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Energy stability of a natural convective flow in a horizontal annular space

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The conditions for global asymptotic stability are determined by energy theory. The basic flow is obtained using the perturbation method. The results are compared to those obtained by the linear theory and to those derived from various experimental investigations.

The linear and nonlinear stability of a Poiseuille flow between two isothermal cylinders has been the subject of many publications[1,2] but little has been published concerning the stability of a natural convection flow between two isothermal cylinders, maintained at temperature T_i on the inner cylinder of radius r_i and temperature T_0 on the outer cylinder of radius r_0, with $T_i > T_0$. Linear stability was first studied in a porous medium[3] and more recently in a fluid.[4]

This note studies the energy stability of the basic flow when it is induced only by natural convection.

We first find the solution of the basic flow. The governing equations, for the basic stationary two-dimensional flow, in dimensionless form, are

$$\Pr\nabla^4\psi = U\frac{\partial}{\partial r}(\nabla^2\psi) + \frac{V}{r}\frac{\partial}{\partial\phi}(\nabla^2\psi) - \mathrm{Ra}\,\Pr\left(\frac{\cos\phi}{r}\frac{\partial T}{\partial\phi} + \sin\phi\frac{\partial T}{\partial r}\right), \tag{1}$$

$$\nabla^2 T = \frac{1}{r}\left(\frac{\partial\psi}{\partial\phi}\frac{\partial T}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial T}{\partial\phi}\right), \tag{2}$$

where ψ, T, U, and V denote, respectively, the stream function, temperature, radial and tangential velocity components of the basic flow. Ra denotes the Rayleigh number based on r_i and Pr denotes the Prandtl number. ϕ has been referenced against the ascending vertical. The boundary conditions are

$$r=1;\quad \psi=\frac{\partial\psi}{\partial r}=0;\quad T=1, \tag{3}$$

$$r=R=\frac{r_0}{r_i};\quad \psi=\frac{\partial\psi}{\partial r}=0;\quad T=0, \tag{4}$$

$$\phi=0,\ \pi;\quad \psi=\frac{\partial^2\psi}{\partial\phi^2}=0;\quad \frac{\partial T}{\partial\phi}=0. \tag{5}$$

To solve Eqs. (1) and (2), we develop T and ψ in a power series expansion of the Rayleigh number up to order two. The convergence and the analytical expression of this development are given in Ref. 5.

To study the nonlinear stability of this basic flow, we use the energy method as extended by Ref. 6. We assume the perturbation to be axisymmetric. Developing the temperature and radial velocity perturbations as periodic wavefunctions on the axial coordinate z: $\phi = \theta(r)e^{isz}$ and $u = u(r)e^{isz}$ and inserting them in the Euler–Lagrange equations[6]

$$R_\lambda\left(\lambda^{1/2}\frac{\partial T}{\partial r} - \frac{1}{\lambda^{1/2}}\right)u - \mathcal{L}(\theta) - \frac{\theta}{r^2} = 0, \tag{6}$$

$$R_\lambda\left[\lambda^{1/2}\left(s^2\theta\frac{\partial T}{\partial r} + \frac{D\theta}{r}\frac{\partial^2 T}{\partial\phi^2}\right) + \frac{1}{\lambda^{1/2}}\left(\frac{D\theta}{r} - s^2\theta\right)\right] + \frac{1}{2r^2}\frac{\partial}{\partial r}\left[u\left(\frac{\partial^2 U}{\partial\phi^2} + \frac{r\partial^2 V}{\partial r\partial\phi} - \frac{\partial V}{\partial\phi}\right)\right] + s^2 u\frac{\partial U}{\partial r} - \frac{u}{r^2}\frac{\partial^3 U}{\partial r\partial\phi^2} + \Pr\mathcal{L}^2(u) = 0, \tag{7}$$

where $D = d/dr$, $\mathcal{L} = D^2 + r^{-1}D - [s^2 + (r^2)^{-1}]$, and

$$\mathcal{L}^2 = D^4 + \frac{2}{r}D^3 - \left(2s^2 + \frac{3}{r^2}\right)D^2 - \frac{1}{r}\left(2s^2 - \frac{3}{r^2}\right)D + \left(s^2 + \frac{1}{r^2}\right)^2 - \frac{4}{r^4}.$$

TABLE I. Critical Rayleigh and wavenumbers obtained by energetic theory and calculated for $R = 1.2$, $\sqrt{2}$, 2 and with approximations $N = 1, 2, 3, 4$.

	$N=1$		$N=2$		$N=3$		$N=4$	
R	Ra_e	s_{ce}	Ra_e	s_{ce}	Ra_e	s_{ce}	Ra_e	s_{ce}
1.2	1740.734	3.1191	1699.230	3.1189	1698.415	3.1189	1698.414	3.1189
$\sqrt{2}$	1710.683	3.1293	1669.723	3.1280	1668.818	3.1280	1668.818	3.1280
2	1615.189	3.1642	1576.291	3.1597	1575.167	3.1599	1575.165	3.1599

