leq (1 + C_*^{-1})\Bigg(\frac{C_1}{\nu_{min}}\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^2(\langle u_h\rangle^{n+1})\|^2\Big)^{1/2}\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)^{1/2} \\
&\quad + t^{*1/2}\Big(\nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)^{1/2} \\
&\quad + t^{*1/2}\Big(\frac{C_1 C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\nu\rangle^{1/2}\nabla(2u_h^n - u_h^{n-1})\|^2\Big)^{1/2} \\
&\quad + \frac{C_P^2|\beta g|t^{*1/2}}{\kappa_{min}^{1/2}}\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\kappa\rangle^{1/2}\nabla(2T_h^n - T_h^{n-1})\|^2\Big)^{1/2} + t^{*1/2}\Big(\Delta t\sum_{n=1}^{N-1}\|f_1^{n+1}\|_{-1}^2\Big)^{1/2}\Bigg).
\end{aligned}$$

Proof. For the pressure, consider equation (4.40), isolate the discrete time-derivative, and let $v_h \in V_h$. Then,

$$\begin{aligned}
(\partial_{\Delta t}^i(u_h^{n+1}), v_h) = &-b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) - b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h) - (\langle\nu\rangle\nabla u_h^{n+1}, \nabla v_h) \\
&- (\nu'\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h) - (\langle\Lambda\rangle \times u_h^{n+1}, v_h) - (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) \\
&+ (\beta g\mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h). \quad (4.82)
\end{aligned}$$

The following estimates hold,

$$-b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) \leq \frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\|\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|\|\nabla v_h\|, \quad (4.83)$$
$$-b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h) \leq C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\|\|\nabla v_h\|, \quad (4.84)$$
$$-(\langle\nu\rangle\nabla u_h^{n+1}, \nabla v_h) \leq \nu_{max}^{1/2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|\|\nabla v_h\|, \quad (4.85)$$
$$-(\nu'\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h) \leq \nu_{max}\left|\frac{\nu'}{\langle\nu\rangle}\right|\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\|\|\nabla v_h\|, \quad (4.86)$$
$$-(\langle\Lambda\rangle \times u_h^{n+1}, v_h) \leq \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|\|\nabla v_h\|, \quad (4.87)$$

$$-(\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) \leq \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}} \|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\|\|\nabla v_h\|, \tag{4.88}$$

$$(\beta g \mathscr{E}^i(T_h^{n+1}), v_h) \leq \frac{C_P^2|\beta g|}{\kappa_{min}^{1/2}} \|\langle\kappa\rangle^{1/2}\nabla\mathscr{E}^i(T_h^{n+1})\|\|\nabla v_h\|, \tag{4.89}$$

$$(f_1^{n+1}, v_h) \leq \|f_1^{n+1}\|_{-1}\|\nabla v_h\|. \tag{4.90}$$

Using the above estimates in equation (4.82), dividing both sides by $0 \neq \|\nabla v_h\|$, taking a supremum over $v_h \in V_h$, and applying Lemma 5 and conditions (4.52) and (4.53) yields

$$\begin{aligned}\|\partial_{\Delta t}^i(u_h^{n+1})\|_{X_h^*} \leq C_*^{-1}\Bigg(&\Big(\frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\| + \nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| \\ &+ \Big(\frac{C_1 C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\| \\ &+ \frac{C_P^2|\beta g|}{\kappa_{min}^{1/2}}\|\langle\kappa\rangle^{1/2}\nabla\mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1}\Bigg).\end{aligned} \tag{4.91}$$

Reconsider equation (4.40), isolate the pressure term, and use the estimates (4.83) - (4.90). Then,

$$\begin{aligned}(p_h^{n+1}, \nabla\cdot v_h) \leq{}& (\partial_{\Delta t}^i(u_h^{n+1}), v_h) \\ &+ \Bigg(\Big(\frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\| + \nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| \\ &+ \Big(\frac{C_1 C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\| \\ &+ \frac{C_P^2|\beta g|}{\kappa_{min}^{1/2}}\|\langle\kappa\rangle^{1/2}\nabla\mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1}\Bigg)\|\nabla v_h\|.\end{aligned} \tag{4.92}$$

Divide by $0 \neq \|\nabla v_h\|$, take a supremum over $v_h \in X_h$, and use the inf-sup condition (2.23). This yields

$$\begin{aligned}\alpha\|p_h^{n+1}\| \leq{}& \|\partial_{\Delta t}^i(u_h^{n+1})\|_{X_h^*} \\ &+ \Big(\frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\| + \nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| \\ &+ \Big(\frac{C_1 C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\|\end{aligned}$$

$$+ \frac{C_P^2|\beta g|}{\kappa_{min}^{1/2}}\|\langle\kappa\rangle^{1/2}\nabla\mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1}. \quad (4.93)$$

Use estimate (4.91), multiply by Δt, and sum over n from $n = i-1$ to $n = N-1$. Then,

$$\begin{aligned}
&\alpha\Delta t\sum_{n=i-1}^{N-1}\|p_h^{n+1}\| \\
&\leq (1+C_*^{-1})\Delta t\sum_{n=i-1}^{N-1}\Bigg(\Big(\frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\| + \nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| \\
&\qquad + \Big(\frac{C_1C_\dagger h}{\Delta t} + C_{\dagger\dagger}\nu_{max} + \frac{C_P^2|\Lambda'|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(u_h^{n+1})\| \\
&\qquad + \frac{C_P^2|\beta g|}{\kappa_{min}^{1/2}}\|\langle\kappa\rangle^{1/2}\nabla\mathscr{E}^i(T_h^{n+1})\| + \|f_1^{n+1}\|_{-1}\Bigg). \quad (4.94)
\end{aligned}$$

The result follows by application of the Cauchy-Schwarz inequality and regrouping. $\square$

In all of the above estimates, a discrete Gronwall inequality, Lemma 3, was used to prove stability. As we saw in Chapter 3, it is possible to remove the exponential growth factor under a condition on the mesh.

Theorem 8. *Suppose the hypotheses of Theorem 7 hold. Further, suppose that $\delta = \mathcal{O}(Ra^{-1})$ and the following condition holds:*

$$\max_{1\leq j\leq J}\max_{K\in\mathcal{T}_h}\frac{\Delta t}{h_K}\|\nabla\mathscr{E}^i(u_h'^{n+1})\|^2_{L^2(K)} \leq C_\dagger\min\{\langle\nu\rangle,\langle\kappa\rangle\}.$$

Then, there exists $C > 0$, independent of t^*, such that

$$\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1}-u_h^n\|^2 + \|T_h^{n+1}-T_h^n\|^2\Big) \\
&\qquad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big) \\
&\qquad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big) + \alpha\Delta t\sum_{n=0}^{N-1}\|p_h^{n+1}\| \leq Ct^*
\end{aligned}$$

and

$$\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\|T_h^{n+1}-2T_h^n+T_h^{n-1}\|^2\Big)\\
&+\frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big)+2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N}\|^2\Big)\\
&+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big)+\alpha\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\|\le Ct^*.
\end{aligned}
$$

Proof. These estimates follow from techniques used in Theorem 5 (Chapter 3) and Theorem 7. □

The mesh condition $\delta=\mathcal{O}(Ra^{-1})$ is removable if the temperature satisfies homogeneous boundary conditions on the entire Dirichlet boundary [8].

Theorem 9. *Suppose the hypotheses of Theorem 7 hold. Further, suppose that $T|_{\Gamma_D}=0$. Then, there exists $C>0$, independent of t^*, such that*

$$
\begin{aligned}
&\|u_h^N\|^2+\|T_h^N\|^2+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1}-u_h^n\|^2+\|T_h^{n+1}-T_h^n\|^2\Big)\\
&+\frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big)\\
&+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N}\|^2+\|\langle\kappa\rangle^{1/2}\nabla T_h^{N}\|^2\Big)+\alpha\Delta t\sum_{n=0}^{N-1}\|p_h^{n+1}\|\le C
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_h^N\|^2+\|T_h^N\|^2+\|2u_h^N-u_h^{N-1}\|^2+\|2T_h^N-T_h^{N-1}\|^2\\
&+\frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2+\|T_h^{n+1}-2T_h^n+T_h^{n-1}\|^2\Big)\\
&+\frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2\Big)+2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N}\|^2+\frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N}\|^2\Big)\\
&+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big)+\alpha\Delta t\sum_{n=1}^{N-1}\|p_h^{n+1}\|\le C.
\end{aligned}
$$

Proof. We note that $T_h^{n+1} \in W_{\Gamma_D,h}$. Consequently, stability of the temperature approximation follows immediately from estimates (4.57), (4.58), and (4.60) with θ_h^{n+1} replaced with T_h^{n+1}. Moreover, the buoyancy term $(\beta g \mathscr{E}^i(T_h^{n+1}), u_h^{n+1})$ appearing in (4.56) is easily dispatched as follows

$$\Delta t \sum_{n=i-1}^{N-1} (\beta g \mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) \leq \frac{|\beta g|^2 C_P^4 \Delta t}{2\nu_{min}\sigma} \sum_{n=i-1}^{N-1} \|\nabla \mathscr{E}^i(^i(T_h^{n+1})\|^2 + \frac{\sigma \Delta t}{2} \sum_{n=i-1}^{N-1} \|\langle \nu \rangle^{1/2} \nabla u_h^{n+1}\|^2. \quad (4.95)$$

The first term on the right-hand side is bounded and the second can be subsumed into the diffusive term on the left-hand side of equation (4.56). The remainder is routine. □

The above result corresponds to flow driven by body or heat forces. It is not surprising that, provided $f_1 = f_2 = 0$, the above result implies $(u_h^{n+1}, T_h^{n+1}, p_h^{n+1}) \to (0,0,0)$ as $n \to \infty$. Analogs of both Theorems 8 and 9 hold for **PEA**, **ACE**, and **ACE-T**. We will not state them in the interest of brevity. Utilizing techniques from Theorem 7, we can prove analogous results for **PEA**.

Theorem 10. *Consider **PEA** (4.43) - (4.44). Suppose $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ and $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. If conditions (4.52) and (4.53) hold, then there exists $C_\#$, $C_\triangle > 0$ such that,*

$$\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1} - u_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2\Big) \\
&\quad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + 4\epsilon\|p_h^{n+1}\|^2\Big) \\
&+ \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big) \leq \exp(C_\# t^*)\Bigg(C_I^2\Big(\frac{16|\beta g|^2 C_P^2}{\nu_{min}} + \frac{12\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2 \\
&\quad + 4\Delta t \sum_{n=0}^{N-1}\Big(\frac{4}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{3}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big) \\
&\quad + \|u_h^0\|^2 + 2\|T_h^0\|^2 + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg) \\
&\quad + \Big(1 + 2\exp(C_\# t^*) + \frac{\langle\kappa\rangle t^*}{4} + \frac{\big(1 + 2\exp(C_\# t^*)\big)\langle\kappa\rangle\Delta t}{2}\Big)\|\tau\|_1^2
\end{aligned}$$

and

$$
\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2 \\
&\qquad + \frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2\Big) \\
&\qquad + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + 2\epsilon\|p_h^{n+1}\|^2\Big) \\
&\qquad + 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big) \\
&+ \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big) \leq \exp(C_\triangle t^*)\Bigg(32C_I^2\Big(\frac{|\beta g|^2 C_P^2}{\nu_{min}} + \frac{\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2 \\
&\qquad + 32\Delta t\sum_{n=1}^{N-1}\Big(\frac{1}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{1}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big) \\
&\qquad + \|u_h^1\|^2 + 2\|T_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 + 2\|2T_h^1 - T_h^0\|^2 \\
&\quad + 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^1\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^1\|^2\Big) + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg) \\
&\qquad + \Big(2 + 4\exp(C_\triangle t^*) + \frac{\langle\kappa\rangle t^*}{2} + \big(3 + 6\exp(C_\triangle t^*)\big)\langle\kappa\rangle\Delta t\Big)\|\tau\|_1^2.
\end{aligned}
$$

Proof. Consider equation (4.43) and use (4.45) to rewrite $\frac{1}{\epsilon}(\nabla\cdot u_h^{n+1}, \nabla\cdot v_h) = -(p_h^{n+1}, \nabla\cdot v_h)$. Letting $v_h = \Delta t u_h^{n+1} \in X_h$, adding $0 = \Delta t b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), \mathscr{E}^i(u_h^{n+1}))$, and reorganizing yields

$$
\begin{aligned}
&(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + \Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 - \Delta t(p_h^{n+1}, \nabla\cdot u_h^{n+1}) \\
&\qquad = -\Delta t b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) - \Delta t(\nu'\nabla\mathscr{E}^i(u_h^{n+1}), \nabla u_h^{n+1}) \\
&\qquad\quad - \Delta t(\Lambda'\times\mathscr{E}^i(u_h^{n+1}), u_h^{n+1}) + \Delta t(\beta g\mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) + \Delta t(f_1^{n+1}, u_h^{n+1}). \qquad (4.96)
\end{aligned}
$$

We must deal with the pressure term, which does not vanish. Take the $L^2(\Omega)$ inner product of equation (4.45) with $\Delta t p_h^{n+1} \in Q_h$. This yields

$$
\epsilon\Delta t\|p_h^{n+1}\|^2 = -\Delta t(\nabla\cdot u_h^{n+1}, p_h^{n+1}). \qquad (4.97)
$$

Add equation (4.96) to (4.97). Then,

$$
(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + \Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \epsilon\Delta t\|p_h^{n+1}\|^2
$$

$$= -\Delta t b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) - \Delta t(\nu'\nabla \mathscr{E}^i(u_h^{n+1}), \nabla u_h^{n+1})$$
$$- \Delta t(\Lambda' \times \mathscr{E}^i(u_h^{n+1})) + \Delta t(\beta g \mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) + \Delta t(f_1^{n+1}, u_h^{n+1}). \quad (4.98)$$

The result follows by similar techniques used in Theorem 7. $\square$

Alternatively, we could have kept $\frac{1}{\epsilon}(\nabla \cdot u_h^{n+1}, \nabla \cdot v_h)$ within equation (4.43). In this case, we can show $\epsilon \Delta t \|p_h^{n+1}\|^2 = \frac{\Delta t}{\epsilon}\|\nabla \cdot u_h^{n+1}\|^2 \leq C(data)$, as needed. However, the same techniques used in the above theorem can be utilized for **ACE** and in the upcoming error analysis.

Theorem 11. *Consider **ACE** (4.46) - (4.47). Suppose $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ and $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. If conditions (4.52) and (4.53) hold, then there exists $C_\#$, $C_\triangle > 0$ such that,*

$$\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \epsilon\|p_h^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1} - u_h^n\|^2 + \|T_h^{n+1} - T_h^n\|^2\Big)$$
$$+ \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + \frac{4\Delta t}{\epsilon}\|\nabla \cdot u_h^{n+1}\|^2\Big)$$
$$+ \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big) \leq \exp(C_\# t^*)\Bigg(C_I^2\Big(\frac{16|\beta g|^2 C_P^2}{\nu_{min}} + \frac{12\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2$$
$$+ 4\Delta t\sum_{n=0}^{N-1}\Big(\frac{4}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{3}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big)$$
$$+ \|u_h^0\|^2 + 2\|T_h^0\|^2 + \epsilon\|p_h^0\|^2 + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg)$$
$$+ \Big(1 + 2\exp(C_\# t^*) + \frac{\langle\kappa\rangle t^*}{4} + \frac{\big(1 + 2\exp(C_\# t^*)\big)\langle\kappa\rangle\Delta t}{2}\Big)\|\tau\|_1^2$$

and

$$\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \epsilon\|p_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2$$
$$+ \frac{1}{2}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1} - 2u_h^n + u_h^{n-1}\|^2 + \|T_h^{n+1} - 2T_h^n + T_h^{n-1}\|^2\Big)$$
$$+ \frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + \frac{2\Delta t}{\epsilon}\|\nabla \cdot u_h^{n+1}\|^2\Big)$$

$$
\begin{aligned}
&+ 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big)\\
&+ \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big) \le \exp(C_\triangle t^*)\Bigg(32C_I^2\Big(\frac{|\beta g|^2 C_P^2}{\nu_{min}} + \frac{\kappa_{max}^2}{\kappa_{min}}\Big)t^*\|\tau\|_1^2\\
&+ 32\Delta t\sum_{n=1}^{N-1}\Big(\frac{1}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{1}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big)\\
&+ \|u_h^1\|^2 + 2\|T_h^1\|^2 + \epsilon\|p_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 + 2\|2T_h^1 - T_h^0\|^2\\
&+ 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^1\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^1\|^2\Big) + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg)\\
&+ \Big(2 + 4\exp(C_\triangle t^*) + \frac{\langle\kappa\rangle t^*}{2} + \big(3 + 6\exp(C_\triangle t^*)\big)\langle\kappa\rangle\Delta t\Big)\|\tau\|_1^2.
\end{aligned}
$$

Proof. Consider equation (4.46) and use (4.48) to rewrite $\frac{\Delta t}{\epsilon}(\nabla\cdot u_h^{n+1}, \nabla\cdot v_h) - (p_h^n, \nabla\cdot v_h) = -(p_h^{n+1}, \nabla\cdot v_h)$. Letting $v_h = \Delta t u_h^{n+1} \in X_h$, adding $0 = \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(u_h^{n+1}), \mathscr{E}^i(u_h^{n+1}))$, and reorganizing yields

$$
\begin{aligned}
&(\partial^i_{\Delta t}(u_h^{n+1}), \Delta t u_h^{n+1}) + \Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 - \Delta t(p_h^{n+1}, \nabla\cdot u_h^{n+1})\\
&= -\Delta t b(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) - \Delta t(\nu'\nabla\mathscr{E}^i(u_h^{n+1}), \nabla u_h^{n+1})\\
&- \Delta t(\Lambda'\times\mathscr{E}^i(u_h^{n+1}), u_h^{n+1}) + \Delta t(\beta g\mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) + \Delta t(f_1^{n+1}, u_h^{n+1}). \quad (4.99)
\end{aligned}
$$

As in the penalty case, we must deal with the non-vanishing pressure term. Take the $L^2(\Omega)$ inner product of equation (4.48) with $\Delta t p_h^{n+1} \in Q_h$. This yields

$$
\frac{\epsilon}{2}\Big(\|p_h^{n+1}\|^2 - \|p_h^n\|^2 + \|p_h^{n+1} - p_h^n\|^2\Big) = -\Delta t(\nabla\cdot u_h^{n+1}, p_h^{n+1}). \quad (4.100)
$$

Add equation (4.99) to (4.100). Then,

$$
\begin{aligned}
&(\partial^i_{\Delta t}(u_h^{n+1}), \Delta t u_h^{n+1}) + \Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{\epsilon}{2}\Big(\|p_h^{n+1}\|^2 - \|p_h^n\|^2 + \|p_h^{n+1} - p_h^n\|^2\Big)\\
&= -\Delta t b(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) - \Delta t(\nu'\nabla\mathscr{E}^i(u_h^{n+1}), \nabla u_h^{n+1})\\
&- \Delta t(\Lambda'\times\mathscr{E}^i(u_h^{n+1})) + \Delta t(\beta g\mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) + \Delta t(f_1^{n+1}, u_h^{n+1}). \quad (4.101)
\end{aligned}
$$

We see that, $\sum_{n=i-1}^{N-1}\frac{\epsilon}{2}\Big(\|p_h^{n+1}\|^2 - \|p_h^n\|^2 + \|p_h^{n+1} - p_h^n\|^2\Big) = \frac{\epsilon}{2}\Big(\|p_h^N\|^2 - \|p_h^{i-1}\|^2\Big)$ $+\frac{\epsilon}{2}\sum_{n=i-1}^{N-1}\|p_h^{n+1} - p_h^n\|^2$. Consequently, the result follows by similar techniques used in Theorem 7. □

Note that $\frac{\epsilon}{2}\sum_{n=i-1}^{N-1}\|p_h^{n+1}-p_h^n\|^2 = \frac{\Delta t}{2\epsilon}\Big(\Delta t\sum_{n=i-1}^{N-1}\|\nabla\cdot u_h^{n+1}\|^2\Big)$. Consequently, for $\epsilon = \mathcal{O}(\Delta t)$, $\nabla\cdot u_h \in L^2(0,t^*;L^2(\Omega)^d)$ and for $\epsilon = \mathcal{O}(\Delta t^2)$, $\nabla\cdot u_h \in L^\infty(0,t^*;L^2(\Omega)^d)$. In other words, pressure stability weakens while velocity becomes more divergence free. Lastly, we prove stability of **ACE-T**.

Theorem 12. *Consider* ***ACE-T*** *(4.49) - (4.50). Suppose* $f_1 \in L^2(0,t^*;H^{-1}(\Omega)^d)$ *and* $f_2 \in L^2(0,t^*;H^{-1}(\Omega))$. *If conditions (4.53) and (4.54) hold, then there exists* $C,\ C_\#,\ C_\triangle > 0$ *such that,*

$$
\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \epsilon\|p_h^N\|^2 + \frac{1}{32}\sum_{n=0}^{N-1}\Big(\|u_h^{n+1}-u_h^n\|^2 + \|T_h^{n+1}-T_h^n\|^2\Big)\\
&\quad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + \frac{4\Delta t}{\epsilon}\|\nabla\cdot u_h^{n+1}\|^2\Big)\\
&\quad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big)\\
&\leq C\exp(C_\# t^*)\Bigg(\bigg(1 + \langle\kappa\rangle + \big(\langle\kappa\rangle + \nu_{min}^{-1} + \frac{\kappa_{max}^2}{\kappa_{min}}\big)t^*\bigg)\|\tau\|_1^2\\
&\quad + \Delta t\sum_{n=0}^{N-1}\Big(\frac{1}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2 + \frac{1}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2\Big)\\
&\quad + \|u_h^0\|^2 + \|T_h^0\|^2 + \epsilon\|p_h^0\|^2 + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla T_h^0\|^2\Big)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|u_h^N\|^2 + \frac{1}{2}\|T_h^N\|^2 + \epsilon\|p_h^N\|^2 + \|2u_h^N - u_h^{N-1}\|^2 + \frac{1}{2}\|2T_h^N - T_h^{N-1}\|^2\\
&\quad + \frac{1}{32}\sum_{n=1}^{N-1}\Big(\|u_h^{n+1}-2u_h^n+u_h^{n-1}\|^2 + \|T_h^{n+1}-2T_h^n+T_h^{n-1}\|^2\Big)\\
&\quad + \frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{n+1}\|^2 + \frac{2\Delta t}{\epsilon}\|\nabla\cdot u_h^{n+1}\|^2\Big)\\
&\quad + 2\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^N\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^N\|^2\Big)\\
&\quad + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{N-1}\|^2 + \frac{1}{2}\|\langle\kappa\rangle^{1/2}\nabla T_h^{N-1}\|^2\Big)\\
&\leq C\exp(C_\triangle t^*)\Bigg(\bigg(1 + \langle\kappa\rangle + \big(\langle\kappa\rangle + \nu_{min}^{-1} + \frac{\kappa_{max}^2}{\kappa_{min}}\big)t^*\bigg)\|\tau\|_1^2
\end{aligned}
$$

$$
+ \Delta t \sum_{n=1}^{N-1} \Big(\frac{1}{\nu_{min}} \|f_1^{n+1}\|_{-1}^2 + \frac{1}{\kappa_{min}} \|f_2^{n+1}\|_{-1}^2 \Big)
$$

$$
+ \|u_h^1\|^2 + \|T_h^1\|^2 + \epsilon \|p_h^1\|^2 + \|2u_h^1 - u_h^0\|^2 + \|2T_h^1 - T_h^0\|^2
$$

$$
+ \Delta t \Big(\|\langle\nu\rangle^{1/2} \nabla u_h^1\|^2 + \|\langle\kappa\rangle^{1/2} \nabla T_h^1\|^2 \Big) + \Delta t \Big(\|\langle\nu\rangle^{1/2} \nabla u_h^0\|^2 + \|\langle\kappa\rangle^{1/2} \nabla T_h^0\|^2 \Big) \Bigg).
$$

Proof. From Theorem 11, we arrive at

$$
\begin{aligned}
(\partial_{\Delta t}^i(u_h^{n+1}), \Delta t u_h^{n+1}) + \frac{\epsilon}{2} \Big(\|p_h^{n+1}\|^2 - \|p_h^n\|^2 + \|p_h^{n+1} - p_h^n\|^2 \Big) + \Delta t \|\langle\nu\rangle^{1/2} \nabla u_h^{n+1}\|^2 \\
+ \Delta t \|{\nu_{turb}}^{1/2} \nabla u_h^{n+1}\|^2 = -\Delta t b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) \\
- \Delta t (\nu' \nabla \mathscr{E}^i(u_h^{n+1}), \nabla u_h^{n+1}) - \Delta t (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), u_h^{n+1}) \\
+ \Delta t (\beta g \mathscr{E}^i(T_h^{n+1}), u_h^{n+1}) + \Delta t (f_1^{n+1}, u_h^{n+1}). \quad (4.102)
\end{aligned}
$$

and

$$
\begin{aligned}
(\partial_{\Delta t}^i(\theta_h^{n+1}), \Delta t \theta_h^{n+1}) + \Delta t \|\langle\kappa\rangle^{1/2} \nabla \theta_h^{n+1}\|^2 + \Delta t \|\big(\frac{\nu_{turb}}{\sigma_{turb}}\big)^{1/2} \nabla \theta_h^{n+1}\|^2 \\
= -\Delta t b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(\theta_h^{n+1}), \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})) - \Delta t b^*(\mathscr{E}^i(u_h^{n+1}), I_h \tau, \theta_h^{n+1}) \\
- \Delta t \Big(\big(\kappa + \frac{\nu_{turb}}{\sigma_{turb}}\big) \nabla I_h \tau, \nabla \theta_h^{n+1} \Big) - \Delta t (\kappa' \nabla \mathscr{E}^i(\theta_h^{n+1}), \nabla \theta_h^{n+1}) + \Delta t (f_2^{n+1}, \theta_h^{n+1}). \quad (4.103)
\end{aligned}
$$

First notice that, by skew-symmetry,

$$
\begin{aligned}
-\Delta t b(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(u_h^{n+1}), u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})) = \Delta t b(\mathscr{E}^i({u'}_h^{n+1}), u_h^{n+1}, u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})), \\
-\Delta t b^*(\mathscr{E}^i({u'}_h^{n+1}), \mathscr{E}^i(\theta_h^{n+1}), \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})) = \Delta t b^*(\mathscr{E}^i({u'}_h^{n+1}), \theta_h^{n+1}, \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Delta t b^*(\mathscr{E}^i({u'}_h^{n+1}), \theta_h^{n+1}, \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})) = (\mathscr{E}^i({u'}_h^{n+1}) \cdot \nabla \theta_h^{n+1}, \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})) \\
+ \frac{1}{2} ((\nabla \cdot \mathscr{E}^i({u'}_h^{n+1})) \theta_h^{n+1}, \theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})) \\
\leq \Delta t^2 \Big(\frac{\sigma_{10,1}}{2} \|\mathscr{E}^i({u'}_h^{n+1}) \cdot \nabla \theta_h^{n+1}\|^2 + \frac{\sigma_{10,2}}{4} \|(\nabla \cdot \mathscr{E}^i({u'}_h^{n+1})) \theta_h^{n+1}\|^2 \Big) \\
+ \big(\frac{1}{2\sigma_{10,1}} + \frac{1}{4\sigma_{10,2}} \big) \|\theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})\|^2 \\
\leq \Delta t^2 \Big(\frac{\sigma_{10,1}}{2} \int_\Omega |\mathscr{E}^i({u'}_h^{n+1})|^2 |\nabla \theta_h^{n+1}|^2 dx + \frac{\sigma_{10,2}}{4} \|\langle\kappa\rangle^{-1/2} \nabla \cdot \mathscr{E}^i({u'}_h^{n+1})\|_{L^4}^2 \|\langle\kappa\rangle^{1/2} \theta_h^{n+1}\|_{L^4}^2 \Big)
\end{aligned}
$$

$$+ (\frac{1}{2\sigma_{10,1}} + \frac{1}{4\sigma_{10,2}})\|\theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})\|^2,$$

$$\leq \Delta t^2\Big(\frac{\sigma_{10,1}}{2}\int_\Omega |\mathscr{E}^i(u'^{n+1}_h)|^2|\nabla\theta_h^{n+1}|^2 dx + \frac{C_S^2\sigma_{10,2}}{4}\|\langle\kappa\rangle^{-1/2}\nabla\cdot\mathscr{E}^i(u'^{n+1}_h)\|^2_{L^4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big)$$
$$+ (\frac{1}{2\sigma_{10,1}} + \frac{1}{4\sigma_{10,2}})\|\theta_h^{n+1} - \mathscr{E}^i(\theta_h^{n+1})\|^2, \quad (4.104)$$

$$\Delta t b(\mathscr{E}^i(u'^{n+1}_h), u_h^{n+1}, u_h^{n+1} - \mathscr{E}^i(u_h^{n+1}))$$
$$\leq \Delta t^2\Big(\frac{\sigma_{11,1}}{2}\int_\Omega |\mathscr{E}^i(u'^{n+1}_h)|^2|\nabla u_h^{n+1}|^2 dx + \frac{C_S^2\sigma_{11,2}}{4}\|\langle\nu\rangle^{-1/2}\nabla\cdot\mathscr{E}^i(u'^{n+1}_h)\|^2_{L^4}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)$$
$$+ (\frac{1}{2\sigma_{11,1}} + \frac{1}{4\sigma_{11,2}})\|u_h^{n+1} - \mathscr{E}^i(u_h^{n+1})\|^2, \quad (4.105)$$

$$-\Delta t(\frac{\nu_{turb}}{\sigma_{turb}}\nabla I_h\tau, \nabla\theta_h^{n+1}) \leq \frac{C_I^2\Delta t}{2\sigma_{12}}\|\tau\|_1^2 + \frac{\sigma_{12}\Delta t}{2}\|(\frac{\nu_{turb}}{\sigma_{turb}})^{1/2}\nabla\theta_h^{n+1}\|^2. \quad (4.106)$$

Let $i = 1$ and use the above estimates as well as estimates (4.58) - (4.61) in equation (4.103). Then,

$$\frac{1}{2}\Big(\|\theta_h^{n+1}\|^2 - \|\theta_h^n\|^2\Big) + \Big(\frac{1}{2} - \frac{1}{2\sigma_{10,1}} - \frac{1}{4\sigma_{10,2}}\Big)\|\theta_h^{n+1} - \theta_h^n\|^2$$
$$+ \Big(\frac{5}{8} - \frac{\sigma_0}{4} - \frac{\sigma_1+\sigma_2}{2} - \frac{C_S^2\sigma_{10,2}\Delta t}{4}\|\nabla\cdot u'^n_h\|^2_{L^4}\Big)\Delta t\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2$$
$$+ \frac{\Delta t}{4}\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 - \|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\Big) + \Big(1 - 2\Big|\frac{\kappa'}{\langle\kappa\rangle}\Big|^2\Big)\frac{\Delta t}{4}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2$$
$$+ \Delta t\int_\Omega\Big(\frac{\langle\kappa\rangle}{8} + (1 - \frac{\sigma_{12}}{2})\frac{\nu_{turb}}{\sigma_{turb}} - \frac{\sigma_{10,1}\Delta t}{2}|u'^n_h|^2\Big)|\nabla\theta_h^{n+1}|^2 dx$$
$$\leq \frac{(1+C_P)^2C_I^2\Delta t}{4\kappa_{min}\sigma_0}\|\tau\|_1^2\|u_h^n\|^2 + (\frac{\kappa_{max}^2\Delta t}{\kappa_{min}\sigma_1} + \frac{1}{\sigma_{12}})\frac{C_I^2\Delta t}{2}\|\tau\|_1^2 + \frac{\Delta t}{2\kappa_{min}\sigma_2}\|f_2^{n+1}\|^2_{-1}. \quad (4.107)$$

Choosing $8\sigma_0 = 32\sigma_1 = 32\sigma_2 = \frac{16}{15}\sigma_{10,1} = 16\sigma_{10,2} = 8\sigma_{12} = 1$, multiplying by 2, and rearranging yields

$$\|\theta_h^{n+1}\|^2 + \frac{1}{32}\|\theta_h^{n+1} - \theta_h^n\|^2 + \Big(1 - 64C_S^2\Delta t\|\nabla\cdot u'^n_h\|^2_{L^4}\Big)\frac{\Delta t}{8}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2$$
$$+ \frac{\Delta t}{2}\Big(\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 - \|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2\Big) + \Big(1 - 2\Big|\frac{\kappa'}{\langle\kappa\rangle}\Big|^2\Big)\frac{\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^n\|^2$$
$$+ \Delta t\int_\Omega\Big(\frac{\langle\kappa\rangle}{4} + (1 - \frac{128}{225})\frac{15C_\nu\Delta t}{8\sigma_{turb}}|u'^n_h|^2\Big)|\nabla\theta_h^{n+1}|^2 dx$$

$$\leq \|\theta_h^n\|^2 + \frac{4(1+C_P)^2C_I^2\Delta t}{\kappa_{min}}\|\tau\|_1^2\|u_h^n\|^2 + 8\big(\frac{4\kappa_{max}^2\Delta t}{\kappa_{min}\sigma_1}+1\big)\frac{C_I^2\Delta t}{2}\|\tau\|_1^2 + \frac{32\Delta t}{\kappa_{min}}\|f_2^{n+1}\|_{-1}^2. \quad (4.108)$$

For the velocity equation 4.102, using the estimate (4.105) and estimates (4.65) - (4.69), and rearranging yields

$$\frac{1}{2}\Big(\|u_h^{n+1}\|^2-\|u_h^n\|^2\Big)+\Big(\frac{1}{2}-\frac{1}{2\sigma_{11,1}}-\frac{1}{4\sigma_{11,2}}\Big)\|u_h^{n+1}-u_h^n\|^2+\frac{\epsilon}{2}\Big(\|p_h^{n+1}\|^2-\|p_h^n\|^2+\|p_h^{n+1}-p_h^n\|^2\Big)$$
$$+\Big(\frac{5}{8}-\frac{\sigma_3+\sigma_4+\sigma_4+\sigma_6}{2}-\frac{C_S^2\sigma_{11,2}\Delta t}{4}\|\nabla\cdot u_h'^n\|_{L^4}^2\Big)\Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2$$
$$+\frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2-\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\Big)+\Big(1-2\Big|\frac{\nu'}{\langle\nu\rangle}\Big|^2\Big)\frac{\Delta t}{4}\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2$$
$$+\Delta t\int_\Omega\Big(\frac{\langle\nu\rangle}{8}+\nu_{turb}-\frac{\sigma_{11,1}\Delta t}{2}|u_h'^n|^2\Big)|\nabla u_h^{n+1}|^2dx \leq \frac{C_P^2\Delta t}{2\nu_{min}\sigma_3}|\Lambda'|^2\|u_h^n\|^2+\frac{|\beta g|^2C_P^2\Delta t}{2\nu_{min}\sigma_4}\|\theta_h^n\|^2$$
$$+\frac{|\beta g|^2C_P^2C_I^2\Delta t}{2\nu_{min}\sigma_5}\|\tau\|_1^2+\frac{\Delta t}{2\nu_{min}\sigma_6}\|f_1^{n+1}\|_{-1}^2. \quad (4.109)$$

Choose $32\sigma_3 = 32\sigma_4 = 32\sigma_5 = \sigma_6 = \frac{16}{15}\sigma_{11,1} = 16\sigma_{11,2} = 1$, multiply by 2, and rearrange. Then,

$$\|u_h^{n+1}\|^2+\epsilon\|p_h^{n+1}\|^2+\frac{1}{32}\|u_h^{n+1}-u_h^n\|^2+\|p_h^{n+1}-p_h^n\|^2$$
$$+\Big(1-64C_S^2\Delta t\|\nabla\cdot u_h'^n\|_{L^4}^2\Big)\frac{\Delta t}{8}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2$$
$$+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2-\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\Big)+\Big(1-2\Big|\frac{\nu'}{\langle\nu\rangle}\Big|^2\Big)\frac{\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2$$
$$+\Delta t\int_\Omega\Big(\frac{\langle\nu\rangle}{4}+2\big(1-\frac{8}{15}\big)C_\nu|u_h'^n|^2\Big)|\nabla u_h^{n+1}|^2dx \leq \big(1+\frac{32C_P^2\Delta t}{\nu_{min}}|\Lambda'|^2\big)\|u_h^n\|^2+\epsilon\|p_h^n\|^2$$
$$+\frac{32|\beta g|^2C_P^2\Delta t}{\nu_{min}}\|\theta_h^n\|^2+\frac{32|\beta g|^2C_P^2C_I^2\Delta t}{\nu_{min}}\|\tau\|_1^2+\frac{32\Delta t}{\nu_{min}}\|f_1^{n+1}\|_{-1}^2. \quad (4.110)$$

Add inequalities (4.108) and (4.110) together, use conditions (4.53) and (4.54), sum over n from $n=0$ to $n=N-1$, use Lemma 3 and the relation $T_h^n=\theta_h^n+I_h\tau$, and the triangle inequality. The second-order case, $i=2$, follows by similar arguments. □

4.3 ERROR ANALYSIS

Herein, we state and prove convergence estimates for the proposed algorithms. In particular, Theorems 13 and 14 guarantee first- and second-order velocity and temperature accuracy for **eBDF**, respectively. Corollary 4 states that the pressure approximation is of the same order of accuracy. In Theorem 15, we prove that **PEA** is first- and second-order accurate in velocity and temperature provided $\epsilon = \mathcal{O}(\Delta t^2)$ and $\epsilon = \mathcal{O}(\Delta t^4)$, respectively. Similarly, **ACE** is proven first- and second-order provided $\epsilon = \mathcal{O}(\Delta t)$ and $\epsilon = \mathcal{O}(\Delta t^3)$, respectively, in Theorem 16. We leave improvement of these conditions as an open problem.

