Discrete IDA-PBC control law for Newtonian mechanical port-Hamiltonian systems

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Abstract— **This paper deals with the stability of discrete closed-loop dynamics arising from digital IDA-PBC controller design. This work concerns the class of Newtonian mechanical port-Hamiltonian systems (PHSs), that is those having separable energy being quadrating in momentum (with constant mass matrix). We first introduce a discretization scheme which ensures a passivity equation relatively to the same storage and dissipation functions as the continuous-time PHS. A discrete controller is then obtained following the IDA-PBC design procedure applied to the discrete PHS system. This method guarantees that, from an energetic viewpoint, the discrete closed-loop behavior is similar to the continuous one. Under zero-state observability assumption, closed-loop stability then follows from LaSalle principle. The method is illustrated on an inertia wheel pendulum model.**

I. INTRODUCTION

Whether from control theory, analysis or modeling point of view, it appears necessary to preserve the fundamental properties of a model while making its transition from continuous-time to discrete-time. From a control viewpoint, the literature provides mainly three approaches to design digital controllers [16] (basically, discretization of the continuous-time controller, discrete-time and sampled-data designs). Regardless the design procedure, the loss of information induced by the (required) approximation step deteriorates the efficiency of the discrete-time controller, potentially up to cause instability of the closed-loop. One motivation here is to frame this loss in order to avoid unexpected discrete behavior.

Since the pioneering work on Dissipative Theory [23], passivity has played a central role in the analysis and control design of continuous-time systems. When considering discrete-time systems it is known that approximation schemes may break off passivity. In the linear time-invariant case, a complete study of dissipativity based on the $\theta-\gamma$ method can be found in [8] (see also references therein). For the general nonlinear case, the influence of discretization schemes remains an open issue. As a consequence, stability analysis (intrinsically related to the passivity equation) is no longer straightforward.

In [3], it is shown that an output feedback control law locally asymptotically stabilizes a discrete-time lossless passive system if and only if the system is locally zero-state observable. Note that this very interesting result is not concerned with the procedure yielding the discrete-time (lossless passive) system from the continuous-time one. We first address this issue for a particular subclass in the port-Hamiltonian framework.

More precisely, we are concerned with nonlinear systems arising in Newtonian mechanics, *i.e.* systems with separable energy being quadratic in momentum. This class can be described by the port-Hamiltonian equations.

Port-Hamiltonian systems (PHSs) [14], [22] are known to be *composable* (in the sense that the system resulting of the interconnection of PHSs belongs to the class) and to satisfy a *passivity equality* whenever the energy function is bounded from below.

Approximation of PHSs has been investigated either from a discretization viewpoint (*e.g.* structure discretization [6], [20] and energy discretization [12], [7]) or from a sampled-data viewpoint (*e.g.* [19], [21]). These results only guarantee a weak passivity equation (or equivalently an almost energy balance equation). This is a major drawback regarding passivity-based control design.

From a control viewpoint, port-Hamiltonian framework necessarily calls passivity-based control design. In particular, we shall consider the IDA-PBC design procedure [18]. A direct IDA-PBC digital controller can be derived from a discrete PHS [11] or from sample-data PHS [21], [15]. Although these results are relevant in terms of efficiency with regard to the *emulation* controller (evaluation of the continuous-time controller by Euler scheme), they can be enhanced. Indeed, closed-loop stability relies on a weak passive relation depending on the time step. In other words, the approximation weakens discrete system properties and impacts closed-loop behavior.

Summarizing, as passivity is the fundamental underlying property of IDA-PBC design within port-Hamiltonian framework, it guides the approximation scheme for the continuous dynamics and frames acceptable information losses.

In contrast with the literature, we introduce a discrete dynamics that approximates the class while ensuring a passivity equality with respect to the same storage function and the same dissipation rate. It involves an energetic integrator [9] that allows to define *conjugate* port-output. Then, a discrete controller is derived following the IDA-PBC procedure. As a result, the closed-loop dynamics is asymptotically stable under zero-state observability assumption.

The document is organized as follows. Section II presents the Hamiltonian form of Newtonian mechanics and recalls the IDA-PBC design procedure for separable energy. Sec-

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tion III concerns the main result. A discrete PHS is defined and the discrete IDA-PBC design is processed. Closed-loop stability is then addressed. Finally, the discrete controller is applied to the inertia wheel pendulum and simulation results are given in section IV, the drawback of the weak passivity relation is also illustrated.

