Thabet, Hajer and Ayadi, Mounir and Rotella, Frédéric *Sliding window identification with linear-equality constraints.* (2015) In: 2015 IEEE 12th International Multi-Conference on Systems, Signals & Devices (SSD15), 16 March 2015 - 19 March 2015 (Madhia, Tunisia)

Sliding Window Identification with Linear-Equality Constraints

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***Abstract*—In this paper, we present a new algorithm of sliding window identification with linear-equality constraints. This algorithm consists in firstly deleting the oldest set of data and in secondly adding the last set of data. The method developed in this paper allows to consider at every step a set of new data by an extension of their result. The proposed algorithm is based on the recursive calculus of the pseudo-inverse matrix from the forms of Albert and Sittler. A simple and easily implementable initialization of the constrained algorithm is proposed. An improvement is obtained by removing the influence of oldest set of data and by satisfying the linear-equality constraints. It is shown that the solutions of the sliding window identification algorithm converge to the true parameter that satisfies the equality constraints. Numerical example is provided to show the effectiveness of the proposed method.**

***Index Terms*—Identification, Recursive Algorithms, Sliding Window, Linear-Equality Constraints.**

I. Introduction

The Recursive Least Squares (RLS) algorithm is widely used approach for real-time applications in various areas such as signal and data processing, communications and control systems. However, the treatment of linear system leads to an estimation based on infinitely increasing horizon and the main drawback of the Least Squares (LS) solution obtained, directly or recursively, lies in the persistent influence of the first measurements. In order to overcome this drawback, it was suggested the forgetting factor (constant or variable) technique [12], [16], or the sliding window identification with constant length [5], [15], [18], [22]. An intermediate version between these two solutions was proposed in [13]. The purpose of considering a fixed length of window lies not only in the ability to track rapid changes of parameters or building the defect detectors sufficiently and quickly sensitive, it keeps adaptive gains in recursive formulation which does not tend to zero. An advantage of the sliding window identification is its natural forgetting factor which allows to follow slow changes in the parameters. Thus it can be used in adaptive control systems. Some applications of sliding windows algorithms are useful for signal processing [6] or for image processing [17].

In many practical problems in which the sliding window identification method is applied, the resulting solutions must satisfy certain constraints. For this reason, the study of LS problems with constraints has received considerable attention. However, the recursive LS algorithms for constrained problems is available after the work presented in [24]. Indeed, variety works have been studied for solving the LS problem with linear-equality constraints (see [2], [4], [7], [11], [19], [20], [21], [23], [24]). In the work [24], they have shown that the solution of LS problem with linear constraints equations and the solution of linear RLS algorithm without constraints have a recursive identical form. The only difference between these two solutions lies in their initial conditions (initial solutions).

In this paper, we propose a sliding window identification algorithm which satisfied linear-equality constraints. This algorithm improves, on the one hand, the tracking of parameter variation by removing the influence of oldest data and, on the other hand, is robust such that the constraints are always guaranteed to be satisfied no matter how large the numerical errors are.

This paper is organized as follows. Section 2 is devoted to the sliding window identification without constraints by presenting the two steps of update. In Section 3, the sliding window identification with linear-equality constraints is presented and analyzed. In addition, the solution to the linear LS problem with linear-equality constraints is given in this section. The simulation results of parameters identification of DC motor are given in Section 4. Finally, some concluding remarks are drawn in Section 5.

II. Sliding Window Identification Without Constraints

In order to identify the model described in discrete-time by the following linear form:

$$y_k = h_k \theta \tag{1}$$

where θ is the vector of unknown parameters to be identified of dimension $(n \times 1)$, y_k is the system output at time k-dimensional (1×1) and h_k is the data vector of dimension $(1 \times n)$, we are faced with the resolution of a global linear system grouping N batches of measures, defined as:

$$Y_N = H_N \theta \tag{2}$$

where H_N and Y_N are respectively a matrix and a vector obtained from measurements and their dimensions are respectively $(m \times n)$ et $(m \times 1)$:

$$Y_N = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad H_N = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix} \tag{3}$$

Let us consider the case where the estimation was made on a sliding window of fixed length horizon L of batch measurements which satisfy the following relationship:

$$Y_{N,L} = H_{N,L}\theta \tag{4}$$

with $Y_{N,L}$ and $H_{N,L}$ having $(mL \times 1)$ and $(mL \times n)$ size:

$$Y_{N,L} = \begin{bmatrix} y_{N-L-1} \\ \vdots \\ y_N \end{bmatrix}, H_{N,L} = \begin{bmatrix} h_{N-L-1} \\ \vdots \\ h_N \end{bmatrix} \tag{5}$$