Denote u^n, p^n, and T^n as the true solutions at time $t^n = n\Delta t$. Assume the solutions satisfy the following regularity assumptions:

$$\begin{aligned} u \in L^\infty(0,t^*;X\cap H^{k+1}(\Omega)^d),\ u_t \in L^2(0,t^*;H^{k+1}(\Omega)^d),\ u_{tt} \in L^2(0,t^*;H^{k+1}(\Omega)^d),\\ T,\tau \in L^\infty(0,t^*;W\cap H^{k+1}(\Omega)),\ T_t \in L^2(0,t^*;H^{k+1}(\Omega)),\ T_{tt} \in L^2(0,t^*;H^{k+1}(\Omega)),\\ p \in L^2(0,t^*;Q\cap H^m(\Omega)),\ p_t \in L^2(0,t^*;Q). \end{aligned} \tag{4.111}$$

Remark: Regularity of the auxiliary temperature solution θ follows since $\theta = T - \tau$. Convergence is proven for θ first. The result will follow for the primitive variable T via the triangle inequality and interpolation estimates.

The errors for the solution variables are denoted

$$e_u^n = (u^n - I_h u^n) - (u_h^n - I_h u^n) = \eta^n - \phi_h^n, \tag{4.112}$$

$$e_\theta^n = (\theta^n - I_h \theta^n) - (\theta_h^n - I_h \theta^n) = \zeta^n - \psi_h^n, \tag{4.113}$$

$$e_p^n = (p^n - I_h p^n) - (p_h^n - I_h p^n) = \lambda^n - \pi_h^n. \tag{4.114}$$

Definition 1. *(Consistency error). The consistency errors are defined as*

$$\begin{aligned} \varsigma_u^i(u^n;v_h) &:= \big(\partial^i_{\Delta t}(u_h^n) - u_t^n, v_h\big) + (\Lambda' \times (u^n - \mathscr{E}^i(u^n)), v_h) + (\beta g(T^n - \mathscr{E}^i(T^n)), v_h)\\ &\quad - B_u^n + D_u^n,\\ \varsigma_T^i(T^n;S_h) &:= \big(\partial^i_{\Delta t}(T_h^{n+1}) - T_t^n, S_h\big) - B_T^n + D_T^n,\\ \varsigma_p(p^n;q_h) &= \epsilon\big(\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} p_t(s)ds, q_h\big), \end{aligned}$$

where

$$
\begin{aligned}
B_u^n &:= b(u^n - \mathscr{E}^i(u^n), u_h^n, v_h) + b(\mathscr{E}^i(u'^n_h), u^n - \mathscr{E}^i(u^n), v_h),\\
B_T^n &:= b^*(u^n - \mathscr{E}^i(u^n), T_h^n, S_h) + b^*(\mathscr{E}^i(u'^n_h), T^n - \mathscr{E}^i(T^n), S_h) + b^*(u^n - \mathscr{E}^i(u^n), \tau, S_h),\\
D_u^n &:= (\nu' \nabla(u^n - \mathscr{E}^i(u^n)), \nabla v_h),\\
D_T^n &:= (\kappa' \nabla(T^n - \mathscr{E}^i(T^n)), \nabla S_h).
\end{aligned}
$$

Lemma 9. *Provided u and T satisfy the regularity assumptions (4.111), then $\forall \sigma,\ r > 0$*

$$
\begin{aligned}
|\varsigma_u^1(u^n; v_h)| \leq{}& \frac{6 C_r C_P^2 \Delta t}{\nu_{min}\sigma}\Big(\|u_{tt}\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + |\Lambda'|^2 \|u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)}\\
&+ |\beta g|^2 \|T_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}\Big) + \frac{6 C_r C_1^2 \Delta t}{\sigma}\Big(\frac{1}{\nu_{min}^2} \|\langle \nu\rangle^{1/2} \nabla u_h^n\|^2 + \|\langle \nu\rangle^{-1/2}\nabla u'^{n-1}_h\|^2\\
&+ \frac{|\nu'|^2}{C_1^2}\Big) \|\nabla u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + \frac{\sigma}{r}\|\langle \nu\rangle^{1/2}\nabla v_h\|^2,\\
|\varsigma_T^1(T^n; S_h)| \leq{}& \frac{5 C_r C_P^2 \Delta t}{\kappa_{min}\sigma}\|T_{tt}\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))}\\
&+ \frac{5 C_r C_4^2 \Delta t}{\sigma}\Big(\frac{1}{\kappa_{min}^2}\|\langle \kappa\rangle^{1/2}\nabla T_h^n\|^2 + \|\langle \kappa\rangle^{-1/2}\nabla u'^{n-1}_h\|^2\\
&+ \frac{1}{\kappa_{min}}\|\tau\|_1^2 + \frac{|\kappa'|^2}{C_4^2}\Big)\|\nabla u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + \frac{\sigma}{r}\|\langle \kappa\rangle^{1/2}\nabla S_h\|^2.
\end{aligned}
$$

Moreover, for the second-order case ($i = 2$), we have

$$
\begin{aligned}
|\varsigma_u^2(u^n; v_h)| \leq{}& \frac{6 C_r C_P^2 \Delta t^3}{\nu_{min}\sigma}\Big(\|u_{ttt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} + |\Lambda'|^2 \|u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)}\\
&+ |\beta g|^2 \|T_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega))}\Big)\\
&+ \frac{6 C_r C_1^2 \Delta t^3}{\sigma}\Big(\frac{1}{\nu_{min}^2}\|\langle \nu\rangle^{1/2}\nabla u_h^n\|^2 + \|\langle \nu\rangle^{-1/2}\nabla \mathscr{E}^2(u'^n_h)\|^2\\
&+ \frac{|\nu'|^2}{C_1^2}\Big)\|\nabla u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} + \frac{\sigma}{r}\|\langle \nu\rangle^{1/2}\nabla v_h\|^2,\\
|\varsigma_T^2(T^n; S_h)| \leq{}& \frac{5 C_r C_P^2 \Delta t^3}{\kappa_{min}\sigma}\|T_{ttt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega))}\\
&+ \frac{5 C_r C_4^2 \Delta t^3}{\sigma}\Big(\frac{1}{\kappa_{min}^2}\|\langle \kappa\rangle^{1/2}\nabla T_h^n\|^2 + \|\langle \kappa\rangle^{-1/2}\nabla \mathscr{E}^2(u'^n_h)\|^2\\
&+ \frac{1}{\kappa_{min}}\|\tau\|_1^2 + \frac{|\kappa'|^2}{C_4^2}\Big)\|\nabla u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} + \frac{\sigma}{r}\|\langle \kappa\rangle^{1/2}\nabla S_h\|^2.
\end{aligned}
$$

Lastly, if $p_t \in L^\infty(t^{n-1}, t^n; L^2(\Omega))$, then

$$
|\varsigma_p(p^n; q_h)| \leq \frac{C_r \epsilon \Delta t}{\sigma}\|p_t\|^2_{L^\infty(t^{n-1},t^n;L^2(\Omega))} + \frac{\epsilon\sigma}{r}\|\nabla q_h\|^2.
$$

Proof. We consider only $\varsigma_u^i(u^n; v_h)$ since the results for $\varsigma_T^i(T^n; S_h)$ and $\varsigma_p(p^n; q_h)$ follow by similar arguments. Consider the first three terms, applying the Cauchy-Schwarz inequality, Taylor's Theorem with integral remainder, Poincaré-Friedrichs inequality, and Young's inequality yields

$$\Big(\frac{u^n - u^{n-1}}{\Delta t} - u_t^n, v_h\Big) \leq \frac{C_r C_P^2 \Delta t}{\nu_{min}\sigma_1}\|u_{tt}\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + \frac{\sigma_1}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.115}$$

$$(\Lambda' \times (u^n - u^{n-1}), v_h) \leq \frac{C_r C_P^2 |\Lambda'|^2 \Delta t}{\nu_{min}\sigma_2}\|u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + \frac{\sigma_2}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.116}$$

$$(\beta g(T^n - T^{n-1}), v_h) \leq \frac{C_r C_P^2 |\beta g|^2 \Delta t}{\nu_{min}\sigma_3}\|T_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega))} + \frac{\sigma_3}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2. \tag{4.117}$$

For the skew-symmetric terms B_u^n, apply Lemma 1, Taylor's Theorem with integral remainder, and Young's inequality. Then,

$$b(u^n - u^{n-1}, u_h^n, v_h) \leq \frac{C_r C_1^2 \Delta t}{\nu_{min}^2 \sigma_4}\|\langle\nu\rangle^{1/2}\nabla u_h^n\|^2\|\nabla u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} \tag{4.118}$$

$$+ \frac{\sigma_4}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.119}$$

$$b(u_h'^{n-1}, u^n - u^{n-1}, v_h) \leq \frac{C_r C_1^2 \Delta t}{\sigma_5}\|\langle\nu\rangle^{-1/2}\nabla u_h'^{n-1}\|^2\|\nabla u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} \tag{4.120}$$

$$+ \frac{\sigma_5}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2. \tag{4.121}$$

Consider the viscous term D_u^n. Apply the Cauchy-Schwarz inequality, Taylor's Theorem with integral remainder, and Young's inequality. Then,

$$(\nu'\nabla(u^n - u^{n-1}), \nabla v_h) \leq \frac{\Delta t}{\sigma_6}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2\|\nabla u_t\|^2_{L^2(t^{n-1},t^n;L^2(\Omega)^d)} + \frac{\sigma_6}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2. \tag{4.122}$$

Letting $\sigma_l = \frac{\sigma}{6}$ for $1 \leq l \leq 6$ and regrouping yields the first result. For the second, $i = 2$, the following estimates hold,

$$\Big(\frac{3u^n - 4u^{n-1} + u^{n-2}}{2\Delta t} - u_t^n, v_h\Big) \leq \frac{C_r C_P^2 \Delta t^3}{\sigma_7}\|u_{ttt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} \tag{4.123}$$

$$+ \frac{\sigma_7}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.124}$$

$$(\Lambda' \times (u^n - 2u^{n-1} + u^{n-2}), v_h) \leq \frac{C_r C_P^2 |\Lambda'|^2 \Delta t^3}{\sigma_8}\|u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} \tag{4.125}$$

$$+ \frac{\sigma_8}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2. \tag{4.126}$$

$$(\beta g(T^n - 2T^{n-1} + T^{n-2}), v_h) \leq \frac{C_r C_P^2 |\beta g|^2 \Delta t^3}{\sigma_9}\|T_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega))} \tag{4.127}$$

$$+\frac{\sigma_9}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.128}$$

$$b(u^n-2u^{n-1}+u^{n-2},u_h^n,v_h)\leq\frac{C_rC_1^2\Delta t^3}{\sigma_{10}}\|\nabla u_h^n\|^2\|\nabla u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} \tag{4.129}$$

$$+\frac{\sigma_{10}}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2, \tag{4.130}$$

$$b(\mathscr{E}^2(u'^n_h),u^n-2u^{n-1}+u^{n-2},v_h)\leq\frac{C_rC_1^2\Delta t^3}{\sigma_{11}}\|\nabla\mathscr{E}^2(u'^n_h)\|^2\|\nabla u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} \tag{4.131}$$

$$+\frac{\sigma_{11}}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2,$$

$$(\nu'\nabla(u^n-2u^{n-1}+u^{n-2}),\nabla v_h)\leq\frac{|\nu'|^2\Delta t^3}{\sigma_{12}}\|\nabla u_{tt}\|^2_{L^2(t^{n-2},t^n;L^2(\Omega)^d)} \tag{4.132}$$

$$+\frac{\sigma_{12}}{r}\|\langle\nu\rangle^{1/2}\nabla v_h\|^2. \tag{4.133}$$

Letting $\sigma_l=\frac{\sigma}{6}$ for $7\leq l\leq 12$ and regrouping yields the second result. $\square$

We are now in a position to prove convergence. We first begin with proving **eBDF** is first-order convergent when $i=1$.

Theorem 13. *Consider first-order **eBDF**. For (u,p,T) satisfying (4.14) - (4.17), suppose that $(u_h^0,p_h^0,T_h^0)\in(X_h,Q_h,W_h)$ are approximations of (u^0,p^0,T^0) to within the accuracy of the interpolant. Further, suppose that conditions (4.52) and (4.53) hold. Then, there exists constants C, $C_{\#}>0$ such that*

$$\begin{aligned}
&\|e_u^N\|^2+\|e_T^N\|^2+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\|e_u^{n+1}-e_u^n\|^2+\|e_T^{n+1}-e_T^n\|^2\Big)\\
&\quad+\frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2\Big)+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2\Big)\\
&\quad\leq C\exp(C_{\#}t^*)\Big\{\inf_{S_h\in W_h}\Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1}|||\nabla(T-S_h)|||^2_{\infty,0}+\kappa_{max}|||\nabla(T-S_h)|||^2_{2,0}\\
&\qquad+\kappa_{min}^{-1}\|(T-S_h)_t\|^2_{L^2(0,t^*;L^2(\Omega))}+h\Delta t\|\nabla(T-S_h)_t\|^2_{L^2(0,t^*;L^2(\Omega))}\Big)\\
&+\inf_{v_h\in X_h}\Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u-v_h)|||^2_{\infty,0}+\big(\kappa_{min}^{-1}+\nu_{max}+(|\langle\Lambda\rangle|^2+|\Lambda'|^2)+|\beta g|^2\big)|||\nabla(u-v_h)|||^2_{2,0}\\
&\qquad+\nu_{min}^{-1}\|(u-v_h)_t\|^2_{L^2(0,t^*;L^2(\Omega)^d)}+h\Delta t\|\nabla(u-v_h)_t\|^2_{L^2(0,t^*;L^2(\Omega)^d)}\Big)\\
&+\inf_{q_h\in Q_h}\nu_{min}^{-1}|||p-q_h|||^2_{2,0}+t^*\inf_{S_h\in W_h}\Big(|\beta g|^2\nu_{min}^{-1}\|\tau-S_h\|^2+(1+\kappa_{min}^{-1}+\kappa_{max})\|\nabla(\tau-S_h)\|^2\Big)\\
&\qquad+h\Delta t+\big(\nu_{min}^{-1}(1+|\Lambda'|^2+|\beta g|^2)+|\kappa'|^2+|\nu'|^2\big)\Delta t^2\Big\}\\
&\qquad\qquad+\|e_u^0\|^2+\|e_T^0\|^2+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2\Big).
\end{aligned}$$

Proof. Let $T^n = \theta^n + \tau$. The true solutions satisfy for all $n = i-1, ..., N-1$:

$$(\partial^i_{\Delta t}(u^{n+1}), v_h) + b(u^{n+1}, u^{n+1}, v_h) + \nu(\nabla u^{n+1}, \nabla v_h) + (\Lambda \times u^{n+1}, v_h) - (p^{n+1}, \nabla \cdot v_h)$$
$$= (\beta g(\theta^{n+1} + \tau), v_h) + (f_1^{n+1}, v_h) + (\partial^i_{\Delta t}(u^{n+1}) - u_t^{n+1}, v_h) \;\; \forall v_h \in X_h, \quad (4.134)$$

$$(\nabla \cdot u^{n+1}, q_h) = 0 \;\; \forall q_h \in Q_h, \quad (4.135)$$

$$(\partial^i_{\Delta t}(\theta^{n+1}), S_h) + b^*(u^{n+1}, \theta^{n+1}, S_h) + b^*(u^{n+1}, \tau, S_h) + \kappa(\nabla \theta^{n+1}, \nabla S_h) + \kappa(\nabla \tau, \nabla S_h)$$
$$= (f_2^{n+1}, S_h) + (\partial^i_{\Delta t}(\theta^{n+1}) - \theta_t^{n+1}, S_h) \;\; \forall S_h \in W_{\Gamma_D,h}. \quad (4.136)$$

Subtract (4.40) from (4.134), add and subtract $b(u^{n+1} - \mathscr{E}^i(u^{n+1}), u_h^{n+1}, v_h)$, $b(\mathscr{E}^i(u'^{n+1}_h), u^{n+1} - \mathscr{E}^i(u^{n+1}), v_h)$, $\nu'(\nabla \mathscr{E}^i(u^{n+1}), \nabla v_h)$, and $(\Lambda' \times \mathscr{E}^i(u^{n+1}), v_h)$, and rearrange. Then, the error equation for velocity is

$$(\partial^i_{\Delta t}(e_u^{n+1}), v_h) + b(u^{n+1}, e_u^{n+1}, v_h) + b(\mathscr{E}^i(e_u^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i(u'^{n+1}_h), e_u^{n+1} - \mathscr{E}^i(e_u^{n+1}), v_h)$$
$$+ \langle \nu \rangle (\nabla e_u^{n+1}, \nabla v_h) + \nu'(\nabla \mathscr{E}^i(e_u^{n+1}), \nabla v_h) + (\langle \Lambda \rangle \times e_u^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(e_u^{n+1}), v_h)$$
$$- (e_p^{n+1}, \nabla \cdot v_h) = \varsigma_u(u^{n+1}, v_h) \;\; \forall v_h \in X_h. \quad (4.137)$$

Similarly, the error equation for temperature follows by subtracting (4.42) from (4.136), adding and subtracting $b^*(u^{n+1} - \mathscr{E}^i(u^{n+1}), \theta_h^{n+1}, S_h)$, $b^*(\mathscr{E}^i(u'^{n+1}_h), \theta^{n+1} - \mathscr{E}^i(\theta^{n+1}), S_h)$, $b^*(\mathscr{E}^i(u^{n+1}), \tau - I_h\tau, S_h)$, and $\kappa'(\nabla \mathscr{E}^i(\theta^{n+1}), \nabla S_h)$, and rearranging. Then,

$$(\partial^i_{\Delta t}(e_\theta^{n+1}), S_h) + b^*(u^{n+1}, e_\theta^{n+1}, S_h) + b^*(\mathscr{E}^i(e_u^{n+1}), \theta_h^{n+1}, S_h) + b^*(\mathscr{E}^i(u'^{n+1}_h), e_\theta^{n+1} - \mathscr{E}^i(e_\theta^{n+1}), S_h)$$
$$+ b^*(\mathscr{E}^i(u^{n+1}), \tau - I_h\tau, S_h) + b^*(\mathscr{E}^i(e_u^{n+1}), I_h\tau, S_h) + \langle \kappa \rangle (\nabla e_\theta^{n+1}, \nabla S_h)$$
$$+ \kappa(\nabla(\tau - I_h\tau), \nabla S_h) + \kappa'(\nabla \mathscr{E}^i(e_\theta^{n+1}), \nabla S_h) = \varsigma_T(\theta^{n+1}, S_h) \;\; \forall S_h \in W_{\Gamma_D,h}. \quad (4.138)$$

Use the substitutions (4.112) - (4.114) in equations (4.137) and (4.138). Then,

$$(\partial^i_{\Delta t}(\phi_h^{n+1}), v_h) + b(u^{n+1}, \phi_h^{n+1}, v_h) + \langle \nu \rangle (\nabla \phi_h^{n+1}, \nabla v_h) = (\partial^i_{\Delta t}(\eta^{n+1}), v_h) + b(u^{n+1}, \eta^{n+1}, v_h)$$
$$+ b(\mathscr{E}^i(\eta^{n+1}), u_h^{n+1}, v_h) - b(\mathscr{E}^i(\phi_h^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i(u'^{n+1}_h), \eta^{n+1} - \mathscr{E}^i(\eta^{n+1}), v_h)$$
$$- b(\mathscr{E}^i(u'^{n+1}_h), \phi_h^{n+1} - \mathscr{E}^i(\phi_h^{n+1}), v_h) + \langle \nu \rangle (\nabla \eta^{n+1}, \nabla v_h) + \nu'(\nabla \mathscr{E}^i(\eta^{n+1}), \nabla v_h)$$

$$
- \nu'(\nabla \mathscr{E}^i(\phi_h^{n+1}), \nabla v_h) + (\langle \Lambda \rangle \times \eta^{n+1}, v_h) + (\Lambda' \times \eta^n, v_h) - (\Lambda' \times \phi_h^n, v_h) - (e_p^{n+1}, \nabla \cdot v_h)
+ (\beta g \mathscr{E}^i(\zeta^{n+1}), v_h) - (\beta g \mathscr{E}^i(\psi_h^{n+1}), v_h) + (\beta g(\tau - I_h\tau), v_h) - \varsigma_u(u^{n+1}, v_h) \ \ \forall v_h \in X_h
$$

and

$$
(\partial_{\Delta t}^i(\psi_h^{n+1}), S_h) + b(u^{n+1}, \psi_h^{n+1}, S_h) + \langle \kappa \rangle (\nabla \psi_h^{n+1}, \nabla S_h) = (\partial_{\Delta t}^i(\zeta^{n+1}), S_h) + b^*(u^{n+1}, \zeta^{n+1}, S_h)
+ b^*(\mathscr{E}^i(\eta^{n+1}), \theta_h^{n+1}, S_h) - b^*(\mathscr{E}^i(\phi_h^{n+1}), \theta_h^{n+1}, S_h) + b^*(\mathscr{E}^i(u'^{n+1}_h), \zeta^{n+1} - \mathscr{E}^i(\zeta^{n+1}), S_h)
- b^*(\mathscr{E}^i(u'^{n+1}_h), \psi_h^{n+1} - \mathscr{E}^i(\psi_h^{n+1}), S_h) + b^*(\mathscr{E}^i(u^{n+1}), \tau - I_h\tau, S_h) + b^*(\mathscr{E}^i(\eta^{n+1}), I_h\tau, S_h)
+ b^*(\mathscr{E}^i(\phi_h^{n+1}), I_h\tau, S_h) + \kappa(\nabla(\tau - I_h\tau), \nabla S_h) + \langle \kappa \rangle (\nabla \zeta^{n+1}, \nabla S_h) + \kappa'(\nabla \mathscr{E}^i(\zeta^{n+1}), \nabla S_h)
- \kappa'(\nabla \mathscr{E}^i(\psi_h^{n+1}), \nabla S_h) - \varsigma_\theta(\theta^{n+1}, S_h) \ \ \forall S_h \in W_{\Gamma_D,h}.
$$

Letting $v_h = \Delta t \phi_h^{n+1} \in X_h$ and $S_h = \Delta t \psi_h^{n+1} \in W_{\Gamma_D,h}$ yields

$$
(\partial_{\Delta t}^i(\phi_h^{n+1}), \Delta t \phi_h^{n+1}) + \Delta t \|\langle \nu \rangle^{1/2} \Delta t \phi_h^{n+1}\|^2 = (\partial_{\Delta t}^i(\eta^{n+1}), \Delta t \phi_h^{n+1}) + \Delta t b(u^{n+1}, \eta^{n+1}, \phi_h^{n+1})
+ \Delta t b(\mathscr{E}^i(\eta^{n+1}), u_h^{n+1}, \phi_h^{n+1}) - \Delta t b(\mathscr{E}^i(\phi_h^{n+1}), u_h^{n+1}, \phi_h^{n+1}) + \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \eta^{n+1} - \mathscr{E}^i(\eta^{n+1}), \phi_h^{n+1})
- \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \phi_h^{n+1} - \mathscr{E}^i(\phi_h^{n+1}), \phi_h^{n+1}) + \langle \nu \rangle \Delta t (\nabla \eta^{n+1}, \nabla \phi_h^{n+1}) + \nu' \Delta t (\nabla \mathscr{E}^i(\eta^{n+1}), \nabla \phi_h^{n+1})
- \nu' \Delta t (\nabla \mathscr{E}^i(\phi_h^{n+1}), \nabla \phi_h^{n+1}) + \Delta t (\Lambda \times \eta^n, \phi_h^{n+1}) - \Delta t (\Lambda \times \phi_h^n, \phi_h^{n+1}) - \Delta t (\lambda^{n+1}, \nabla \cdot \phi_h^{n+1})
+ \Delta t (\beta g \mathscr{E}^i(\zeta^{n+1}), \phi_h^{n+1}) - \Delta t (\beta g \mathscr{E}^i(\psi_h^{n+1}), \phi_h^{n+1})
+ \Delta t (\beta g(\tau - I_h\tau), \phi_h^{n+1}) - \Delta t \varsigma_u(u^{n+1}, \phi_h^{n+1}) \quad (4.139)
$$

and

$$
(\partial_{\Delta t}^i(\psi_h^{n+1}), \Delta t \psi_h^{n+1}) + \Delta t \|\langle \kappa \rangle^{1/2} \nabla \psi_h^{n+1}\|^2 = (\partial_{\Delta t}^i(\zeta^{n+1}), \Delta t \psi_h^{n+1}) + \Delta t b^*(u^{n+1}, \zeta^{n+1}, \psi_h^{n+1})
+ \Delta t b^*(\mathscr{E}^i(\eta^{n+1}), \theta_h^{n+1}, \psi_h^{n+1}) - \Delta t b^*(\mathscr{E}^i(\phi_h^{n+1}), \theta_h^{n+1}, \psi_h^{n+1})
+ \Delta t b^*(\mathscr{E}^i(u'^{n+1}_h), \zeta^{n+1} - \mathscr{E}^i(\zeta^{n+1}), \psi_h^{n+1}) - \Delta t b^*(\mathscr{E}^i(u'^{n+1}_h), \psi_h^{n+1} - \mathscr{E}^i(\psi_h^{n+1}), \psi_h^{n+1})
+ \Delta t b^*(\mathscr{E}^i(u^{n+1}), \tau - I_h\tau, \psi_h^{n+1}) + \Delta t b^*(\mathscr{E}^i(\eta^{n+1}), I_h\tau, \psi_h^{n+1})
- \Delta t b^*(\mathscr{E}^i(\phi_h^{n+1}), I_h\tau, \psi_h^{n+1}) + \kappa \Delta t (\nabla(\tau - I_h\tau), \nabla \psi_h^{n+1}) + \langle \kappa \rangle \Delta t (\nabla \zeta^{n+1}, \nabla \psi_h^{n+1})
+ \kappa' \Delta t (\nabla \mathscr{E}^i(\zeta^{n+1}), \nabla \psi_h^{n+1}) - \kappa' \Delta t (\nabla \mathscr{E}^i(\psi_h^{n+1}), \nabla \psi_h^{n+1}) - \Delta t \varsigma_\theta(\theta^{n+1}, \psi_h^{n+1}). \quad (4.140)
$$

We seek to now estimate all terms on the right-hand sides in such a way that we may subsume the terms involving unknown pieces ψ_h^k and ϕ_h^k into the left-hand sides. Consider equation

(4.139) and let $i=1$ (first-order). The following estimates are formed using skew-symmetry, Lemma 1, and the Cauchy-Schwarz-Young inequality,

$$\begin{aligned}
\Delta t b(u^{n+1},\eta^{n+1},\phi_h^{n+1}) &= \Delta t b^*(\langle\nu\rangle^{-1/2}u^{n+1},\eta^{n+1},\langle\nu\rangle^{1/2}\phi_h^{n+1}) \\
&\leq C_1\Delta t\|\langle\nu\rangle^{-1/2}\nabla u^{n+1}\|\|\nabla\eta^{n+1}\|\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\| \\
&\leq \frac{C_rC_1^2\Delta t}{\nu_{min}\sigma_1}\|\nabla u^{n+1}\|^2\|\|\nabla\eta^{n+1}\|^2+\frac{\sigma_1\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \qquad (4.141)\\
\Delta t b(\eta^n,u_h^{n+1},\phi_h^{n+1}) &\leq \frac{C_rC_1^2}{\nu_{min}^2\sigma_2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\|\nabla\eta^n\|^2+\frac{\sigma_2}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \qquad (4.142)
\end{aligned}$$

Applying Lemma 1, the Cauchy-Schwarz-Young inequality, Taylor's theorem, and condition (4.52) yields,

$$\begin{aligned}
&-\Delta t b(u'^n_h,\eta^{n+1}-\eta^n,\phi_h^{n+1}) \leq C_1\|\langle\nu\rangle^{-1/2}\nabla u'^n_h\|\|\nabla(\eta^{n+1}-\eta^n)\|\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\| \\
&\quad\leq \frac{C_rC_1^2\Delta t^2}{\sigma_4}\|\langle\nu\rangle^{-1/2}\nabla u'^n_h\|^2\|\nabla\eta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}+\frac{\sigma_4\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 \\
&\quad\leq \frac{C_rC_1^2h\Delta t}{C_\dagger\sigma_4}\|\nabla\eta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}+\frac{\sigma_4\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \qquad (4.143)
\end{aligned}$$

