A direct design procedure for linear state functional observers ☆

Frédéric Rotella, Irène Zambettakis

Laboratoire de Génie de Production, Ecole d'Ingénieurs de Tarbes, 47 avenue d'Azereix, 65016 Tarbes CEDEX, France

A B S T R A C T

We propose a constructive procedure to design a Luenberger observer to estimate a linear multiple linear state functional for a linear time-invariant system. Among other features the proposed design algorithm is not based on the solution of a Sylvester equation nor on the use of canonical state space forms. The design is based on the solution set of a linear equation and a realization method. The consistency of this equation and the stability of the observer can be used as a functional observability test.

Keywords:
Time-invariant system
Linear system
State functional observer
Luenberger observer

1. Introduction

Since Luenberger's works (Luenberger, 1963, 1964, 1966) a significant amount of research has been devoted to the problem of observing a linear functional of the state of a linear time-invariant system. The main developments are detailed in O'Reilly (1983), in Aldeen and Trinh (1999), Trinh and Fernando (2007) and Tsui (1985, 1998) and, in the recent books Korovin and Fomichev (2009) and Trinh and Fernando (2012) and the reference therein. The problem at first glance can be formulated as follows. For the linear state-space model

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \tag{1}$$

where, for every time t in $\mathbb{R}^+$, $x(t)$ is the n-dimensional state vector, $u(t)$ the p-dimensional measured input, $y(t)$ the m-dimensional measured output, and, A, B and C are constant matrices of adapted dimensions, the objective is to get,

$$v(t) = Lx(t), \tag{2}$$

where L is a constant $(l \times n)$ matrix. The observation of $v(t)$ can be carried out with the design of a Luenberger observer

$$\begin{aligned}\dot{z}(t) &= Fz(t) + Gu(t) + Hy(t),\\ w(t) &= Pz(t) + Vy(t),\end{aligned} \tag{3}$$

where $z(t)$ is a q-dimensional state vector. Constant matrices F, G, H, P and V are determined such that

$$\lim_{t\to\infty} (v(t) - w(t)) = 0.$$

We know from Fortmann and Williamson (1972) and Fuhrmann and Helmke (2001) that the observable linear functional observer (3) exists if and only if there exists a $(q \times n)$ matrix T such that:

$$G = TB,$$

$$TA - FT = HC, \tag{4}$$

$$L = PT + VC, \tag{5}$$

where F is an Hurwitz matrix, namely, when all the real parts of the eigenvalues of F are strictly negative. When these conditions are fulfilled we have $\lim_{t\to\infty}(z(t) - Tx(t)) = 0$. It is also well known that when rank $\left(\begin{bmatrix} L^\top & C^\top \end{bmatrix}\right) = m + l$ the order q of the multiple state functional observer is such that $q \geq l$ (Roman & Bullock, 1975; Sirisena, 1979). Darouach in Darouach (2000) has then proposed existence conditions for a Luenberger observer of the functional (2) with a minimum order l. Moreover, when the model (1) is detectable we have $q \leq n - m$. Indeed, $n - m$ is the order of the reduced-order observer or Cumming–Gopinath observer (Cumming, 1969; Gopinath, 1971) which can be built to observe $x(t)$ and, consequently, $v(t)$.

Until now the direct design of a minimal observer of a given linear functional is an open question. Since Fortmann and Williamson (1972), a lot of design schemes have been proposed to reduce the order of the observer (3). One way is to determine the matrices

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E-mail addresses: rotella@enit.fr (F. Rotella), izambettakis@iut-tarbes.fr (I. Zambettakis).

T and F such that the Sylvester equation (4) is fulfilled (Trinh, Nahavandi, & Tran, 2008; Tsui, 2004). Unfortunately, F and T are unknown in (4) and some conditions are added to get a solution (e.g. fixed eigenvalues for the observer, canonical state space forms). Another way, implied by the functional observability notion (Fernando, Jennings, & Trinh, 2010a,b; Fernando, Trinh, & Jennings, 2010; Fernando & Trinh, 2013; Jennings, Fernando, & Trinh, 2011), consists in expanding the matrix L with a matrix R such that there exists a s-order Luenberger observer of the linear state functional $Sx(t)$ where

$$S = \begin{bmatrix} R \\ L \end{bmatrix},$$

and $s = \operatorname{rank}(S)$. It is underlined in Trinh and Fernando (2012, p. 66) that "how to find a matrix R with the smallest number of rows is an intriguing and challenging problem". Nevertheless, based on singular value decompositions (Fernando et al., 2010b), on eigenspace projections (Jennings et al., 2011), on canonical forms (Korovin, Medvedev, & Fomichev, 2010), or on computations for row-space extensions (Fernando & Trinh, 2013), some procedures have been proposed to tackle the observer design where the matrix F can have arbitrary eigenvalues. Indeed, the design of an observer must be thought in two different frameworks: on the one hand, the fixed-pole observer problem where the poles are fixed at the outset and, on the other hand, the stable observer problem where the poles are permitted to lie anywhere in the left half-plane. The main contributions on the design of functional observers tackle the first problem.