Assume that $H_{N,L}$ is a full columns rank matrix, then $H_{N,L}^{+}$ the pseudo-inverse or the Moore-Penrose generalized inverse of $H_{N,L}$, is defined by this formula [3], [14]:

$$H_{N,L}^{+} = \left(H_{N,L}^T H_{N,L}\right)^{-1} H_{N,L}^T \tag{6}$$

If the system (2) is compatible, then there exists a set of least squares (LS) solutions which minimizes the Euclidean norm and is written as:

$$\hat{\theta}_{N,L} = H_{N,L}^{+} Y_{N,L} + \left(I - H_{N,L}^{+} H_{N,L}\right) z \tag{7}$$

where z is an arbitrary vector. For $z = 0$, we obtain the unique solution of minimal Euclidean norm:

$$\hat{\theta}_{N,L} = H_{N,L}^{+} Y_{N,L} \left(H_{N,L}^T H_{N,L}\right)^{-1} H_{H,L}^T Y_{N,L} \tag{8}$$

if and only if the matrix $\left(H_{N,L}^T H_{N,L}\right)$ is invertible. This is a batch solution and it is thus not suitable for real-time applications because its computational complexity increases with N.

In the case where $H_{N,L}$ and $H_{N+1,L}$ are a full columns rank, we have:

$$H_{N+1,L} = \begin{bmatrix} h_{N-L} \\ \vdots \\ h_{N+1} \end{bmatrix}, \quad Y_{N+1,L} = \begin{bmatrix} y_{N-L} \\ \vdots \\ y_{N+1} \end{bmatrix} \tag{9}$$

the estimation is then:

$$\hat{\theta}_{N+1,L} = \left(H_{N+1,L}^T H_{N+1,L}\right)^{-1} H_{N+1,L}^T Y_{N+1,L} \tag{10}$$

By considering the following partitioning:

$$H_{N,L} = \begin{bmatrix} h_{N-L-1} \\ H_{N,L-1} \end{bmatrix}, \quad Y_{N,L} = \begin{bmatrix} y_{N-L-1} \\ y_{N,L-1} \end{bmatrix},$$
$$H_{N+1,L} = \begin{bmatrix} H_{N,L-1} \\ h_{N+1} \end{bmatrix}, \quad Y_{N+1,L} = \begin{bmatrix} Y_{N,L-1} \\ y_{N+1} \end{bmatrix}$$

the sliding window identification is effected in two steps of update:

- Removing the oldest set of data.
- Adding the new set of data.

A. *First Step: Removing the Oldest Set of Data*

Assume that the matrices are always of full rank in columns. To pass from $(H_{N,L}, Y_{N,L})$ to $(H_{N+1,L}, Y_{N+1,L})$, we will use an intermediate set of data, which is obtained by removing the oldest data (h_{N-L-1}, y_{N-L-1}). Assuming that the length L of the sliding window remains constant, the intermediate estimate $\hat{\theta}_{N,L-1}$ is defined by:

$$\hat{\theta}_{N,L-1} = \left(H_{N,L-1}^T H_{N,L-1}\right)^{-1} H_{N,L-1}^T Y_{N,L-1} \tag{11}$$

where the matrix $(H_{N,L-1}^T H_{N,L-1})$ is assumed non singular. Singularity will constitute an entry point of the non regular algorithm. We can then write:

$$\begin{aligned} \hat{\theta}_{N,L-1} = &\left(H_{N,L}^T H_{N,L} - h_{N-L-1}^T h_{N-L-1}\right)^{-1} \\ &\left(H_{N,L}^T Y_{N,L} - h_{N-L-1}^T h_{N-L-1}\right) \end{aligned} \tag{12}$$

where $P_{i,j} = \left(H_{i,j}^T H_{i,j}\right)^{-1}$ and $P_{N,L}$ is considered non singular. The matrix inversion lemma [7] led to the following update of the covariance matrix:

$$\begin{aligned} P_{N,L-1} = &P_{N,L} + \Phi_{N,L} h_{N-L-1}^T \\ &\left(I_m - h_{N-L-1} P_{N,L} h_{N-L-1}^T\right)^{-1} h_{N-L-1} P_{N,L} \end{aligned} \tag{13}$$