Apply the triangle inequality, Lemma 1, and the Cauchy-Schwarz-Young inequality twice. This yields

$$\begin{aligned}
&-\Delta t b(\phi_h^n,u_h^{n+1},\phi_h^{n+1}) \leq C_4\Delta t\|\nabla u_h^{n+1}\|\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|\sqrt{\|\langle\nu\rangle^{-3/2}\phi_h^n\|\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|} \\
&\quad\leq \sigma_3\Delta t\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2+\frac{C_4^2\Delta t}{4\sigma_3}\|\nabla u_h^{n+1}\|^2\|\langle\nu\rangle^{-3/2}\phi_h^n\|\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\| \\
&\quad\leq \sigma_3\Delta t\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2+\frac{C_4^2\Delta t}{8\delta_3\sigma_3\nu_{min}^3}\|\nabla u_h^{n+1}\|^2\|\phi_h^n\|^2 \\
&\qquad+\frac{C_4^2\delta_3\Delta t}{8\sigma_3}\|\nabla u_h^{n+1}\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2. \qquad (4.144)
\end{aligned}$$

Use Lemma 7 and the Cauchy-Schwarz-Young inequality,

$$-\Delta t b(u'^n_h,\phi_h^{n+1}-\phi_h^n,\phi_h^n) \leq \frac{2C_\star^2\Delta t^2}{h}\|\langle\nu\rangle^{-1/2}\nabla u'^n_h\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2+\frac{1}{4}\|\phi_h^{n+1}-\phi_h^n\|^2.$$

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality, and Taylor's theorem yield

$$(\eta^{n+1}-\eta^n,\phi_h^{n+1}) \leq \frac{C_P^2C_r}{\nu_{min}\sigma_0}\|\eta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}+\frac{\sigma_0\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \qquad (4.145)$$

Use the Cauchy-Schwarz-Young inequality,

$$\langle\nu\rangle\Delta t(\nabla\eta^{n+1},\nabla\phi_h^{n+1}) \leq \frac{C_r\nu_{max}\Delta t}{\sigma_6}\|\nabla\eta^{n+1}\|^2 + \frac{\sigma_6\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.146}$$

$$\nu'\Delta t(\nabla\eta^{n},\nabla\phi_h^{n+1}) \leq \frac{C_r\nu_{max}\Delta t}{\sigma_7}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2\|\nabla\eta^{n}\|^2 + \frac{\sigma_7\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.147}$$

$$-\nu'\Delta t(\nabla\phi_h^{n},\nabla\phi_h^{n+1}) \leq \frac{\Delta t}{2}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n}\|^2 + \frac{\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \tag{4.148}$$

The following estimates are formed using skew-symmetry, Lemma 1, and the Cauchy-Schwarz-Young inequality,

$$\Delta t(\langle\Lambda\rangle\times\eta^{n+1},\phi_h^{n+1}) \leq \frac{|\langle\Lambda\rangle|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_9}\|\eta^{n+1}\|^2 + \frac{\sigma_9\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.149}$$

$$\Delta t(\Lambda'\times\eta^{n},\phi_h^{n+1}) \leq \frac{|\Lambda'|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_{10}}\|\eta^{n}\|^2 + \frac{\sigma_{10}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.150}$$

$$-\Delta t(\Lambda'\times\phi_h^{n},\phi_h^{n+1}) \leq \frac{|\Lambda'|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_{11}}\|\phi_h^{n}\|^2 + \frac{\sigma_{11}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.151}$$

$$-\Delta t(\lambda^{n+1},\nabla\cdot\phi_h^{n+1}) \leq \frac{dC_r\Delta t}{\nu_{min}\sigma_{12}}\|\lambda^{n+1}\|^2 + \frac{\sigma_{12}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.152}$$

$$-\Delta t(\beta g\eta^{n},\phi_h^{n+1}) \leq \frac{|\beta g|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_{13}}\|\eta^{n}\|^2 + \frac{\sigma_{13}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.153}$$

$$\Delta t(\beta g\psi_h^{n},\phi_h^{n+1}) \leq \frac{|\beta g|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_{14}}\|\psi_h^{n}\|^2 + \frac{\sigma_{14}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \tag{4.154}$$

$$-\Delta t(\beta g(\tau - I_h\tau),\phi_h^{n+1}) \leq \frac{|\beta g|^2C_P^2C_r\Delta t}{\nu_{min}\sigma_{15}}\|\tau - I_h\tau\|^2 + \frac{\sigma_{15}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \tag{4.155}$$

Similarly, for the temperature equation, the following estimates hold

$$\Delta t b^*(u^{n+1},\zeta^{n+1},\psi_h^{n+1}) \leq \frac{C_rC_4^2\Delta t}{\kappa_{min}\sigma_{18}}\|\nabla u^{n+1}\|^2\|\|\nabla\zeta^{n+1}\|^2 + \frac{\sigma_{18}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2, \tag{4.156}$$

$$\Delta t b^*(\eta^{n},\theta_h^{n+1},\psi_h^{n+1}) \leq \frac{C_rC_4^2}{\kappa_{min}^2\sigma_{19}}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\|\nabla\eta^{n}\|^2 + \frac{\sigma_{19}}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \tag{4.157}$$

Applying Lemma 1, the Cauchy-Schwarz-Young inequality, Taylor's theorem, and condition (4.52) yields,

$$\begin{aligned}-\Delta t b^*(u'^{n}_h,\zeta^{n+1}-\zeta^{n},\psi_h^{n+1}) \leq{}& \frac{C_rC_4^2h\Delta t}{C_\dagger\sigma_{21}}\|\nabla\zeta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\\ &+ \frac{\sigma_{21}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2.\end{aligned} \tag{4.158}$$

Apply the triangle inequality, Lemma 1, and the Cauchy-Schwarz-Young inequality twice. This yields

$$- \Delta t b^*(\phi_h^n, \theta_h^{n+1}, \psi_h^{n+1}) \leq \sigma_{20} \Delta t \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 \Delta t}{8\delta_{20}\sigma_{20}\kappa_{min}^2 \nu_{min}} \|\nabla \theta_h^{n+1}\|^2 \|\phi_h^n\|^2 + \frac{C_4^2 \delta_{20} \Delta t}{8\sigma_{20}} \|\nabla \theta_h^{n+1}\|^2 \|\langle\nu\rangle^{1/2} \nabla \phi_h^n\|^2 \quad (4.159)$$

and

$$- \Delta t b^*(\phi_h^n, I_h\tau, \psi_h^{n+1}) \leq \sigma_{25} \Delta t \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2 + \frac{C_4^2 C_I^2 \Delta t}{2\kappa_{min}^2 \nu_{min} \delta_{25}\sigma_{25}} \|\tau\|_1^2 \|\phi_h^n\|^2 + \frac{C_4^2 C_I^2 \delta_{25} \Delta t}{2\sigma_{25}} \|\tau\|_1^2 \|\langle\nu\rangle^{1/2} \nabla \phi_h^n\|^2. \quad (4.160)$$

Use Lemma 7 and the Cauchy-Schwarz-Young inequality. Then,

$$- \Delta t b^*({u'}_h^n, \psi_h^{n+1} - \psi_h^n, \psi_h^{n+1}) \leq \frac{2C_{\star\star}^2 \Delta t^2}{h} \|\langle\kappa\rangle^{-1/2} \nabla {u'}_h^n\|^2 \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2 + \frac{1}{4} \|\psi_h^{n+1} - \psi_h^n\|^2. \quad (4.161)$$

Use Lemma 1 and the Cauchy-Schwarz-Young inequality on both terms. Also, use interpolant estimates on the second, then

$$\Delta t b^*(u^n, \tau - I_h\tau, \psi_h^{n+1}) \leq \frac{C_r C_4^2 \Delta t}{\kappa_{min}\sigma_{23}} \|\nabla u^n\|^2 \|\nabla(\tau - I_h\tau)\|^2 + \frac{\sigma_{23}\Delta t}{r} \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2, \quad (4.162)$$

$$\Delta t b^*(\eta^n, I_h\tau, \psi_h^{n+1}) \leq \frac{C_r C_4^2 C_I^2 \Delta t}{\kappa_{min}\sigma_{24}} \|\tau\|_1^2 \|\nabla \eta^n\|^2 + \frac{\sigma_{24}\Delta t}{r} \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2. \quad (4.163)$$

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality and Taylor's theorem yield

$$(\zeta^{n+1} - \zeta^n, \psi_h^{n+1}) \leq \frac{C_P^2 C_r}{\kappa_{min}\sigma_{17}} \|\zeta_t\|_{L^2(t^n, t^{n+1}; L^2(\Omega))}^2 + \frac{\sigma_{17}\Delta t}{r} \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2. \quad (4.164)$$

Lastly, use the Cauchy-Schwarz-Young inequality,

$$\Delta t \kappa (\nabla(\tau - I_h\tau), \nabla \psi_h^{n+1}) \leq \frac{C_r(1+\kappa_{max})\Delta t}{\sigma_{26}} \left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2 \|\nabla(\tau - I_h\tau)\|^2 + \frac{\sigma_{26}\Delta t}{r} \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2, \quad (4.165)$$

$$\langle\kappa\rangle \Delta t (\nabla \zeta^{n+1}, \nabla \psi_h^{n+1}) \leq \frac{C_r \kappa_{max} \Delta t}{\sigma_{27}} \|\nabla \zeta^{n+1}\|^2 + \frac{\sigma_{27}\Delta t}{r} \|\langle\kappa\rangle^{1/2} \nabla \psi_h^{n+1}\|^2, \quad (4.166)$$

$$\kappa'\Delta t(\nabla\zeta^n, \nabla\psi_h^{n+1}) \leq \frac{C_r\kappa_{max}\Delta t}{\sigma_{28}}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\|\nabla\zeta^n\|^2 + \frac{\sigma_{28}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2, \quad (4.167)$$

$$-\kappa'\Delta t(\nabla\psi_h^n, \nabla\psi_h^{n+1}) \leq \frac{\Delta t}{2}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\|\langle\kappa\rangle^{1/2}\nabla\psi_h^n\|^2 + \frac{\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \quad (4.168)$$

Add equations (4.139) and (4.140) together and use all of the above estimates. For $0 \leq k \leq 16$, let $r = 80$ and $\sigma_k = 1$ and, for $17 \leq k \leq 29$, let $r = 96$ and $\sigma_k = 1$. Letting $C_\# = C\nu_{min}^{-1}\max\{\nu_{min}^{-2}, |\beta g|^2, |\Lambda'|^2, \kappa_{min}^{-2}\}$ and reorganizing yields

$$\begin{aligned}
&\frac{1}{2}\Big(\|\phi_h^{n+1}\|^2 - \|\phi_h^n\|^2 + \|\psi_h^{n+1}\|^2 - \|\psi_h^n\|^2\Big) + \frac{1}{4}\Big(\|\phi_h^{n+1} - \phi_h^n\|^2 + \|\psi_h^{n+1} - \psi_h^n\|^2\Big) \\
&\quad + \frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2\Big) \\
&\quad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 - \|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 - \|\langle\kappa\rangle^{1/2}\nabla\psi_h^n\|^2\Big) \\
&\leq C_\#\Delta t\Big(\|\phi_h^n\|^2 + \|\psi_h^n\|^2\Big) \\
&\quad + C\Delta t\Bigg(\nu_{min}^{-1}\|\nabla u^{n+1}\|^2\|\|\nabla\eta^{n+1}\|^2 + \kappa_{min}^{-1}\|\nabla u^{n+1}\|^2\|\|\nabla\zeta^{n+1}\|^2 \\
&\quad + \nu_{min}^{-2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\|\nabla\eta^n\|^2 + \kappa_{min}^{-2}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\|\nabla\zeta^n\|^2 + hC_\dagger^{-1}\Big(\|\nabla\zeta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\quad + \|\nabla\eta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big) + \nu_{min}^{-1}\Delta t^{-1}\|\eta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + \kappa_{min}^{-1}\Delta t^{-1}\|\zeta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} \\
&\quad + \kappa_{min}^{-1}\|\nabla u^n\|^2\|\nabla(\tau - I_h\tau)\|^2 + \kappa_{min}^{-1}\|\nabla\eta^n\|^2 + \nu_{max}\|\nabla\eta^{n+1}\|^2 \\
&\quad + \kappa_{max}\|\nabla\zeta^{n+1}\|^2 + (1+\kappa_{max})C_{\dagger\dagger}\|\nabla(\tau - I_h\tau)\|^2 + C_{\dagger\dagger}\nu_{max}\|\nabla\eta^n\|^2 + C_{\dagger\dagger}\kappa_{max}\|\nabla\zeta^n\|^2 \\
&\quad + |\langle\Lambda\rangle|^2\nu_{min}^{-1}\|\eta^{n+1}\|^2 + (|\Lambda'|^2 + |\beta g|^2)\nu_{min}^{-1}\|\eta^n\|^2 + \nu_{min}^{-1}\|\lambda^{n+1}\|^2 + |\beta g|^2\nu_{min}^{-1}\|\tau - I_h\tau\|^2 \\
&\quad + \nu_{min}^{-1}\Delta t\Big(\|u_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\Lambda'|^2\|u_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\beta g|^2\|\theta_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big) \\
&\quad + \kappa_{min}^{-1}\Delta t\|\theta_{tt}\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \Delta t\Big(\nu_{min}^{-1}\|\nabla u_h^{n+1}\|^2 + C_\dagger h\Delta t^{-1} + |\nu'|^2\Big)\|\nabla u_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} \\
&\quad + \Delta t\Big(\kappa_{min}^{-1}\|\nabla\theta_h^{n+1}\|^2 + C_\dagger h\Delta t^{-1} + \kappa_{min}^{-1}\|\tau\|_1^2 + |\kappa'|^2\Big)\|\nabla u_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\Bigg). \quad (4.169)
\end{aligned}$$

Sum over n from $n = 0$ to $n = N - 1$ and apply Lemma 3. Then,

$$\begin{aligned}
&\frac{1}{2}\Big(\|\phi_h^N\|^2 + \|\psi_h^N\|^2\Big) + \frac{1}{4}\sum_{n=0}^{N-1}\Big(\|\phi_h^{n+1} - \phi_h^n\|^2 + \|\psi_h^{n+1} - \psi_h^n\|^2\Big) \\
&\quad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^N\|^2\Big) \leq C\exp(C_\# t^*)\Bigg(\nu_{min}^{-1}|||\nabla u|||_{2,0}^2|||\nabla\eta|||_{\infty,0}^2 \\
&+ \kappa_{min}^{-1}|||\nabla u|||_{2,0}^2|||\nabla\zeta|||_{\infty,0}^2 + \nu_{min}^{-2}\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)|||\nabla\eta|||_{\infty,0}^2 \\
&+ \kappa_{min}^{-2}\Big(\Delta t\sum_{n=0}^{N-1}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big)|||\nabla\zeta|||_{\infty,0}^2 + h\Delta t C_\dagger^{-1}\Big(\|\nabla\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + \|\nabla\zeta_t\|_{L^2(0,t^*;L^2(\Omega))}^2\Big) \\
&+ \nu_{min}^{-1}\|\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + \kappa_{min}^{-1}\|\zeta_t\|_{L^2(0,t^*;L^2(\Omega))}^2 + \kappa_{min}^{-1}|||\nabla u|||_{2,0}^2\|\nabla(\tau - I_h\tau)\|^2 + \kappa_{min}^{-1}|||\nabla\eta|||_{2,0}^2 \\
&+ \nu_{max}|||\nabla\eta|||_{2,0}^2 + \kappa_{max}|||\nabla\zeta|||_{2,0}^2 + (1+\kappa_{max})C_{\dagger\dagger}t^*\|\nabla(\tau - I_h\tau)\|^2 + C_{\dagger\dagger}\nu_{max}|||\nabla\eta|||_{2,0}^2 \\
&+ C_{\dagger\dagger}\kappa_{max}|||\nabla\zeta|||_{2,0}^2 + (|\langle\Lambda\rangle|^2 + |\Lambda'|^2 + |\beta g|^2)\nu_{min}^{-1}|||\nabla\eta|||_{2,0}^2 + \nu_{min}^{-1}|||\lambda|||_{2,0}^2 \\
&+ |\beta g|^2\nu_{min}^{-1}t^*\|\tau - I_h\tau\|^2 + \nu_{min}^{-1}\Delta t^2\Big(\|u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + |\Lambda'|^2\|u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 \\
&+ |\beta g|^2\|\theta_t\|_{L^2(0,t^*;L^2(\Omega))}^2\Big) + \kappa_{min}^{-1}\Delta t^2\|\theta_{tt}\|_{L^2(0,t^*;L^2(\Omega))}^2 \\
&+ \Big(\nu_{min}^{-1}|||\nabla u_h|||_{\infty,0}^2 + C_\dagger h\Delta t^{-1} + |\nu'|^2\Big)\Delta t^2\|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 \\
&+ \Big(\kappa_{min}^{-1}|||\nabla\theta_h|||_{\infty,0}^2 + C_\dagger h\Delta t^{-1} + \kappa_{min}^{-1}\|\tau\|_1^2 + |\kappa'|^2\Big)\Delta t^2\|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 \\
&+ \frac{1}{2}\Big(\|\phi_h^0\|^2 + \|\psi_h^0\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^0\|^2\Big)\Bigg). \quad (4.170)
\end{aligned}
$$

Apply the triangle inequality and the identity $\theta^n = T^n - \tau$, take infimums over V_h, W_h, and Q_h, apply Lemma 4, and collect constants. The result follows. □

We move now to the second-order algorithm, $i = 2$.

Theorem 14. *Consider second-order* ***eBDF****. Suppose that the hypotheses of Theorem 13 and that $(u_h^1, p_h^1, T_h^1) \in (X_h, Q_h, W_h)$ are approximations of (u^1, p^1, T^1) to within the accuracy of the interpolant. Then, there exists constants C, $C_\# > 0$ such that*

$$
\begin{aligned}
&\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2}\|e_T^N\|^2 + \frac{1}{2}\|2e_T^N - e_T^{N-1}\|^2 \\
&\quad + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2 + \|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2\Big) \\
&+ \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2\Big) \\
&\quad + \frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{N-1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{N-1}\|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\exp(C_{\#}t^*)\Big\{\inf_{S_h\in W_h}\Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1}|||\nabla(T-S_h)|||_{\infty,0}^2+\kappa_{max}|||\nabla(T-S_h)|||_{2,0}^2\\
&\qquad+\kappa_{min}^{-1}\|(T-S_h)_t\|_{L^2(0,t^*;L^2(\Omega))}^2+h\Delta t^3\|\nabla(T-S_h)_{tt}\|_{L^2(0,t^*;L^2(\Omega))}^2\Big)\\
&+\inf_{v_h\in X_h}\Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u-v_h)|||_{\infty,0}^2+\big(\kappa_{min}^{-1}+\nu_{max}+(|\langle\Lambda\rangle|^2+|\Lambda'|^2)+|\beta g|^2)\big)|||\nabla(u-v_h)|||_{2,0}^2\\
&\qquad+\nu_{min}^{-1}\|(u-v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2+h\Delta t^3\|\nabla(u-v_h)_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\Big)\\
&+\inf_{q_h\in Q_h}\nu_{min}^{-1}|||p-q_h|||_{2,0}^2+t^*\inf_{S_h\in W_h}\Big(|\beta g|^2\nu_{min}^{-1}\|\tau-S_h\|^2+(1+\kappa_{min}^{-1}+\kappa_{max})\|\nabla(\tau-S_h)\|^2\Big)\\
&\qquad+h\Delta t^3+\big(\nu_{min}^{-1}(1+|\Lambda'|^2+|\beta g|^2)+|\kappa'|^2+|\nu'|^2\big)\Delta t^4\Big\}\\
&\qquad+\|e_u^1\|^2+\|2e_u^1-e_u^0\|^2+\|e_T^1\|^2+\|2e_T^1-e_T^0\|^2\\
&\qquad+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^1\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^1\|^2\Big)+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2\Big).
\end{aligned}
$$

Proof. The following estimates are formed using skew-symmetry, Lemma 1, and the Cauchy-Schwarz-Young inequality,

$$
\Delta tb(2\eta^n-\eta^{n-1},u_h^{n+1},\phi_h^{n+1})\leq\frac{8C_rC_1^2}{\nu_{min}^2\sigma_2}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big(\|\nabla\eta^n\|^2+\|\nabla\eta^{n-1}\|^2\Big)+\frac{\sigma_2}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \quad (4.171)
$$

Applying Lemma 1, the Cauchy-Schwarz-Young inequality, Taylor's theorem, and condition (4.52) yields,

$$
-\Delta tb(\mathscr{E}^2(u_h'^{n+1}),\eta^{n+1}-2\eta^n+\eta^{n-1},\phi_h^{n+1})\leq\frac{C_rC_1^2h\Delta t}{C_\dagger\sigma_4}\|\nabla\eta_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)}^2+\frac{\sigma_4\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2.
$$

Apply the triangle inequality, Lemma 1 and the Cauchy-Schwarz-Young inequality twice. This yields

$$
\begin{aligned}
-\Delta tb(2\phi_h^n-\phi_h^{n-1},u_h^{n+1},\phi_h^{n+1})&\leq\sigma_{3,1}\Delta t\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2\\
&+\frac{C_4^2\Delta t}{2\delta_{3,1}\sigma_{3,1}\nu_{min}^3}\|\nabla u_h^{n+1}\|^2\|\phi_h^n\|^2+\frac{C_4^2\delta_{3,1}\Delta t}{2\sigma_{3,1}}\|\nabla u_h^{n+1}\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2\\
&+\sigma_{3,2}\Delta t\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2+\frac{C_4^2\Delta t}{8\delta_{3,2}\sigma_{3,2}\nu_{min}^3}\|\nabla u_h^{n+1}\|^2\|\phi_h^{n-1}\|^2\\
&+\frac{C_4^2\delta_{3,2}\Delta t}{8\sigma_{3,2}}\|\nabla u_h^{n+1}\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n-1}\|^2. \quad (4.172)
\end{aligned}
$$

Using Lemma 7 and the Cauchy-Schwarz-Young inequality yields

$$-\Delta t b(\mathscr{E}^2(u_h'^{n+1}), \phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}, \phi_h^n) \leq \frac{4C_\star^2 \Delta t^2}{h} \|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^2(u_h'^{n+1})\|^2 \|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 + \frac{1}{8}\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2. \quad (4.173)$$

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality and Taylor's theorem yield

$$(\frac{3\eta^{n+1} - 4\eta^n + \eta^{n-1}}{2}, \phi_h^{n+1}) \leq \frac{C_P^2 C_r}{\nu_{min}\sigma_0}\|\eta_t\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)} + \frac{\sigma_0 \Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \quad (4.174)$$

Use the Cauchy-Schwarz-Young inequality,

$$\langle\nu\rangle\Delta t(\nabla\eta^{n+1}, \nabla\phi_h^{n+1}) \leq \frac{C_r \nu_{max}\Delta t}{\sigma_6}\|\nabla\eta^{n+1}\|^2 + \frac{\sigma_6\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.175)$$

$$\nu'\Delta t(\nabla(2\eta^n - \eta^{n-1}), \nabla\phi_h^{n+1}) \leq \frac{8C_r\nu_{max}\Delta t}{\sigma_7}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2 \Big(\|\nabla\eta^n\|^2 + \|\nabla\eta^{n-1}\|^2\Big) + \frac{\sigma_7\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.176)$$

$$-\nu'\Delta t(\nabla(2\phi_h^n - \phi_h^{n-1}), \nabla\phi_h^{n+1}) \leq \Delta t\left|\frac{\nu'}{\langle\nu\rangle}\right|^2 \|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2 + \frac{\Delta t}{2}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2 \|\langle\nu\rangle^{1/2}\nabla\phi_h^{n-1}\|^2 + \frac{3\Delta t}{2}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \quad (4.177)$$

The following estimates are formed using skew-symmetry, Lemma 1, and the Cauchy-Schwarz-Young inequality,

$$\Delta t(\langle\Lambda\rangle \times \eta^{n+1}, \phi_h^{n+1}) \leq \frac{|\langle\Lambda\rangle|^2 C_P^2 C_r \Delta t}{\nu_{min}\sigma_9}\|\eta^{n+1}\|^2 + \frac{\sigma_9\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.178)$$

$$\Delta t(\Lambda' \times (2\eta^n - \eta^{n-1}), \phi_h^{n+1}) \leq \frac{8|\Lambda'|^2 C_P^2 C_r \Delta t}{\nu_{min}\sigma_{10}}\Big(\|\eta^n\|^2 + \|\eta^{n-1}\|^2\Big) + \frac{\sigma_{10}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.179)$$

$$-\Delta t(\Lambda' \times (2\phi_h^n - \phi_h^{n-1}), \phi_h^{n+1}) \leq \frac{|\Lambda'|^2 C_P^2 C_r \Delta t}{\nu_{min}\sigma_{11}}\|2\phi_h^n - \phi_h^{n-1}\|^2 + \frac{\sigma_{11}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.180)$$

$$-\Delta t(\beta g(2\eta^n - \eta^{n-1}), \phi_h^{n+1}) \leq \frac{8|\beta g|^2 C_P^2 C_r \Delta t}{\nu_{min}\sigma_{12}}\Big(\|\eta^n\|^2 + \|\eta^{n-1}\|^2\Big) + \frac{\sigma_{12}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2, \quad (4.181)$$

$$\Delta t(\beta g(2\psi_h^n - \psi_h^{n-1}), \phi_h^{n+1}) \leq \frac{|\beta g|^2 C_P^2 C_r \Delta t}{\nu_{min}\sigma_{13}} \|2\psi_h^n - \psi_h^{n-1}\|^2 + \frac{\sigma_{13}\Delta t}{r}\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2. \tag{4.182}$$

Similarly, for the temperature equation, the following estimates hold

$$\Delta t b^*(2\eta^n - \eta^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) \leq \frac{8C_r C_4^2}{\kappa_{min}^2\sigma_{16}}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big(\|\eta^n\|^2 + \|\eta^{n-1}\|^2\Big) + \frac{\sigma_{16}}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \tag{4.183}$$

Applying Lemma 1, the Cauchy-Schwarz-Young inequality, Taylor's theorem, and condition (4.52) yields,

$$-\Delta t b^*(\mathscr{E}^2(u'^{n+1}_h), \zeta^{n+1} - 2\zeta^n + \zeta^{n-1}, \psi_h^{n+1}) \leq \frac{C_r C_4^2 h\Delta t}{C_\dagger \sigma_{18}}\|\nabla\zeta_{tt}\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \frac{\sigma_{18}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \tag{4.184}$$

Apply the triangle inequality, Lemma 1 and the Cauchy-Schwarz-Young inequality twice. This yields

$$\begin{aligned}
-\Delta t b^*(2\phi_h^n - \phi_h^{n-1}, \theta_h^{n+1}, \psi_h^{n+1}) &\leq \sigma_{17,1}\Delta t\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 \\
&+ \frac{C_4^2\Delta t}{2\delta_{17,1}\sigma_{17,1}\kappa_{min}^2\nu_{min}}\|\nabla\theta_h^{n+1}\|^2\|\phi_h^n\|^2 + \frac{C_4^2\delta_{17,1}\Delta t}{2\sigma_{17,1}}\|\nabla\theta_h^{n+1}\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2 \\
&+ \sigma_{17,2}\Delta t\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 + \frac{C_4^2\Delta t}{8\delta_{17,2}\sigma_{17,2}\kappa_{min}^2\nu_{min}}\|\nabla\theta_h^{n+1}\|^2\|\phi_h^{n-1}\|^2 \\
&+ \frac{C_4^2\delta_{17,2}\Delta t}{8\sigma_{17,2}}\|\nabla\theta_h^{n+1}\|^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n-1}\|^2,
\end{aligned} \tag{4.185}$$

$$\begin{aligned}
-\Delta t b^*(2\phi_h^n - \phi_h^{n-1}, I_h\tau, \psi_h^{n+1}) &\leq \sigma_{22}\Delta t\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 + \frac{C_4^2 C_I^2\Delta t}{2\kappa_{min}^2\nu_{min}\delta_{22}\sigma_{22}}\|\tau\|_1^2\|\phi_h^n\|^2 \\
&+ \frac{C_4^2 C_I^2\delta_{22}\Delta t}{2\sigma_{22}}\|\tau\|_1^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2 + \sigma_{22,2}\Delta t\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 \\
&+ \frac{C_4^2 C_I^2\Delta t}{8\kappa_{min}^2\nu_{min}\delta_{22,2}\sigma_{22,2}}\|\tau\|_1^2\|\phi_h^n\|^2 + \frac{C_4^2 C_I^2\delta_{22,2}\Delta t}{8\sigma_{22,2}}\|\tau\|_1^2\|\langle\nu\rangle^{1/2}\nabla\phi_h^n\|^2.
\end{aligned} \tag{4.186}$$

Use Lemma 7 and the Cauchy-Schwarz-Young inequality. Then,

$$-\Delta t b^*(\mathscr{E}^2(u'^{n+1}_h), \psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}, \psi_h^{n+1}) \leq \frac{4C_{\star\star}\Delta t^2}{h}\|\langle\kappa\rangle^{-1/2}\nabla\mathscr{E}^2(u'^{n+1}_h)\|^2\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2 + \frac{1}{8}\|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2. \quad (4.187)$$

Use the Cauchy-Schwarz-Young inequality on the first term. Apply Lemma 1, interpolant estimates, and Taylor's theorem on the remaining. Then,

$$\Delta t b^*(2u^n - u^{n-1}, \tau - I_h\tau, \psi_h^{n+1}) \leq \frac{8C_r C_4^2 \Delta t}{\kappa_{min}\sigma_{20}}\Big(\|\nabla u^n\|^2 + \|\nabla u^{n-1}\|^2\Big)\|\nabla(\tau - I_h\tau)\|^2 + \frac{\sigma_{20}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2, \quad (4.188)$$

$$\Delta t b^*(2\eta^n - \eta^{n-1}, I_h\tau, \psi_h^{n+1}) \leq \frac{8C_r C_4^2 C_I^2 \Delta t}{\kappa_{min}\sigma_{21}}\|\tau\|_1^2\Big(\|\nabla\eta^n\|^2 + \|\nabla\eta^{n-1}\|^2\Big) + \frac{\sigma_{21}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \quad (4.189)$$

The Cauchy-Schwarz-Young inequality, Poincaré-Friedrichs inequality and Taylor's theorem yield

$$\Big(\frac{3\zeta^{n+1} - 4\zeta^n + \zeta^{n-1}}{2}, \psi_h^{n+1}\Big) \leq \frac{C_P^2 C_r}{\kappa_{min}\sigma_{14}}\|\zeta_t\|^2_{L^2(t^{n-1},t^{n+1};L^2(\Omega))} + \frac{\sigma_{14}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \quad (4.190)$$

Lastly, use the Cauchy-Schwarz-Young inequality,

$$\kappa'\Delta t(\nabla(2\zeta^n - \zeta^{n-1}), \nabla\psi_h^{n+1}) \leq \frac{8C_r\kappa_{max}\Delta t}{\sigma_{25}}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\Big(\|\nabla\zeta^n\|^2 + \|\nabla\zeta^{n-1}\|^2\Big) + \frac{\sigma_{25}\Delta t}{r}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2, \quad (4.191)$$

$$-\kappa'\Delta t(\nabla(2\psi_h^n - \psi_h^{n-1}), \nabla\psi_h^{n+1}) \leq \Delta t\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\|\langle\kappa\rangle^{1/2}\nabla\psi_h^n\|^2 + \frac{\Delta t}{2}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n-1}\|^2 + \frac{3\Delta t}{2}\|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2. \quad (4.192)$$

Add equations (4.139) and (4.140) and use the above estimates. Let for $1 \leq k \leq 14$ let $r = 96$ and $\sigma_k = 1$ and for $15 \leq k \leq 27$ let $r = 80$ and $\sigma_k = 1$. Let $C_\triangle = C\nu_{min}^{-1}\max\{\nu_{min}^{-2}, |\beta g|^2, |\Lambda'|^2, \kappa_{min}^{-2}\}$, sum over n from $n = 1$ to $n = N-1$, apply Lemma 3, and reorganize. Then,

$$\frac{1}{2}\Big(\|\phi_h^N\|^2 + \|2\phi_h^N - \phi_h^{N-1}\|^2 + \|\psi_h^N\|^2 + \|2\psi_h^N - \psi_h^{N-1}\|^2\Big)$$

$$
\begin{aligned}
&+ \frac{1}{4}\sum_{n=1}^{N-1}\Big(\|\phi_h^{n+1} - 2\phi_h^n + \phi_h^{n-1}\|^2 + \|\psi_h^{n+1} - 2\psi_h^n + \psi_h^{n-1}\|^2\Big)\\
&+ \frac{\Delta t}{4}\sum_{n=1}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^{n+1}\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^N\|^2\Big)\\
&+ \frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla\phi_h^{N-1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^{N-1}\|^2\Big) \le C\exp(C_\triangle t^*)\Bigg(\nu_{min}^{-1}|||\nabla u|||_{2,0}^2|||\nabla\eta|||_{\infty,0}^2\\
&+ \kappa_{min}^{-1}|||\nabla u|||_{2,0}^2|||\nabla\zeta|||_{\infty,0}^2 + \nu_{min}^{-2}\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2\Big)|||\nabla\eta|||_{\infty,0}^2\\
&+ \kappa_{min}^{-2}\Big(\Delta t\sum_{n=1}^{N-1}\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2\Big)|||\nabla\zeta|||_{\infty,0}^2\\
&+ h\Delta t C_\dagger^{-1}\Big(\|\nabla\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + \|\nabla\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\Big) + \nu_{min}^{-1}\|\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\\
&+ \kappa_{min}^{-1}\|\zeta_t\|_{L^2(0,t^*;L^2(\Omega))}^2 + \kappa_{min}^{-1}|||\nabla u|||_{2,0}^2\|\nabla(\tau - I_h\tau)\|^2 + \kappa_{min}^{-1}|||\nabla\eta|||_{2,0}^2\\
&+ \nu_{max}|||\nabla\eta|||_{2,0}^2 + \kappa_{max}|||\nabla\zeta|||_{2,0}^2 + (1+\kappa_{max})C_{\dagger\dagger}t^*\|\nabla(\tau - I_h\tau)\|^2 + C_{\dagger\dagger}\nu_{max}|||\nabla\eta|||_{2,0}^2\\
&+ C_{\dagger\dagger}\kappa_{max}|||\nabla\zeta|||_{2,0}^2 + (|\Lambda|^2 + |\beta g|^2)\nu_{min}^{-1}|||\nabla\eta|||_{2,0}^2 + \nu_{min}^{-1}|||\lambda|||_{2,0}^2 + |\beta g|^2\nu_{min}^{-1}t^*\|\tau - I_h\tau\|^2\\
&+ \nu_{min}^{-1}\Delta t^4\Big(\|u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + |\Lambda|^2\|u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + |\beta g|^2\|\theta_t\|_{L^2(0,t^*;L^2(\Omega))}^2\Big)\\
&+ \kappa_{min}^{-1}\Delta t^4\|\theta_{tt}\|_{L^2(0,t^*;L^2(\Omega))}^2 + \Big(\nu_{min}^{-1}|||\nabla u_h|||_{\infty,0}^2 + C_\dagger h\Delta t^{-1} + |\nu'|^2\Big)\Delta t^4\|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\\
&+ \Big(\kappa_{min}^{-1}|||\nabla\theta_h|||_{\infty,0}^2 + C_\dagger h\Delta t^{-1} + \kappa_{min}^{-1}\|\tau\|_1^2 + |\kappa'|^2\Big)\Delta t^4\|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\\
&+ \frac{1}{2}\Big(\|\phi_h^1\|^2 + \|2\phi_h^1 - \phi_h^0\|^2 + \|\psi_h^1\|^2 + \|2\psi_h^1 - \psi_h^0\|^2\Big)\\
&+ \frac{\Delta t}{4}\Big(2\|\langle\nu\rangle^{1/2}\nabla\phi_h^1\|^2 + \|\langle\nu\rangle^{1/2}\nabla\phi_h^0\|^2 + 2\|\langle\kappa\rangle^{1/2}\nabla\psi_h^1\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\psi_h^0\|^2\Big)\Bigg). \qquad (4.193)
\end{aligned}
$$

Apply the triangle inequality and the identity $\theta^n = T^n - \tau$. Taking infimums over V_h, W_h, and Q_h, applying Lemma 4, and collecting constants yields the result. □

As a corollary, the pressure approximation is shown to have the same order of accuracy.