II. Newtonian mechanics within Hamiltonian framework & IDA-PBC design

This section is dedicated to the port-Hamiltonian description of Newtonian mechanics and to the IDA-PBC strategy.

A. Port-Hamiltonian formulation of Newtonian mechanics

We are interested in nonlinear systems having separable energy H being quadratic in momentum, such as

$$H(q,p) = V(q) + K(p), \quad K(p) = \frac{1}{2} p^T M^{-1} p, \tag{1}$$

where the (constant) inertia matrix $M \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, and described by the following port-Hamiltonian equations

$$\left\{ \begin{array}{l} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & -R(q,p) \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u \\ y = B^T(q) \nabla_p H \end{array} \right. . \tag{2}$$

$q \in \mathbb{R}^n$ is the generalized displacement and $p \in \mathbb{R}^n$ the generalized momenta. I_n denotes the identity $n \times n$ matrix and ∇ the gradient operator. The dissipation matrix $R(q,p) \in \mathbb{R}^{n\times n}$ is symmetric positive definite and the input force matrix $B(q) \in \mathbb{R}^{n\times m}$ full-column rank ($m \leq n$). $B(q)u$ denotes the generalized forces resulting from the control inputs $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ is the conjugate port-output of the system.

The energy balance of the system, integrating $\frac{\mathrm{d}}{\mathrm{d}t} H$ from t_0 to $t_1 \geq t_0$, expresses

$$H(t_1) = H(t_0) + \int_{t_0}^{t_1} y^T(\tau) u(\tau) \mathrm{d}\tau - \int_{t_0}^{t_1} \nabla K^T R \nabla K(\tau) \mathrm{d}\tau. \tag{3}$$

Equation (3) is a *passivity equation* with storage function H and dissipation rate $\nabla K^T R \nabla K$. It simply states that the energy stored at t_1 equals the amount of energy at t_0 increased by the power supplied through the port variables during the interval $[t_0, t_1]$ and decreased by the amount of dissipated power during this interval.

Since we are concerned with passivity-based control design, we pay a particular attention to translate this equation while deriving a discrete Hamiltonian dynamics.

Remark 1.1: As pointed out by a reviewer, this paper contains similarities with the authors' paper [2]. Differences are as follows. Driven by simulating controlled Hamiltonian systems, [2] illustrates unexpected discrete closed-loop behavior relative to the design error. Limited to the elementary 2D case, it is there proposed an energetic integrator approximating SISO Hamiltonian system. The current paper extends this energetic integrator to the MIMO 2nD case. In particular, an underactuated system is considered as an illustration of the IDA-PBC design. This class of systems does not fit with [2].

B. Continuous-time IDA-PBC design

Let us recall the IDA-PBC design [18] for lossless system with separable energy, *i.e.* H given by (1) and $R \equiv 0$ in (2).

The desired closed-loop dynamics will belong to the same class, hence the separable closed-loop energy

$$H_d(q,p) = \frac{1}{2} p^T M_d^{-1} p + V_d(q), \tag{4}$$

where V_d has an isolated equilibrium at $q^\star$, together with the closed-loop dynamics [17]

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} J_d - R_d \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}, \tag{5}$$

with

$$J_d = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \quad \text{and} \quad R_d = \begin{bmatrix} 0 & 0 \\ 0 & B k_{\mathrm{di}} B^T \end{bmatrix}, \tag{6}$$

where k_{di} is a control parameter to be tuned. The IDA-PBC design procedure consists of the following two steps.

The first step, called *energy shaping*, steers the closed-loop system to the desired energy H_d. The associated control input u_{es} is obtained by solving the model matching (2) = (5) without the dissipative matrix

$$\begin{bmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H \\ \nabla_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u_{\mathrm{es}} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & 0 \end{bmatrix} \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix}. \tag{7}$$

The energy shaping controller u_{es} has to satisfy

$$B(q)\ u_{\mathrm{es}} = \left\{ \nabla_q H - M_d M^{-1} \nabla_q H_d \right\}. \tag{8}$$