After well-known algebraic manipulations, we are led to the following adaptive algorithm which gives the intermediate estimation written as follows:

$$\hat{\theta}_{N,L-1} = \hat{\theta}_{N,L} - K_{N,L-1}^1 \varepsilon_{N,L-1}^1 \tag{14}$$

where the prediction error $\varepsilon_{N,L-1}^1$ and the adaptation gain $K_{N,L-1}^1$ are given by:

$$\begin{aligned} \varepsilon_{N,L-1}^1 &= y_{N-L-1} - h_{N-L-1}\hat{\theta}_{N,L} \\ K_{N,L-1}^1 &= P_{N,L} h_{N-L-1}^T \\ &\quad \left(I_m - h_{N-L-1} P_{N,L} h_{N-L-1}^T\right)^{-1} \end{aligned} \tag{15}$$

B. *Second Step: Adding the New Set of Data*

The next step consists in adding the new data (h_{N+1}, y_{N+1}) to $H_{N,L-1}$ and $Y_{N,L-1}$. Assuming here that the matrices are always of full rank in columns, the estimation is defined as follows:

$$\begin{aligned} \hat{\theta}_{N+1,L} = &\left(H_{N,L-1}^T H_{N,L-1} + h_{N+1}^T h_{N+1}\right)^{-1} \\ &\left(H_{N,L-1}^T Y_{N,L-1} + h_{N+1}^T y_{N+1}\right) \end{aligned} \tag{16}$$

The matrix inversion lemma and the same manipulations lead to the adaptive algorithm:

$$\hat{\theta}_{N+1,L} = \hat{\theta}_{N,L-1} + K_{N+1,L}^2 \varepsilon_{N+1,L}^2 \tag{17}$$

where the prediction error $\varepsilon_{N+1,L}^2$ and the adaptation gain $K_{N+1,L}^2$ are given by :

$$\begin{aligned} \varepsilon_{N+1,L}^2 &= y_{N+1} - h_{N+1}\hat{\theta}_{N,L-1} \\ K_{N+1,L}^2 &= P_{N,L-1} h_{N+1}^T \left(I_m + h_{N+1} P_{N,L-1} h_{N+1}^T\right)^{-1} \end{aligned} \tag{18}$$

and the update of the covariance matrix is obtained as follows:

$$\begin{aligned} P_{N+1,L} = &P_{N,L-1} - P_{N,L-1} h_{N+1}^T \\ &\left(I_m + h_{N+1} P_{N,L-1} h_{N+1}^T\right)^{-1} h_{N+1} P_{N,L-1} \end{aligned} \tag{19}$$

III. SLIDING WINDOW IDENTIFICATION WITH LINEAR-EQUALITY CONSTRAINTS

A. Linear-Equality Constraints

In practice, it is often necessary to impose additional linear-equality constraint on the parameter values. The solution θ has to satisfy the following system of linear algebraic equations:

$$A\theta = B \tag{20}$$

where A and B are respectively a matrix and a vector of respective dimensions $(d \times n)$ and $(d \times 1)$. We assume that (20) is consistent and underdetermined. Hence, applying the theory of pseudo-inverse matrix [14], we have:

$$AA^{+}B = B$$

Theorem 1: If there exists a matrix $A^{\{1\}}$ such that $AA^{\{1\}}B = B$, then $\forall A^{\{1\}}$, we obtain $AA^{\{1\}}B = B$.

Theorem 2: If we have $AA^{+}B = B$, then $\forall A^{\{1\}}$, we obtain $AA^{\{1\}}B = B$ such that $A^{+} \in A\{1\}$.

In the compatible case, the general solution of system (20) is:

$$\hat{\theta} = A^{+}B + \left(I - A^{+}A\right)\xi \tag{21}$$

where ξ is an arbitrary vector of size $(n \times 1)$.
Let:

$$P = I - A^{+}A \tag{22}$$

an orthogonal projection (since $P^2 = P = P^T$) and is not full rank (since $\text{rank}\,(P) = n - \text{rank}\,(A)$, where n is the columns number of A). By using an orthogonal basis of the null space of A, the LS problem with linear-equality constraints has been shown to have a general solution defined by [8], [11]:

$$\hat{\theta}_N = A^{+}B + \left(PH_N^T H_N P\right)^{+} H_N^T \left(Y_N - H_N A^{+}B\right) + PZ \tag{23}$$

where $Z\,(n \times 1)$ is an arbitrary vector satisfying:

$$H_N P Z = 0 \tag{24}$$

The unique solution is then given by:

$$\hat{\theta}_N = A^{+}B + \left(PH_N^T H_N P\right)^{+} H_N^T \left(Y_N - H_N A^{+}B\right) \tag{25}$$