Corollary 4. *Suppose the hypotheses of Theorem 13 hold. Then, the pressure approximation satisfies, for $i = 1$,*

$$
\begin{aligned}
\alpha\Delta t\sum_{n=0}^{N-1}\|e_p^{n+1}\| \le (1 + C_*^{-1})\Bigg(&2\Big(C_1\nu_{min}^{-1/2}|||\nabla u|||_{2,0} + C_1C_\dagger^{1/2}(Nh)^{1/2}\\
&+ (\nu_{max}t^*)^{1/2} + C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}t^{*1/2} + (\nu_{max}t^*)^{1/2}\left|\frac{\nu'}{\langle\nu\rangle}\right|t^{*1/2}\Big)\Big|\Big|\Big|\langle\nu\rangle^{1/2}\nabla e_u\Big|\Big|\Big|_{2,0}
\end{aligned}
$$

$$
\begin{aligned}
&+ (\alpha + d^{1/2}) t^{*1/2} \inf_{q_h \in Q_h} |||p - q_h|||^2_{2,0} + C(\alpha) C_P \sqrt{t^*} \inf_{v_h \in X_h} \|(u - v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)} \\
&+ C\Delta t \Big(t^{*1/2} \Big(\|u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\Lambda'|^2 \|u_t\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\beta g| \|T_t\|_{0,t^*;L^2(\Omega))} \Big) \\
&+ \Big(\nu_{min}^{-1} \Big(\Delta t \sum_{n=0}^{N-1} \|\langle \nu \rangle^{1/2} \nabla u_h^{n+1}\|^2 \Big)^{1/2} + (C_\dagger h)^{1/2} + |\nu'| \ \Delta t^{1/2} \Big) \|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)} \Big) \Big).
\end{aligned}
$$

Further, let $i = 2$ and suppose the hypotheses of Theorem 14 hold. Then,

$$
\begin{aligned}
\alpha \Delta t \sum_{n=1}^{N-1} \|e_p^{n+1}\| \leq (1 + C_*^{-1}) \Bigg(&4 \Big(C_1 \nu_{min}^{-1/2} |||\nabla u|||_{2,0} + C_1 C_\dagger^{1/2} (Nh)^{1/2} \\
&+ (\nu_{max} t^*)^{1/2} + C_P^2 |\langle \Lambda \rangle| \nu_{min}^{-1/2} t^{*1/2} + (\nu_{max} t^*)^{1/2} \left| \frac{\nu'}{\langle \nu \rangle} \right| t^{*1/2} \Big) \Big|\Big|\Big| \langle \nu \rangle^{1/2} \nabla e_u \Big|\Big|\Big|_{2,0} \\
&+ (\alpha + d^{1/2}) t^{*1/2} \inf_{q_h \in Q_h} |||p - q_h|||^2_{2,0} + C(\alpha) C_P \sqrt{t^*} \inf_{v_h \in X_h} \|(u - v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)} \\
&+ C\Delta t^2 \Big(t^{*1/2} \Big(\|u_{ttt}\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\Lambda'| \|u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\beta g| \|T_{tt}\|_{L^2(0,t^*;L^2(\Omega))} \Big) \\
&+ \Big(\nu_{min}^{-1} \Big(\Delta t \sum_{n=0}^{N-1} \|\langle \nu \rangle^{1/2} \nabla u_h^{n+1}\|^2 \Big)^{1/2} + (C_\dagger h)^{1/2} + |\nu'| \ \Delta t^{1/2} \Big) \|\nabla u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)} \Big).
\end{aligned}
$$

Proof. Recall the error equation for velocity (4.137): for all $v_h \in X_h$,

$$
\begin{aligned}
(\partial^i_{\Delta t}(e_u^{n+1}), v_h) &+ b(u^{n+1}, e_u^{n+1}, v_h) + b(\mathscr{E}^i(e_u^{n+1}), u_h^{n+1}, v_h) + b(\mathscr{E}^i(u_h'^{n+1}), e_u^{n+1} - \mathscr{E}^i(e_u^{n+1}), v_h) \\
&+ \langle \nu \rangle (\nabla e_u^{n+1}, \nabla v_h) + \nu' (\nabla \mathscr{E}^i(e_u^{n+1}), \nabla v_h) + (\langle \Lambda \rangle \times e_u^{n+1}, v_h) \\
&+ (\Lambda' \times \mathscr{E}^i(e_u^{n+1}), v_h) - (e_p^{n+1}, \nabla \cdot v_h) = \varsigma_u^i(u^{n+1}, v_h).
\end{aligned}
$$

Decompose $(\partial^i_{\Delta t}(e_u^{n+1}), v_h)$ into $(\partial^i_{\Delta t}(\eta^{n+1}), v_h) + (\partial^i_{\Delta t}(\phi_h^{n+1}), v_h)$, let $v_h \in V_h$, and rearrange. Then,

$$
\begin{aligned}
(\partial^i_{\Delta t}(\phi_h^{n+1}), v_h) &= (\partial^i_{\Delta t}(\eta^{n+1}), v_h) + b(u^{n+1}, e_u^{n+1}, v_h) + b(\mathscr{E}^i(e_u^{n+1}), u_h^{n+1}, v_h) \\
&+ b(\mathscr{E}^i(u_h'^{n+1}), e_u^{n+1} - \mathscr{E}^i(e_u^{n+1}), v_h) + \langle \nu \rangle (\nabla e_u^{n+1}, \nabla v_h) + \nu' (\nabla \mathscr{E}^i(e_u^{n+1}), \nabla v_h) \\
&+ (\langle \Lambda \rangle \times e_u^{n+1}, v_h) + (\Lambda' \times \mathscr{E}^i(e_u^{n+1}), v_h) - (\lambda^{n+1}, \nabla \cdot v_h) - \varsigma_u^i(u^{n+1}, v_h).
\end{aligned}
$$

The following estimates hold,

$$
b(u^{n+1}, e_u^{n+1}, v_h) \leq C_1 \nu_{min}^{-1/2} \|\nabla u^{n+1}\| \|\langle \nu \rangle^{1/2} \nabla e_u^{n+1}\| \|\nabla v_h\|, \tag{4.194}
$$

$$
b(\mathscr{E}^i(e_u^{n+1}), u_h^{n+1}, v_h) \leq C_1 \nu_{min}^{-1/2} \|\nabla u^{n+1}\| \|\langle \nu \rangle^{1/2} \nabla \mathscr{E}^i(e_u^{n+1})\| \|\nabla v_h\|, \tag{4.195}
$$

$$\langle\nu\rangle(\nabla e_u^{n+1},\nabla v_h) \leq \nu_{max}^{1/2}\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|\|\nabla v_h\|, \tag{4.196}$$

$$\nu'(\nabla\mathscr{E}^i(e_u^{n+1}),\nabla v_h) \leq \nu_{max}^{1/2}\left|\frac{\nu'}{\langle\nu\rangle}\right|\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(e_u^{n+1})\|\|\nabla v_h\|, \tag{4.197}$$

$$(\langle\Lambda\rangle\times e_u^{n+1}, v_h) \leq C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|\|\nabla v_h\|, \tag{4.198}$$

$$(\Lambda'\times\mathscr{E}^i(e_u^{n+1}), v_h) \leq C_P^2|\Lambda'|\nu_{min}^{-1/2}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(e_u^{n+1})\|\|\nabla v_h\|, \tag{4.199}$$

$$-(\lambda^{n+1},\nabla\cdot v_h) \leq d^{1/2}\|\lambda^{n+1}\|\|\nabla v_h\|. \tag{4.200}$$

Also,

$$\begin{aligned} &b(\mathscr{E}^i(u_h'^{n+1}), e_u^{n+1}-\mathscr{E}^i(e_u^{n+1}), v_h) \\ &\qquad\leq C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|\|\langle\nu\rangle^{1/2}\nabla(e_u^{n+1}-\mathscr{E}^i(e_u^{n+1}))\|\|\nabla v_h\|. \end{aligned} \tag{4.201}$$

Now, consider $(\partial_{\Delta t}^i(\eta^{n+1}), v_h)$ and $\varsigma_u^i(u^{n+1}, v_h)$. For $i=1$,

$$(\frac{\eta^{n+1}-\eta^n}{\Delta t}, v_h) \leq C_P\Delta t^{-1/2}\|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\|\nabla v_h\|, \tag{4.202}$$

$$\begin{aligned} -\varsigma_u^1(u^{n+1},v_h) \leq C\Delta t^{1/2}\bigg(&\Big(\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\Lambda'|^2\|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} \\ &+|\beta g|\|T_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big) + \Big(\nu_{min}^{-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| + \|\langle\nu\rangle^{-1/2}\nabla u_h'^{n}\| \\ &+|\nu'|\Big)\|\nabla u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\bigg)\|\nabla v_h\|, \end{aligned} \tag{4.203}$$

and for $i=2$,

$$(\frac{3\eta^{n+1}-4\eta^n+\eta^{n-1}}{2\Delta t}, v_h) \leq CC_P\Delta t^{-1/2}\|\eta_t\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)}\|\nabla v_h\|, \tag{4.204}$$

$$\begin{aligned} -\varsigma_u^2(u^{n+1},v_h) \leq C\Delta t^{3/2}\bigg(&\Big(\|u_{ttt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)} + |\Lambda'|\|u_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)} \\ &+|\beta g|\|T_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega))}\Big) \\ &+\Big(\nu_{min}^{-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| + \|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^2(u_h'^{n+1})\| \\ &+|\nu'|\Big)\|\nabla u_{tt}\|_{L^2(t^{n-1},t^{n+1};L^2(\Omega)^d)}\bigg)\|\nabla v_h\|. \end{aligned} \tag{4.205}$$

Apply the above estimates, divide by $\|\nabla v_h\|$, and take an infimum over V_h. Then,

$$\begin{aligned} &\|\partial_{\Delta t}^i(\phi_h^{n+1})\|_{V_h^*} \\ &\leq \Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\| + C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\| + \nu_{max}^{1/2} + C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\| \end{aligned}$$

$$+\Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\|+C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|+\nu_{max}^{1/2}\left|\frac{\nu'}{\langle\nu\rangle}\right|+C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(e_u^{n+1})\|$$

$$+ d^{1/2}\|\lambda^{n+1}\| + C_P\Delta t^{-1/2}\|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}$$

$$+ C\Delta t^{1/2}\Bigg(\Big(\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\Lambda'|^2\|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\beta g|\|T_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big)$$

$$+ \Big(\nu_{min}^{-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| + \|\langle\nu\rangle^{-1/2}\nabla u_h'^{n}\| + |\nu'|\Big)\|\nabla u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\Bigg).$$

Lemma 5 implies

$$\|\partial_{\Delta t}^i(\phi_h^{n+1})\|_{X_h^*} \leq (1+C_*^{-1})\Bigg(\Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\| + C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|$$

$$+ \nu_{max}^{1/2} + C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|$$

$$+\Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\|+C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|+\nu_{max}^{1/2}\left|\frac{\nu'}{\langle\nu\rangle}\right|+C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(e_u^{n+1})\|$$

$$+ d^{1/2}\|\lambda^{n+1}\| + C_P\Delta t^{-1/2}\|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}$$

$$+ C\Delta t^{1/2}\Bigg(\Big(\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\Lambda'|^2\|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\beta g|\|T_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big)$$

$$+ \Big(\nu_{min}^{-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| + \|\langle\nu\rangle^{-1/2}\nabla u_h'^{n}\| + |\nu'|\Big)\|\nabla u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\Bigg)\Bigg). \quad (4.206)$$

Reconsider the error equation (4.137) and rewrite $-(e_p^{n+1}, \nabla\cdot v_h) = -(\lambda^{n+1}, \nabla\cdot v_h) + (\pi_h^{n+1}, \nabla\cdot v_h)$. Isolating $(\pi_h^{n+1}, \nabla\cdot v_h)$, applying the estimates (4.194) - (4.203), dividing by $\|\nabla v_h\|$, taking a supremum over $v_h \in X_h$, and using the discrete inf-sup condition (2.23) and estimate (4.206) yields

$$\beta\|\pi_h^{n+1}\| \leq (1+C_*^{-1})\Bigg(\Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\| + C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|$$

$$+ \nu_{max}^{1/2} + C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|$$

$$+\Big(C_1\nu_{min}^{-1/2}\|\nabla u^{n+1}\|+C_1\|\langle\nu\rangle^{-1/2}\nabla\mathscr{E}^i(u_h'^{n+1})\|+\nu_{max}^{1/2}\left|\frac{\nu'}{\langle\nu\rangle}\right|+C_P^2|\langle\Lambda\rangle|\nu_{min}^{-1/2}\Big)\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(e_u^{n+1})\|$$

$$+ d^{1/2}\|\lambda^{n+1}\| + C_P\Delta t^{-1/2}\|\eta_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}$$

$$+ C\Delta t^{1/2}\Bigg(\Big(\|u_{tt}\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\Lambda'|^2\|u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)} + |\beta g|\|T_t\|_{L^2(t^n,t^{n+1};L^2(\Omega))}\Big)$$

$$+ \Big(\nu_{min}^{-1}\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\| + \|\langle\nu\rangle^{-1/2}\nabla u_h'^{n}\| + |\nu'|\Big)\|\nabla u_t\|_{L^2(t^n,t^{n+1};L^2(\Omega)^d)}\Bigg)\Bigg).$$

Multiply by Δt, sum over n from $n=0$ to $n=N-1$, and apply both the Cauchy-Schwarz inequality and condition (4.52). Then,

$$\beta \Delta t \sum_{n=0}^{N-1} \|\pi_h^{n+1}\| \leq (1+C_*^{-1}) \Bigg(2\Big(C_1 \nu_{min}^{-1/2} |||\nabla u|||_{2,0} + C_1 C_\dagger^{1/2} (Nh)^{1/2}$$

$$+ (\nu_{max} t^*)^{1/2} + C_P^2 |\langle \Lambda \rangle| \nu_{min}^{-1/2} t^{*1/2} + (\nu_{max} t^*)^{1/2} \left| \frac{\nu'}{\langle \nu \rangle} \right| t^{*1/2} \Big) |||\langle \nu \rangle^{1/2} \nabla e_u|||_{2,0}$$

$$+ (dt^*)^{1/2} |||\lambda|||_{2,0} + C_P \sqrt{t^*} \|\eta_t\|_{L^2(0,t^*;L^2(\Omega)^d)}$$

$$+ C\Delta t \Bigg(t^{*1/2} \Big(\|u_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\Lambda'|^2 \|u_t\|_{L^2(0,t^*;L^2(\Omega)^d)} + |\beta g| \|T_t\|_{0,t^*;L^2(\Omega))} \Big)$$

$$+ \Big(\nu_{min}^{-1} \Big(\Delta t \sum_{n=0}^{N-1} \|\langle \nu \rangle^{1/2} \nabla u_h^{n+1}\|^2 \Big)^{1/2} + (C_\dagger h)^{1/2} + |\nu'| \ \Delta t^{1/2} \Big) \|\nabla u_t\|_{L^2(0,t^*;L^2(\Omega)^d)} \Bigg) \Bigg).$$

Lastly, apply the triangle inequality, take infimums over Q_h and V_h, and use Lemma 4. This yields the first result. The second follows similarly, utilizing estimates (4.204) and (4.205) in place of (4.202) and (4.203). □

Although we do not prove it here, it is possible to prove both stability and error estimates for the pressure in $L^2(0,t^*;L^2(\Omega))$; see, e.g., [41, 69]. It will be useful to specify the explicit dependencies on the mesh parameter h and timestep Δt after common choices of finite elements.

Corollary 5. *Suppose the assumptions of Theorem 13 hold with $k=m=1$. Further suppose that the finite element spaces (X_h,Q_h,W_h) are given by P1b-P1-P1b (MINI), then the errors in velocity, temperature, and pressure satisfy*

$$\|e_u^N\|^2 + \|e_T^N\|^2 + \frac{1}{2} \sum_{n=0}^{N-1} \Big(\|e_u^{n+1} - e_u^n\|^2 + \|e_T^{n+1} - e_T^n\|^2 \Big)$$

$$+ \frac{\Delta t}{4} \sum_{n=0}^{N-1} \Big(\|\langle \nu \rangle^{1/2} \nabla e_u^{n+1}\|^2 + \|\langle \kappa \rangle^{1/2} \nabla e_T^{n+1}\|^2 \Big) + \frac{\Delta t}{2} \Big(\|\langle \nu \rangle^{1/2} \nabla e_u^N\|^2 + \|\langle \kappa \rangle^{1/2} \nabla e_T^N\|^2 \Big)$$

$$+ \alpha \Delta t \sum_{n=1}^{N-1} \|e_p^{n+1}\| \leq C\big(h^2 + h\Delta t + \Delta t^2 + \textit{initial errors} \big).$$

Furthermore, if the assumptions of Theorem 14 hold with $k=m=2$ and the finite element spaces (X_h,Q_h,W_h) are given by P2-P1-P2 (Taylor-Hood), then the errors in velocity, temperature, and pressure satisfy

$$
\begin{aligned}
&\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2}\|e_T^N\|^2 + \frac{1}{2}\|2e_T^N - e_T^{N-1}\|^2 \\
&\qquad + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2 + \|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2\Big) \\
&\quad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2\Big) \\
&+\frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{N-1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^{N-1}\|^2\Big)+\alpha\Delta t\sum_{n=1}^{N-1}\|e_p^{n+1}\| \le C\big(h^4+h\Delta t^3+\Delta t^4+\mathit{initial\ errors}\big).
\end{aligned}
$$

Interestingly, these results can be extended to the averages of the error quantities.

Corollary 6. *Suppose the assumptions of Theorem 13 hold with $k = m = 1$. Further suppose that the finite element spaces (X_h,Q_h,W_h) are given by P1b-P1-P1b (MINI), then the errors in velocity, temperature, and pressure satisfy*

$$
\begin{aligned}
&\|\langle e_u^N\rangle\|^2 + \|\langle e_T^N\rangle\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|\langle e_u^{n+1} - e_u^n\rangle\|^2 + \|\langle e_T^{n+1} - e_T^n\rangle\|^2\Big) \\
&+ \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla\langle e_u^{n+1}\rangle\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\langle e_T^{n+1}\rangle\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\langle e_u^N\rangle\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\langle e_T^N\rangle\|^2\Big) \\
&\qquad\qquad + \alpha\Delta t\sum_{n=1}^{N-1}\|\langle e_p^{n+1}\rangle\| \le C\big(h^2 + h\Delta t + \Delta t^2 + \langle \mathit{initial\ errors}\rangle\big).
\end{aligned}
$$

Furthermore, if the assumptions of Theorem 14 hold with $k = m = 2$ and the finite element spaces (X_h,Q_h,W_h) are given by P2-P1-P2 (Taylor-Hood), then the errors in velocity, temperature, and pressure satisfy

$$
\begin{aligned}
&\|\langle e_u^N\rangle\|^2 + \|\langle 2e_u^N - e_u^{N-1}\rangle\|^2 + \frac{1}{2}\|\langle e_T^N\rangle\|^2 + \frac{1}{2}\|\langle 2e_T^N - e_T^{N-1}\rangle\|^2 \\
&\qquad + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|\langle e_u^{n+1} - 2e_u^n + e_u^{n-1}\rangle\|^2 + \|\langle e_T^{n+1} - 2e_T^n + e_T^{n-1}\rangle\|^2\Big) \\
&+ \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla\langle e_u^{n+1}\rangle\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\langle e_T^{n+1}\rangle\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla\langle e_u^N\rangle\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\langle e_T^N\rangle\|^2\Big) \\
&\qquad + \frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla\langle e_u^{N-1}\rangle\|^2 + \|\langle\kappa\rangle^{1/2}\nabla\langle e_T^{N-1}\rangle\|^2\Big) \\
&\qquad\qquad + \alpha\Delta t\sum_{n=1}^{N-1}\|\langle e_p^{n+1}\rangle\| \le C\big(h^4 + h\Delta t^3 + \Delta t^4 + \langle \mathit{initial\ errors}\rangle\big).
\end{aligned}
$$

Proof. This follows from Corollary 5 and the following sequence of inequalities: For all $\chi \in L^2(\Omega)$,

$$\| < \chi > \| \leq < \|\chi\| > \leq \sqrt{< \|\chi\|^2 >}.$$

□

As in Section 4.2, we will not state analogs of both Corollary 5 and Corollary 6, which hold, for **PEA** and **ACE**.

Theorem 15. *Consider first-order* ***PEA****. For (u,p,T) satisfying (4.14) - (4.17), suppose that* $(u_h^0, p_h^0, T_h^0) \in (X_h, Q_h, W_h)$ *are approximations of* (u^0, p^0, T^0) *to within the accuracy of the interpolant. Further, suppose that conditions (4.52) and (4.53) hold and* $p_t \in L^\infty(0, t^*; L^2(\Omega))$*. Then, there exists constants* C, $C_\# > 0$ *such that*

$$\begin{aligned}
&\|e_u^N\|^2 + \|e_T^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big(\|e_u^{n+1} - e_u^n\|^2 + \|e_T^{n+1} - e_T^n\|^2\Big) \\
&\qquad + \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2 + 4\epsilon\|e_p^{n+1}\|^2\Big) \\
&\qquad\qquad + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2\Big) \\
&\leq C\exp(C_\# t^*)\Big\{\inf_{S_h \in W_h}\Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1}|||\nabla(T - S_h)|||_{\infty,0}^2 + \kappa_{max}|||\nabla(T - S_h)|||_{2,0}^2 \\
&\qquad + \kappa_{min}^{-1}\|(T - S_h)_t\|_{L^2(0,t^*;L^2(\Omega))}^2 + h\Delta t\|\nabla(T - S_h)_t\|_{L^2(0,t^*;L^2(\Omega))}^2\Big) \\
&+ \inf_{v_h \in X_h}\Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u - v_h)|||_{\infty,0}^2 + \big(\kappa_{min}^{-1} + \nu_{max} + (|\langle\Lambda\rangle|^2 + |\Lambda'|^2) + |\beta g|^2)\big)|||\nabla(u - v_h)|||_{2,0}^2 \\
&\qquad + \nu_{min}^{-1}\|(u - v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2 + h\Delta t\|\nabla(u - v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\Big) \\
&+ \inf_{q_h \in Q_h}\epsilon^2\nu_{min}^{-1}|||p - q_h|||_{2,0}^2 + t^*\inf_{S_h \in W_h}\Big(|\beta g|^2\nu_{min}^{-1}\|\tau - S_h\|^2 + (1 + \kappa_{min}^{-1} + \kappa_{max})\|\nabla(\tau - S_h)\|^2\Big) \\
&\qquad + h\Delta t + \epsilon + \big(\nu_{min}^{-1}(1 + |\Lambda'|^2 + |\beta g|^2) + |\kappa'|^2 + |\nu'|^2\big)\Delta t^2\Big\} \\
&\qquad\qquad + \|e_u^0\|^2 + \|e_T^0\|^2 + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2\Big).
\end{aligned}$$

Moreover, for second-order ***PEA****, there exists constants* C, $C_\# > 0$ *such that*

$$\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2}\|e_T^N\|^2 + \frac{1}{2}\|2e_T^N - e_T^{N-1}\|^2$$

$$+\frac{1}{2}\sum_{n=0}^{N-1}\Big(\|e_u^{n+1}-2e_u^n+e_u^{n-1}\|^2+\|e_T^{n+1}-2e_T^n+e_T^{n-1}\|^2\Big)$$
$$+\frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2+4\epsilon\|e_p^{n+1}\|^2\Big)+\frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{N}\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^{N}\|^2\Big)$$
$$+\frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{N-1}\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^{N-1}\|^2\Big)$$
$$\leq C\exp(C_\# t^*)\Big\{\inf_{S_h\in W_h}\Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1}|||\nabla(T-S_h)|||_{\infty,0}^2+\kappa_{max}|||\nabla(T-S_h)|||_{2,0}^2$$
$$+\kappa_{min}^{-1}\|(T-S_h)_t\|_{L^2(0,t^*;L^2(\Omega))}^2+h\Delta t^3\|\nabla(T-S_h)_{tt}\|_{L^2(0,t^*;L^2(\Omega))}^2\Big)$$
$$+\inf_{v_h\in X_h}\Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u-v_h)|||_{\infty,0}^2+\big(\kappa_{min}^{-1}+\nu_{max}+(|\langle\Lambda\rangle|^2+|\Lambda'|^2)+|\beta g|^2)\big)|||\nabla(u-v_h)|||_{2,0}^2$$
$$+\nu_{min}^{-1}\|(u-v_h)_t\|_{L^2(0,t^*;L^2(\Omega)^d)}^2+h\Delta t^3\|\nabla(u-v_h)_{tt}\|_{L^2(0,t^*;L^2(\Omega)^d)}^2\Big)$$
$$+\inf_{q_h\in Q_h}\epsilon^2\nu_{min}^{-1}|||p-q_h|||_{2,0}^2+t^*\inf_{S_h\in W_h}\Big(|\beta g|^2\nu_{min}^{-1}\|\tau-S_h\|^2+(1+\kappa_{min}^{-1}+\kappa_{max})\|\nabla(\tau-S_h)\|^2\Big)$$
$$+h\Delta t^3+\epsilon+\big(\nu_{min}^{-1}(1+|\Lambda'|^2+|\beta g|^2)+|\kappa'|^2+|\nu'|^2\big)\Delta t^4\Big\}$$
$$+\|e_u^1\|^2+\|2e_u^1-e_u^0\|^2+\|e_T^1\|^2+\|2e_T^1-e_T^0\|^2$$
$$+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^1\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^1\|^2\Big)+\Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2+\|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2\Big).$$

Proof. Our strategy is to consider the error equation for the continuity equation, utilize the Stokes projection (2.24) - (2.25) to negate additional problem terms, and augment the techniques and estimates of Theorems 13 and 14. The error equation for continuity and momentum are,

$$(\partial_{\Delta t}^i(e_u^{n+1}),v_h)+b(u^{n+1},e_u^{n+1},v_h)+b(\mathscr{E}^i(e_u^{n+1}),u_h^{n+1},v_h)+b(\mathscr{E}^i(u_h'^{n+1}),e_u^{n+1}-\mathscr{E}^i(e_u^{n+1}),v_h)$$
$$+\langle\nu\rangle(\nabla e_u^{n+1},\nabla v_h)+\nu'(\nabla\mathscr{E}^i(e_u^{n+1}),\nabla v_h)+(\langle\Lambda\rangle\times e_u^{n+1},v_h)+(\Lambda'\times\mathscr{E}^i(e_u^{n+1}),v_h)$$
$$-(e_p^{n+1},\nabla\cdot v_h)=\varsigma_u(u^{n+1},v_h)\ \ \forall v_h\in X_h,\quad(4.207)$$

$$\epsilon(e_p^{n+1},q_h)+(\nabla\cdot e_u^{n+1},q_h)=\epsilon(p^{n+1},q_h),\tag{4.208}$$

where $\frac{1}{\epsilon}(\nabla\cdot u_h^{n+1},\nabla\cdot v_h)=(p_h^{n+1},\nabla\cdot v_h)$ was used in (4.43). Use the relations (4.112) and (4.114), where the velocity and pressure interpolant is chosen to be the Stokes projection. Let $q_h=\Delta t\pi_h^{n+1}\in Q_h$, and rearrange. Then,

$$\epsilon\Delta t\|\pi_h^{n+1}\|^2+\Delta t(\nabla\cdot\phi_h^{n+1},\pi_h^{n+1})=\epsilon\Delta t(\lambda^{n+1},\pi_h^{n+1})-\epsilon\Delta t(p^{n+1},\pi_h^{n+1}).\tag{4.209}$$

Note that in the velocity error equation, $(e_p^{n+1}, \nabla \cdot v_h) = -(\pi_h^{n+1}, \nabla \cdot v_h)$ for all $v_h \in X_h$.

Consider equation (4.208). The Cauchy-Schwarz-Young inequality yields

$$\epsilon\Delta t(\lambda^{n+1}, \pi_h^{n+1}) \leq \epsilon\Delta t\|\lambda^{n+1}\|^2 + \frac{\epsilon\Delta t}{4}\|\pi_h^{n+1}\|^2, \tag{4.210}$$

$$-\epsilon\Delta t(p^{n+1}, \pi_h^{n+1}) \leq \epsilon\Delta t\|p^{n+1}\|^2 + \frac{\epsilon\Delta t}{4}\|\pi_h^{n+1}\|^2. \tag{4.211}$$

Set $v_h = \Delta t\phi_h^{n+1}$ in equation (4.207), combine with equation (4.208), and rearrange. Then,

$$\begin{aligned}
(\partial^i_{\Delta t}(\phi_h^{n+1}), \Delta t\phi_h^{n+1}) + \Delta t\|\langle\nu\rangle^{1/2}\Delta t\phi_h^{n+1}\|^2 + \epsilon\Delta t\|\pi_h^{n+1}\|^2 &= (\partial^i_{\Delta t}(\eta^{n+1}), \Delta t\phi_h^{n+1}) \\
+ \Delta t b(u^{n+1}, \eta^{n+1}, \phi_h^{n+1}) + \Delta t b(\mathscr{E}^i(\eta^{n+1}), u_h^{n+1}, \phi_h^{n+1}) &- \Delta t b(\mathscr{E}^i(\phi_h^{n+1}), u_h^{n+1}, \phi_h^{n+1}) \\
+ \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \eta^{n+1} - \mathscr{E}^i(\eta^{n+1}), \phi_h^{n+1}) &- \Delta t b(\mathscr{E}^i(u'^{n+1}_h), \phi_h^{n+1} - \mathscr{E}^i(\phi_h^{n+1}), \phi_h^{n+1}) \\
+ \langle\nu\rangle\Delta t(\nabla\eta^{n+1}, \nabla\phi_h^{n+1}) &+ \nu'\Delta t(\nabla\mathscr{E}^i(\eta^{n+1}), \nabla\phi_h^{n+1}) \\
- \nu'\Delta t(\nabla\mathscr{E}^i(\phi_h^{n+1}), \nabla\phi_h^{n+1}) + \Delta t(\Lambda \times \eta^n, \phi_h^{n+1}) &- \Delta t(\Lambda \times \phi_h^n, \phi_h^{n+1}) \\
- \Delta t(\lambda^{n+1}, \nabla \cdot \phi_h^{n+1}) + \epsilon\Delta t(\lambda^{n+1}, \pi_h^{n+1}) &- \epsilon\Delta t(p^{n+1}, \pi_h^{n+1}) \\
+ \Delta t(\beta g\mathscr{E}^i(\zeta^{n+1}), \phi_h^{n+1}) &- \Delta t(\beta g\mathscr{E}^i(\psi_h^{n+1}), \phi_h^{n+1}) \\
+ \Delta t(\beta g(\tau - I_h\tau), \phi_h^{n+1}) &- \Delta t\varsigma_u(u^{n+1}, \phi_h^{n+1}).
\end{aligned} \tag{4.212}$$

Use the estimates (4.210) and (4.211) together with, e.g., estimates (4.141) - (4.155), from Theorem 13, on the above. The result then follows using the techniques of Theorems 13 and 14. $\square$

Lastly, we prove convegence estimates for **ACE**.