When B is invertible (fully-actuated case) the control law is easily computed by left multiplying both sides by B^{-1}. Otherwise, the system is underactuated and the matrix B is not invertible. So, the following set of constraint equations must be satisfied

$$B^\perp \left\{ \nabla_q H - M_d M^{-1} \nabla_q H_d \right\} = 0, \tag{9}$$

where $B^\perp$ is a full rank left annihilator of B (*i.e.* $B^\perp B = 0$). The partial differential equation (9), called *matching equation*, has to be solvable in order to derive the energy shaping controller using the Moore-Penrose pseudo-inverse. In such a case, the energy shaping controller writes

$$u_{\mathrm{es}} = B^\dagger \left\{ \nabla_q H - M_d M^{-1} \nabla_q H_d \right\}, \tag{10}$$

where $B^\dagger$ is the Moore-Penrose pseudo-inverse of B, that is $B^\dagger = \left(B^T B\right)^{-1} B^T$.

Note that solvability of (9) is a main issue when dealing with IDA-PBC design. It only concerns few references and is beyond the scope of this work.

The second step, called *damping injection*, consists of adding friction to the system in order to achieve asymptotic stabilization of the desired equilibrium. The damping injection controller u_{di} is constructed as the output feedback

$$u_{\mathrm{di}} = -k_{\mathrm{di}}\, y_d = -k_{\mathrm{di}}\, B(q)^T \nabla_p H_d, \quad k_{\mathrm{di}} > 0. \tag{11}$$

Finally, the complete control law writes $u = u_{\mathrm{es}} + u_{\mathrm{di}}$.

III. MAIN RESULT

We now introduce a discrete dynamics that approximates the separable Hamiltonian system (2) while ensuring a *passivity equation* with respect to the same storage function and dissipation rate. This approximation involves an energetic integrator [9] (introduced there for unforced systems) and a suitable definition of conjugate port-output. The resulting discrete energy balance is the discrete counterpart of (3). This passive discrete dynamics thus allows to properly set the IDA-PBC design procedure. This approach sounds consistent since the discrete controller coincides with the discrete approximation of the continuous one.

The key point here is the use of an energetic integrator. Indeed, it allows to define a discrete conjugate port-output, and conjugacy is fundamental to derive the passivity equation. This is the slight difference with approximation schemes of PHSs proposed in the literature (*e.g.* structure discretization [20] and time-discretization [12]) where the discrete port-output emulates the continuous one, only ensuring a passive relation up to a certain order. As we shall see in section IV, this loss of system's intrinsic property may generate undesired closed-loop behavior.

A. Discrete gradient for separable energy H

Canonical unforced Hamiltonian systems discretization divides in two categories: energetic and geometric integrators, meaning that either the Hamiltonian or the volume is preserved. It has been shown that, except for the linear case, it is not possible to guarantee both [5].

Among the collection of available energetic schemes (see *e.g.* [13], [10], [4] to overview possibilities), we make use of the energetic integrator presented in [9] and introduce $\Delta H(k)$, the discrete gradient of H at stage k, defined as the column vector $\Delta H(k) = \begin{bmatrix}\Delta V^T(k) & \Delta K^T(k)\end{bmatrix}^T$.

The discrete potential energy rate is computed as a difference gradient, its i-th component $\Delta_i V(k)$ writes

$$\begin{aligned}\Delta_i V(k) = \frac{1}{q_i(k+1)-q_i(k)} \times \\ \Big[V(\cdots, q_i(k+1), q_{i+1}(k), \cdots) \\ -V(\cdots, q_{i-1}(k+1), q_i(k), \cdots)\Big].\end{aligned} \tag{12}$$

where $(\cdots, q_i(k+1), q_{i+1}(k), \cdots)$ denotes the state whose components from q_1 to q_i are at stage $n+1$ and the remaining ones from q_{i+1} to q_n at stage k.

The discrete kinetic energy rate is computed by a midpoint scheme

$$\Delta K(k) = M^{-1}\frac{p(k+1)+p(k)}{2} = \big[\Delta_i K(k)\big]_i. \tag{13}$$

B. Discrete-time separable port-Hamiltonian systems

As already mentioned, analysis and control synthesis within continuous-time port-Hamiltonian framework relies on passivity. We make use of the previous discrete gradient to define the discrete conjugate port-output. Therefore, the approximation encodes two fundamental properties: energy conservation and passivity.

Discrete conjugacy thus improves existing results [11] based on a modified Euler scheme to achieve *weak passivity* (*i.e.* truncated Taylor expansion of the passive relation).