if and only if $\begin{pmatrix} A^T & H_N^T \end{pmatrix}^T$ is full columns rank.
Note that if $H_N^T H_N$ is nonsingular then $\begin{pmatrix} A^T & H_N^T \end{pmatrix}^T$ has full rank but the inverse is not true in general. In addition, if $A = 0$ and $B = 0$, then $P = I$, and (25) reduces to the unconstrained solution $\hat{\theta}_N = \left(H_N^T H_N\right)^{-1} H_N^T Y_N$. If $\begin{pmatrix} A^T & H_N^T \end{pmatrix}^T$ does not have full rank, then (25) is the minimum-norm solution.
The unique solution given in (25) is equivalent to:

$$\hat{\theta}_N = A^{+}B + \left(H_N P\right)^{+} \left(Y_N - H_N A^{+}B\right) \tag{26}$$

with:

$$\begin{aligned}\left(PH_N^T H_N P\right)^{+} P &= \left[\left(PH_N^T H_N P\right)^T \left(PH_N^T H_N P\right)\right]^{+} \\ &\quad \left(PH_N^T H_N P\right)^T P = \left(PH_N^T H_N P\right)^{+}\end{aligned}$$

$$\begin{aligned}\left(H_N P\right)^{+} &= \left[\left(H_N P\right)^T \left(H_N P\right)\right]^{+} \left(H_N P\right)^T \\ &= \left(PH_N^T H_N P\right)^{+} H_N^T\end{aligned}$$

In this paper, we are now faced to obtain a recursive sliding window estimation satisfying a linear-equality constraints. The estimation algorithm is then decomposed in two steps, the oldest removed set of data and the newest added one.

B. Deleting the Oldest Set of Data

For a sliding window estimation, the solution given in (25) with equality constraints becomes in the form of the following relationship:

$$\hat{\theta}_{N,L} = A^{+}B + \left(H_{N,L}P\right)^{+} \left(Y_{N,L} - H_{N,L}A^{+}B\right) \tag{27}$$

if and only if $\begin{pmatrix} A^T & H_{N,L}^T \end{pmatrix}^T$ is full columns rank. The intermediate estimation $\hat{\theta}_{N,L-1}$ becomes:

$$\begin{aligned}\hat{\theta}_{N,L-1} &= A^{+}B + \left(H_{N,L-1}P\right)^{+} \\ &\quad \left(Y_{N,L-1} - H_{N,L-1}A^{+}B\right) \\ &= A^{+}B + \begin{pmatrix} I + K_{N,L-1}^1 h_{N-L-1} & -K_{N,L-1}^1 \end{pmatrix} \\ &\quad \begin{pmatrix} \left(H_{N,L}P\right)^{+} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{N,L} - H_{N,L}A^{+}B \\ y_{N-L-1} - h_{N-L-1}A^{+}B \end{pmatrix} \\ &= A^{+}B + \begin{pmatrix} I + K_{N,L-1}^1 h_{N-L-1} & -K_{N,L-1}^1 \end{pmatrix} \\ &\quad \begin{pmatrix} \left(H_{N,L}P\right)^{+} Y_{N,L} - H_{N,L}A^{+}B \\ y_{N-L-1} - h_{N-L-1}A^{+}B \end{pmatrix} \\ &= A^{+}B + \left(H_{N,L}P\right)^{+} \left(Y_{N,L} - H_{N,L}A^{+}B\right) \\ &\quad - K_{N,L-1}^1 y_{N-L-1} + K_{N,L-1}^1 h_{N-L-1}A^{+}B \\ &\quad - K_{N,L-1}^1 \left(H_{N,L}P\right)^{+} \left(Y_{N,L} - H_{N,L}A^{+}B\right) \\ &= \hat{\theta}_{N,L} - K_{N,L-1}^1 \left(y_{N-L-1} - h_{N-L-1}\hat{\theta}_{N,L}\right)\end{aligned} \tag{28}$$

The recursive algorithm of deleting the oldest data with linear-equality constraints is:

$$\hat{\theta}_{N,L-1} = \hat{\theta}_{N,L} - K_{N,L-1}^1 \varepsilon_{N,L-1}^1 \tag{29}$$

$$\varepsilon_{N,L-1}^1 = y_{N-L-1} - h_{N-L-1}\hat{\theta}_{N,L} \tag{30}$$

$$K_{N,L-1}^1 = P_{N,L} h_{N-L-1}^T \left(I_m - h_{N-L-1} P_{N,L} h_{N-L-1}^T\right)^{-1} \tag{31}$$

$$\begin{aligned}P_{N,L-1} &= \left(PH_{N,L-1}^T H_{N,L-1} P\right)^{-1} \\ &= \left(I + K_{N,L-1}^1 h_{N-L-1}\right) P_{N,L}\end{aligned} \tag{32}$$