In the opposite, in order to seek for minimality of the observer order we focus here on the stable observer problem and develop a constructive procedure to design a Luenberger observer of the functional (2) for the system (1). Our algorithm is based on the solution set of a linear equation and linear algebraic operations in a state space setting. It can be seen as an extension of the algorithm proposed in Rotella and Zambettakis (2011) for single linear functional observers. With respect to other procedures the design procedure does not require the solution of the Sylvester equation. Moreover, the proposed solution exhibits free design parameters in the candidate observer to achieve asymptotic stability. Let us mention that we do not suppose any canonical form for the system neither for the observer. The main objectives of the paper are thus to provide a simple test for functional observability and a constructive procedure to design a stable Luenberger observer of a linear functional for a linear system. The paper is organized as follows. In the first section the procedure to design the observer structure is detailed. From the consistency condition of a linear equation are deduced a minimal index and the state space equation of the observer. The second section is devoted to analyze the stabilizability of the matrix F. The proposed procedure is exemplified in a third section. Finally, a method is proposed in the fourth section to reduce the order of the stable observer and to get it minimal. It has been underlined in Tsui (1998), that the calculus of the matrix T is not a necessary step. This point is a specific feature of the procedure we propose. Indeed, we exhibit the closed-form of this matrix as a collateral result.

2. A constructive procedure

Let us suppose that there exists an integer ν such that

$$\operatorname{rank}\left(\begin{bmatrix} LA^{\nu} \\ \Sigma_{\nu} \end{bmatrix}\right) = \operatorname{rank}\left(\Sigma_{\nu}\right), \tag{6}$$

where the matrix Σ_{ν} is defined as $\Sigma_{\nu} = C$ when $\nu = 0$ and,

$$\Sigma_{\nu} = \begin{bmatrix} CA^{\nu} \\ LA^{\nu-1} \\ CA^{\nu-1} \\ \vdots \\ LA \\ CA \\ L \\ C \end{bmatrix}, \tag{7}$$

when $\nu > 0$. In other words, the linear equation

$$LA^{\nu} = \Phi \Sigma_{\nu}, \tag{8}$$

is consistent, namely, there exist matrices $F_{L,i}$, $i = 0$ to $\nu - 1$, and $F_{C,i}$, $i = 0$ to ν, such that

$$LA^{\nu} = \sum_{i=0}^{\nu-1} F_{L,i} LA^{i} + \sum_{i=0}^{\nu} F_{C,i} CA^{i}. \tag{9}$$

Remark 1. Due to Cayley–Hamilton theorem the hypothesis (6) can always be fulfilled.

Let us notice as well that we have, for $k = 0, 1, \ldots$

$$v^{(k)}(t) = LA^{k}x(t) + \sum_{i=0}^{k-1} LA^{k-1-i}Bu^{(i)}(t),$$

so, from (9) we can write

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i}LA^{i}x(t) + \sum_{i=0}^{\nu} F_{C,i}CA^{i}x(t) + \sum_{i=0}^{\nu-1} LA^{\nu-1-i}Bu^{(i)}(t). \tag{10}$$

2.1. Structural design of the observer

This section is devoted to the determination of matrices F, G, H, P and, V in (3) from the existence of the relationship (9). Firstly, to eliminate $x(t)$ in (10) we use