Definition 2.1: A *discrete Newtonian mechanical PHS* with separable energy $H(q,p) = V(q) + K(p)$ as in (1) is defined by the equations

$$\frac{q_i(k+1)-q_i(k)}{\Delta t} = \Delta_i K(k) \tag{14a}$$

$$\frac{p_i(k+1)-p_i(k)}{\Delta t} = -\Delta_i V(k) - \big[R(k)\Delta K(k)\big]_i + \big[B(k)u\big]_i \tag{14b}$$

$$y(k) = B^T(k)\Delta K(k) \tag{14c}$$

where $R(k)$ and $B(k)$ denote the dissipation and the input matrices evaluated at $(q(k), p(k))$.

Proposition 2.1: The discrete PHS defined by (14) is an approximation of (2).

Proof 2.1: The discrete gradient ΔH is actually a discrete derivative, hence the result.

Proposition 2.2: The discrete PHS defined by (14) is passive relatively to the storage function H and the dissipation rate $\Delta K^T R \Delta K$.

Proof 2.2: Notice first that, thanks to (12) and (13), subenergy variation respectively writes

$$\delta K(k) = \sum_i \Delta_i K(k) \underbrace{(p_i(k+1)-p_i(k))}_{\delta p_i(k)}, \tag{15}$$

and

$$\delta V(k) = \sum_i \Delta_i V(k) \underbrace{(q_i(k+1)-q_i(k))}_{\delta q_i(k)}. \tag{16}$$

The energy variation $\delta H(k) = H(k+1) - H(k)$ thus becomes

$$\begin{aligned}\delta H(k) = &\ (\delta K + \delta V)(k) \\ = &\ \textstyle\sum_i (\Delta_i K \delta p_i + \Delta_i V \delta q_i)(k) \\ = &\ \textstyle\sum_i \Delta t \Big(\Delta_i K\big(-\Delta_i V - [R\Delta K]_i + [Bu]_i\big) \\ &\ +\Delta_i V(\Delta_i K)\Big)(k)\end{aligned} \tag{17}$$

where the last equality is obtained using equations (14a) and (14b). Finally, simplifying and using (14c), one obtains the passivity equation

$$\delta H(k) = \Delta t(y^T u)(k) - \Delta t(\Delta K^T R \Delta K)(k). \tag{18}$$

which is the discrete counterpart of the continuous-time passivity equation (3).

As a consequence, we shall see that the unforced discrete dynamics is intrinsically stable (*i.e.* regardless the time step). This is valuable since *weak passivity* is usually considered in the literature and therefore stability criteria relates time step and dissipation rate. In other words, stability is no longer an intrinsic property and is achieved only when the dissipation rate compensates the discrete energy drift

Moreover, note that the approximation scheme (14) is passive and lossless (*conservative*) whenever the original system is lossless.

C. Discrete-time IDA-PBC design

Once the discrete Newtonian mechanical PHS intrinsically encodes the (exact) passivity equation, one expects to directly process the IDA-PBC design presented in section II while ensuring the desired continuous-time closed-loop system behavior.

First consider the discrete approximations of the open-loop (2) and the closed-loop (5) dynamics by (14).

Following the procedure, the discrete (lossless) model matching yields

$$-\Delta_i V(k) + \left[B(k)u_{\text{es}}\right]_i = -M_d M^{-1}\Delta_i V_d(k) \tag{19}$$

Assuming the matching equation (9) is solvable, the discrete *energy shaping* controller $u_{\text{es}}(k)$ thus writes

$$u_{\text{es}}(k) = B^{\dagger}(k)\{\Delta V(k) - M_d M^{-1}\Delta V_d(k)\}\,. \tag{20}$$

Noting that the discrete output now is $y(k) = B^T(k)\Delta K_d(k)$, the output feedback control law generates the following discrete damping injection controller $u_{\text{di}}(k)$

$$u_{\text{di}}(k) = -k_{\text{di}}y(k) = -k_{\text{di}}B^T(k)\Delta K_d(k)\,, k_{\text{di}} > 0\,. \tag{21}$$

Finally, the discrete control law writes $u(k) = (u_{\text{es}} + u_{\text{di}})(k)$.