Theorem 16. *Consider **ACE**. For (u,p,T) satisfying (1) - (5), suppose that $(u_h^0, p_h^0, T_h^0) \in (X_h, Q_h, W_h)$ are approximations of (u^0, p^0, T^0) to within the accuracy of the interpolant. Further, suppose that conditions (4.52) and (4.53) hold and $p_t \in L^\infty(0, t^*; L^2(\Omega))$. Then, there exists constants C, $C_\triangle > 0$ such that*

$$\begin{aligned}
&\frac{1}{2}\|e_T^N\|^2 + \|e_u^N\|^2 + \epsilon\|e_p^N\|^2 + \frac{1}{2}\sum_{n=0}^{N-1}\Big\{\|e_T^{n+1} - e_T^n\|^2 + \|e_u^{n+1} - e_u^n\|^2 + 2\epsilon\|e_p^{n+1} - e_p^n\|^2\Big\} \\
&+ \frac{\Delta t}{4}\sum_{n=0}^{N-1}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2\Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq Cexp(C_\triangle t^*)\Big\{ \inf_{S_h\in W_h} \Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1} |||\nabla(T-S_h)|||_{\infty,0}^2 + \kappa_{max}|||\nabla(T-S_h)|||_{2,0}^2 \\
&\quad + \kappa_{min}^{-1}\|(T-S_h)_t\|^2_{L^2(0,t^*;L^2(\Omega))} + h\Delta t\|\nabla(T-S_h)_t\|^2_{L^2(0,t^*;L^2(\Omega))}\Big) \\
&+ \inf_{v_h\in X_h} \Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u-v_h)|||_{\infty,0}^2 + \big(\kappa_{min}^{-1}+\nu_{max}+(|\langle\Lambda\rangle|^2+|\Lambda'|^2)+|\beta g|^2)\big)|||\nabla(u-v_h)|||_{2,0}^2 \\
&\quad + \nu_{min}^{-1}\|(u-v_h)_t\|^2_{L^2(0,t^*;L^2(\Omega)^d)} + h\Delta t\|\nabla(u-v_h)_t\|^2_{L^2(0,t^*;L^2(\Omega)^d)}\Big) \\
&\quad + t^* \inf_{S_h\in W_h} \Big(|\beta g|^2\nu_{min}^{-1}\|\tau - S_h\|^2 + (1+\kappa_{min}^{-1}+\kappa_{max})\|\nabla(\tau-S_h)\|^2 \Big) \\
&\quad + h\Delta t + \epsilon\Delta t + \big(\nu_{min}^{-1}(1+|\Lambda'|^2+|\beta g|^2) + |\kappa'|^2 + |\nu'|^2\big)\Delta t^2 \Big\} \\
&\quad + \|e_u^0\|^2 + \|e_T^0\|^2 + \epsilon\|e_p^0\|^2 + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2 \Big).
\end{aligned}
$$

*Moreover, for second-order **ACE**, exists constants C, $C_\triangle > 0$ such that*

$$
\begin{aligned}
&\|e_u^N\|^2 + \|2e_u^N - e_u^{N-1}\|^2 + \frac{1}{2}\|e_T^N\|^2 + \frac{1}{2}\|2e_T^N - e_T^{N-1}\|^2 + \epsilon\|e_p^N\|^2 \\
&\quad + \frac{1}{2}\sum_{n=0}^{N-1} \Big(\|e_u^{n+1} - 2e_u^n + e_u^{n-1}\|^2 + \|e_T^{n+1} - 2e_T^n + e_T^{n-1}\|^2 + 2\epsilon\|e_p^{n+1} - e_p^n\|^2 \Big) \\
&+ \frac{\Delta t}{4}\sum_{n=0}^{N-1} \Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{n+1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{n+1}\|^2 \Big) + \frac{\Delta t}{2}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^N\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^N\|^2 \Big) \\
&\quad + \frac{\Delta t}{4}\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^{N-1}\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^{N-1}\|^2 \Big) \\
&\leq C\exp(C_\triangle t^*)\Big\{ \inf_{S_h\in W_h} \Big((1+\kappa_{min}^{-1})\kappa_{min}^{-1} |||\nabla(T-S_h)|||_{\infty,0}^2 + \kappa_{max}|||\nabla(T-S_h)|||_{2,0}^2 \\
&\quad + \kappa_{min}^{-1}\|(T-S_h)_t\|^2_{L^2(0,t^*;L^2(\Omega))} + h\Delta t^3\|\nabla(T-S_h)_{tt}\|^2_{L^2(0,t^*;L^2(\Omega))}\Big) \\
&+ \inf_{v_h\in X_h} \Big((1+\nu_{min}^{-1})\nu_{min}^{-1}|||\nabla(u-v_h)|||_{\infty,0}^2 + \big(\kappa_{min}^{-1}+\nu_{max}+(|\langle\Lambda\rangle|^2+|\Lambda'|^2)+|\beta g|^2)\big)|||\nabla(u-v_h)|||_{2,0}^2 \\
&\quad + \nu_{min}^{-1}\|(u-v_h)_t\|^2_{L^2(0,t^*;L^2(\Omega)^d)} + h\Delta t^3\|\nabla(u-v_h)_{tt}\|^2_{L^2(0,t^*;L^2(\Omega)^d)}\Big) \\
&\quad + t^* \inf_{S_h\in W_h} \Big(|\beta g|^2\nu_{min}^{-1}\|\tau - S_h\|^2 + (1+\kappa_{min}^{-1}+\kappa_{max})\|\nabla(\tau-S_h)\|^2 \Big) \\
&\quad + h\Delta t^3 + \epsilon\Delta t + \big(\nu_{min}^{-1}(1+|\Lambda'|^2+|\beta g|^2) + |\kappa'|^2 + |\nu'|^2\big)\Delta t^4 \Big\} \\
&\quad + \|e_u^1\|^2 + \|2e_u^1 - e_u^0\|^2 + \|e_T^1\|^2 + \|2e_T^1 - e_T^0\|^2 + \epsilon\|e_p^0\|^2 \\
&\quad + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^1\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^1\|^2 \Big) + \Delta t\Big(\|\langle\nu\rangle^{1/2}\nabla e_u^0\|^2 + \|\langle\kappa\rangle^{1/2}\nabla e_T^0\|^2 \Big).
\end{aligned}
$$

Proof. We follow similarly as in Theorem 15. The error equation for continuity is

$$
\epsilon(\partial_{\Delta t}^i(e_p^{n+1}), q_h) + (\nabla\cdot e_u^{n+1}, q_h) = \varsigma_p(p^{n+1}, q_h). \tag{4.213}
$$

Use the relations (4.112) and (4.114), where the interpolant is chosen to be the Stokes projection. Let $q_h = \Delta t \pi_h^{n+1} \in Q_h$, and rearrange. Then,

$$\frac{\epsilon}{2}\Big\{\|\pi_h^{n+1}\|^2 - \|\pi_h^n\|^2 + \|\pi_h^{n+1} - \pi_h^n\|^2\Big\} + \Delta t(\nabla \cdot \phi_h^{n+1}, \pi_h^{n+1}) \\ = \epsilon(\lambda^{n+1} - \lambda^n, \pi_h^{n+1}) - \Delta t \varsigma_p(p^{n+1}, \pi_h^{n+1}). \quad (4.214)$$

Note that in the velocity error equation, $\Delta t(e_p^{n+1}, \nabla \cdot v_h) = -(\pi_h^{n+1}, \nabla \cdot v_h)$ for all $v_h \in X_h$.

Consider equation (4.214). Add and subtract $\epsilon(\lambda^{n+1} - \lambda^n, \pi_h^n)$ and $-\Delta t \upsilon_p(p^{n+1}, \pi_h^n)$. Use Taylor's theorem and the Cauchy-Schwarz-Young inequality. This leads to

$$\begin{aligned} 2\epsilon(\lambda^{n+1} - \lambda^n, \pi_h^{n+1}) &= 2\epsilon(\lambda^{n+1} - \lambda^n, \pi_h^{n+1} - \pi_h^n) + 2\epsilon(\lambda^{n+1} - \lambda^n, \pi_h^n) \\ &\leq \frac{4\epsilon C_r \Delta t^2}{\delta_{26}}\|\lambda_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \frac{\epsilon\delta_{26}}{r}\|\pi_h^{n+1} - \pi_h^n\|^2 \\ &+ \frac{4\epsilon C_r \Delta t}{\delta_{27}}\|\lambda_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \frac{\epsilon\delta_{27}\Delta t}{r}\|\pi_h^n\|^2, \quad (4.215) \\ -2\Delta t \varsigma_p(p^{n+1}, \pi_h^{n+1}) &\leq \frac{4\epsilon C_r \Delta t^2}{\delta_{28}}\|p_t\|^2_{L^2(t^n,t^{n+1};L^2(\Omega))} + \frac{\epsilon\delta_{28}}{r}\|\pi_h^{n+1} - \pi_h^n\|^2 \\ &+ \frac{4\epsilon C_r \Delta t}{\delta_{29}}\|p_t\|^2_{L^\infty(t^n,t^{n+1};L^2(\Omega))} + \frac{\epsilon\delta_{29}\Delta t}{r}\|\pi_h^n\|^2. \quad (4.216) \end{aligned}$$

The remainder is routine: use the above estimates together with estimates and techniques from Theorems 13 and 14 to yield the result. □

4.4 NUMERICAL TESTS

In this section, we illustrate proven qualities of the proposed algorithms. In particular, convergence rates are calculated and speed comparisons are provided. The numerical experiments include a convergence experiment with an analytical solution devised through the method of manufactured solutions and the double pane window benchmark [136]. The software platform used is FreeFem++ [60].

4.4.1 Stability condition

Recall, each algorithm is stable provided conditions (4.52) and (4.53) hold:

$$\frac{\Delta t}{h}\max_{1\leq j\leq J}\|\nabla\mathscr{E}^i(u_h'^{n+1})\|^2 \leq C_\dagger \min\{\langle\nu\rangle,\langle\kappa\rangle\},$$
$$\max\left\{\max_{1\leq j\leq J}\left|\frac{\nu'}{\langle\nu\rangle}\right|^2,\max_{1\leq j\leq J}\left|\frac{\kappa'}{\langle\kappa\rangle}\right|^2\right\}\leq C_{\dagger\dagger}.$$

Moreover, for **ACE-T**, condition (4.52) can be replaced with condition (4.54):

$$\Delta t\max_{1\leq j\leq J}\|\nabla\cdot\mathscr{E}^i(u_h'^{n+1})\|^2_{L^4(\Omega)} \leq C_\dagger \min\{\langle\nu\rangle,\langle\kappa\rangle\},$$

provided $C_\nu \geq C_{\dagger\dagger\dagger}$ and $\frac{C_\nu}{\sigma_{turb}} \geq C_{\dagger\dagger\dagger\dagger}$. In general, the stability constant $C_\dagger$ is determined via pre-computations; see Appendix B for theoretical determinations. Herein, we estimate it for the double pane window benchmark; it is set to 1. Condition (4.52) is checked at each timestep. The timestep is halved and the timestep is repeated if violated. The timestep is never increased. Moreover, the condition (4.53) can be checked once before any computations are performed. If violated, the ensemble set can be broken into smaller subsets which satisfy the condition.

4.4.2 Perturbation generation

The bred vector (BV) algorithm [135] is used to generate perturbations in Section 4.4.4 and Chapter 5. The BV algorithm simulates growth errors due to uncertainty in the initial conditions; for practical problems, this is necessary and *random perturbations are not sufficient* [135]. With the BV algorithm, the nonlinear error growth in the ensemble average is reduced, which is witnessed in Chapter 5. Our experimental results are drastically different when using BVs compared to random perturbations, consistent with the above. In particular, predictability calculations in 5 Section 5.1 are more pessimistic (smaller average effective Lyapunov exponents and variance) when using BVs over random perturbations.

To begin, an initial random positive and negative perturbation pair is generated, $\pm\epsilon = \pm(\delta_1,\delta_2,...,\delta_M)$; $\delta_i\in(0,0.01)$ or $(0,0.1)$ $\forall 1\leq i\leq M$, for the double pane window and manufactured solution problems of Sections 4.4.4 and 4.4.3, respectively. Denoting the control and perturbed numerical approximations χ_h^n and $\chi_{p,h}^n$, respectively, a bred vector $bv(\chi;\delta_i)$ is generated via:

Algorithm: BV

Step one: Given χ_h^0 and δ_i, put $\chi_{p,h}^0 = \chi_h^0 + \delta_i$. Select time reinitialization interval $\delta t \geq \Delta t$ and let $t^k = k\delta t$ with $0 \leq k \leq k^* \leq N$.

Step two: Compute χ_h^k and $\chi_{p,h}^k$. Calculate $bv(\chi^k;\delta_i) = \frac{\delta_i}{\|\chi_{p,h}^k - \chi_h^k\|}(\chi_{p,h}^k - \chi_h^k)$.

Step three: Put $\chi_{p,h}^k = \chi_h^k + bv(\chi^k;\delta_i)$.

Step four: Repeat from **Step two** with $k = k+1$.

Step five: Put $bv(\chi;\delta_i) = bv(\chi^{k^*};\delta_i)$.

The bred vector pair generates a pair of initial conditions via $\chi_\pm = \chi^0 + bv(\chi;\pm\delta_i)$. We let $k^* = 5$ and choose $\delta t = \Delta t$ for all tests. A perturbation pair is associated with each component of velocity and the temperature. If a pressure initial condition is needed, as in **ACE** and **ACE-T**, then a perturbation pair is also prescribed.

4.4.3 Convergence Tests

We now illustrate convergence rates for the proposed algorithms. Typically, a solution is specified, inserted into the set of governing equations, and the forcing terms are calculated. This technique is known as the "method of manufactured solutions". The known solution is then compared to the numerical approximation at successive refinements of the mesh and/or timestep. Rates of convergence are then calculated. The calculated rates *prove nothing.* They suggest convergence rates for numerical methods indicating what can possibly be proven and are used to illustrate theoretical conclusions.

The domain and unperturbed parameters are $\Omega = (0,1)^2$ and $\nu = \kappa = \beta = \Lambda = 1$. The unperturbed solution is given by

$$
\begin{aligned}
u(x,y,t) &= (A(t)x^2(x-1)^2y(y-1)(2y-1), -A(t)x(x-1)(2x-1)y^2(y-1)^2)^T,\\
T(x,y,t) &= u_1(x,y,t) + u_2(x,y,t),\\
p(x,y,t) &= A(t)(2x-1)(2y-1),
\end{aligned}
$$

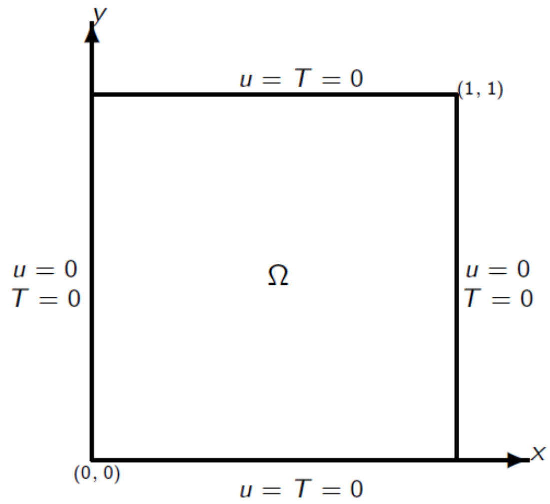

Figure 4.1: Domain & BCs: manufactured solution problem.

with $A(t) = 100\cos(t)$. We select random perturbations of $\mathcal{O}(10^{-1})$ for each of the parameters and initial conditions (σ_j, below), see Table 4.1. Letting $B(x,y) = x(x-1)y(y-1)$, the perturbed solutions are given by

$$u(x,y,t;\omega_j) = (u_1(x,y,t) + \sigma_1(\omega_j)B(x,y), u_2(x,y,t) + \sigma_2(\omega_j)B(x,y))^T \tag{4.217}$$

$$T(x,y,t;\omega_j) = T(x,y,t) + \sigma_3(\omega_j)B(x,y), \tag{4.218}$$

$$p(x,y,t;\omega_j) = p(x,y,t), \tag{4.219}$$

for $j = 1,2,3$. Forcings are adjusted as needed. Notice that $u \in X\bigcap P_7(\Omega)^2$, $T \in H_0^1(\Omega)\bigcap P_7(\Omega)$, and $p \in Q\bigcap P_4(\Omega)$; the domain together with the boundary conditions are presented in Figure 4.1.

The finite element mesh is constructed via Delaunay triangulation generated from m points on each side of the domain; see Figure 4.2. We set $\epsilon = 100\Delta t^i$ for **PEA** methods and $\epsilon = \Delta t^i$ for **ACE** methods. Errors in approximations of the average velocity and temperature are calculated with the $L^\infty(0,t^*;L^2(\Omega))$ and $L^2(0,t^*;H^1(\Omega))$ norms. For the average pressure, $L^1(0,t^*;L^2(\Omega))$, and $L^2(0,t^*;L^2(\Omega))$ norms are used for **eBDF** and **PEA**. For

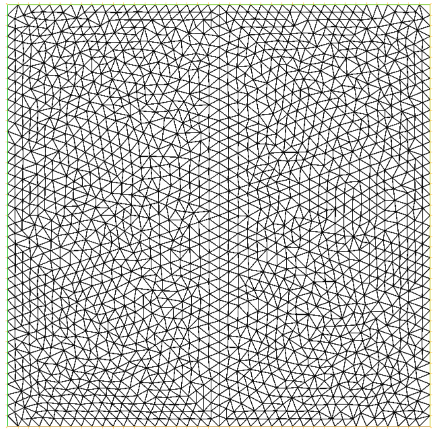

Figure 4.2: Mesh: manufactured solution problem.

ACE, average pressure errors are calculated using the $L^{\infty}(0, t^*; L^2(\Omega))$ and $L^2(0, t^*; L^2(\Omega))$ norms. Rates are calculated from the errors at two successive $\Delta t_{1,2}$ via

$$\frac{\log_2(e_\chi(\Delta t_1)/e_\chi(\Delta t_2))}{\log_2(\Delta t_1/\Delta t_2)},$$

respectively, with $\chi = u, T, p$. We set $\Delta t = 0.5/m$ and vary m between 4, 8, 16, 24, and 32. Results are presented in Tables 4.2 - 4.7.

Optimal-order convergence is observed for velocity and temperature and these results are consistent with the results of our theoretical analyses. Further, all pressure results are consistent with the theoretical analyses when considering the $\sqrt{\epsilon}$ scaling of our estimates. This indicates that the $\sqrt{\epsilon}$ factor cannot be removed.

We also see an anomaly arise in the last row and column of Table 4.7. It is uncertain what this is due to, however, it should be recalled that the choice of ϵ can strongly effect the behavior and accuracy of the method for either **PEA** or **ACE**. As a last comment, these numerical results suggest that **eBDF** provides more accurate results across the board. Moreover, both **PEA** and **ACE** produce substantially inferior pressure accuracy, with **PEA** being the worst performer. Therefore, for practical computing, one should consider both timestep/mesh requirements and whether the pressure is a desired quantity or not when selecting a method.

Table 4.1: Perturbations to associated parameters and initial conditions.

	Parameters					
j	$u(x,0)$	$T(x,0)$	ν	Λ	β	κ
1	(0.038293264, 0.0461225485)	0.01199364526	0.01551310425	0.01481403912	0.0864345507	0.0464799222
2	(0.0510703744, 0.02141882264)	0.0740124158	0.0561074383	0.01743837107	0.013325773	0.01888897295
3	(0.01736815896, 0.0680989749)	0.01669886031	0.01594498955	0.01520503142	0.030090834	0.01835103811

Table 4.2: eBDF (1st-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle\nabla e_u\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle\nabla e_T\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{1,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{2,0}$	Rate
4	8.75E-02	-	1.71	-	6.96E-02	-	1.49	-	1.29	-	1.32	-
8	2.43E-02	1.85	8.57E-01	1.00	1.93E-02	1.85	8.60E-01	0.79	5.05E-01	1.36	5.13E-01	1.36
16	4.98E-03	2.29	3.68E-01	1.22	3.94E-03	2.29	3.64E-01	1.24	1.37E-01	1.88	1.39E-01	1.88
24	2.49E-03	1.71	2.65E-01	0.81	1.96E-03	1.72	2.61E-01	0.82	1.01E-01	0.77	1.02E-01	0.77
32	1.29E-03	1.95	1.86E-01	0.98	1.02E-03	2.27	1.83E-01	1.24	6.15E-02	1.71	6.23E-02	1.71

Table 4.3: eBDF (2nd-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle\nabla e_u\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle\nabla e_T\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{1,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{2,0}$	Rate
4	3.00E-02	-	5.42E-01	-	5.15E-03	-	9.01E-02	-	8.11E-01	-	8.81E-01	-
8	4.54E-03	2.72	1.51E-01	1.84	5.24E-04	3.30	1.80E-02	2.32	1.77E-01	2.20	1.86E-01	2.25
16	5.36E-04	3.08	3.37E-02	2.16	5.06E-05	3.37	3.59E-03	2.33	3.33E-02	2.41	3.44E-02	2.43
24	2.09E-04	2.32	1.70E-02	1.69	1.86E-05	2.47	1.99E-03	1.45	1.66E-02	1.72	1.70E-02	1.73
32	1.35E-04	1.52	8.46E-03	2.42	6.85E-06	3.47	9.17E-04	2.69	9.62E-03	1.90	9.83E-03	1.90

Table 4.4: PEA (1st-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_u \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_T \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{1,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{2,0}$	Rate
4	4.45E-01	-	3.48	-	6.99E-02	-	1.49	-	19.79	-	20.14	-
8	3.72E-01	0.26	2.61	0.41	2.00E-02	1.81	8.60E-01	0.79	13.50	0.55	13.72	0.55
16	2.24E-01	0.73	1.56	0.75	5.10E-03	1.97	3.64E-01	1.24	7.22	0.90	7.34	0.90
24	1.59E-01	0.85	1.12	0.81	3.05E-03	1.27	2.62E-01	0.81	4.77	1.03	4.84	1.03
32	1.21E-01	0.93	8.60E-01	0.92	2.08E-03	1.34	1.83E-01	1.24	4.09	0.53	4.15	0.53

Table 4.5: PEA (2nd-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_u \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_T \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{1,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{2,0}$	Rate
4	1.23E-01	-	9.27E-01	-	5.49E-03	-	8.95E-02	-	10.64	-	11.56	-
8	3.20E-02	1.94	3.02E-01	1.62	8.41E-04	2.71	1.83E-02	2.29	6.18	0.78	6.48	0.84
16	8.06E-03	1.99	8.54E-02	1.82	1.66E-04	2.34	3.71E-03	2.30	3.09	1.00	3.19	1.02
24	3.58E-03	2.00	4.14E-02	1.79	6.70E-05	2.23	2.04E-03	1.48	2.07	0.99	2.12	1.00
32	2.02E-03	1.98	2.64E-02	1.56	3.60E-05	2.16	9.49E-04	2.66	1.59	0.91	1.63	0.92

Table 4.6: ACE (1st-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_u \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_T \rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle e_p \rangle\|\|\|_{2,0}$	Rate
4	4.10E-01	-	2.56	-	6.96E-02	-	1.49	-	12.40	-	6.86	-
8	3.66E-01	0.16	1.91	0.42	1.93E-02	1.85	8.61E-01	0.79	8.41	0.56	4.55	0.59
16	2.02E-01	0.86	1.05	0.86	3.96E-03	2.29	3.64E-01	1.24	4.22	0.99	2.40	0.92
24	1.32E-01	1.04	7.36E-01	0.89	1.98E-03	1.70	2.62E-01	0.82	2.71	1.10	1.61	0.99
32	1.05E-01	0.79	5.96E-01	0.74	1.19E-03	1.77	1.83E-01	1.24	2.26	0.63	1.35	0.60

Table 4.7: ACE (2nd-order): Errors and rates for average velocity, temperature, and pressure in corresponding norms.

m	$\|\|\|\langle e_u\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_u\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_T\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle \nabla e_T\rangle\|\|\|_{2,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{\infty,0}$	Rate	$\|\|\|\langle e_p\rangle\|\|\|_{2,0}$	Rate
4	1.18E-01	-	6.55E-01	-	5.36E-03	-	9.16E-02	-	7.23	-	4.20	-
8	3.03E-02	1.96	1.95E-01	1.75	5.59E-04	3.26	1.82E02	2.33	4.05	0.56	2.11	0.99
16	7.61E-03	2.00	5.00E-02	1.96	7.76E-05	2.85	3.63E-03	2.33	1.86	2.17	9.78E-01	1.11
24	3.33E-03	2.04	2.32E-02	1.89	3.13E-05	2.24	2.01E-03	1.47	1.12	3.27	5.46E-01	1.44
32	1.82E-03	2.10	1.29E-02	2.05	1.62E-05	2.29	9.26E-04	2.69	1.06	3.26	5.77E-01	-0.19

4.4.4 The double pane window problem

The double pane window problem is a classic test for numerical methods designed for natural convection [137]. The problem is the flow of air, $Pr = 0.71$, in a unit square cavity subject to no-slip boundary conditions. The horizontal walls are adiabatic and vertical wall temperature is maintained at constant temperature [136]; see Figure 4.3.

This problem setup simulates a window consisting of two glass walls with a column of air between: a double pane window. From the practical viewpoint, the objective is to calculate the average Nusselt numbers at the vertical (glass) walls. The Nusselt number measures the flux of heat and, therefore, is a measure of the quality of the window as an insulator. For our purposes, the quantities of interest are: $\max_{y\in\Omega_h} u_1(0.5, y, t^*)$, $\max_{x\in\Omega_h} u_2(x, 0.5, t^*)$, the local Nusselt number at vertical walls, and average Nusselt number at the hot wall. The latter two are calculated via

$$Nu(x,t) = -n \cdot \nabla T,$$
$$Nu_{avg} = \int_{\Gamma_{D_1}} Nu(x,t)ds.$$

We first validate each of ensemble algorithms. We set $J = 2$ and vary $Ra \in \{10^3, 10^4, 10^5, 10^6\}$. In this range of Ra, the fluid possesses a core flow enveloped by a boundary layer of thickness $\mathcal{O}(Ra^{-1/4})$ [49]. Consequently, we specify the finite element mesh as a division of $(0,1)^2$ into 64^2 squares with diagonals connected with a line within each square in the same direction; see Figure 4.4.

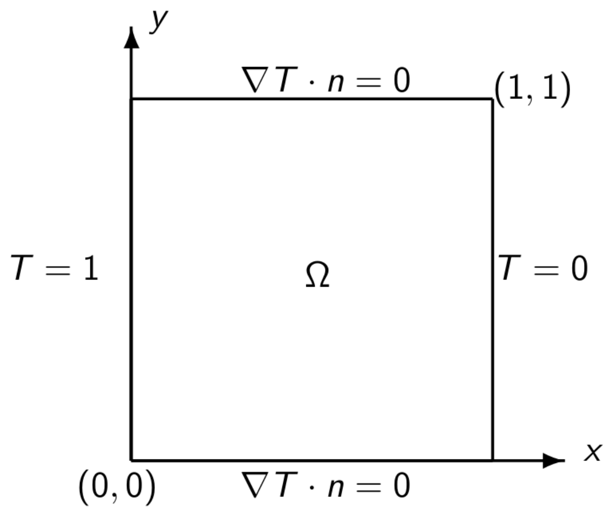

Figure 4.3: Domain & BCs: double pane window problem.

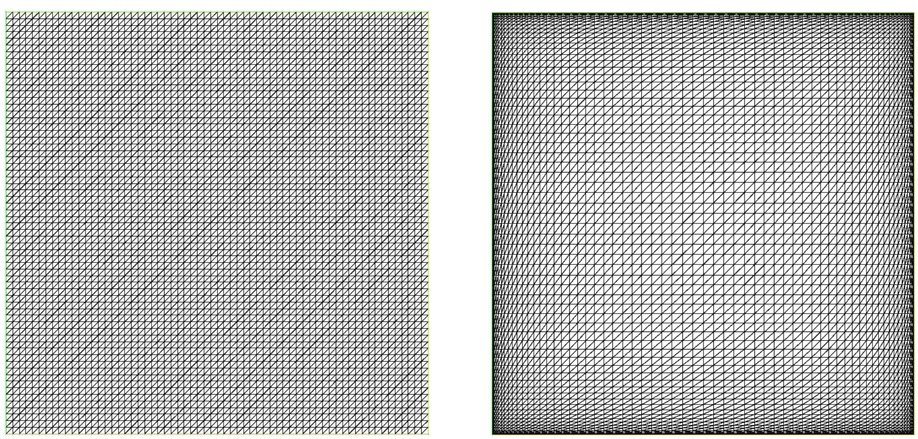

Figure 4.4: Mesh: double pane window problem, $10^3 \leq Ra \leq 10^6$ (left) and $10^7 \leq Ra \leq 10^8$ (right).

This problem has two fundamentally important timescales. The conductive timescale $t_{\Delta} = \frac{L^2}{\nu}$, where L is the length of the square and ν is the fluid viscosity, determines the time to reach steady state. The convective timescale $t_{\nabla} = t_{\Delta} Ra^{-1/2}$ determines the required timestep [61]. Thus, we set the timestep $\Delta t = C(Ra)Ra^{-1/2}$ with $C(Ra) = \{0.025, 0.1, 0.025, 0.1\}$, respectively. We set $\epsilon = 100\Delta t^i$ for **PEA** and $\epsilon = 0.01\Delta t$ and $\epsilon = \Delta t^2$ for first- and second-oder **ACE** and **ACE-T**, respectively. Moreover, for **ACE-T**, the tuning parameters C_ν and σ_T are set to $\frac{\sqrt{2}}{2}$ and 1, respectively.

The initial conditions are generated via the BV algorithm,

$$
\begin{aligned}
u_{\pm}(x, y, 0) &:= u(x, y, 0; \omega_{1,2}) = (u_1^{prev} + bv(u_1; \pm\delta_1), u_2^{prev} + bv(u_2; \pm\delta_2))^T, \\
T_{\pm}(x, y, 0) &:= T(x, y, 0; \omega_{1,2}) = T^{prev} + bv(T; \pm\delta_3), \\
p_{\pm}(x, y, 0) &:= p(x, y, 0; \omega_{1,2}) = p^{prev} + bv(p; \pm\delta_4),
\end{aligned}
$$

where the subscript *prev* denotes the solution from the previous value of Ra; for $Ra = 10^3$, the previous values are all set to 0 (rest). The BV, $bv(T; +\delta_3)$, is presented in Figure 4.5 for first-order **ACE** and varying Ra. Forcings are identically zero for $j = 1, 2$. Since the fluid reaches a steady state in this setting, a stopping condition is prescribed:

$$
\max_{0 \le n \le N-1} \left\{ \frac{\|u_h^{n+1} - u_h^n\|}{\|u_h^{n+1}\|}, \frac{\|T_h^{n+1} - T_h^n\|}{\|T_h^{n+1}\|} \right\} \le 10^{-5}. \tag{4.220}
$$

Plots of Nu at the hot and cold walls are presented in Figures 4.6 and 4.7. Computed values of the remaining quantities are presented, alongside several of those seen in the literature, in Tables 4.8 - 4.10. Figures 4.8 and 4.9 present the velocity streamlines and temperature isotherms for the averages. All results are consistent with benchmark values in the literature [22, 99, 136, 138, 145].