- for $i = 1$ to $\nu - 1$, $v^{(i)}(t) = \sum_{j=0}^{i-1} LA^{j}Bu^{(i-1-j)}(t) + LA^{i}x(t)$, thus

$$LA^{i}x(t) = v^{(i)}(t) - \sum_{j=0}^{i-1} LA^{i-1-j}Bu^{(j)}(t); \tag{11}$$

- for $i = 1$ to ν, $y^{(i)}(t) = \sum_{j=0}^{i-1} CA^{j}Bu^{(i-1-j)}(t) + CA^{i}x(t)$, thus

$$CA^{i}x(t) = y^{(i)}(t) - \sum_{j=0}^{i-1} CA^{i-1-j}Bu^{(j)}(t). \tag{12}$$

Taking into account (11) and (12) in (10) we get then

$$v^{(\nu)}(t) = \sum_{i=0}^{\nu-1} F_{L,i}v^{(i)}(t) + \sum_{i=0}^{\nu} F_{C,i}y^{(i)}(t) + \sum_{i=0}^{\nu-1} G_{i}u^{(i)}(t), \tag{13}$$

where $G_{\nu-1} = \left(L - F_{C,\nu}C\right)B$ and, for $\nu \geq 2$ and $j = 0$ to $\nu - 2$,

$$G_{j} = \left(LA^{\nu-1-j} - \sum_{i=j+1}^{\nu-1} F_{L,i}LA^{i-1-j} - \sum_{i=j+1}^{\nu} F_{C,i}CA^{i-1-j}\right)B. \tag{14}$$

The next step consists in the realization of the differential equations (13). Following Kailath (1980), (13) can be written as

$$\begin{aligned} v(t) &= F_{C,\nu}y(t) \\ &\quad + p^{-1}\left[F_{L,\nu-1}v(t) + F_{C,\nu-1}y(t) + G_{\nu-1}u(t)\right. \\ &\quad \vdots \\ &\quad + p^{-1}\left[F_{L,1}v(t) + F_{C,1}y(t) + G_1u(t)\right. \\ &\quad + p^{-1}\left.\left.\left[F_{L,0}v(t) + F_{C,0}y(t) + G_0u(t)\right]\right]\cdots\right], \end{aligned}$$

where p stands for the derivative operator and, p^{-1} for the integration. Let us introduce the vectors

$$\begin{aligned} z_0(t) &= p^{-1}\left[F_{L,0}v(t) + F_{C,0}y(t) + G_0u(t)\right], \\ z_1(t) &= p^{-1}\left[F_{L,1}v(t) + F_{C,1}y(t) + G_1u(t) + z_0(t)\right], \\ &\vdots \\ z_{\nu-1}(t) &= p^{-1}\left[F_{L,\nu-1}v(t) + F_{C,\nu-1}y(t) + G_{\nu-1}u(t) + z_{\nu-2}(t)\right]. \end{aligned}$$

With $v(t) = z_{\nu-1}(t) + F_{C,\nu}y(t)$ we obtain

$$\begin{aligned} z_0(t) &= p^{-1}\left[F_{L,0}z_{\nu-1}(t) + H_{C,0}y(t) + G_0u(t)\right], \\ z_1(t) &= p^{-1}\left[F_{L,1}z_{\nu-1}(t) + H_{C,1}y(t) + G_1u(t) + z_0(t)\right], \\ &\vdots \\ z_{\nu-1}(t) &= p^{-1}\left[F_{L,\nu-1}z_{\nu-1}(t) + H_{C,\nu-1}y(t) + G_{\nu-1}u(t)\right. \\ &\quad \left. + z_{\nu-2}(t)\right], \end{aligned}$$

where, for $i = 0$ to $\nu - 1$, $H_{C,i} = F_{C,i} + F_{L,i}F_{C,\nu}$. The vector

$$z(t) = \begin{bmatrix} z_0(t)^\top & z_1(t)^\top & \cdots & z_{\nu-1}(t)^\top \end{bmatrix}^\top$$

is the state of the Luenberger observer structure (3) with

$$F = \begin{bmatrix} & & & & F_{L,0} \\ I_l & & & & F_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1} \end{bmatrix}, \qquad G = \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{\nu-2} \\ G_{\nu-1} \end{bmatrix}, \tag{15}$$
$$H = \begin{bmatrix} H_{C,0}^\top & H_{C,1}^\top & \cdots & H_{C,\nu-2}^\top & H_{C,\nu-1}^\top \end{bmatrix}^\top,$$
$$P = \begin{bmatrix} 0 & \cdots & 0 & I_l \end{bmatrix}, \qquad V = F_{C,\nu}.$$

Remark 2. Notice that the realization (15) is observable.