It is worth noting that the discrete IDA-PBC controllers (20) and (21) coincide with the approximations of the continuous-time controllers (10) and (11) following the discrete gradient defined by (13) and (12). Hence, with the approximation scheme proposed here, the discrete controller $u(k)$ is invariantly obtained by discretizing the continuous-time controller $u(t)$ or by a direct discrete design based on the discrete dynamics (14).

For stability analysis, let us recall the following result on passive lossless systems [3].

Proposition 3.1: Suppose a discrete-time passive control affine system is lossless with storage function positive definite. Let $\phi : \mathbb{R}^m \to \mathbb{R}^m$ be any smooth mapping such that $\phi(0) = 0$ and $y^T\phi(y) > 0, \forall y \neq 0$. Then the smooth output feedback control law

$$u = -\phi(y) \tag{22}$$

locally asymptotically stabilizes the equilibrium $x = 0$ if and only if the system is zero-state observable.

Proposition 3.2: Consider a discrete *lossless* ($R \equiv 0$) open-loop dynamics (2) and a discrete closed-loop dynamics (5) both derived using (14). Suppose V_d has an isolated minimum at $q^\star$ and (2) is zero-state observable. Then

1) $(q^\star, 0)$ is a (locally) stable equilibrium of the discrete closed-loop system with $u(k) = u_{\text{es}}(k)$ given by (20),
2) $(q^*, 0)$ is a locally asymptotically stable equilibrium of the discrete closed-loop system with $u(k) = (u_{\text{es}} + u_{\text{di}})(k)$ given by (20) and (21).

Proof 3.1: Notice first that by construction the control law $u(k)$ steers the system to the desired discrete closed-loop dynamics. Let $L(q,p) = H_d(q,p) - H_d(q^\star,0)$. Then L is a positive definite storage function in a neighborhood of the desired equilibrium $(q^\star,0)$, and $\delta L(k) = \delta H_d(k)$. It follows

1) u_{es} generates a lossless closed-loop dynamics, $\delta L(k) \equiv 0$ by (18), and stability follows from Lyapunov theory.
2) Applying $u(k) = (u_{\text{es}} + u_{\text{di}})(k)$, one gets

$$\delta L(k) = -\Delta t\, y^T(k) k_{\text{di}}\, y(k)$$

which is negative definite with respect to y. With zero-state observability, one concludes on the asymptotic stability by virtue of proposition 3.2.

Next section illustrates and compares efficiency of this discrete design control laws on the inertia wheel pendulum example.

IV. The Inertia Wheel Pendulum

The inertia wheel pendulum is a planar inverted pendulum with a revolving wheel at its end. This mechanical system is an interesting example of underactuated separable Hamiltonian system. It has two degrees of freedom and only one actuator located at the disk (see, Figure 1).

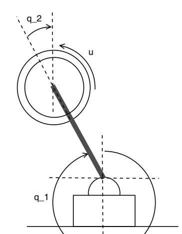


Fig. 1. The inertia wheel pendulum

The variable q_1 represents the pendulum angular deviation from the vertical axis and q_2 represents the angular deviation of the disk with respect to the pendulum. The control u is the torque applied to the disk. This system can be put into the Hamiltonian form (2) with energy function defined by the inertia matrix $M = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$ where J_1 and J_2 are the moments of inertia of the pendulum and the disk respectively, and the potential energy $V(q_1,q_2) = m_3\cos(q_1 - 1)$ with $m_3 = mg\ell$, m being the pendulum mass, ℓ its length and g the gravity constant. The input matrix is $B = [-1\ 1]^T$ and no dissipative effects are considered $R = 0$.

Stabilization of this system has been proposed in [17] following IDA-PBC design as follows. The desired inertia matrix M_d is chosen as

$$M_d = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}, \quad a_1 > 0\,, \quad a_1 a_3 > a_2^2\,. \tag{23}$$