C. Adding the New Set of Data

The estimation of the vector $\hat{\theta}_{N+1,L}$ becomes:

$$\hat{\theta}_{N+1,L} = A^+B + (H_{N+1,L}P)^+ \left(Y_{N+1,L} - H_{N+1,L}A^+B\right) \tag{33}$$

Besides, as the expression (33) verifies the relationship (17), we have:

$$\begin{aligned}
\hat{\theta}_{N+1,L} &= A^+B + (H_{N+1,L}P)^+ \left(Y_{N+1,L} - H_{N+1,L}A^+B\right) \\
&= A^+B + \begin{pmatrix} I - K^2_{N+1,L}h_{N+1} & K^2_{N+1,L} \end{pmatrix} \\
&\quad \begin{pmatrix} (H_{N,L-1}P)^+ & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Y_{N,L-1} - H_{N,L-1}A^+B \\ y_{N+1} - h_{N+1}A^+B \end{pmatrix} \\
&= A^+B + \begin{pmatrix} I - K^2_{N+1,L}h_{N+1} & K^2_{N+1,L} \end{pmatrix} \\
&\quad \begin{pmatrix} (H_{N,L-1}P)^+ Y_{N,L-1} - H_{N,L-1}A^+B \\ y_{N+1} - h_{N+1}A^+B \end{pmatrix} \\
&= A^+B + (H_{N,L-1}P)^+ \left(Y_{N,L-1} - H_{N,L-1}A^+B\right) \\
&\quad + K^2_{N+1,L}y_{N+1} - K^2_{N+1,L}h_{N+1}A^+B \\
&\quad + K^2_{N+1,L} \left(H_{N,L-1}P\right)^+ \left(Y_{N,L-1} - H_{N,L-1}A^+B\right) \\
&= \hat{\theta}_{N,L-1} + K^2_{N+1,L} \left(y_{N+1} - h_{N+1}\hat{\theta}_{N,L-1}\right)
\end{aligned} \tag{34}$$

The recursive algorithm of adding the recent data with linear-equality constraints is as follows:

$$\hat{\theta}_{N+1,L} = \hat{\theta}_{N,L-1} + K^2_{N+1,L}\varepsilon^2_{N+1,L} \tag{35}$$

$$\varepsilon^2_{N,L-1} = y_{N+1} - h_{N+1}\hat{\theta}_{N,L-1} \tag{36}$$

$$K^2_{N+1,L} = P_{N,L-1}h^T_{N+1} \left(I + h_{N+1}P_{N,L-1}h^T_{N+1}\right)^{-1} \tag{37}$$

$$\begin{aligned}
P_{N+1,L} &= \left(PH^T_{N+1,L}H_{N+1,L}P\right)^{-1} \\
&= \left(I + K^2_{N+1,L}h_{N+1}\right) P_{N,L-1}
\end{aligned} \tag{38}$$

D. Proposed Adaptive Algorithm

It is possible to group the two previous steps by eliminating the intermediate estimation. This point of view leads to the one step adaptive algorithm :

$$\hat{\theta}_{N+1,L} = \hat{\theta}_{N,L} - K^r_{N+1,L}\varepsilon^r_{N+1,L} + K^a_{N+1,L}\varepsilon^a_{N+1,L} \tag{39}$$

by using the following notations:

$$\varepsilon^r_{N+1,L} = y_{N-L-1} - h_{N-L-1}\hat{\theta}_{N,L} \tag{40}$$

$$K^r_{N+1,L} = K^1_{N,L-1} - K^2_{N+1,L}h_{N+1}K^1_{N,L-1} \tag{41}$$

$$\varepsilon^a_{N+1,L} = y_{N+1} - h_{N+1}\hat{\theta}_{N,L} \tag{42}$$

$$K^a_{N+1,L} = K^2_{N+1,L} \tag{43}$$

In the next section, a simple initialization of constrained algorithm is presented to facilitate the convergence towards the true parameters (the exact solution).

E. Initialization of Constrained Algorithm

The only difference between the two solutions without constraints and with linear-equality constraints, in the sliding window identification algorithm, lies in their initial values. If the initial values are $\hat{\theta}_{N_0,L} = \left(H^T_{N_0,L}H_{N_0,L}\right)^{-1} H^T_{N_0,L}Y_{N_0,L}$ and $P_{N_0,L} = \left(H^T_{N_0,L}H_{N_0,L}\right)^{-1}$, $\hat{\theta}_{N,L}$ is the solution without constraints. If the initial values are $\hat{\theta}_{N_0,L} = A^+B + \left(PH^T_{N_0,L}H_{N_0,L}P\right)^+ H^T_{N_0,L}\left(Y_{N_0,L} - H_{N_0,L}A^+B\right)$ and $P_{N_0,L} = \left(PH^T_{N_0,L}H_{N_0,L}P\right)^+$, $\hat{\theta}_{N,L}$ is then the solution with linear-equality constraints.
In practice this initialization is undesirable or even unacceptable. For this raison, using a simple initialization which was proposed in [24]. Defining:

$$\tilde{H}_{N,L} \stackrel{\text{def}}{=} [H_{0,L}, H_{N,L}], \quad \tilde{Y}_{N,L} \stackrel{\text{def}}{=} [Y_{0,L}, Y_{N,L}]$$

and considering the following LS problem subject to linear constraints (20):

$$\begin{aligned}
\min_{\tilde{\theta}_{N,L}} \tilde{S}_N &= \left(\tilde{Y}_{N,L} - \tilde{H}_{N,L}\tilde{\theta}_{N,L}\right)\left(\tilde{Y}_{N,L} - \tilde{H}_{N,L}\tilde{\theta}_{N,L}\right)^T \\
&= \left(\tilde{Y}_{0,L} - \tilde{H}_{0,L}\tilde{\theta}_{N,L}\right)\left(\tilde{Y}_{0,L} - \tilde{H}_{0,L}\tilde{\theta}_{N,L}\right)^T \\
&\quad + \left(Y_{N,L} - H_{N,L}\tilde{\theta}_{N,L}\right)\left(Y_{N,L} - H_{N,L}\tilde{\theta}_{N,L}\right)^T
\end{aligned} \tag{44}$$

where $\tilde{\theta}_{N,L}$ is the solution of this problem which is not the exact recursive solution $\hat{\theta}_{N,L}$. Note that $H_{0,L}$ and $Y_{0,L}$ are chosen in a way to obtain this simple initialization:

$$P_{0,L} = \left(PH^T_{N_0,L}H_{N_0,L}P\right)^+ = (PRP)^+ \tag{45}$$

$$\begin{aligned}
\tilde{\theta}_{0,L} &= A^+B + \left(PH^T_{0,L}H_{0,L}P\right)^+ H^T_{0,L}\left(Y_{0,L} - H_{0,L}A^+B\right) \\
&= A^+B + P\tilde{\xi}
\end{aligned} \tag{46}$$

where $\tilde{\xi}$ is an arbitrary vector of suitable dimension and R is any Hermitian positive definite matrix. For reasons of simplicity, we can choose:

$$H^T_{0,L}H_{0,L} = R = \alpha I, \quad \alpha > 0 \tag{47}$$

Clearly, $H^T_{0,L}H_{0,L}$ is non singular. Based on (45), (47) and $P^2 = P$ (since P is a projector), we have $P^+_{0,L} = \alpha P$ and $P_{0,L} = \alpha^{-1}P^+$. Therefore, the solution $\tilde{\theta}_{N,L}$ converges to the solution $\hat{\theta}_{N,L}$ with linear-equality constraints when N increases, and $\hat{\theta}_{N,L}$ converges to the true parameters.

IV. NUMERICAL SIMULATIONS

In order to show the performance comparison of the constrained estimators discussed above, we consider an example of a DC motor (adopted in [9] and [10]) whose dynamic behavior can be described using the two following equations:

$$u = K_{em}\omega + Ri + L\frac{di}{dt} \tag{48}$$

$$J\frac{d\omega}{dt} = K_{em}i - f\omega \tag{49}$$

TABLE I
ALGEBRAIC SYMBOLS DEFINITIONS.

Symbol	Unit	Definition
u	V	Electric terminal voltage
i	A	Electric armature current
ω	1/s	Rotational frequency
R	Ω	Ohmic ferrule resistor
K_{em}	NmA^{-1}	Generator constant
L	H	Inductivity
J	kgm^2	Moment of inertia
f	Nms	Sliding friction

The algebraic symbols are represented in the following table I.

The transition to the Laplace domain of temporal equations (48) and (49), gives the transfer function of DC motor defined in the following form:

$$H(s) = \frac{K_{em}}{R + {K_{em}}^2 + (RJ + Lf)\, s + LJs^2} \tag{50}$$

where s is the Laplace operator. The sampling period T_e of the discrete-time model is chosen such that:

$$T_e \leqslant \frac{T}{2}$$

where T is the time constant of system. For a sampling period, $T_e = 0.1\mathrm{s}$, the discrete-time linear model of a DC motor is defined by the following transfer function:

$$H\left(z^{-1}\right) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{51}$$

where z is the discrete-time operator. The real parameters values of the discrete model ($a_1 = -1.1753$, $a_2 = 0.8153$, $b_1 = 0.0072$ and $b_2 = 0.0054$) are obtained with the numerical values of the linear model (50) given in the table II.