Remark 3. The matrix H can be written as

$$H = \begin{bmatrix} F_{C,0} \\ F_{C,1} \\ \vdots \\ F_{C,\nu-2} \\ F_{C,\nu-1} \end{bmatrix} + \begin{bmatrix} F_{L,0} \\ F_{L,1} \\ \vdots \\ F_{L,\nu-2} \\ F_{L,\nu-1} \end{bmatrix} F_{C,\nu}. \tag{16}$$

Remark 4. For sake of completeness, let us consider here the cases $\nu = 0$ and $\nu = 1$. When $\nu = 0$ there exists Λ such that $L = \Lambda C$. The observer of the functional (2) is $w(t) = \Lambda y(t)$. When $\nu = 1$, namely,

$$\operatorname{rank}\left(\begin{bmatrix} LA \\ CA \\ L \\ C \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} CA \\ L \\ C \end{bmatrix}\right),$$

there exist $F_{C,1}$, $F_{L,0}$ and $F_{C,0}$ such that

$$LA = F_{C,1}CA + F_{L,0}L + F_{C,0}C.$$

This case has been detailed in Rotella and Zambettakis (2011) and leads to the Darouach–Luenberger observer structure (3) with

$$F = F_{L,0}, \qquad G = \left(L - F_{C,1}C\right)B, \qquad H = F_{C,0} + F_{L,0}F_{C,1},$$
$$P = I_l, \qquad V = F_{C,1}.$$

The following section ensures that, when F is an Hurwitz matrix, we have designed an asymptotically stable Luenberger observer.

2.2. Main result

Theorem 5. *If there exist $\nu \in \mathbb{N}$, and constant matrices $F_{L,i}$, $i = 0$ to $\nu - 1$, and $F_{C,i}$, $i = 0$ to ν, such that (9) is fulfilled and the matrix $F = F_{L,0}$, for $\nu = 1$, or*

$$F = \begin{bmatrix} & & & & F_{L,0} \\ I_l & & & & F_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1} \end{bmatrix}, \tag{17}$$

for $\nu > 1$, is an Hurwitz matrix then the Luenberger observer (3) defined by (15) is an asymptotic observer of $v(t)$.

Proof. The proof lies on the determination of T which fulfills the necessary conditions (4) and (5). For sake of simplicity let us consider the case $\nu \geq 2$. Firstly, let us remark that the relationship $G = TB$ with

$$G = \begin{bmatrix} G_0^\top & G_1^\top & \cdots & G_{\nu-2}^\top & G_{\nu-1}^\top \end{bmatrix}^\top$$

where the matrices G_j are defined in (14), leads to the hypothesis

$$T = \begin{bmatrix} T_1^\top & T_2^\top & \cdots & T_{\nu-1}^\top & T_\nu^\top \end{bmatrix}^\top,$$

where, for $j = 1$ to $\nu - 1$

$$T_j = LA^{\nu-j} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i-j} - \sum_{i=j}^{\nu} F_{C,i}CA^{i-j},$$

and $T_\nu = L - F_{C,\nu}C$. In the following, we state this matrix T is a solution of $TA - FT = HC$ where F and H are defined in (15). Let us denote

$$TA = \begin{bmatrix} (TA)_1 \\ (TA)_2 \\ \vdots \\ (TA)_{\nu-1} \\ (TA)_\nu \end{bmatrix} \quad \text{and} \quad FT = \begin{bmatrix} (FT)_1 \\ (FT)_2 \\ \vdots \\ (FT)_{\nu-1} \\ (FT)_\nu \end{bmatrix},$$

where, for $j = 1$ to ν, the blocks $(TA)_j$ and $(FT)_j$ have l rows. On the one hand, we have

$$(TA)_\nu = LA - F_{C,\nu}CA,$$

and, for $j = 1$ to $\nu - 1$,

$$(TA)_j = LA^{\nu-j+1} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i-j+1} - \sum_{i=j}^{\nu} F_{C,i}CA^{i-j+1}.$$

On the other hand, we have

$$(FT)_1 = F_{L,0}L - F_{L,0}F_{C,\nu}C,$$

and, for $j = 2$ to ν,

$$(FT)_j = T_{j-1} + F_{L,j-1}T_\nu,$$
$$= LA^{\nu+1-j} - \sum_{i=j-1}^{\nu-1} F_{L,i}LA^{i+1-j} + F_{L,j-1}L$$
$$- \sum_{i=j-1}^{\nu} F_{C,i}CA^{i+1-j} - F_{L,j-1}F_{C,\nu}C. \quad (18)$$