We also compare the run times of each the algorithms. Standard GMRES, with residual tolerance $TOL = 10^{-7}$, is used for the velocity and temperature solves. Results are presented in Figure 4.10. **ACE** and **ACE-T** are comparable, as expected. We see that **ACE** is at least 1.5 times faster than its **eBDF** counterpart. The speed gain of first-order **PEA** and **ACE** over first-order **eBDF** is most dramatic with a 2.5 to 22.5 speed up. Interestingly, they do not suffer from increased run time, over this range of Ra, as **eBDF** does. Further, the penalty and artificial compression based second-order algorithms do not see the same

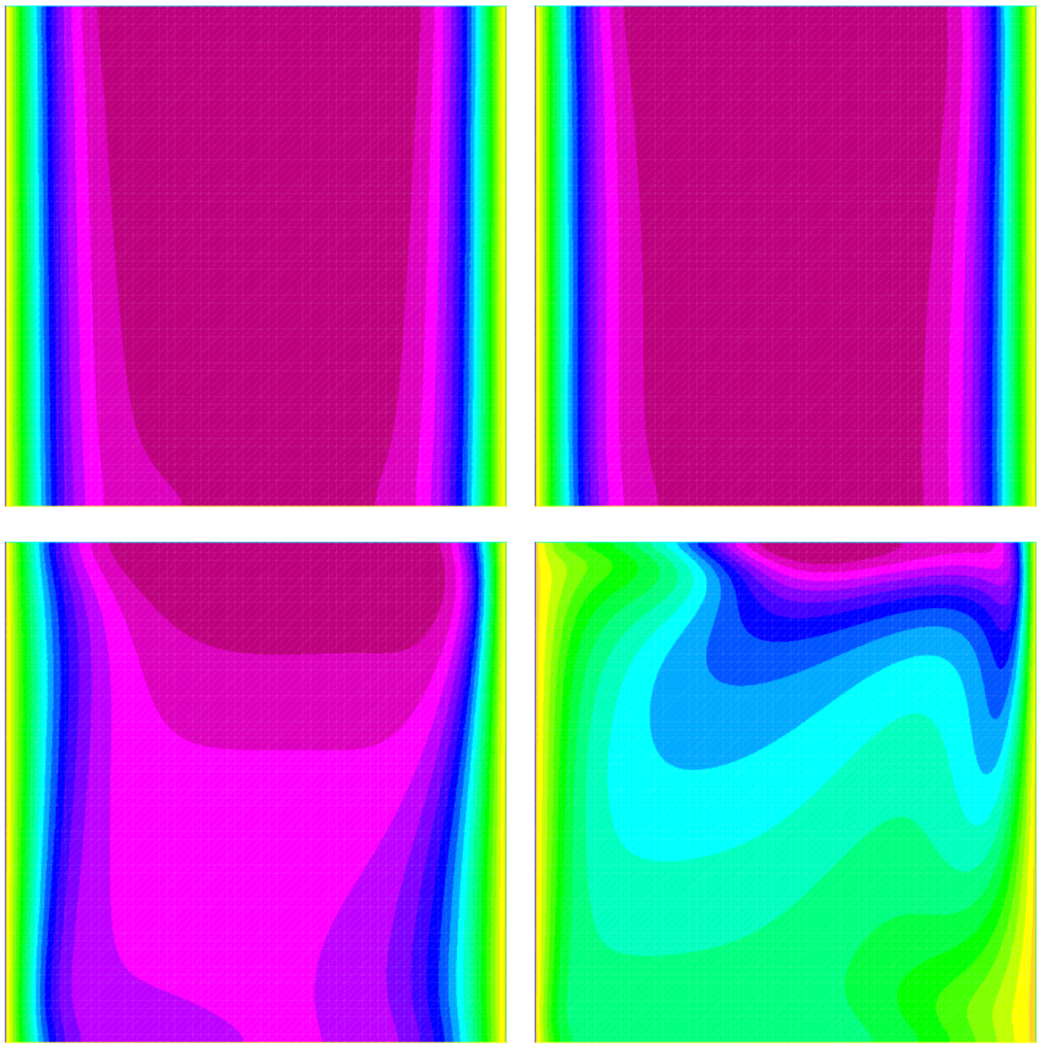

Figure 4.5: BV ($bv(T;+\delta_3)$): $Ra = 10^3, 10^4$ (top row), 10^5, and 10^6 (bottom row), left to right.

gains as their first-order counterparts over **eBDF**. This is likely due to the $\frac{1}{\epsilon}$ and $\frac{\Delta t}{\epsilon}$ scaling in front of the grad-div matrix.

For the Rayleigh number range specified above, the fluid reached a steady state. If Ra is increased, the fluid transitions to non-stationary behavior and eventually turbulent. As mentioned earlier in this chapter, our numerical methods are designed for laminar flows; near and into the onset of turbulence, these algorithms will breakdown owing to condition (4.52). This issue motivated, impart, the development of **ACE-T**. Thus, we use this algorithm to quantify the behavior for $Ra \in \{10^7,\ 10^8\}$. We utilize the graded mesh seen in Figure 4.4 to resolve the boundary layer. Results are reported for second-order **ACE-T**. The local variation of the Nusselt number is presented in Figure 4.11. Streamlines and isotherms are presented in Figure 4.12 and pertinent quantities are presented in Table 4.11. Once again, all results are consistent with the literature [2, 104, 138].

4.5 CONCLUSION

In this chapter, we presented eight new ensemble time-stepping schemes for numerically simulating Boussinseq flow subject to uncertain data. The first pair of algorithms **eBDF** (4.40) - (4.42) were based upon linearly implicit BDF schemes. The convective, diffusive, rotation, and conductive terms were treated in implicit-explicit fashion by decompositions of the convective velocity, viscosity, rotation rate, and thermal conductivity into ensemble averages and fluctuations. The resulting algorithms require J linear solves with the same coefficient matrix but different right-hand sides, at each timestep, for the velocity and temperature equations. Therefore, storage requirements and turnaround times are reduced. Nonlinear, energy stability and optimal-order convergence were proven for each algorithm under both a CFL-type condition (condition (4.52)), involving fluctuations of the velocity, and a condition involving the ratio between fluctuations of viscosity and thermal conductivity and their means (condition (4.53)).

Building upon this, **PEA** (4.43) - (4.44) and **ACE** (4.46) - (4.47) were developed using the penalty and artificial compressibility methods to further reduce complexity and com-

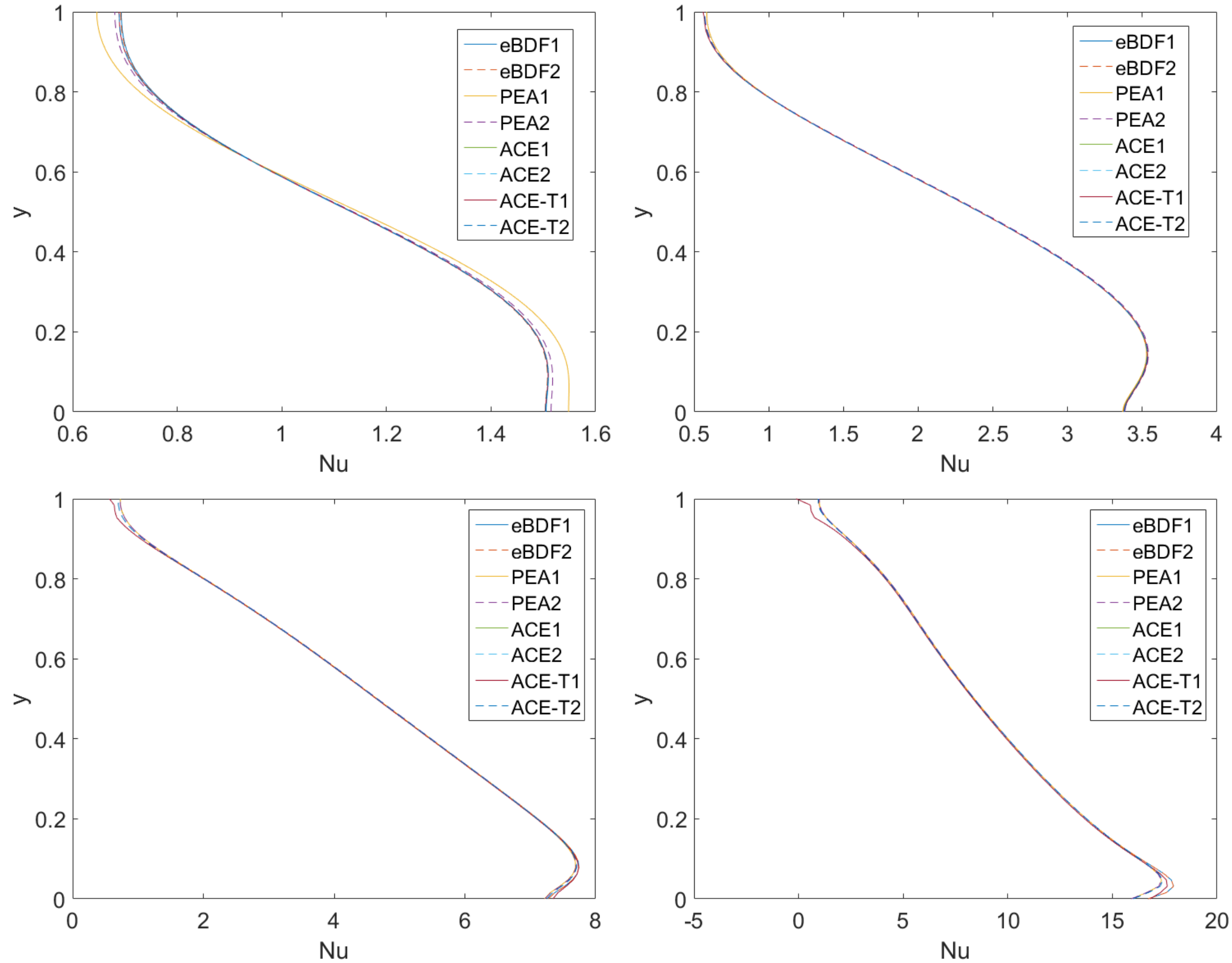

Figure 4.6: Variation of the local Nusselt number at the hot wall: $Ra = 10^3, 10^4$ (top row), 10^5, and 10^6 (bottom row), left to right.

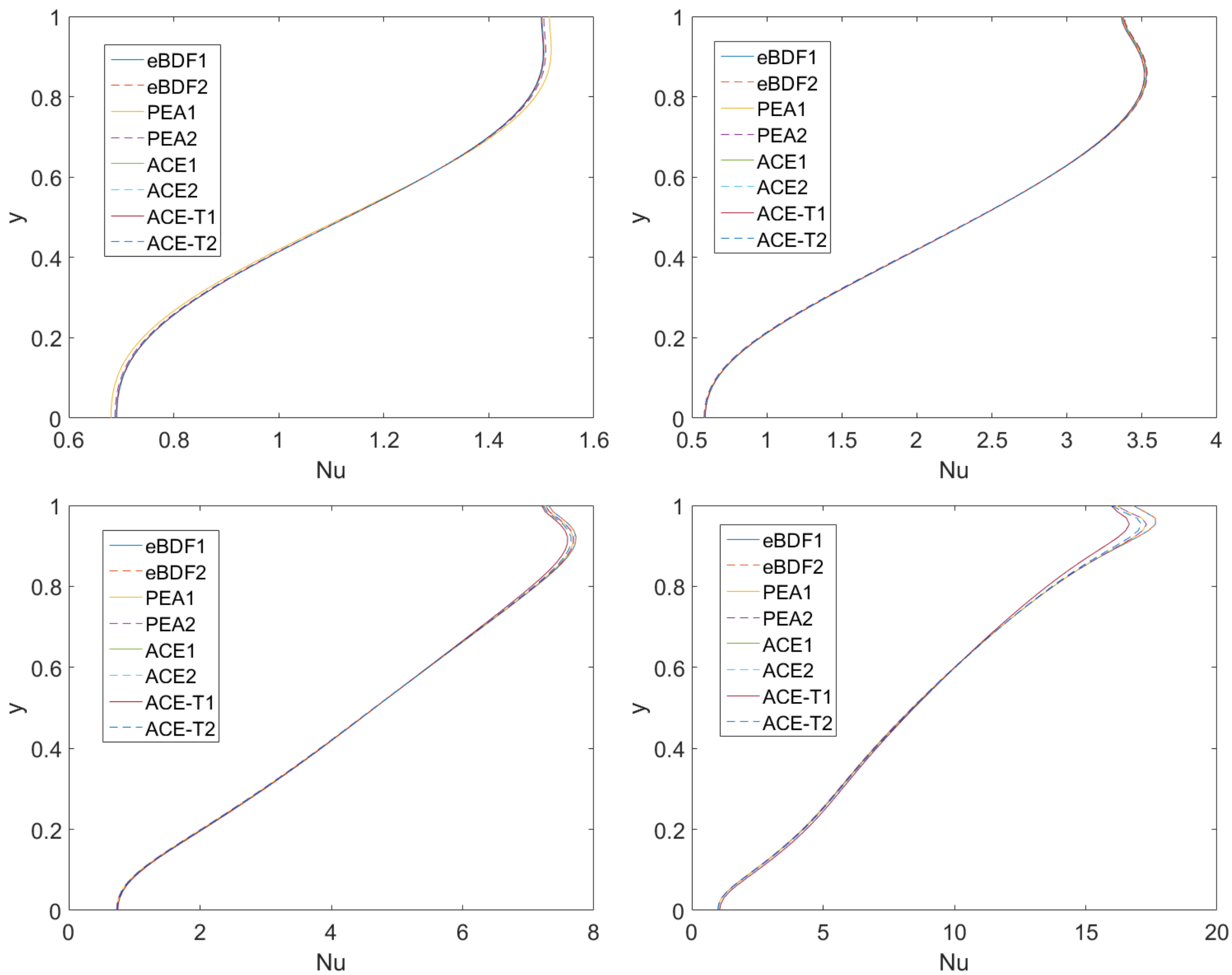

Figure 4.7: Variation of the local Nusselt number at the cold wall: $Ra = 10^3, 10^4$ (top row), 10^5, and 10^6 (bottom row), left to right.

Table 4.8: Comparison: maximum horizontal velocity at x = 0.5 & mesh size, double pane window problem.

Ra	eBDF1	eBDF2	PEA1	PEA2	ACE1	ACE2	ACE-T1	ACE-T2
10^3	3.65	3.65	3.64	3.65	3.64	3.64	3.64	3.64
10^4	16.18	16.18	16.18	16.18	16.16	16.16	16.16	16.16
10^5	34.76	34.73	34.65	34.62	34.69	34.62	34.69	34.62
10^6	64.80	64.79	63.84	63.86	65.26	64.25	65.26	64.25

Table 4.9: Comparison: maximum vertical velocity at y = 0.5 & mesh size, double pane window problem.

Ra	eBDF1	eBDF2	PEA1	PEA2	ACE1	ACE2	ACE-T1	ACE-T2
10^3	3.70	3.70	3.82	3.73	3.70	3.70	3.70	3.70
10^4	19.60	19.60	19.63	19.66	19.65	19.64	16.65	19.64
10^5	68.53	68.52	68.78	68.79	68.89	68.79	68.89	68.79
10^6	215.96	215.79	217.19	217.20	218.35	217.45	218.35	217.45

Table 4.10: Comparison: average Nusselt number at the hot wall.

Ra	eBDF1	eBDF2	PEA1	PEA2	ACE1	ACE2	ACE-T1	ACE-T2
10^3	1.12	1.12	1.13	1.12	1.12	1.12	1.12	1.12
10^4	2.24	2.24	2.24	2.25	2.24	2.24	2.24	2.24
10^5	4.53	4.53	4.52	4.52	4.50	4.51	4.50	4.51
10^6	8.88	8.88	8.83	8.83	8.76	8.82	8.76	8.82

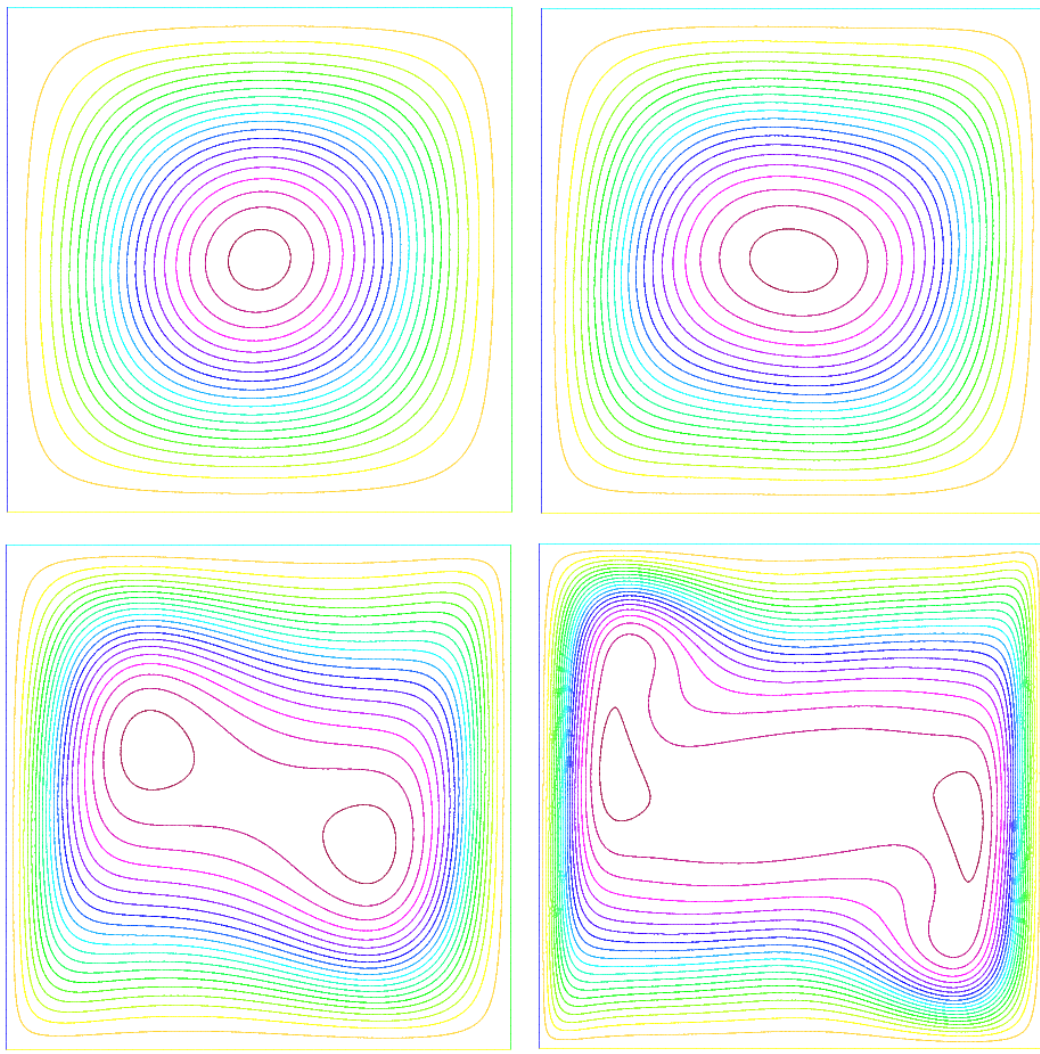

Figure 4.8: Streamlines: $Ra = 10^3, 10^4$ (top row), 10^5, and 10^6 (bottom row), left to right.

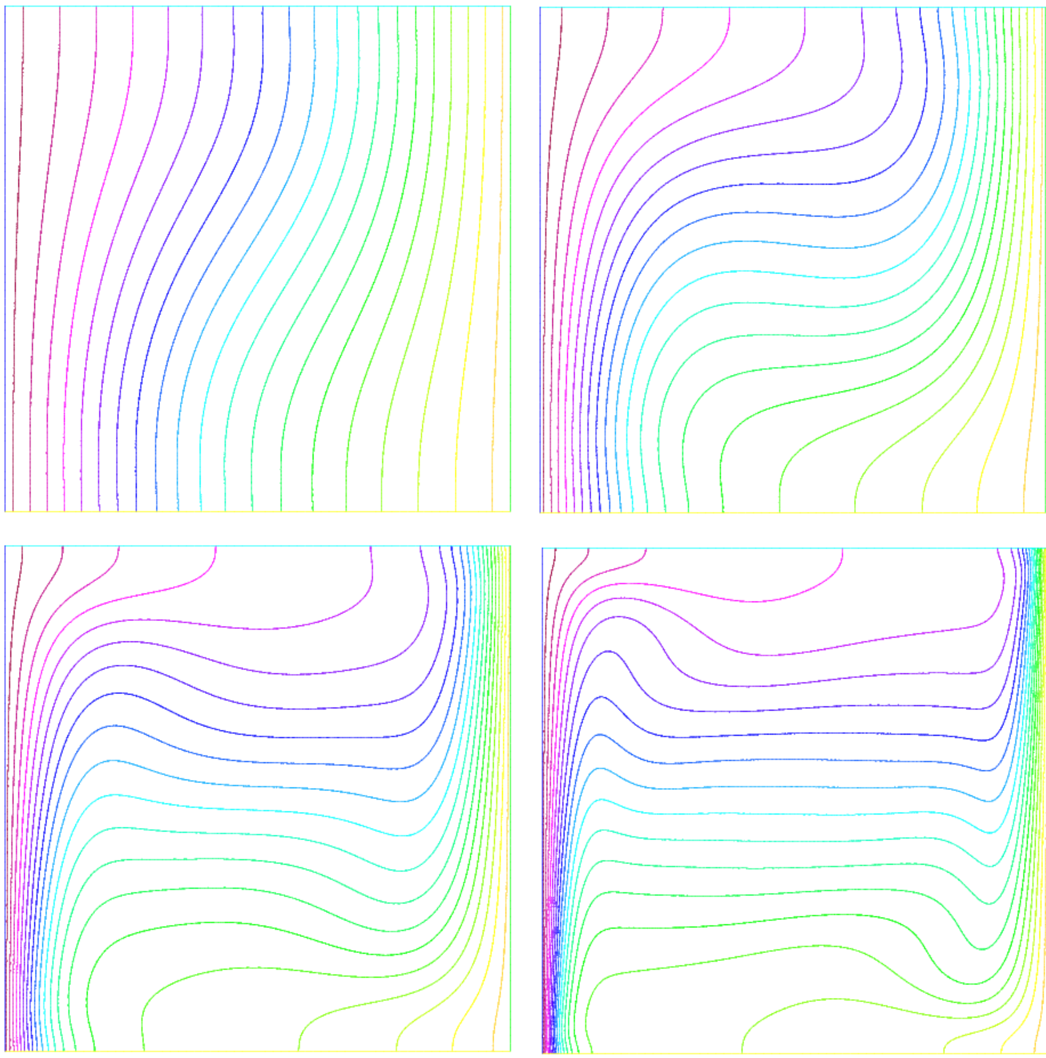

Figure 4.9: Isotherms: $Ra = 10^3, 10^4$ (top row), 10^5, and 10^6 (bottom row), left to right.

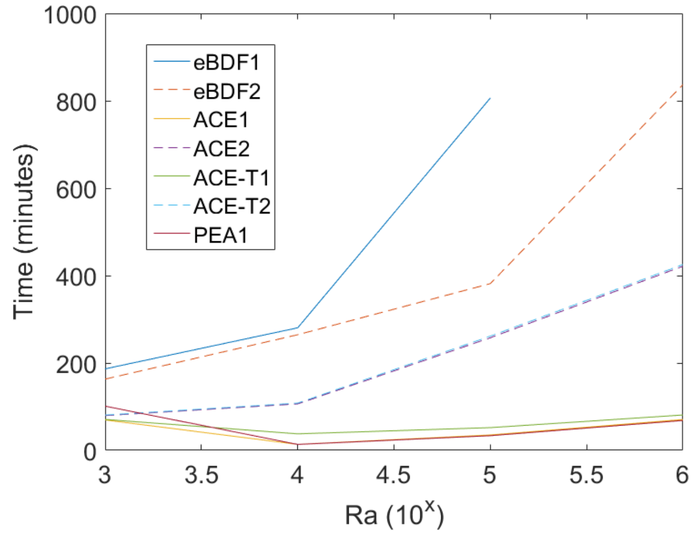

Figure 4.10: Time to steady state: **ACE** performs best followed by **eBDF**.

putation time by effectively decoupling the velocity and pressure solves. Nonlinear, energy stability and optimal-order convergence were proven for each algorithm, under appropriate choice of ϵ, under the same conditions. Lastly, the **ACE-T** (4.49) - (4.50) algorithm was developed for turbulent flow. It was proven stable under a less restrictive condition (condition (4.54)) on the velocity fluctuations.

Numerical experiments were performed to verify and validate each algorithm. In particular, we illustrated the expected convergence rates developed by our theoretical analyses. It was found that our convergence estimates are sub-optimal for the pressure solution with respect to the penalty/artificial compression parameter ϵ. In particular, Theorem 15 required $\epsilon = \mathcal{O}(\Delta t^{2i})$ for first- and second-order convergence of the pressure approximation in $\sqrt{\epsilon}|||\cdot|||_{2,0}$. Further, in Theorem 16, second-order **ACE** required $\epsilon = \mathcal{O}(\Delta t^{3})$ for second-order convergence of pressure in $\sqrt{\epsilon}|||\cdot|||_{\infty,0}$. Otherwise, our estimates are consistent with what is seen experimentally; that is, optimal-order convergence of velocity and temperature in $|||\cdot|||_{\infty,0}$ and $|||\cdot|||_{2,1}$.

We also utilized these algorithms on a problem of technological significance: the double pane window problem. All algorithms produced accurate results but at varying turnaround

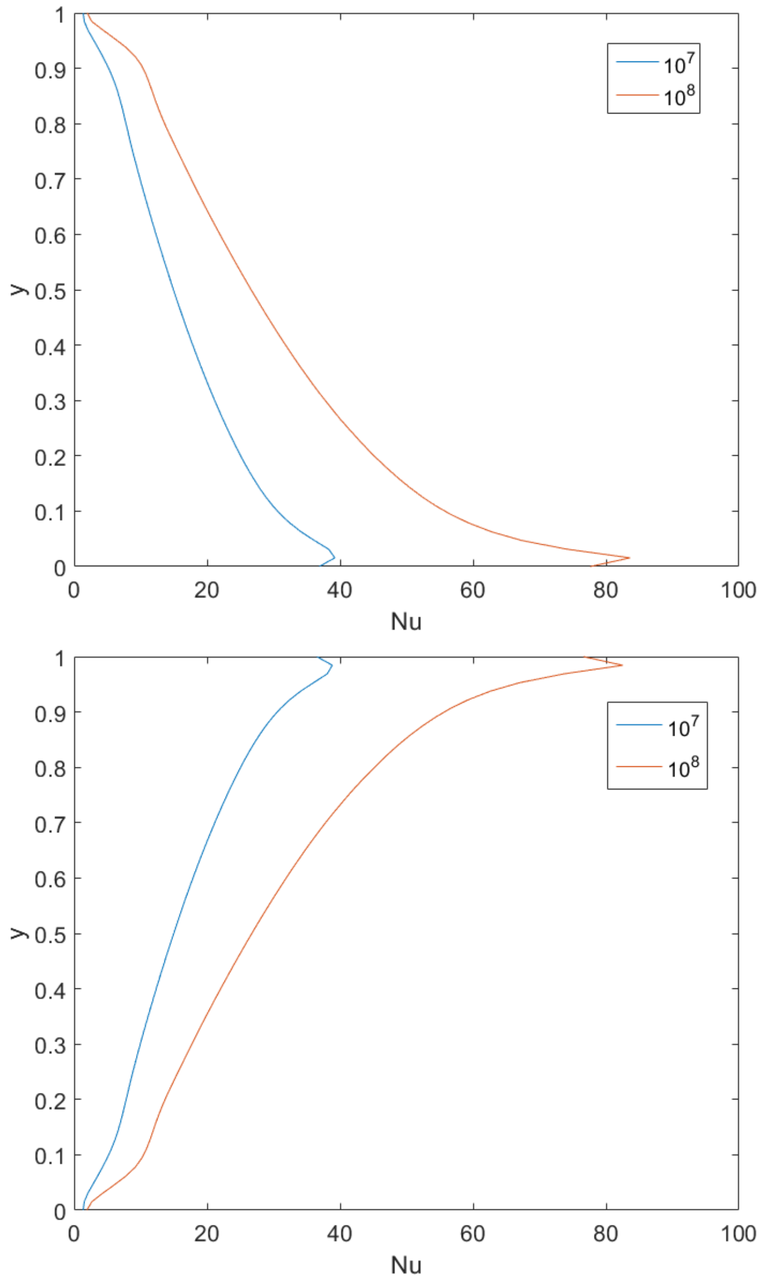

Figure 4.11: Variation of the local Nusselt number: hot wall (top) and cold wall (bottom).

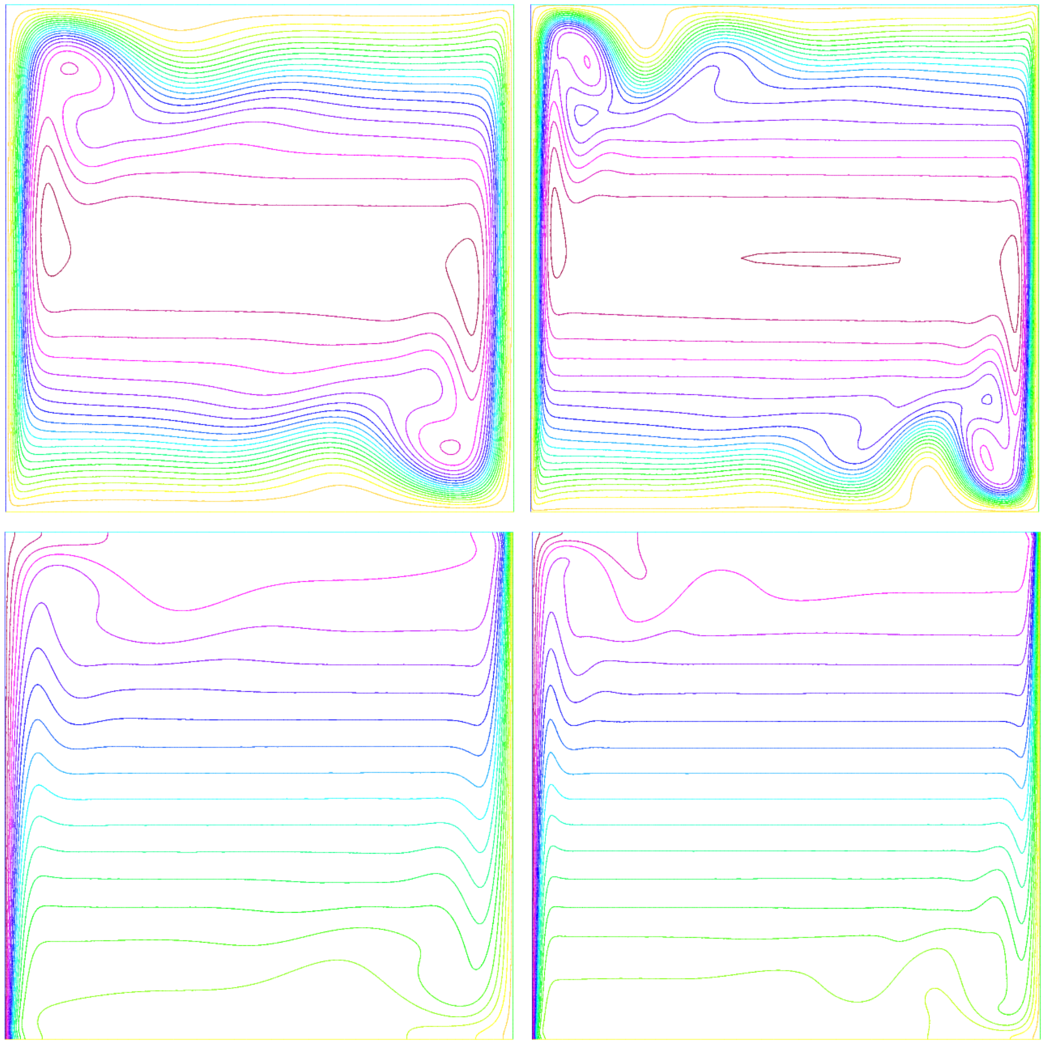

Figure 4.12: Streamlines (top row) and isotherms (bottom row), $Ra = 10^7, 10^8$, left to right.

Table 4.11: Second-order **ACE-T** is consistent with literature.

Ra	$\max_{y\in\Omega_h} u_1(0.5, y, t^*)$				$\max_{x\in\Omega_h} u_2(x, 0.5, t^*)$				Nu_{avg}		
-	Present study	Ref. [138]	Ref. [104]	Ref. [2]	Present study	Ref. [138]	Ref. [104]	Ref. [2]	Present study	Ref. [138]	Ref. [2]
10^7	146.23	143.56	145.27	148.60	698.45	714.48	703.25	699.20	16.51	16.66	16.52
10^8	311.10	296.71	283.69	322.7	2209.34	2223.44	2259.08	2223.00	30.16	31.49	30.31

times. **PEA** and **ACE** are fastest, **ACE-T** next, and **eBDF** is last. Second-order methods tended to be slower than their first-order counterparts but required fewer timesteps. Furthermore, second-order **PEA** was too ill-conditioned to operate effectively with standard GMRES. Moreover, for $\Delta t << 1$, **PEA** and **ACE** are both subject to potential solver breakdown.

A myriad of open questions still exist, which we collect in Chapter 6. For example, it is expected that Theorems 15 and 16 can be improved so that they reflect the numerical experiments: $\epsilon = \mathcal{O}(\Delta t^i)$ yields optimal-order convergence in appropriate norms. Additionally, recent developments suggest that it is possible to alleviate solver breakdown due to the grad-div matrix [36]. Lastly, uncertain boundary conditions and domain have not been included, but would be an important next step.