The coefficients a_1, a_2 and a_3 are imposed to ensure M_d is positive definite. Solving the matching equation

$$\left(\frac{a_1 + a_2}{J_1}\right)\nabla_{q_1}V_d + \left(\frac{a_2 + a_3}{J_2}\right)\nabla_{q_2}V_d = -m_3 sin(q_1)\,, \tag{24}$$

the desired potential energy is given by

$$
\begin{aligned}
V_d(q) &= \frac{J_1 m_3}{a_1 + a_2} cos(q_1) + \phi\left(z(q)\right) \\
z(q) &= q_2 - \frac{J_1(a_2 + a_3)}{J_2(a_1 + a_2)} q_1
\end{aligned}
\qquad (25)
$$

for any differentiable function ϕ. With $\phi = \frac{1}{2}z^2$, a straightforward calculus leads to the energy shaping controller

$$
u_{\text{es}}(t) = 30\ sin(q_1) + 5(q_2 + q_1)\,, \qquad (26)
$$

and to the damping injection controller

$$
u_{\text{di}}(t) = -5\left(-2p_1 - p_2\right)\,. \qquad (27)
$$

Remind that the *emulation* controller consists on evaluating (26) and (27) *via* an Euler integration of the dynamics. It is well-known that its relevance is questionable. In [11], a discrete controller based on a modified Euler scheme enhances closed-loop system behavior. It will be referred as *Euler improved* controller.

Finally, following the IDA-PBC design based on discrete passivity proposed in this paper, the energy shaping controller (20) expresses as

$$
\begin{aligned}
u_{\text{es}}(k) = &-30\frac{\cos(q_1(k+1)) - \cos(q_1(k))}{q_1(k+1) - q_1(k)} - \frac{15}{2} q_1(k+1) \\
&+ \frac{25}{2} q_1(k) - 10 q_2(k+1) + 15 q_2(k)\,,
\end{aligned}
\qquad (28)
$$

(leading to a lossless system which is zero-state observable) and the damping injection controller (21) as

$$
\begin{aligned}
u_{\text{di}}(k) = -k_{\text{di}} \Big\{ &\frac{J_2(a_1 + a_2)}{a_1 a_3 - a_2^2} \times \frac{q_2(k+1) - q_2(k)}{\Delta t} \\
&- \frac{J_1(a_2 + a_3)}{a_1 a_3 - a_2^2} \times \frac{q_1(k+1) - q_1(k)}{\Delta t} \Big\}\,.
\end{aligned}
\qquad (29)
$$

Simulation results gather closed-loop pendulum behaviors with the *continuous*, *emulation*, *Euler improved*, and *passive* controllers respectively obtained with `Matlab` solver with smallest tolerance, with evaluation by Euler integration, with evaluation by enhanced Euler scheme [11], and with the IDA-PBC design derived from passive discrete dynamics proposed here. Discrete controllers computations are done with respect to the same time step Δt.

Simulation parameters, taken from [11], are $J_1 = 0.1$, $J_2 = 0.2$, $m_3 = 10$, $a_1 = 2$, $a_2 = -3$, $a_3 = 5$ and with initial conditions $(q_1, q_2, p_1, p_2) = (2, 0, 0, 0)$.

Displacements q_1 and q_2 of the closed-loop are depicted in Fig. 2 with integration step $\Delta t = 0.033s$. One notices that the *emulation* controller does not generate relevant behavior and that the remaining controllers steer the system to the desired equilibrium. One can also notice that the *passive* controller mimics the *continuous* one whereas the *Euler improved* is slower.

To explain these differences, we draw the phase diagram $q_1 \times p_1$ and the energy H_d with the same time step