TABLE II
PARAMETER VALUES.

Parameter	Unit	Value
R	Ω	1.3658
L	mH	0.63
K_{em}	NmA^{-1}	$45.15\ 10^{-3}$
J	kgm^2	10^{-4}
f	Nms	$2.5\ 10^{-6}$

In order to show the benefits of the proposed algorithm, a comparison between the sliding window identification algorithm with linear-equality constraints (SWLE) and the Recursive Least Squares algorithm (RLS) is carried. The difference between the solution of SWLE algorithm and that of the RLS algorithm lies in their initial conditions.

The sliding window identification algorithm $\tilde{\theta}_{N,L}$ is initialized at $N = 0$, according to (45) and (46), by:

$$Y_{0,L} = 0,$$

$$\tilde{\theta}_{0,L} = \left(I - \alpha^2 P\right) A^+ B,$$

$$P_{0,L} = \alpha^{-1} P^+$$

with $\alpha = 10^{-4}$ and the sliding window $L = 7$ s. In this example, we show the convergence of the SWLE algorithm when the true parameter θ satisfies $A\theta = B$. The following were used:

$$\theta = \left[\begin{array}{cccc} 0.0072 & 0.0054 & -1.1753 & 0.8513 \end{array}\right]^T,$$

$$A = \left[\begin{array}{cccc} 3 & -2 & -4.5 & 9 \end{array}\right],$$

$$B = 12.6374.$$

In the simulations, the system is excited by a Pseudo Random Binary Sequence (PRBS) signal. The simulation results are given in the following figures. The figures 1 and 2 clearly indicate the good tracking performance of the identified system output with respect to the real system output. The shapes of measured errors between the actual and estimated outputs are given in the figures 3 and 4. We can see that the variations of the system error in the case of RLS algorithm are more important than those in the case of SWLE algorithm.

The figure 5 shows the four elements of the vector $\hat{\theta}\,(be_1, be_2, ae_1, ae_2)$ converge quickly to the true parameters $\theta\,(b_1, b_2, a_1, a_2)$ in around 20 s. This indicates that the linear equality constraints $A\tilde{\theta}_{N,L} = B$ are always satisfied. Nevertheless, in the case of RLS algorithm, the convergence of the estimated parameters $\hat{\theta}$ to the true parameters is very slow compared to that of the SWLE algorithm. The figure 6 shows that the estimated parameters converge to the actual parameters in approximately 70 s.

Thus, it is demonstrated that the equality constrained algorithm developed in this paper for simple initial values is more convergent and efficient than the RLS algorithm. The table III shows a comparison between the latest estimated and true parameter values of system.

TABLE III
VALUES OF TRUE AND ESTIMATED PARAMETERS.

Parameter	b_1	b_2	a_1	a_2
True value	0.0072	0.0054	-1.1753	0.8153
Latest estimated value (RLS)	0.0072	0.0054	-1.1712	0.8121
Latest estimated value (SWLE)	0.0072	0.0054	-1.1753	0.8153

V. CONCLUSION

We have proposed in this paper a new algorithm of sliding window identification which uses a batch procedure. The main benefit of this proposed algorithm is that it satisfies the linear-equality constraints no matter how large the numerical errors are. The advantage of the presented procedure for sliding window identification consists its natural forgetting factor, which allows to follow the slow parameter changes. It is not necessary to consider a forgetting factor outside the

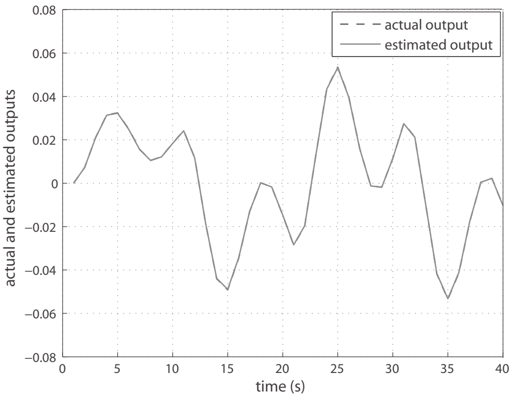

Fig. 1. Actual and estimated system outputs in the case of SWLE algorithm.

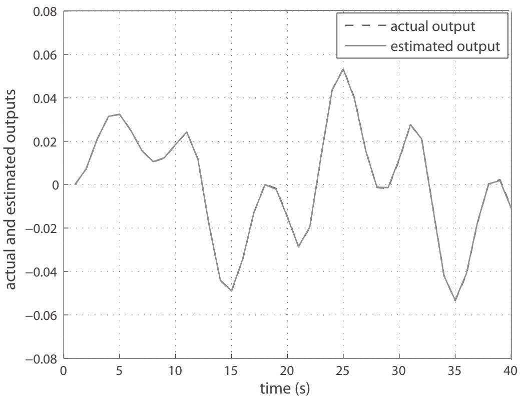

Fig. 2. Actual and estimated system outputs in the case of RLS algorithm.