5.0 PREDICTABILITY

It is difficult to make predictions, especially about the future.

Unknown (Danish proverb)

Predictability of a flow is the extent to which it is possible to accurately predict the flow given a theoretically complete knowledge of the governing equations [134]. The ensemble algorithms developed in the previous chapter addressed the competition between mesh density and ensemble size. We recall that ensemble calculations are necessary since uncertainties in the initial data or model can destroy the fidelity of the approximate solutions. Potential solution fidelity can be quantified, in a sense, by the notion of predictability.

Some of the early pioneering works surrounding this discovery (of predictability) are recalled below. Notably, it is the weather community that has been at the forefront; see, e.g., [80, 91]. Early meteorological studies using computing systems encountered the issue of predictability and paved a path toward understanding and, consequently, developing tools for quantifying it. Charney [15], noting that the geostrophic approximation could produce poor results, revisited the primitive equations. In this setting, he noted that inaccurate initial values of wind and pressure gave rise to spurious oscillations that obscured meteorologically significant motions.

Later, Philips' [112] computations exhibited "explosive" kinetic energy growth after a certain simulation time while studying the general circulation of the atmosphere. Thompson [134] surmises that the growth of inherent errors produce increasing errors in wind predictions over a period of a few days. Moreover, this error depends on the final simulation time and perturbations to the initial value of wind, among others.

Lorenz [92] noticed that if present and past states are not known with absolute accuracy, the fidelity of the numerical approximation will severely degrade. Investigating further [93–95], he concludes that even if the perfect models and observations were known, the state of the atmosphere could be predictable up to about two weeks. Since the days of this startling conclusion, many tools have been utilized/devised to quantify and reach this predictability horizon. Leading Lyapunov exponents and ensemble calculations are examples of such tools.

In more recent years, it has been argued that the leading Lyapunov exponent is inadequate for quantifying predictability [9]. For example, it is a global quantity associated with a large time limit; that is, the maximal time averaged (exponential) rate of divergence of nearby trajectories. Thus, it does not account for local fluctuations of this rate which can be important: exponential divergence need not occur everywhere at all times. As a solution, researchers have proposed finite time leading Lyapunov exponents [9, 59], local Lyapunov exponents [29], etc. We utilize the former.

Recent works involving applications to predictability of ensemble algorithms include [72, 83]. In particular, Jiang [72] studied predictability of 2d iso-thermal flow between two offset cylinders. The flow is driven by a counter-clockwise rotation that decays to zero after a prescribed time. The average effective Lyapunov exponent was utilized to estimate the predictability horizon.

Khankan [83] studied the predictability of temperature spatial averages for 2d non-isothermal flow within an annulus; a 2d representation of the earth's atmosphere. Average effective Lyapunov exponents were calculated and it was found that the average (in space) temperature distribution has an infinite predictability horizon, for the selected parameter choices. Moreover, predictability increased as the size of the domain was increased.

Herein, we will utilize second-order **ACE** to investigate predictability horizons of two numerical experiments. The numerical experiments considered are the double pane window problem and manufactured solution problem in Chapter 4 Sections 4.4.3 and 4.4.4. Altogether, we will quantify the predictability of solution quantities and their spatial averages subject to rotational effects.

5.1 NUMERICAL TESTS

As stated in the introduction, we seek to quantify the predictability of certain natural convection problems. The following quantities will play a central role towards this goal. We define the relative energy fluctuation, average effective Lyapunov exponent, and predictability horizon as follows.

Definition 2. *Let $\chi = u,\ T,\ p$ be the solution to the Boussinesq equations (4.14) - (4.17). Denote $\chi_\pm$ as solutions generated with a positive/negative pair of perturbed initial conditions. Then, the relative energy fluctuation is defined as*

$$r(t) := \frac{\|\chi_+ - \chi_-\|^2}{\|\chi_+\|\|\chi_-\|},$$

and the average effective Lyapunov exponent is

$$\gamma_\tau(t) := \frac{1}{2\tau}\log\big(\frac{r(t+\tau)}{r(t)}\big),$$

with $0 < t+\tau \le t^*$. Let $tol > \|(\chi_+ - \chi_-)(0)\|$, then the tol-predictability horizon is

$$t_p := \frac{1}{\gamma_{t^*}(0)}\log\Big(\frac{tol}{\|(\chi_+ - \chi_-)(0)\|}\Big).$$

The definitions above indicate that the average effective Lyapunov depends on the time t that a perturbation is introduced and the delay or time that the system is measured τ. Further, the tol-predictability horizon is dependent on the prescribed tolerance *tol*; that is, the acceptable level of deviation of a prediction from the true state.

Recall, the discrete differential filter is: Given $\chi_h \in L^2(\Omega)^d$ or $L^2(\Omega)$, find $\overline{\chi_h} \in X_h$ or W_h satisfying

$$\delta^2(\nabla\overline{\chi_h}, \nabla v_h) + (\overline{\chi_h}, v_h) = (\chi_h, v_h), \quad \forall v_h \in X_h \text{ or } W_{\Gamma_D,h}.$$

We utilize the new second-order **ACE** algorithm presented in Chapter 4 and apply a filter to each member of the ensemble at each timestep. For instance, consider the following algorithm: For each $1 \le j \le J$,

Step zero: Compute $\chi_\pm = \chi^0 + bv(\chi; \pm\delta_i)$ for $\chi = u, T, p$ using the BV algorithm.

Step one: Compute $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1})$ with second-order **ACE** (4.46) - (4.47).

Step two: Find $\overline{\chi_h}^{n+1} = \overline{u_h}^{n+1}$ or $\overline{T_h}^{n+1}$ via

$$\delta^2(\nabla\overline{\chi_h}^{n+1}, \nabla v_h) + (\overline{\chi_h}^{n+1}, v_h) = (\chi_h^{n+1}, v_h) \ \ \forall v_h \in X_h \ or \ W_{\Gamma_D,h}. \tag{5.1}$$

where δ is the filter radius. Below, we will focus on predictability of temperature averages; that is, $\chi = T$. For each value of $\frac{1}{m} \le \delta \le 1$, we can compute a filtered temperature $\overline{T_h}$ and calculate average effective Lyapunov exponents and predictability horizons defined above.

For our first tests, we consider the double pane window problem. The effects of increasing domain size and spatial filtering on the predictability of the flow are studied. For the former, the domains are defined as $(0, L)^2$ with $L \in \{1, 1.2, 1.4, 1.6\}$ and associated Rayleigh number $Ra_L = Ra_1 L^3$; herein, $Ra_1 = 10^3$. The spatial mesh is constructed via Delaunay triangulation generated from mL points on each side of the domain. The timesteps are chosen the same as in Section 4.4.4. The final simulation time t^* is defined as the time for which the steady state criterion (4.220) is met. For simplicity, we set $tol = e\|(\chi_+ - \chi_-)(0)\|$, $\chi = u$, T, and p. Thus, $\frac{1}{\gamma_{t^*}(0)}$ corresponds to a solutions predictability horizon associated with this tolerance.

The average effective Lyapunov exponent $\gamma_{t^*}(0)$ is presented together with domain size in Table 5.1. We see that, as the length of the square domain increases, the average effective Lyapunov exponent becomes increasingly more negative. Thus, the temperature becomes increasingly predictable.

To study the effect of spatial filtering, the differential filter is applied to each temperature ensemble member, at each timestep. Filter radii of $\delta \in \{0, \frac{1}{30L}, \frac{1}{10}, \frac{3L}{10}, L\}$ are selected; the second corresponding to the mesh length h. These filtered quantities $\overline{T_h}^{n+1}(x;\omega_j)$ are used to calculate average effective Lyapunov exponents. The results are also tabulated in Table 5.1. As the filter radius increases, the temperature averages become increasingly predictable. Our results are consistent with those presented by Khankan [83].

The test problem with manufactured solution is now considered. We first consider $Ra \in \{10^2, 10^3, 10^4\}$ and calculate average effective Lyapunov exponents and both energies and variances, with the aim of providing an alternative viewpoint. The results are presented in Figures 5.1 - 5.4. Although we do not present results for filtered values, the conclusion is reversed: temperature averages become decreasingly predictable with increasing filter radii. The energy and variance are defined below.

Definition 3. *The energy is given by*

$$Energy := \|T\| + \frac{1}{2}\|u\|^2.$$

Definition 4. *The variance of* χ *is*

$$V(\chi) := \langle \|\chi\|^2 \rangle - \|\langle \chi \rangle\|^2 = \langle \|\chi'\|^2 \rangle.$$

In Figures 5.1 and 5.2, we see that as the Rayleigh number increases, velocity and temperature predictability decreases while pressure maintains high predictability. In Figure 5.3, we compare the energy of the approximate solutions associated with the positive and negative bred vectors and the ensemble average with the true solution for increasing Ra. It is clear that the ensemble average performs best. Evidently, the bred vector algorithm is doing as it should: generating highly divergent solutions which, when averaged, mitigate nonlinear error growth.

The variances of each solution are plotted in Figure 5.4. We see that higher Rayleigh numbers are associated with increased variance. Equivalently, approximate solutions become decreasingly reliable for faster flows. The variance of the pressure is especially interesting. Essentially, the approximate pressure rapidly deviates to a different "solution". However, once it reaches this solution, it does not deviate much. This is consistent with Figures 5.1 and 5.2 and our convergence tests in Chapter 4; that is, artificial compressibility methods tend to produce good velocity and temperature approximations, but pressure approximations can be grossly inaccurate.

Lastly, we fix the Rayleigh number to 10^4 and consider rotations such that $10^7 \leq Ta \leq 10^{13}$. The average effective Lyapunov exponent is calculated and the results are plotted in Figure 5.5. For this test problem, rotations have little to no effect on predictability until $Ta \approx 10^{10}$. Interestingly, we see that sufficiently large rotation rates can turn a flow with a finite predictability horizon into one with an infinite predictability horizon. This appears to be consistent with improvements in solution stability with rotation rate, reported in the literature [5, 17, 100, 130].

Table 5.1: $\gamma_{t^*}(0)$: Larger domain sizes and filter radius increase predictability.

δ/L	1	1.2	1.4	1.6
0	**-0.46**	**-0.52**	**-0.60**	**-0.65**
$\frac{1}{30L}$	**-6.84**	-5.56	-4.89	-4.39
$\frac{1}{10}$	**-7.57**	-6.26	-5.58	-5.07
$\frac{3L}{10}$	**-8.30**	-6.97	-6.29	-5.77
L	**-8.77**	-7.35	-6.63	-6.08

5.2 CONCLUSION

Predictability of non-isothermal fluid flow was studied. We considered two test problems, the double pane window benchmark and a problem with manufactured solution. Second-order **ACE** was used in conjunction with the BV algorithm to quantify predictability. In particular, average effective Lyapunov exponent, predictability horizons, and variance were defined and calculated. These quantities indicate how predictable a flow is and therefore the potential reliability of the numerical approximation.

From the first test, it was concluded that larger domains are more predictable. Moreover, filtering out small spatial scales increased predictability. In the second test, it was found that the ensemble average is the most likely temperature distribution and variance gives an estimate of prediction reliability. Further, sufficiently large rotations increase the predictability of a flow. Lastly, filtering out small spatial scales *decreased* predictability. Evidently, predictability is complex and highly problem-dependent. Additional tests are needed to draw more robust conclusions.

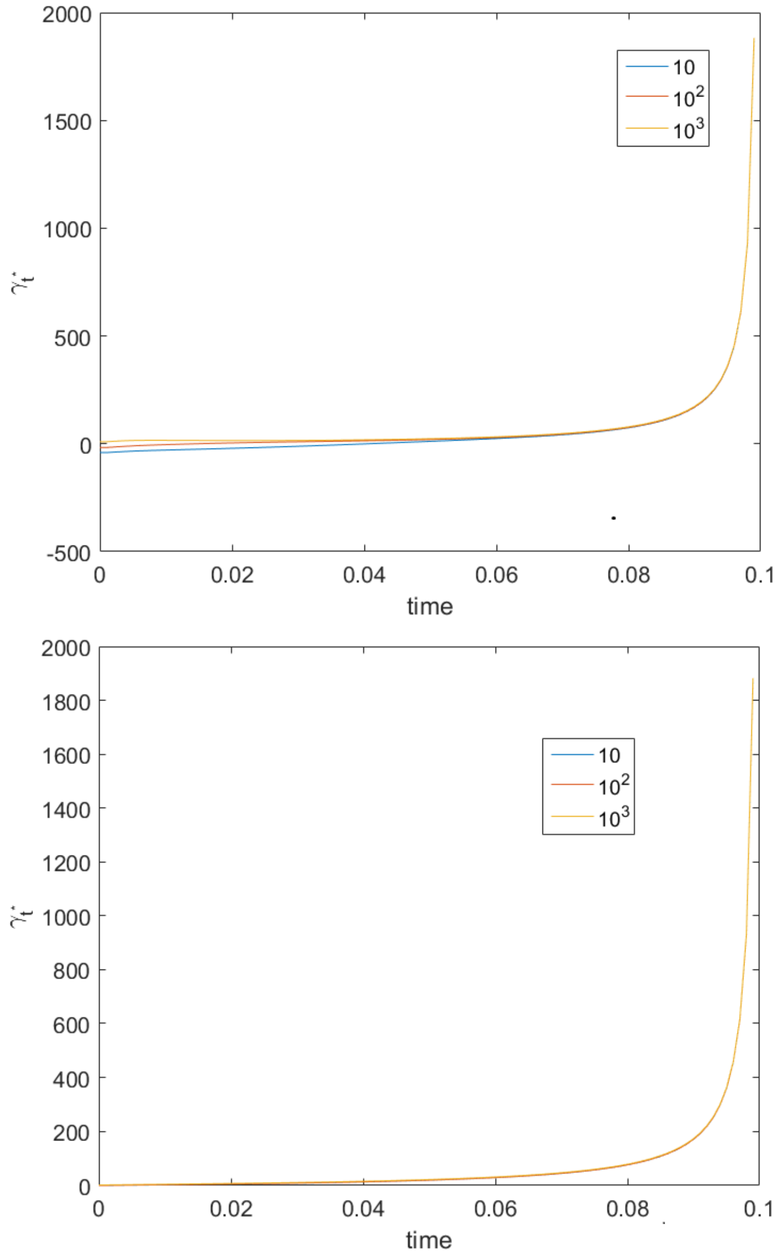

Figure 5.1: Lyapunov exponent: Increasing Ra reduces predictability; velocity (top) and temperature (bottom).

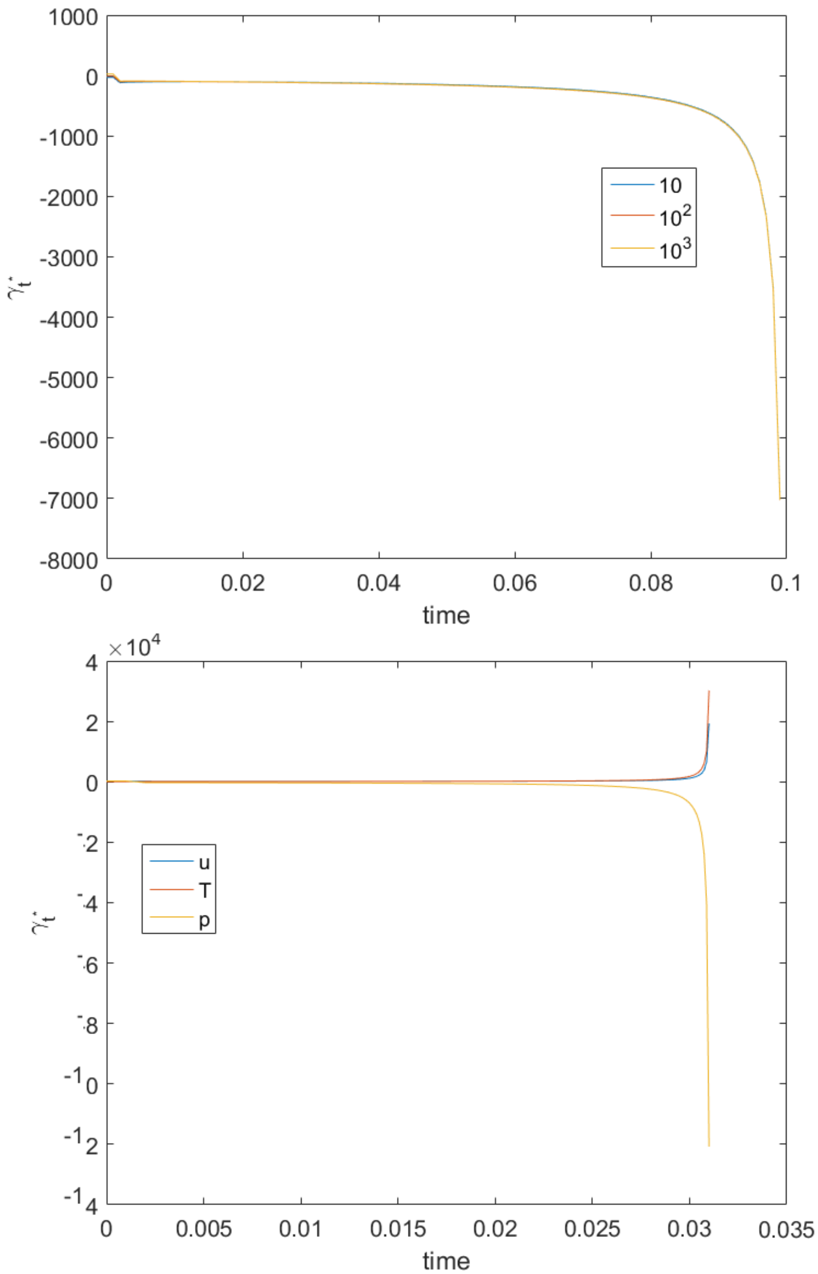

Figure 5.2: Lyapunov exponent: Increasing Ra reduces predictability; pressure (top) and all solutions for $Ra = 10^4$ (bottom).

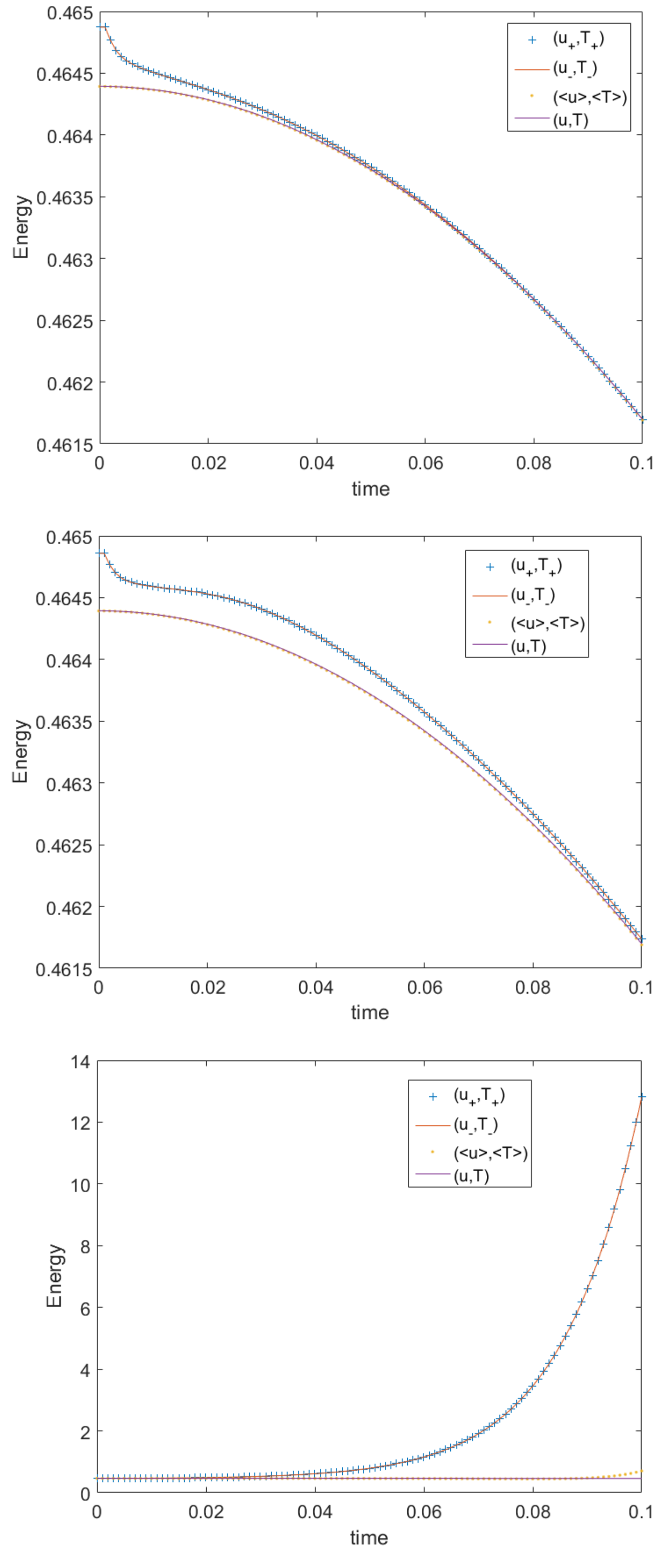

Figure 5.3: Energy: $Ra = 10^2$ to 10^4, top to bottom.

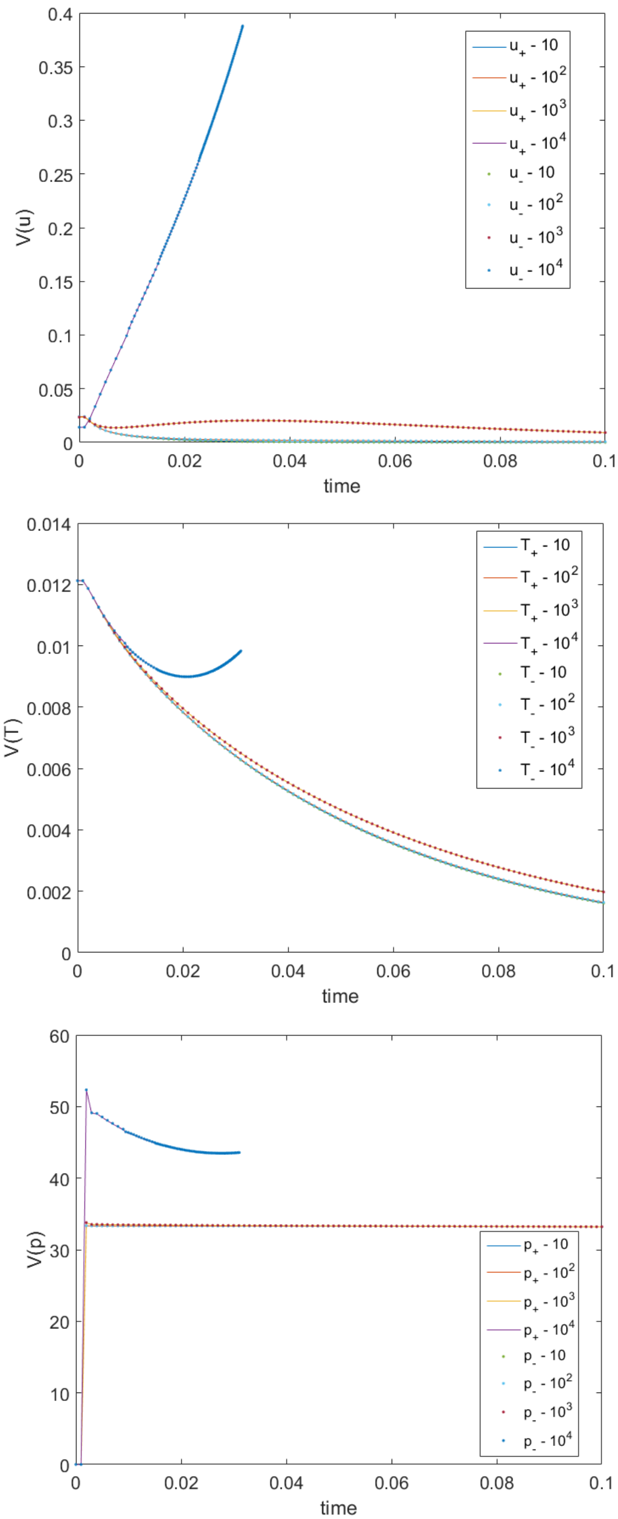

Figure 5.4: Variance: Increasing Ra reduces predictability; velocity, temperature, and pressure, top to bottom.

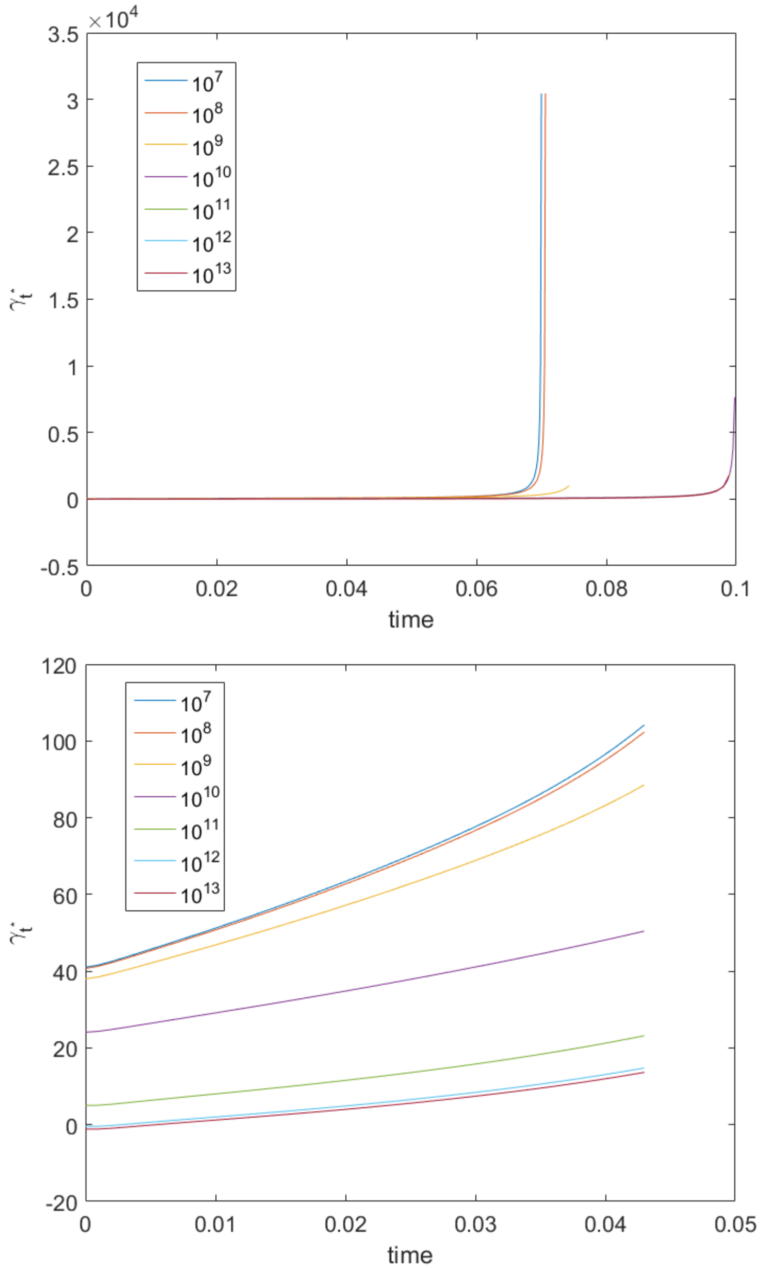

Figure 5.5: Lyapunov exponent: Large rotations can stabilize and increase predictability; varying Ta (top) and zoomed in (bottom).

6.0 CONCLUSIONS AND OPEN QUESTIONS

I open at the close.

Albus Dumbledore, Harry Potter and the Deathly Hallows [120]

The underlying motivation of this thesis was practical computing. Our concerns were three fold. Firstly, producing approximations that reproduce stability properties of the simulated physical phenomenon. Secondly, the development of efficient numerical methods that address the uncertainty in initial data. Lastly, we sought to apply our numerical methods to study predictability. In our pursuit, much was learned, failures were many, successes few, and the journey worth the effort. The mathematician can choose to pursue topics without immediate perceivable consequence, however, it is satisfying to perceive the tiniest sliver of the natural world previously unseen.

Recounting, longtime stability of approximate solutions to the Boussinesq equations using FEM in space and the BDF family in time was studied in Chapter 3. The discrete Hopf interpolant was introduced as a mathematical tool. It was then shown that, the velocity and temperature approximations can exhibit, at most, sub-linear growth in the final simulation time t^* under a mesh condition. The pressure approximations could grow at most linearly. The mesh condition required that, at the hot wall, the first mesh-line must be within $\mathcal{O}(Ra^{-1})$. It was noted that practitioners carry out numerical simulations on graded meshes, typically of $\mathcal{O}(Ra^{-1/4})$ near the boundaries, to resolve the boundary layer and, thereby, improve accuracy. Thus, our condition is more restrictive, possibly owing to a gap in the analysis, however, it is indicative of the value of graded meshes for both stability and accuracy.

In Chapter 4, we developed efficient algorithms addressing the need to run multiple realizations of a code with perturbed initial data. Understanding that ensemble calculations are essential, our aim was to improve upon them by decreasing storage requirements and turnaround time. Our tactic was to first decompose certain parameters (viscosity, thermal conductivity, and rotation rate) and the convective velocity into ensemble mean and fluctuating components. We then applied an IMEX discretization to the associated terms. Further, we introduced the artificial compression and penalty methods to remove the existing saddle point structure. We concluded with introducing a turbulence model based on the eddy viscosity hypothesis, Boussinesq assumption, Kolmogorov-Prandtl relation, and gradient-diffusion hypothesis. The fully discrete algorithms resulted in linear systems that share the same coefficient matrix, reducing storage and computation time. We proved that these algorithms were conditionally, nonlinearly, energy stable and optimally-order accurate in appropriate norms. Numerical experiments illustrated these properties.

We utilize our flagship algorithm, second-order **ACE**, to illustrate the use of ensembles in Chapter 5. We introduced several quantities including the average effective Lyapunov exponent, predictability horizon, and variance, to quantify predictability of flow variables. With these quantities, we studied the effect of domain size, averages with respect to spatial scales, and rotations on predictability. It was found that fixing all other variables, increasing the domain size increases predictability and sufficiently large rotations increase predictability. Filtering out smaller spatial scales could either increase or decrease predictability.

The remainder of this chapter, is devoted to open questions that have arisen in our pursuits. We hope that others will be inspired to tackle them. In Chapter 3, we saw that the interpolant appearing within the buoyancy term, $PrRa(\xi\tau, v_h)$, prevented a uniform in time result. Perhaps, under certain circumstances, e.g., Rayleigh-Bénard flow within the unit square, improvements could be made. In particular, if there exists $\phi \in Q_h$ such that

$$(\xi\tau, v_h) = (\nabla\phi, v_h) = (-\phi, \nabla \cdot v_h),$$

then uniform in time stability would follow. Such a result would require Q_h to contain piecewise continuous quadratics.

Alternatively, the mesh condition that arose, $\delta = \mathcal{O}(Ra^{-1})$, is extremely restrictive for most practical flows. Practitioners often use the laminar boundary layer scaling $\mathcal{O}(Ra^{-1/4})$ when constructing meshes, with good results. Thus, it would be interesting to see if our results could be improved to $\delta = \mathcal{O}(Ra^{-1/a})$ for $a > 0$. We conjecture that $a = 1/2$ is achievable. Experiments indicate that the boundary layer scales with $a = 1/4$, $2/7$, $1/3$, and $1/2$, as we approach and enter the turbulence regime; see, e.g., [49, 62, 63]. One entry point would be to improve the estimate (3.1) of Theorem 3:

$$|b^*(\chi_1, \tau, \chi_2)| \leq C\delta^b \Big(\epsilon^{-1}\|\nabla \chi_1\|^2 + \epsilon\|\nabla \chi_2\|^2\Big), \tag{6.1}$$

with $b > 1$. This, in turn, may lead one to consider a variant of the discrete Hopf interpolant, e.g., using piecewise quadratics or cubics instead of linears; non-conforming spaces could even allow piecewise constants.

Another interesting problem would be to use the discrete Hopf interpolant to quantify the energy dissipation rate and heat flux through the hot wall. In this direction, a first step would be to consider the semi-discrete (FEM space + continuous time) problem first. The "background flow" technique can be used followed by a time average; see, e.g., [77]. If the former is successful, the fully-discrete setting should then be considered. The discrete analogs of the "background flow" (discrete Hopf interpolant) and time average ($\frac{1}{N}\sum_{n=i-1}^{N-1}$) can then be employed.

In Chapter 4, all presented algorithms were conditionally, nonlinearly, energy stable. Condition (4.53) was a condition on the parameters, but was not especially restrictive since the ensemble set could be broken into several sets for which the condition held. Condition (4.54) was not very restrictive, however, the condition (4.52) could be and motivated the turbulence model used. It is an open question as to whether this condition could be improved through analysis or by modifying the algorithms. Regarding the latter, operator splitting seems to be a potential path forward. If exactly divergence-free elements [58, 78] are used, **ACE-T** is unconditionally stable.