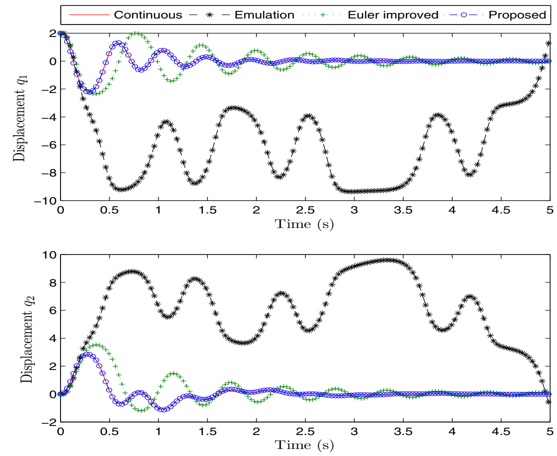


Fig. 2. Displacements q_1 and q_2 of the pendulum

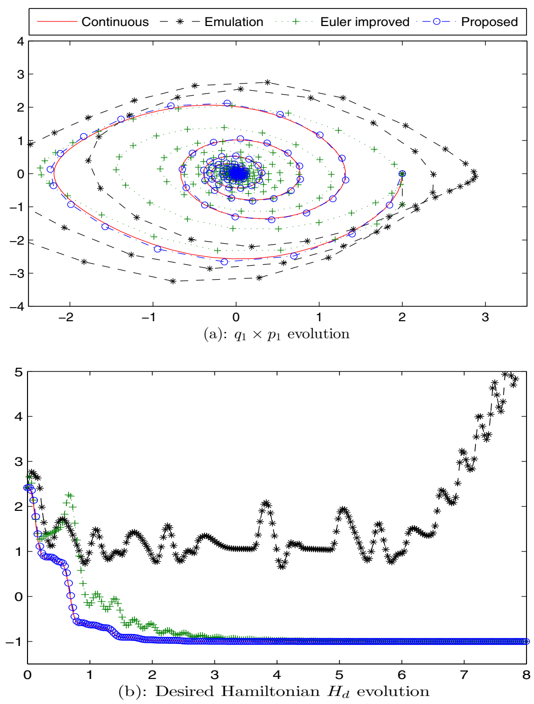
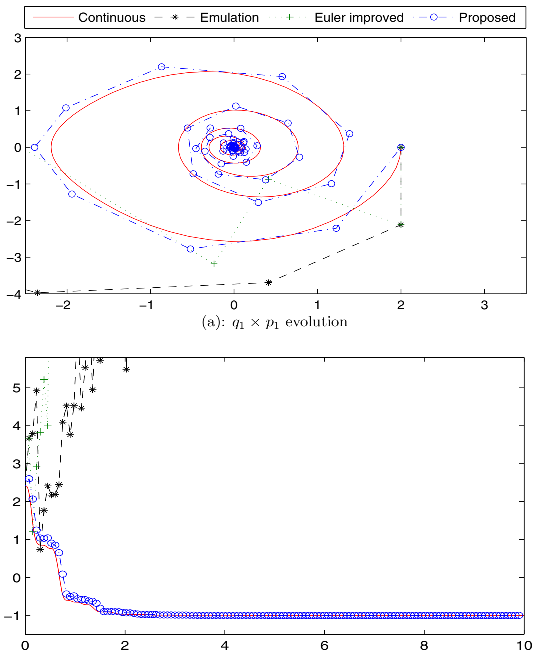


Fig. 3. Performances of Inertia Wheel Pendulum, $\Delta t = 33ms$

$\Delta t = 0.033s$ in Fig. 3. The graphs confirm that the *passive* controller proposed in this paper exactly behaves as the *continuous* controller. Lower convergence rate of *Euler improved* is explained by its associated closed-loop energy behavior depicted in Fig. 3 (b): compared to the *continuous* and *passive* cases, the closed-loop energy has an undesired time evolution. This is due to the almost passive relation (truncated Taylor expansion) leading to numerical energy drift. Note that, despite of this numerical drift, the closed-loop system still converges.

However, as stability is no longer an intrinsic property when *weak* passivity is considered, the closed-loop system may have unexpected behavior. Stabilization is only achieved whenever the numerical drift is compensated by the damping injection controller. This obviously depends on the time step.



(a): $q_1 \times p_1$ evolution

(b): Desired Hamiltonian H_d evolution

Fig. 4. Performances of Inertia Wheel Pendulum, $\Delta t = 75ms$

As depicted in Fig. 4, for a time step $\Delta t = 0.075$, the *emulation* and *Euler improved* controllers generate unstable closed-loop behaviors whereas the *passive* one remains stable by construction.

V. Conclusion

This paper is concerned with the stability of discrete-time closed-loop system arising from digital IDA-PBC design. An energetic approximation scheme for the class of Newtonian mechanical PHSs is introduced. The resulting discrete dynamics encodes energy conservation and passivity, thanks to discrete conjugate port-output. Following the IDA-PBC strategy, a discrete controller that ensures closed-loop stability regardless the time step is derived. Our procedure sounds consistent since discretizing the continuous-time or designing the discrete-time invariantly leads to the same discrete control law. This result contributes to improve discrete-time controller design within port-Hamiltonian framework proposed in the literature where closed-loop stability resides on compensating the numerical energy drift during a time mesh. Numerical simulations illustrate the relevance of the result on the underactuated inertia wheel pendulum model.

Regarding approximation, a more general framework is studied in [1]. Stabilization issue is under progress.

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