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MULTISYMBOL WITH MEMORY NONCOHERENT DETECTION OF CPFSK

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ABSTRACT

Multisymbol receiver is an effective method to demodulate noncoherent sequences. However it is necessary to correlate an important number of symbols in a noncoherent scheme to reach the performances carried out by optimal coherent Maximum a Posteriori (MAP) detectors such as BCJR. In this paper, we propose an advanced multisymbol receiver by adding some memory to the decision process. The advanced receiver, called here Multisymbol With Memory (MWM) takes into account the cumulative phase information unlike multisymbol algorithm and thus it can be seen as a truncated BCJR. An exact mathematical derivation is performed for this truncated BCJR. Then an implementation of the MWM detector applied to a continuous phase frequency shift keying modulation is presented. Finally an asymptotic analysis is carried out based on the achievable Symmetric Mutual Information rate. The proposed system exhibits good performances compared to classical multisymbol receivers at the expense of increased complexity and can approach the performances of a coherent receiver.

Index Terms— Multisymbol, noncoherent, BCJR, continuous phase modulation, mutual information, channel capacity, continuous phase frequency shift keying.

1. INTRODUCTION

Continuous phase modulation (CPM) [1],[2], [3] is a particular modulation characterised by a constant envelop waveform that means the transmitted carrier power is constant. This special feature leads to excellent power efficiency. A second important aspect of CPM is the phase continuity yielding high spectral efficiency. The phase of a CPM signal for a given symbol is determined by the cumulative phase of previous transmitted symbols known as the phase memory. Hence the decision taken on the current symbol must take into account the previous ones. An other important element of CPM is the modulation index which could restrain, in a particular case, the set of the phase memory to a finite set. A well-known type of CPM is the continuous phase frequency shift keying (CPFSK) [4], [5] described by a rectangular phase response and a memory of one, meaning that the new phase is computed only from one previous symbol. In the coherent case, an efficient method is proposed by Cheng based on the well-knwon BCJR [7]. Cheng [8] implemented the BCJR algorithm for CPFSK. The trellis consists of Q states where Q is the cardinal of the set composed of all possible values taken by the phase

memory. Q is also the denominator of the CPFSK modulation index $h = \frac{P}{Q}$. To get a coherent detector with a finite set of phase memory, it is mandatory having h rational. In noncoherent channel, symbol detection is done through a multisymbol detector well described by Valenti [6].

Multisymbol receiver proposed by Valenti [6] does the correlation between the block of received symbols and all existing combinations of same length blocks. The condition required to use this method is the absence of phase shift between symbols belonging to the same block. As well, in this paper perfect frequency and time synchronisation is assumed. We suppose the phase shift is stable over a block of N symbols. It exists M^N possible combinations for a block of size N (with M the modulation order). The process is conducted as follows. The incoming signal is filtered by a bank of M matched filters. Then the conditional probability of the transmitted symbols block of size N under the condition of one of the possible combination is done for each existing combination. Eventually, the demodulator computes the log-likelihood ratio (LLR) for each bit of the block based on the conditional probability. The literal analytical expression of the conditional probability, given by Valenti, is independent of the initial phase of the block, moreover an other extra advantage of the multisymbol receiver is that it can work for any value of h. It reveals the robustness of the detector against untimely channel phase shift.

In this paper, we propose an advanced multisymbol receiver taking into account the phase memory. Block of symbols and accumulated phase are henceforth included in states. The transition from one state to another is done by the phase memory at the beginning of each state. It can be seen as a truncated BCJR. In [9], Giulio Colavope suggested a decomposition of BCJR states close to the model proposed in this paper. Here, we adopt another point of view starting from the multisymbol classical receiver and adding memory leading to a windowed multisymbol receiver with memory. It conducts to different expressions of the BCJR recursions and a different state space model.

The remainder of this paper is organized as follows. In the next section, we describe the system model. Then, in section III, we carry out an analysis of our model. Thereafter the derivation of the MWM algorithm is given in section IV. An implementation of the MWM applied to CPFSK is presented section V. Finally, the mutual information rate of the system is derived and subsequently used to compute the channel capacity. Section VII will conclude the paper.

2. SYSTEM MODEL

Let $U=\{u_0,...,u_{K-1}\}$ be a set of K independent and identically distributed (i.i.d) symbols belonging to a M-ary alphabet $\{0,...,M-1\}$. Symbols are then sent to a modulator comprised of a memoryless modulator and a continuous phase encoder (CPE). As a reminder, CPFSK modulation is assumed meaning the phase response is rectangular and the memory consists of one symbol.

At the kth symbol interval, the memoryless modulator matches symbol u_k to the signal $x_{u_k}(t)$ corresponding to the u_k^{th} element of $X=\{x_i(t), i=0...M-1\}$ (see [6]) with,

$$x_i(t)=\sqrt{\frac{1}{T}} \cdot e^{j2\pi \frac{iht}{T}}, \quad t \in [0,T) \tag{1}$$

where T is the symbol period and h the modulation index.