We proved optimal-order convergence, in appropriate norms, of **PEA** provided $\epsilon = \mathcal{O}(\Delta t^{2i})$. A similar result was proven for second-order **ACE** with $\epsilon = \mathcal{O}(\Delta t^3)$. Numerical experiments suggested that these results were sub-optimal with respect to ϵ. We leave it as an open problem to determine whether the ϵ scaling can be improved. In particular, proving optimal-order accuracy, in appropriate norms, provided $\epsilon = \mathcal{O}(\Delta t^i)$.

The above results appear to be achievable, at least for **ACE**, owing to the work of Shen [123]. Shen analyzed the NSE with artificial compressibility and variants, drawing connections to projection methods. In particular, he proves an error estimate for the solutions of the model compared to solutions of the NSE; that is, under sufficient regularity [123], the solutions u^ϵ and p^ϵ, of the artificial compressibility model, satisfy

$$\|u(t) - u^\epsilon(t)\| + \Big(\int_{t_0}^{t} \|u(s) - u^\epsilon(s)\|_1^2 ds\Big)^{1/2} + \sqrt{\epsilon}\|p(t) - p^\epsilon(t)\| \leq C\epsilon. \tag{6.2}$$

Thus, by the triangle inequality

$$\begin{aligned}\|(u,p) - (u_h^{n+1}, p_h^{n+1})\| &\leq \|(u^\epsilon, p^\epsilon) - (u,p)\| + \|(u^\epsilon, p^\epsilon) - (u_h^{n+1}, p_h^{n+1})\| \\ &\leq C\epsilon + \|(u^\epsilon, p^\epsilon) - (u_h^{n+1}, p_h^{n+1})\|,\end{aligned}$$

with $\|(u,p)\| := \|u\| + \big(\int_{t_0}^{t} \|u(s)\|_1^2 ds\big)^{1/2} + \sqrt{\epsilon}\|p\|$. Comparing the model and our method, we see that

$$\varsigma_p(p^n; q_h) := \epsilon\big(\frac{p^{n+1} - p^n}{\Delta t} - p_t^{n+1}, q_h\big) \leq \frac{C_r \epsilon \Delta t}{\sigma}\|p_{tt}\|^2_{L^2(t^n, t^{n+1}; L^2(\Omega))} + \frac{\sigma\epsilon}{r}\|q_h\|^2$$

or

$$\varsigma_p(p^n; q_h) \leq \frac{C_r \epsilon \Delta t}{\sigma}\|p_{tt}\|^2_{L^\infty(t^n, t^{n+1}; L^2(\Omega))} + \frac{\sigma\epsilon\Delta t}{r}\|q_h\|^2,$$

which would yield the result. Clearly, details must be worked out for the Boussinesq equations and regularity assumptions required for (6.2) would be imposed in addition to $p_{tt} \in L^\infty(0, t^*; L^2(\Omega))$.

Recall, for **PEA** and **ACE**, a grad-div term arose in the momentum equation (4.14). This term proved to be potentially disastrous for second-order **PEA**. Recently, first-order [33] and second-order [119] modular algorithms were devised that add minimally intrusive modules which implement grad-div stabilization. These algorithms do not suffer from either solver breakdown or debilitating slow down for large values of grad-div parameters. This is precisely the issue we witness with **PEA**; **ACE** is susceptible as well. For simplicity, consider a single realization of the NSE. Then, candidates include:

Modular Penalty NSE

Step 1: Given u^n, find $\hat{u}^{n+1}$ satisfying

$$\frac{\hat{u}^{n+1} - u^n}{\Delta t} + u^n \cdot \nabla \hat{u}^{n+1} - \nu \Delta \hat{u}^{n+1} = f^{n+1}. \tag{6.3}$$

Step 2: Given $\hat{u}^{n+1}$, find u^{n+1} satisfying

$$\frac{u^{n+1} - \hat{u}^{n+1}}{\Delta t} - \frac{1}{\epsilon}\nabla\nabla \cdot u^{n+1} = 0. \tag{6.4}$$

Step 3: Given u^{n+1}, find p^{n+1} satisfying

$$p^{n+1} = -\frac{1}{\epsilon}\nabla \cdot u^{n+1}. \tag{6.5}$$

For artificial compressibility:

Modular AC NSE

Step 1: Given u^n and p^n, find $\hat{u}^{n+1}$ satisfying

$$\frac{\hat{u}^{n+1} - u^n}{\Delta t} + u^n \cdot \nabla \hat{u}^{n+1} - \nu \Delta \hat{u}^{n+1} + \nabla p^n = f^{n+1}. \tag{6.6}$$

Step 2: Given $\hat{u}^{n+1}$, find u^{n+1} satisfying

$$\frac{u^{n+1} - \hat{u}^{n+1}}{\Delta t} - \frac{\Delta t}{\epsilon}\nabla\nabla \cdot u^{n+1} = 0. \tag{6.7}$$

Step 3: Given u^{n+1}, find p^{n+1} satisfying

$$p^{n+1} = p^n - \frac{\Delta t}{\epsilon}\nabla \cdot u^{n+1}. \tag{6.8}$$

The first algorithm can be proven to be unconditionally, nonlinearly, energy stable and is consistent provided $\epsilon = \mathcal{O}(\Delta t)$. The second algorithm is consistent provided $\epsilon = \mathcal{O}(\Delta t)$, however, stability is an open question. Numerical experiments suggest that it is first-order convergent in appropriate norms. Fully-discrete error analyses are open questions for both algorithms.

It is well known that the viscosity and thermal conductivity of a fluid can vary with temperature, pressure, volume fraction of solute, and etc; see, e.g., [33,56,66] and references therein. Some technologically important applications that utilize models for these quantities are metal 3D printing, industrial lubricants between bearings, and biomass transport.

Recently, DeCaria, Khankan, and McLaughlin [24] developed first- and quasi-second-order accurate time-stepping methods for the NSE with viscosity depending explicitly on space and time. In particular, they make the following first-order accurate approximation:

$$\nabla \cdot (\nu(x, t^{n+1})\nabla u^{n+1}) \approx \nabla \cdot (\nu_{max}^n \nabla u^{n+1}) + \nabla \cdot (\sqrt{\nu'^n \nu'^{n-1}}\nabla u^n),$$

where $\nu_{max}^n = \sup_{x\in\Omega} \nu(x, t^n)$ and $\nu'^n = \nu_{max}^n - \nu(x, t^n)$. The algorithm is proven to be unconditionally, nonlinearly, energy stable and first-order accurate. Interestingly, we can modify this algorithm to account for temperature dependent viscosities. In particular,

$$\nabla \cdot (\nu(\theta(x, t^{n+1}; \omega_j))\nabla u^{n+1}) \approx \nabla \cdot (\nu_{max} \nabla u^{n+1}) + \nabla \cdot (\sqrt{\nu'^n \nu'^{n-1}}\nabla u^n),$$

where $\nu_{\max} = \max_{1\le j\le J} \sup\limits_{(x,t)\in\Omega\times[0,t^*]} \nu(T(x, t; \omega_j))$ and $\nu'^n = \nu_{max} - \nu(T(x, t^n))$. Provided $\nu(\omega_j)$ is Lipschitz continuous and bounded uniformly for all $(x, t) \in \Omega \times [0, t^*]$, then the resulting algorithm can similarly be proven stable and convergent [41]; thermal conductivity can be approximated in similar fashion. It is an open question whether provably second-order accurate variants exist.

From the viewpoint of efficient computation, an important next step is to compare the speed of our algorithms using block solvers with deflation [43, 57]. This is a much needed comparison of significant interest. As a first step, one could fix the timestep and compare solution times of our algorithms over alternatives; in general, comparisons for a well-selected array of test problems would be convincing.

Other important next steps include extending these results to the primitive equations and other physical problems. The primitive equations, used for atmosphere and ocean simulations, are based off the Boussinesq equations. The development of fast ensemble algorithms for these equations would draw significant interest. The applicable boundary conditions are more complex and therefore interesting for the mathematician.

Penetrative convection is another important physical phenomenon that occurs in the lower atmosphere, ocean, and lakes [107]. From the mathematical standpoint, this can be modeled with the Boussinesq equations with $\beta T \leftarrow \beta_1 T + \beta_2 T^2$. For the nonlinear term $\beta_2 {T^{n+1}}^2$, a second-order accurate approximation is $\beta_2(2(T^n)^2 - (T^{n-1})^2)$ [117]. Other phenomenon include, doubly diffusive convection, flow through porous media, and additive manufacturing processes.

Boundary conditions define new physical systems as well. Recall, for the velocity and temperature, we prescribed no-slip and mixed boundary conditions. Consideration of alternative boundary conditions would be interesting. For example, slip with friction for velocity and Robin boundary conditions for temperature. This could be a stepping stone towards treating the primitive equations, for instance.

In our algorithms, we used $\langle u\rangle^n = \frac{1}{J}\sum_{j=1}^{J} u(x,t^n;\omega_j)$, however, this can be replaced with the weighted arithmetic mean; that is, $\langle u\rangle_w^n = \sum_{j=1}^{J} w_j u(x,t^n;\omega_j)$ such that $\sum_{j=1}^{J} w_j = 1$. With this weighted arithmetic mean and associated fluctuation, all results proven hold. The weights form an additional parameter that can be used to optimize the stability conditionn. In particular, an additional step can be implemented:

$$\max_{1\leq j\leq J}\ \min_{w\in B^J(0,1)} \|\nabla \mathscr{E}_w^i({u'}_h^{n+1})\|,$$

where $B^J(0,1)$ is the J-dimensional unit ball and $\mathscr{E}_w^i(\cdot)$ is the fluctuation associated with $\langle\cdot\rangle_w$. The aim would be to produce a more stable algorithm allowing for larger timesteps and, therefore, increased efficiency.

In the works [72–75], the authors consider alternative turbulence models and perspectives. For instance, the Prandtl relation is used instead of the Kolmogorov-Prandtl relation. Moreover, Leray regularizations are considered. Each of these can be considered here. In addition, it would be interesting to consider Smagorinsky and deconvolution models for the eddy viscosity and alternative models, such as the algebraic flux and differential flux models, for the turbulent heat flux.

Charnyi, Heister, Olshanskii, and Rebholz [16] study conservation properties of certain numerical methods with several trilinear forms. They consider the convective, explicitly skew-symmetric, rotational, and EMAC formulations of the trilinear form. They found that these formulations can produce very different results for certain test problems. Thus, it would be interesting to study these in the context of ensemble simulations. It would also be interesting to study the rotational form of the NSE and its Boussinesq counterpart in this context.

Table 6.1: CR element: Consistent with literature up to $Ra = 10^5$.

Ra	Time (s)	$\max_{y\in\Omega_h} u_1(0.5, y, t^*)$	$\max_{x\in\Omega_h} u_2(x, 0.5, t^*)$	Nu_{avg}
10^3	1153.67	3.66	3.73	1.12
10^4	705.41	16.21	19.81	2.24
10^5	1210.95	34.45	72.17	4.63
10^6	1829.71	97.78	271.07	10.09

Recall that we considered uncertain initial conditions, forcings, and parameters. However, the boundary conditions and domain boundary cannot be known exactly either. Considering these uncertainties would be an important next step. The latter is a delicate issue. However, a path forward for the former is clear. A first step would be to study ensemble algorithms for shear driven flow where the shear velocity U is perturbed.

In the aim of increased efficiency, the non-conforming Crouzeix-Raviart element (P1nc-P0) has great potential. For **PEA**, **ACE**, and **ACE-T**, the pressure update is a true algebraic update. Preliminary numerical tests indicate optimal-order accuracy and speed ups, over Taylor-Hood (P2-P1), of between 2.1-2.7; see Table 6.1. Theoretical analysis is an open question, however, a clear path forward exists; the non-conformity is dealt with as a "variational crime" [12, 18].

Lastly, in Chapter 5, we concluded that sufficiently large rotations can act to increase predictability, larger domains are more predictable, and that spatial averaging can either act to reduce or increase predictability. Our studies applied to the double pane window problem and a manufactured solution. It would be interesting to study Rayleigh-Bénard convection with rotation on various geometry (in a cube, between two concentric spheres, or planes). These results would have implications for weather prediction.

APPENDIX A

NON-DIMENSIONALIZATION

Recall, the Boussinesq equations are given by: Find $u(x,t) : \Omega \times (0,t^*] \to \mathbb{R}^d$, $p(x,t) : \Omega \times (0,t^*] \to \mathbb{R}$, and $T(x,t) : \Omega \times (0,t^*] \to \mathbb{R}$ satisfying

$$\rho(u_t + u \cdot \nabla u + \Lambda \times u) - \mu\Delta u + \nabla p = \rho\beta g(T - T_{ref}) + f_1 \ \ in\ \Omega, \tag{A.1}$$

$$\nabla \cdot u = 0 \ \ in\ \Omega, \tag{A.2}$$

$$\rho c_V(T_t + u \cdot \nabla T) - \kappa\Delta T = f_2 \ \ in\ \Omega, \tag{A.3}$$

$$u = 0 \ \ on\ \partial\Omega, \quad T = T_H \ \ on\ \Gamma_{D_1}, \quad T = T_C \ \ on\ \Gamma_{D_2}, \quad n \cdot \nabla T = 0 \ \ on\ \Gamma_N. \tag{A.4}$$

Consider the following relationships,

$$x = L\tilde{x},\ t = \tau\tilde{t},\ u = U\tilde{u}, \tag{A.5}$$

$$p = \rho U^2\tilde{p},\ T = (T_H - T_C)\tilde{T} + T_{ref},\ \Lambda = |\Lambda|e_\Lambda, \tag{A.6}$$

where $U = \frac{\kappa}{\rho c_V L}$ is the conductive velocity and $\tau = \frac{L}{U}$ is the associated timescale; ρU^2 is often called the dynamic pressure. Introduce the relations (A.5) - (A.6) into equation (A.1) first:

$$\rho(\frac{U}{\tau}\tilde{u}_{\tilde{t}} + \frac{U^2}{L}\tilde{u} \cdot \tilde{\nabla}\tilde{u} + U|\Lambda|e_\Lambda \times \tilde{u}) - \frac{\mu U}{L^2}\tilde{\Delta}\tilde{u} + \frac{\rho U^2}{L}\tilde{\nabla}\tilde{p} = \rho\beta g(T_H - T_C)\tilde{T} + f_1. \tag{A.7}$$

Dividing both sides by $\frac{\rho U}{\tau} = \frac{\rho U^2}{L}$ yields

$$\tilde{u}_{\tilde{t}} + \tilde{u} \cdot \tilde{\nabla}\tilde{u} + \frac{|\Lambda|L}{U}e_\Lambda \times \tilde{u} - \frac{\nu}{LU}\tilde{\Delta}\tilde{u} + \tilde{\nabla}\tilde{p} = \frac{\beta g L(T_H - T_C)}{U^2}\tilde{T} + \frac{L}{\rho U^2}f_1, \tag{A.8}$$

where $\tilde{\nabla}$ and $\tilde{\Delta}$ denote the non-dimensional del and Laplace operators. Denote $\tilde{\kappa} = \frac{\kappa}{\rho c_V}$ and define the Rayleigh number $Ra = \frac{|g|\beta(T_H - T_C)L^3}{\nu\tilde{\kappa}}$, Prandtl number $Pr = \frac{\nu}{\alpha}$, Taylor number $Ta = \frac{|\Lambda|^2 L^4}{\nu^2}$, and unit vector in the direction of gravity $\xi = \frac{g}{|g|}$. Then, the above is equivalent to

$$\tilde{u}_{\tilde{t}} + \tilde{u} \cdot \tilde{\nabla}\tilde{u} + PrTa^{1/2} e_\Lambda \times \tilde{u} - Pr\tilde{\Delta}\tilde{u} + \tilde{\nabla}\tilde{p} = PrRa\xi\tilde{T} + \tilde{f}_1. \tag{A.9}$$

Similarly, for the temperature equation (A.3), use the relations (A.5) - (A.6), and divide both sides by $\frac{\tau(T_H - T_C)}{\rho C_V}$. Then,

$$\tilde{T}_{\tilde{t}} + \tilde{u} \cdot \tilde{\nabla}\tilde{T} - \tilde{\kappa}\tilde{\Delta}\tilde{T} = \tilde{f}_2. \tag{A.10}$$

The temperature boundary conditions become

$$\tilde{T} = \frac{T_H - T_{ref}}{T_H - T_C} \;\; on\; \Gamma_{D_1}, \quad \tilde{T} = \frac{T_C - T_{ref}}{T_H - T_C} \;\; on\; \Gamma_{D_2}, \quad n \cdot \tilde{\nabla}\tilde{T} = 0 \;\; on\; \Gamma_N. \tag{A.11}$$

Selecting $T_{ref} = T_C$ yields

$$\tilde{T} = 1 \;\; on\; \Gamma_{D_1}, \quad \tilde{T} = 0 \;\; on\; \Gamma_{D_2}. \tag{A.12}$$

Dropping the tilde notation, we have the following non-dimensional form of the Boussinesq equations,

$$u_t + u \cdot \nabla u - Pr\Delta u + PrTa^{1/2} e_\Lambda \times u + \nabla p = PrRa\xi T + f_1 \;\; in\; \Omega, \tag{A.13}$$

$$\nabla \cdot u = 0 \;\; in\; \Omega, \tag{A.14}$$

$$T_t + u \cdot \nabla T - \kappa\Delta T = f_2 \;\; in\; \Omega, \tag{A.15}$$

$$u = 0 \;\; on\; \partial\Omega, \quad T = 1 \;\; on\; \Gamma_{D_1}, \quad T = 0 \;\; on\; \Gamma_{D_2}, \quad n \cdot \nabla T = 0 \;\; on\; \Gamma_N. \tag{A.16}$$

Non-dimensionalization is not unique. For example, choosing $U = \sqrt{|g|\beta L(T_H - T_C)}$, defining the Rossby number $Ro = \frac{U}{|\Lambda|L}$, and using the same techniques as in the above yields,

$$u_t + u \cdot \nabla u - \sqrt{\frac{Pr}{Ra}}\Delta u + Ro^{-1} e_\Lambda \times u + \nabla p = \xi T + f_1 \;\; in\; \Omega, \tag{A.17}$$

$$\nabla \cdot u = 0 \;\; in\; \Omega, \tag{A.18}$$

$$T_t + u \cdot \nabla T - \frac{1}{\sqrt{RaPr}}\Delta T = f_2 \;\; in\; \Omega, \tag{A.19}$$

$$u = 0 \;\; on\; \partial\Omega, \quad T = 1 \;\; on\; \Gamma_{D_1}, \quad T = 0 \;\; on\; \Gamma_{D_2}, \quad n \cdot \nabla T = 0 \;\; on\; \Gamma_N. \tag{A.20}$$

Here, U is the convective velocity and τ the associated timescale.

APPENDIX B

DETERMINATION OF $C_\dagger$

In this section, we concern ourselves with forming estimates for $C_\star$ and $C_{\star\star}$ used in determining $C_\dagger$ for condition (4.52). Recall, for $d = 2$ or 3,

$$\sup_{u,v,w \in X_h} \frac{b(u,v,w)}{\|\nabla u\| \|\nabla v\| \|w\|} \leq C_\star h^{-1/2},$$
$$\sup_{\substack{u \in X_h \\ T,S \in W_h}} \frac{b^*(u,T,S)}{\|\nabla u\| \|\nabla T\| \|S\|} \leq C_{\star\star} h^{-1/2}.$$

Now, consider a uniform mesh on the unit square. The perimeter and area of each element K is $(2+\sqrt{2})h$ and $\frac{h^2}{2}$. The following upper bound hold,

$$
\begin{aligned}
b(u,v,w) &= (u \cdot \nabla v, w) + \frac{1}{2}\big((\nabla \cdot u)v, w\big) && \text{(B.1)}\\
&\leq \|u\|_{L^4} \|\nabla v\| \|w\|_{L^4} + \frac{1}{2}\|\nabla \cdot u\| \|v\|_{L^4} \|w\|_{L^4} && \text{(B.2)}\\
&\leq C_L^2 \sqrt{\|u\| \|\nabla u\|} \|\nabla v\| \sqrt{\|w\| \|\nabla w\|} + \frac{C_L^2 \sqrt{d}}{2} \|\nabla u\| \sqrt{\|v\| \|\nabla v\|} \sqrt{\|w\| \|\nabla w\|} && \text{(B.3)}\\
&\leq C_L^2 C_P^{1/2} (1 + \frac{\sqrt{d}}{2}) \|\nabla u\| \|\nabla v\| \sqrt{\|w\| \|\nabla w\|} && \text{(B.4)}\\
&\leq C_L^2 C_P^{1/2} C_{inv}^{1/2} (1 + \frac{\sqrt{d}}{2}) h^{-1/2} \|\nabla u\| \|\nabla v\| \|w\|, && \text{(B.5)}
\end{aligned}
$$

where $C_L = 2^{1/4}$ is the Ladyzhenskaya constant, $C_P = 1/2$, $C_{inv} = \frac{(4+2\sqrt{2})\sqrt{C_j}}{h}$ [110], and C_j is a constant associated with piecewise polynomials of total degree j. Thus,

$$C_\star \leq \sqrt{4 + 2\sqrt{2}}\, C_j^{1/4} h^{-1/2},$$

where $C_1 = 6$, $C_2 = 45/2$, and $C_3 = 56.8879$. Thus, for Taylor-Hood $C_\star \leq 3.44$ and for the MINI element $C_\star \leq 4.34$.

For a general polyhedral domain in $\mathbb{R}^d$, the Poincaré-Friedrichs inequality holds with constant $C_P = \frac{2diam(\Omega)}{d}$ [115]. Consider the unit cube with uniform mesh using regular tetrahedrons with surface area and volume of $\sqrt{3}h^2$ and $\frac{h^3}{6\sqrt{2}}$. Then,

$$b(u, v, w) \leq \|u\|_{L^6}\|\nabla v\|\|w\|_{L^3} + \frac{1}{2}\|\nabla \cdot u\|\|v\|_{L^6}\|w\|_{L^3} \tag{B.6}$$

$$\leq C_G C_L\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|} + \frac{C_S C_L\sqrt{d}}{2}\|\nabla u\|\|\nabla v\|\sqrt{\|w\|\|\nabla w\|} \tag{B.7}$$

$$\leq C_G C_L C_P^{1/2} C_{inv}^{1/2}(1 + \frac{\sqrt{d}}{2})h^{-1/2}\|\nabla u\|\|\nabla v\|\|w\|, \tag{B.8}$$

where $C_G = \frac{1}{3^{1/4}}$ is the Gagliardo-Nirenberg constant, $C_L = \frac{2}{\sqrt{3}}$ is the Ladyzhenskaya constant, $C_P = \frac{4\sqrt{2}}{3}$, and $C_{inv} = \frac{\sqrt{3}\sqrt{C_j}}{6\sqrt{2}h}$ [110]. Thus,

$$C_\star \leq \frac{2\sqrt{2}}{3}(\frac{1}{2} + \frac{\sqrt{3}}{3})C_j^{1/4}h^{-1/2},$$

where $C_1 = 10$, $C_2 = 63/2$, and $C_3 = 42 + 12\sqrt{7}$. Thus, for Taylor-Hood $C_\star \leq 3.41$ and for the MINI element $C_\star \leq 4.21$.

For $b^*(u, T, S)$, we must be careful as the temperature is not zero on the entirety of the boundary; thus, we cannot extend by zero. Using the extension operator from [32], we find that there exists $C > 0$ such that $C\star\star \leq (1 + \frac{C}{2})C_\star$. Determination of $C > 0$ on certain mesh/domain combinations is left open. However, the estimates above will hold for $T|_{\partial\Omega} = 0$.

APPENDIX C

EXISTENCE AND UNIQUENESS

In Chapter 4, we proposed eight efficient algorithms for computing an ensemble of solutions to the Boussinesq system. We stated but did not prove well-posedness, which we provide herein. The following result is extraordinarily useful.

Theorem 17. *(Lax-Milgram) Consider the problem: Find $u \in H$ such that*

$$a(u, v) = f(v), \ \forall v \in H. \tag{C.1}$$

Let H be a Hilbert space. Suppose $a : H \times H \to \mathbb{R}$ is a bilinear form satisfying

$$\begin{aligned} a(u, v) \leq C_{cont} \|u\|_H \|v\|_H \ (continuous), \\ a(u, u) \geq C_{coer} \|u\|_H^2 \ (coercive), \end{aligned}$$

and $f \in H'$ a linear functional satisfying

$$f(v) \leq C\|v\|_H \ (continuous).$$

Then, the problem (C.1) is well posed; that is, there exists a unique solution u satisfying (C.1). Moreover,

$$\|u\|_H \leq C_{coer}^{-1} \|f\|_{H'}, \ \forall f \in H'.$$

Proof. See Lemma 2.8 on p. 85 of [31]. □

Theorem 18. *Consider the ensemble algorithms:* ***eBDF*** *(4.40) - (4.42),* ***PEA*** *(4.43) - (4.44),* ***ACE*** *(4.46) - (4.47), and* ***ACE-T*** *(4.49) - (4.50). Suppose $f_1^{n+1} \in H^{-1}(\Omega)^d$, $f_2^{n+1} \in H^{-1}(\Omega)$, $u_h^n \in X_h$, $T_h^n \in W_h$, and $p_h^n \in Q_h$(when required). Then, there exists unique solutions u_h^{n+1}, $T_h^{n+1} \in W_h$, and $p_h^{n+1} \in Q_h$.*

Proof. We will provide two proofs. For the first, we note that each algorithm reduces to a finite dimensional linear system after picking a basis. Consider Algorithm **eBDF**, equation (4.40) is equivalent to

$$
\begin{aligned}
\frac{1}{i\Delta t}(u_h^{n+1}, v_h) + b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) + \langle\nu\rangle(\nabla u_h^{n+1}, \nabla v_h) + (\langle\Lambda\rangle \times u_h^{n+1}, v_h) - (p_h^{n+1}, \nabla\cdot v_h)& \\
= \frac{1}{i\Delta t}(u_h^{n+1}, v_h) - (\partial^i_{\Delta t}(u_h^{n+1}), v_h) + (\beta g\mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h)& \\
+ b(\mathscr{E}^i(u'^{n+1}_h), \mathscr{E}^i(u_h^{n+1}), v_h) + \nu'(\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h)& \\
+ (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h) := Data.& \quad \text{(C.2)}
\end{aligned}
$$

Existence is equivalent to uniqueness, thus we must show $(u_h^{n+1}, p_h^{n+1}, T_h^{n+1}) \equiv (0,0,0)$ provided the right-hand sides are zero; that is, $Data \equiv 0$. Let $v_h = u_h^{n+1} \in V_h$ in (C.2), then

$$\|u_h^{n+1}\|^2 + i\Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 = 0,$$

which implies $u_h^{n+1} \equiv 0$. Similarly, rewrite $T_h^{n+1} = \theta_h^{n+1} + I_h\tau$, rearrange (4.42), set the right-hand sides to zero, and let $S_h = \theta_h^{n+1} \in W_{\Gamma_D,h}$. Then,

$$\|\theta_h^{n+1}\|^2 + i\Delta t\|\langle\kappa\rangle^{1/2}\nabla\theta_h^{n+1}\|^2 = 0.$$

Consequently, $T_h^{n+1} - I_h\tau = T_h^{n+1} = \theta_h^{n+1} \equiv 0$. Uniqueness of the pressure follows via the discrete inf-sup condition (2.23). In particular,

$$\beta\|p_h^{n+1}\| \leq (1 + C_*^{-1})\Big(\frac{C_1}{\nu_{min}}\|\langle\nu\rangle^{1/2}\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\| + \nu_{max}^{1/2} + \frac{C_P^2|\langle\Lambda\rangle|}{\nu_{min}^{1/2}}\Big)\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|. \quad \text{(C.3)}$$

Thus, $p_h^{n+1} \equiv 0$ since $u_h^{n+1} \equiv 0$, as needed.

For **PEA**, rearrange (4.43) and set the right-hand sides to zero. Select $v_h = u_h^{n+1} \in X_h$, then

$$\|u_h^{n+1}\|^2 + i\Delta t\|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 + \frac{i\Delta t}{\epsilon}\|\nabla\cdot u_h^{n+1}\|^2 = 0.$$

Consequently, $u_h^{n+1} \equiv 0$ and $p_h^{n+1} \equiv 0$ since $\epsilon\|p_h^{n+1}\|^2 = \frac{1}{\epsilon}\|\nabla\cdot u_h^{n+1}\|^2 = 0$. Temperature follows as in the above. Both **ACE** and **ACE-T** follow similarly.

Alternatively, we can use the Lax-Milgram theorem. Consider **eBDF**. At each timestep, we must find $u_h^{n+1} \in V_h$ satisfying equation (C.2). Define a and f as follows,

$$\begin{aligned} a(u_h^{n+1}, v_h) = \frac{1}{i\Delta t}(u_h^{n+1}, v_h) + b(\mathscr{E}^i(\langle u_h\rangle^{n+1}), u_h^{n+1}, v_h) \\ +\langle\nu\rangle(\nabla u_h^{n+1}, \nabla v_h) + (\langle\Lambda\rangle \times u_h^{n+1}, v_h), \end{aligned} \tag{C.4}$$

$$\begin{aligned} f(v_h) = \frac{1}{i\Delta t}(u_h^{n+1}, v_h) - (\partial_{\Delta t}^i(u_h^{n+1}), v_h) + (\beta g\mathscr{E}^i(T_h^{n+1}), v_h) + (f_1^{n+1}, v_h) \\ + b(\mathscr{E}^i(u_h'^{n+1}), \mathscr{E}^i(u_h^{n+1}), v_h) + \nu'(\nabla\mathscr{E}^i(u_h^{n+1}), \nabla v_h) + (\Lambda' \times \mathscr{E}^i(u_h^{n+1}), v_h). \end{aligned} \tag{C.5}$$

We see that

$$a(u_h^{n+1}, u_h^{n+1}) = \frac{1}{i\Delta t}\|u_h^{n+1}\|^2 + \|\langle\nu\rangle^{1/2}\nabla u_h^{n+1}\|^2 \geq C\|\nabla u_h^{n+1}\|^2, \tag{C.6}$$

$$\begin{aligned} a(u_h^{n+1}, v_h) &\leq \frac{C_P^2}{i\Delta t}\|\nabla u_h^{n+1}\|\|\nabla v_h\| + C_1\|\nabla\mathscr{E}^i(\langle u_h\rangle^{n+1})\|\|\nabla u_h^{n+1}\|\|\nabla v_h\| \\ &\quad +|\langle\nu\rangle|\|\nabla u_h^{n+1}\|\|\nabla v_h\| + |\langle\Lambda\rangle|C_P^2\|\nabla u_h^{n+1}\|\|\nabla v_h\| \\ &\leq C\|\nabla u_h^{n+1}\|\|\nabla v_h\|. \end{aligned} \tag{C.7}$$

By similar arguments,

$$f(v_h) \leq C\|\nabla v_h\|. \tag{C.8}$$

Thus, at each timestep, a is a continuous and coercive bilinear form on $V_h \subset V$ and f is a linear functional on V_h'. Thus, by the Lax-Milgram theorem (17), a solution u_h^{n+1} exists uniquely, and therefore p_h^{n+1}. Applying the same techniques to equation (4.42) yields unique existence of the temperature approximation T_h^{n+1}. It is then routine to apply this technique to the other algorithms. □

APPENDIX D

PUBLICATIONS

1. Y. Rong and J. A. Fiordilino, *Numerical analysis of a BDF2 modular grad-div stabilization method for the Navier-Stokes equations*, submitted.
2. N. Li, J. A. Fiordilino, and X. Feng, *Ensemble Time-stepping Algorithms for the Convection-Diffusion Equation with Random Diffusivity*, submitted.
3. J. A. Fiordilino, *On pressure estimates for the Navier-Stokes equations*, arXiv preprint arXiv:1803.04366 (2018).
4. J. A. Fiordilino and M. McLaughlin, *An Artificial Compressibility Ensemble Timestepping Algorithm for Flow Problems*, submitted.
5. J. A. Fiordilino and A. Pakzad, *A Discrete Hopf Interpolant and Stability of the Fully Discrete Finite Element Method for Natural Convection Problems*, submitted.
6. J. A. Fiordilino, W. J. Layton, and Y. Rong, *An Efficient and Modular Grad-Div Stabilization*, Comput. Methods Appl. Mech. Engrg., 335 (2018), pp. 327-346.
7. J. A. Fiordilino, *Ensemble time-stepping algorithms for the heat equation with uncertain conductivity*, Numer. Methods Partial Differential Eq., 00 (2018), pp. 1-16.
8. J. A. Fiordilino, *A Second Order Ensemble Timestepping Algorithm for Natural Convection*, SIAM J. Numer. Anal., 56 (2018), pp. 816-837.
9. J. A. Fiordilino and S. Khankan, *Ensemble Timestepping Algorithms for Natural Convection*, Int. J. Numer. Anal. Model, 15 (2018), pp. 524-551.
10. J. A. Fiordilino, M. Massoudi, and A. Vaidya, *On the heat transfer and flow of a non-homogenous fluid*, Applied Mathematics and Computation, 243 (2014), pp. 184-196.

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