For $j = 2$ to $\nu - 1$, (18) can be written as

$$(FT)_j = LA^{\nu+1-j} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i+1-j}$$
$$- \sum_{i=j}^{\nu} F_{C,i}CA^{i+1-j} - \left(F_{C,j-1} + F_{L,j-1}F_{C,\nu}\right)C,$$

and, for $j = \nu$,

$$(FT)_\nu = LA - F_{C,\nu}CA - \left(F_{C,\nu-1} + F_{L,\nu-1}F_{C,\nu}\right)C.$$

Let us remark that (9) leads to write

$$LA^\nu - \sum_{i=1}^{\nu-1} F_{L,i}LA^i - \sum_{i=1}^{\nu} F_{C,i}CA^i = \sum_{i=0}^{\nu-1} F_{L,i}LA^i$$
$$+ \sum_{i=0}^{\nu} F_{C,i}CA^i - \sum_{i=1}^{\nu-1} F_{L,i}LA^i - \sum_{i=1}^{\nu} F_{C,i}CA^i$$
$$= F_{L,0}L + F_{C,0}C.$$

Thus, $(TA)_j - (FT)_j$ can be read

- for $j = 1$, $F_{L,0}L + F_{C,0}C - \left(F_{L,0}L - F_{L,0}F_{C,\nu}C\right) = \left(F_{C,0} + F_{L,0}F_{C,\nu}\right)C$;
- for $j = 2$ to $\nu - 1$,

$$LA^{\nu-j+1} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i-j+1} - \sum_{i=j}^{\nu} F_{C,i}CA^{i-j+1}$$
$$- \Bigg(LA^{\nu+1-j} - \sum_{i=j}^{\nu-1} F_{L,i}LA^{i+1-j} - \sum_{i=j}^{\nu} F_{C,i}CA^{i+1-j}$$
$$\times \sum_{i=j}^{\nu} F_{C,i}CA^{i+1-j} - \left(F_{C,j-1} + F_{L,j-1}F_{C,\nu}\right)C\Bigg)$$
$$= \left(F_{C,j-1} + F_{L,j-1}F_{C,\nu}\right)C;$$

- for $j = \nu$, $LA - F_{C,\nu}CA - LA + F_{C,\nu}CA + \left(F_{C,\nu-1} + F_{L,\nu-1}F_{C,\nu}\right)C = \left(F_{C,\nu-1} + F_{L,\nu-1}F_{C,\nu}\right)C$.

Taking into account that, for $j = 0$ to $\nu - 1$, $H_{C,j} = F_{C,j} + F_{L,j}F_{C,\nu}$, we deduce that T fulfills the Sylvester equation $TA - FT = HC$. Moreover,

$$P = \begin{bmatrix} 0 & \cdots & 0 & I_l \end{bmatrix} \quad \text{and} \quad V = F_{C,\nu},$$

lead to

$$PT + VC = L - F_{C,\nu}C + F_{C,\nu}C = L,$$

which ends the proof. $\diamond$

When F is not an Hurwitz matrix, we propose to increase the integer ν. In the following section we see how to get, when it is possible, a stable observer.

Remark 6. We have not supposed, as usually, that

$$\text{rank} \begin{bmatrix} C \\ L \end{bmatrix} = m + l.$$

Indeed, the proposed design procedure includes this case as well.

3. Analysis of the linear system

3.1. The solution set

The design procedure is based on the existence of the decomposition (9) when (8) is consistent. The matrices $F_{L,i}$, $i = 0$ to $\nu - 1$, and $F_{C,i}$, $i = 0$ to ν, are given through the solution set of the consistent linear equation (8) by

$$\begin{bmatrix} F_{C,\nu} & F_{L,\nu-1} & F_{C,\nu-1} & \cdots & F_{L,0} & F_{C,0} \end{bmatrix}$$
$$= LA^\nu \Sigma_\nu^{[1]} + \Omega\left(I_\rho - \Sigma_\nu \Sigma_\nu^{[1]}\right), \quad (19)$$

where $\rho = m + \nu(m + l)$, Ω is an arbitrary $(l \times \rho)$ matrix and, $\Sigma_\nu^{[1]}$ a generalized inverse of Σ_ν (Ben-Israel & Greville, 2003)

$$\Sigma_\nu^{[1]} \in \left\{X \in R^{n\times\rho},\ \Sigma_\nu X \Sigma_\nu = \Sigma_\nu\right\}.$$

Remark 7. If rank $(\Sigma_\nu) = \rho$ the solution (19) is unique and given by $LA^\nu \Sigma_\nu^{[1]}$. This solution is independent of the particular choice of $\Sigma_\nu^{[1]}$ (Ben-Israel & Greville, 2003). A convenient partition of (19) leads then to F (17). An eigenvalues inspection is then required to verify that F is an Hurwitz matrix.