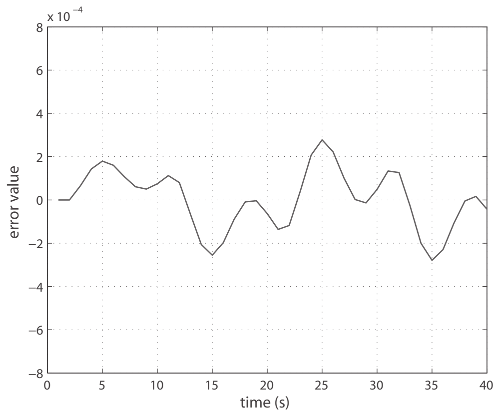

Fig. 3. System error in the case of SWLE algorithm.

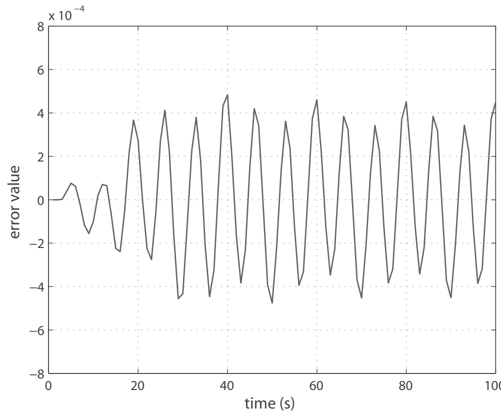

Fig. 4. System error in the case of RLS algorithm.

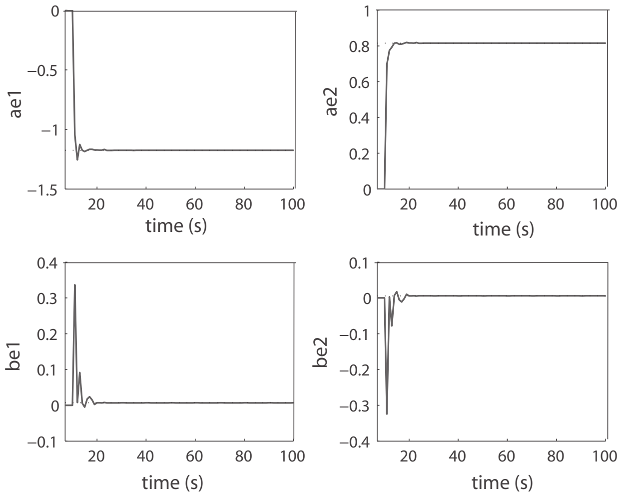

Fig. 5. Convergence of estimated parameters in the case of SWLE algorithm.

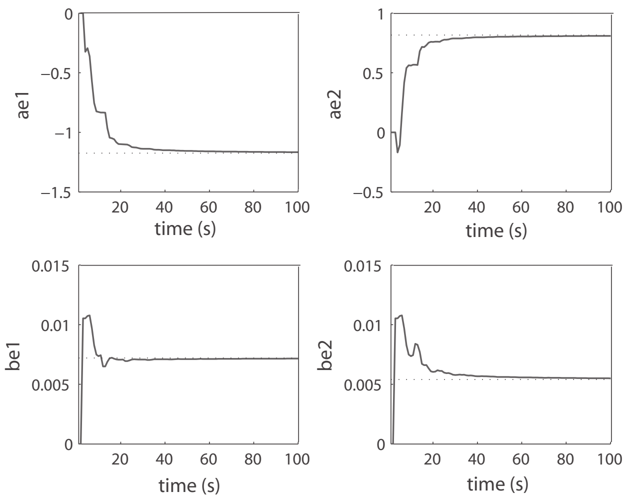

Fig. 6. Convergence of estimated parameters in the case of RLS algorithm.

window [13], because it seems less interesting. Therefore, because, we obtain in this case an infinite horizon algorithm. The unique solution of sliding window algorithm with linear-equality constraints can always be calculated by a recursion which is identical to the unconstrained algorithm solution. Consequently, the proposed algorithm is numerically robust allowing to ensure the recursive solution obtained by sliding window which satisfies the linear-equality constraints. The simple initialization of our algorithm allows to converge the obtained solution with linear-equality constraints to the true parameters. The numerical example considered of DC motor confirms the above analytical results.

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