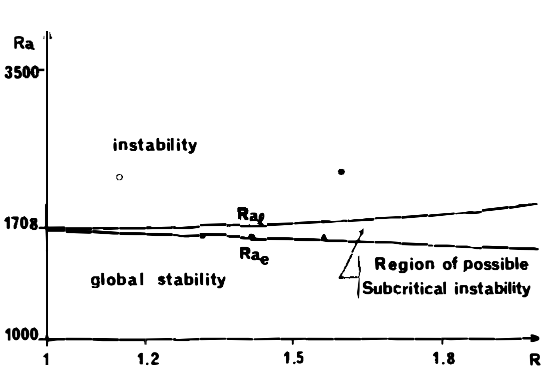

FIG. 1. Comparison of experimental result to those obtained by linear and energetic theory for air (Pr = 0.7). ● Powe *et al.*[7]; ▲ Grigull and Hauf[8]; ○ Liu *et al.*[9]

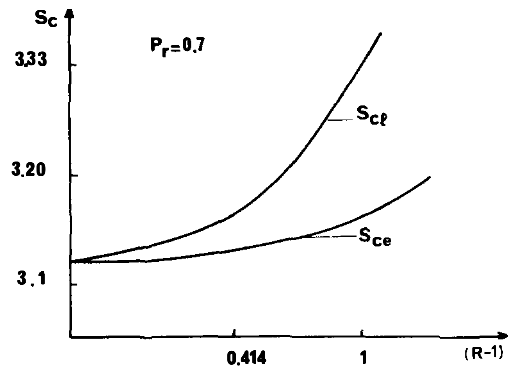

FIG. 2. Critical linear and energetic wavenumbers as a function of the radius ratio for air.

The Galerkin method is applied to solve Eqs. (6) and (7) and consists of representing the perturbations θ and u by a set of functions satisfying the boundary conditions

$$\theta = \sum_{1}^{N} a_k \theta_k(r) \text{ and } u = \sum_{1}^{N} b_k u_k(r) .$$

Introducing these expressions into (6) and (7), multiplying (6) by θ_l and (7) by u_l, and integrating over $(1,R)$ we obtain a homogeneous algebraic system of $2N$ equations with $2N$ unknowns (a_k and b_k). For this system to admit a nontrivial solution, the associated determinant must be zero. This determines the stability condition of this flow,

$$F(R_\lambda, s^2, \lambda^{1/2}) = 0 .$$

We numerically determined this stability curve and derived the wave number s_{ce} and critical Rayleigh number Ra_e $[\mathrm{Ra}_e = \max_\lambda (\min_s R_\lambda)]$ for each value of the radius ratio R and for various degrees of approximation (Table I). To determine Ra_e and the linear critical Rayleigh number Ra_l the same trial functions were used

$$u_k = \frac{[(r-1)(R-r)]^{k+1}}{(R-1)^{2k+2}} \text{ and } \theta_k = \frac{[(r-1)(R-r)]^k}{(R-1)^{2k}} .$$

The critical Rayleigh and wavenumbers cited in Table I and plotted as a function of R in Figs. 1 and 2, are based on the annular thickness $(r_0 - r_i)$.

Table I gives, for three different radius ratios, the values of Ra_e and s_{ce} for air (Pr = 0.7). The good convergence of Galerkin's method is to be noted since, for $N = 3$ and $N = 4$, the difference is only 2×10^{-5}.

In Fig. 1 the experimental results for air obtained by several authors[7-9] are compared with our theoretical work. For R greater than 2, experimental results vary widely and, on the other hand, the perturbation method does not provide a good approximation; so we cannot use it. As R tends toward 1, the upper region of the annular space can be considered as a horizontal plane; Ra_e and s_{ce} simultaneously approach 1708 and 3.12, respectively. These are well known results, they correspond to the onset of convection in a horizontal fluid layer heated from below.

For a flow due to natural convection in a horizontal cylindrical annulus, our treatment of linear and energy stability theories permit us to forecast the conditions for instability in the stationary two dimensional flow for two-dimensional disturbances. Two critical Rayleigh numbers have been presented. For $\mathrm{Ra} < \mathrm{Ra}_e$, the flow is always stable, regardless of the perturbation amplitude. For $\mathrm{Ra} > \mathrm{Ra}_l$, the flow is linearly unstable and always reaches a turbulent state. For $\mathrm{Ra}_e < \mathrm{Ra} < \mathrm{Ra}_l$, the flow may be, but is not necessarily, unstable for perturbations of finite amplitude. However for infinitesimal perturbations the flow remains stable. As Ra increases toward Ra_l, the perturbation amplitude necessary to render the system unstable approaches zero. This explains why Ra_l is difficult to obtain experimentally and accounts for the differences in Fig. 1.

[1]J. E. Mott and D. D. Joseph, Phys. Fluids **11**, 10 (1968).
[2]S. Carmi, Phys. Fluids **13**, 829 (1970).
[3]J. P. Caltagirone, J. Fluid Mech. **76**, 337 (1976).
[4]A. Mojtabi and J. P. Caltagirone, J. Méc. (to be published).
[5]L. R. Mack and E. H. Bishop, J. Mech. Appl. Math. **21**, 223 (1968).
[6]D. D. Joseph, Arch. Rat. Mech. Anal. **22**, 169 (1966).
[7]R. E. Powe, C. T. Carley, and S. L. Carruth, J. Heat Transfer **92**, 210 (1971).
[8]U. Grigull and W. Hauf, in *Proceeding of the Third International Heat Transfer Conference* (American Institute of Chemical Engineers, Chicago, 1966), Vol. 2, p. 182.
[9]C. Y. Liu, W. K. Mueller, and F. Landis, in *International Developments in Heat Transfer* (ASME, Boulder, 1961), p. 976.