3.2. Stabilizability test for F

Let us consider that the matrix Σ_ν defined in (7) is such that $r = \text{rank}\ (\Sigma_\nu) < \rho$. Namely, the matrices $F_{L,i}$, $i = 0$ to $\nu - 1$, and $F_{C,i}$, $i = 0$ are not unique. In order to give a closed-form of these matrices and to test the existence of a matrix Ω such that F is an Hurwitz matrix, let us consider the singular values decomposition (SVD) of Σ_ν (Golub & Van Loan, 1996)

$$\Sigma_\nu = U_\nu S_\nu V_\nu^\top, \quad (20)$$

where U_ν and V_ν are two orthogonal matrices with size $(\rho \times \rho)$ and $(n \times n)$ respectively, and S_ν is the $(\rho \times n)$ diagonal matrix of the singular values of Σ_ν

$$S_\nu = \text{diag}\left\{\sigma_1, \ldots, \sigma_r, 0, \ldots, 0\right\},$$

with $\sigma_1 \geq \cdots \geq \sigma_r > 0$. From the SVD (20) we can build the pseudo-inverse (Ben-Israel & Greville, 2003) of Σ_ν, which is in Σ_ν [1], as

$$\Sigma_\nu^\dagger = V_\nu S_\nu^{-\top} U_\nu^\top,$$

where $S_\nu^{-\top} = \text{diag}\left\{\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots, 0\right\}$. This choice for the generalized inverse of Σ_ν, leads to write

$$I_\rho - \Sigma_\nu \Sigma_\nu^\dagger = I_\rho - U_\nu \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U_\nu^\top = U_\nu \begin{bmatrix} 0 & 0 \\ 0 & I_{\rho-r} \end{bmatrix} U_\nu^\top.$$

Let us denote $U_{2,\nu}^\top$ the $\rho - r$ last lines of $U_\nu^\top$, Γ_2, the $\rho - r$ last columns of the arbitrary matrix $\Gamma = \Omega U_\nu$ and,

$$\Phi^b = \begin{bmatrix} F_{C,\nu}^b & F_{L,\nu-1}^b & F_{C,\nu-1}^b & \cdots & F_{L,0}^b & F_{C,0}^b \end{bmatrix} = LA^\nu \Sigma_\nu^\dagger.$$

We get then

$$\begin{bmatrix} F_{C,\nu} & F_{L,\nu-1} & F_{C,\nu-1} & \cdots & F_{L,0} & F_{C,0} \end{bmatrix} = \Phi^b + \Gamma_2 U_{2,\nu}^\top,$$

after partitioning $U_{2,\nu}^\top$ according to

$$U_{2,\nu}^\top = \begin{bmatrix} \Upsilon_{C,\nu} & \Upsilon_{L,\nu-1} & \Upsilon_{C,\nu-1} & \cdots & \Upsilon_{L,0} & \Upsilon_{C,0} \end{bmatrix},$$

where the sizes of the matrices $\Upsilon_{C,i}$ and $\Upsilon_{C,i}$ are $((\rho - r) \times m)$ and $((\rho - r) \times l)$ respectively, we can write F as

$$F = \begin{bmatrix} & & & & F_{L,0}^b + \Gamma_2 \Upsilon_{L,0} \\ I_l & & & & F_{L,1}^b + \Gamma_2 \Upsilon_{L,1} \\ & \ddots & & & \vdots \\ & & I_l & & F_{L,\nu-2}^b + \Gamma_2 \Upsilon_{L,\nu-2} \\ & & & I_l & F_{L,\nu-1}^b + \Gamma_2 \Upsilon_{L,\nu-1} \end{bmatrix}. \quad (21)$$

So the asymptotic observer exists if it is possible to find a $(\rho \times (\rho - r))$ matrix Γ_2 such that (21) is an Hurwitz matrix. The application of the Routh–Hurwitz stability test on det $(\lambda I_{\nu l} - F)$ allows to build existence conditions for such a Γ_2.

Remark 8. Following Bernstein (2009) and Gantmacher (1960), we can notice that det $(\lambda I_{\nu l} - F)$ is given by

$$\det\left(\Lambda^{\nu} - \sum_{j=0}^{\nu-1}\left(F_{L,j}^{b} + \Gamma_2 \Upsilon_{L,j}\right)\Lambda^{j}\right)$$
$$= \det\left(\left(\Lambda^{\nu} - \sum_{j=0}^{\nu-1} F_{L,j}^{b}\Lambda^{j}\right) - \Gamma_2\left(\sum_{j=0}^{\nu-1}\Upsilon_{L,j}\Lambda^{j}\right)\right)$$

where $\Lambda = \lambda I_l$. So, just the determinant of a $(l \times l)$ matrix has to be calculated.

4. Illustrative example

In order to illustrate the different steps of our design method let us consider an example inspired from Trinh and Fernando (2012) and Tsui (1998), with

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The first step deals with the determination of ν. Denoting $r_\nu = \operatorname{rank}\left(\begin{bmatrix} LA^{\nu} \\ \Sigma_\nu \end{bmatrix}\right) - \operatorname{rank}(\Sigma_\nu)$, we obtain $r_0 = r_1 = 2$, and $r_2 = 0$. Thus we deduce $\nu = 2$, and the linear equation (8) with

$$\Sigma_2 = \begin{bmatrix} 1 & 0 & 0 & -4 & 0 & 1 & 1 \\ -2 & 3 & 0 & 5 & 2 & 0 & -2 \\ 0 & 3 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$LA^2 = \begin{bmatrix} 3 & 0 & 3 & -3 & 4 & -1 & -1 \\ 1 & 1 & 0 & 9 & 1 & -4 & -5 \end{bmatrix}.$$

With $\Sigma_2^{[1]} = \Sigma_2^{\dagger}$ the pseudo-inverse of Σ_2, and

$$\Omega = \begin{bmatrix} -0.4 & 0.3 & -0.6 & 1 & -0.5 & 0 & 0 & 0.4 & 0.5 & 0.9 \\ -0.4 & 0.3 & -0.6 & 1 & -0.5 & 0 & 0 & 0.4 & 0.5 & 0.9 \end{bmatrix},$$

we obtain from (19)

$$F_{L,0} = \begin{bmatrix} -1.00 & -0.90 \\ 0 & -1.87 \end{bmatrix}, \quad F_{L,1} = \begin{bmatrix} -1.15 & 0.56 \\ -1.73 & -1.80 \end{bmatrix},$$

$$F_{C,0} = \begin{bmatrix} -0.42 & 1.73 \\ 0.92 & 2.01 \end{bmatrix}, \quad F_{C,1} = \begin{bmatrix} -0.95 & 4 \\ -3.26 & 0 \end{bmatrix},$$

$$F_{C,2} = \begin{bmatrix} -0.38 & 0.58 \\ -0.44 & -1.36 \end{bmatrix}.$$

Following (17) the second step consists in the definition of

$$F = \begin{bmatrix} 0 & 0 & -1.00 & -0.90 \\ 0 & 0 & 0 & -1.87 \\ 1 & 0 & -1.15 & 0.56 \\ 0 & 1 & -1.73 & -1.80 \end{bmatrix}.$$

The eigenvalues of F are $-1.33 \pm 1.73i$ and $-0.14 \pm 0.61i$. Thus F is an Hurwitz matrix and we can design a fourth-order Luenberger asymptotically stable observer for the linear functional $Lx(t)$.

The observer design is achieved with the third step. Applying the formulas (14) and (16) we get

$$G = \begin{bmatrix} -0.58 \\ 0.08 \\ 0.38 \\ 0.44 \end{bmatrix}, \quad H = \begin{bmatrix} 0.35 & -0.02 \\ 1.73 & -0.40 \\ -0.75 & 4.07 \\ 1.78 & -3.49 \end{bmatrix}.$$

The design of the observer is complete with $V = F_{C,2}$, and

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. Reduction of the observer

When theorem 5 is fulfilled the order of the designed observer is νl. In order to obtain a minimum stable observer an order reduction is needed. Let us suppose that F is an Hurwitz matrix and the matrix T, which depends on the choice of Ω is such that

$\operatorname{rank}(T) = \kappa < l\nu$.

Let us consider then the full rank decomposition

$$T = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \Gamma, \tag{22}$$

where Φ_1 is a $(\kappa \times \kappa)$ non singular matrix and, Γ a full row rank matrix. Finally, let us define the matrices $T_1 = \Phi_1\Gamma$ and $T_2 = \Phi_2\Gamma$, the following partitions

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}, \tag{23}$$

where F_{11}, G_1 and, H_1 are matrices with κ rows, P_1 is a matrix with κ columns and the vectors $\eta(t) = \Phi_1^{-1}z_1(t)$ and $z_2(t) = \Phi_2\eta(t)$. We know that $\lim_{t\to\infty} z_1(t) = T_1x(t)$ and $\lim_{t\to\infty}\eta(t) = \Phi_1^{-1}T_1x(t) = \Gamma x(t)$. On the one hand, the state space equation for $\eta(t)$ is given by

$$\begin{aligned} \dot{\eta}(t) &= \Phi_1^{-1}\dot{z}_1(t), \\ &= \Phi_1^{-1}\left(F_{11}z_1(t) + F_{12}z_2(t) + G_1u(t) + H_1y(t)\right) \\ &= \Phi_1^{-1}\left(F_{11}\Phi_1 + F_{12}\Phi_2\right)\eta(t) + \Phi_1^{-1}G_1u(t) + \Phi_1^{-1}H_1y(t). \end{aligned}$$

On the other hand, we have $w(t) = (P_1\Phi_1 + P_2\Phi_2)\eta(t) + Vy(t)$, and,

$$\begin{aligned} \lim_{t\to\infty} w(t) &= (P_1\Phi_1 + P_2\Phi_2)\Gamma x(t) + Vy(t), \\ &= (P_1T_1 + P_2T_2)x(t) + Vy(t), \\ &= (PT + VC)x(t) = Lx(t) = v(t). \end{aligned}$$

We have proved the following result which leads to a minimal order stable functional observer of $v(t)$.

Corollary 9. *If theorem 5 is fulfilled, the κth order system*

$$\begin{aligned} \dot{\eta}(t) &= \Phi_1^{-1}\left(F_{11}\Phi_1 + F_{12}\Phi_2\right)\eta(t) \\ &\quad + \Phi_1^{-1}G_1u(t) + \Phi_1^{-1}H_1y(t), \\ w(t) &= (P_1\Phi_1 + P_2\Phi_2)\eta(t) + Vy(t), \end{aligned}$$

where the involved matrices are defined in (22) and (23) is a stable functional Luenberger observer of the linear functional (2) for the linear system (1).

6. Conclusion

In this paper, we have proposed a constructive procedure to design a stable Luenberger functional observer for a linear system. Specific features are, on the one hand, the solution of the Sylvester equation is not necessary, and, on the other hand, no canonical form is required. Let us remark that the Sylvester equation is nonlinear due to the fact that F and T are unknown. Our method encompasses this drawback. Indeed, the design is based on the solution set of a linear equation and on a realization method. The consistency of this equation and the stability of the matrix F can be seen as a functional observability test. Future works will deal with the determination of conditions on the matrix Ω to lead to a minimal order stable observer.

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Frédéric Rotella received his Ph.D. in 1983 and his Doctorat d'Etat in 1987 from the University of Science and Technology of Lille-Flandres-Artois, France. During this period, he served at the Ecole Centrale de Lille, France, as assistant professor in automatic control. In 1994, he joined the Ecole Nationale d'Ingénieurs de Tarbes, France, as professor of automatic control and mathematics where he is in charge of the Department of Electrical Engineering. His research fields of interest are about control of nonlinear systems and linear time-varying systems with a particular focus on observation. Professor Rotella has coauthored eight French textbooks in automatic control (e.g. Automatique élémentaire, Hermès-Lavoisier, 2008) or in applied mathematics (e.g. Théorie et pratique du calcul matriciel, Technip, 1995, and, Traitement des systèmes linéaires, Ellipses, 2015).

Irène Zambettakis received her Ph.D. in 1983 and her Doctorat d'Etat in 1987 from the University of Science and Technology of Lille-Flandres-Artois, France. During this period, she served at the Ecole Centrale de Lille, France, first as lecturer and then as assistant professor from 1988 to 1994, in automatic control, image processing and mathematics. In 1994, she joined the Institut Universitaire de Technologie de Tarbes (University Paul Sabatier, Toulouse, France) where she became professor in 2000 in automatic control and signal processing. Her research interests include distributed parameter systems and linear and nonlinear systems control. In 2012, she joined the Laboratoire de Génie de Production where she develops researches on observers. Professor Zambettakis has coauthored seven French textbooks in automatic control and applied mathematics. The last ones, coauthored with Professor Rotella, are entitled Automatique élémentaire (Hermès-Lavoisier, 2008) and Traitement des systèmes linéaires (Ellipses, 2015).