The CPE ensures the continuity between the transmitted continuous-time waveforms by accumulating the phase of each modulated symbol.

$$\phi_{k+1}=\phi_k + 2\pi h u_k \tag{2}$$

ϕ_k is the accumulated phase at the start of the kth symbol.

It leads to a CPFSK complex baseband representation of the transmitted continuous-time waveform during the kth symbol time of the observation interval:

$$s_k(t)=\sqrt{Es} \cdot x_{u_k}(t) \cdot e^{j\phi_k} \tag{3}$$

The transmitted signal undergoes a phase rotation θ and it is transmitted over a complex additive white Gaussian noise (AWGN channel) with noise spectral density N_0. θ is assumed to be unknown, constant over the whole transmission and uniformly distributed between $[0,2\pi[$. The channel is said to be noncoherent. The corresponding complex-baseband received signal is given by

$\forall\, t \in [kT; (k+1)T)$,

$$r_k(t)=s_k(t) \cdot a \cdot e^{j\theta} + n(t) \tag{4}$$

where a is a possible channel attenuation which is assumed known to the receiver. Without loss of generality, we assume here that $a=1$. $n(t)$ in (4) corresponds to the complex AWGN.

Signal $r_k(t)$ received during the kth symbol interval is passed through a bank of M matched filters whose impulse responses are given by $\bar{x}_i(t), i=0,...,M-1$ where $\bar{x}_i(t)$ is the complex conjugate of $x_i(t)$ (see Fig. 1).
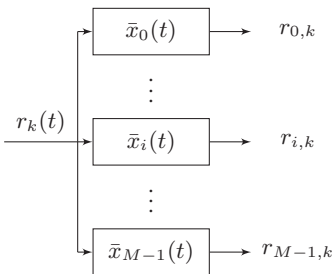

Fig. 1. Complex matched filters of M-CPFSK modulation

Considering a perfect timing synchronisation, $r_{i,k}$ is the element resulting from the correlation between $r_k(t)$ and $x_i(t)$.

$$r_{i,k}=\int_0^T r_k(t)\bar{x}_i(t)dt \tag{5}$$

In the sequel we adopt the following notation $\boldsymbol{r}_k=\{r_{0,k},...,r_{M-1,k}\}$ and the set of observations is given by $\boldsymbol{r}_0^{K-1}=\{\boldsymbol{r}_0,...,\boldsymbol{r}_{K-1}\}$.

3. MULTISYMBOL WITH MEMORY DETECTOR

The proposed MWM detector is based on state trellis representation allowing us to use a modified version of the BCJR algorithm to determine the conditional probability $p(u_k|\boldsymbol{r}_0^{K-1})$. We denote $\delta_k=\{\phi_k, u_k; ...; u_{k+N-2}\}$ as a state of our MWM detector taking into account the accumulated phase ϕ_k and a series of $N-1$ symbols u_k^{k+N-2}. Fig. 2 illustrates the usual BCJR and the way symbols and accumulated phase are gathered to form a MWM state. In this figure, we can notice that transition $\{\phi_k \rightarrow \phi_{k+1}\}$ is done such that $\phi_{k+1}=\phi_k + 2\pi h u_k$.
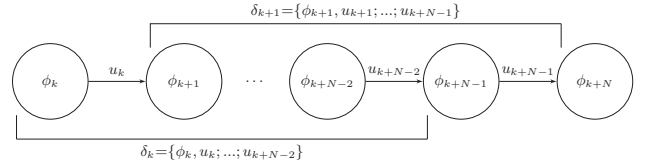

Fig. 2. State Diagram of the usual BCJR

The transition $\{\delta_k \rightarrow \delta_{k+1}\}$ corresponds to the emitted symbol u_{k+N-1}. Finally, BCJR metrics have to be recomputed in order to take into account MWM states.

From Bayes theorem, the conditional probability of symbols given the observations $p(u_{k+N-1}|\boldsymbol{r}_0^{K-1})$ is computed from $p(\boldsymbol{r}_0^{K-1}|u_{k+N-1})$. At first, The conditional probability is developed as follows.

$$
\begin{aligned}
&p(\boldsymbol{r}_0^{K-1}|u_{k+N-1})\\
&=p(\boldsymbol{r}_0^{k-1}, \boldsymbol{r}_k^{k+N-1}, \boldsymbol{r}_{k+N}^{K-1}|u_{k+N-1})\\
&=\sum_{\delta_k} p(\boldsymbol{r}_0^{k-1}, \boldsymbol{r}_k^{k+N-1}, \boldsymbol{r}_{k+N}^{K-1}|\delta_k, u_{k+N-1})p(\delta_k|u_{k+N-1})\\
&=\sum_{\delta_k} p(\boldsymbol{r}_{k+N}^{K-1}|\boldsymbol{r}_0^{k-1}, \boldsymbol{r}_k^{k+N-1}, \delta_k, u_{k+N-1})p(\delta_k|u_{k+N-1}) \quad (6)\\
&\cdot p(\boldsymbol{r}_0^{k-1}|\boldsymbol{r}_k^{k+N-1}, \delta_k, u_{k+N-1})p(\boldsymbol{r}_k^{k+N-1}|\delta_k, u_{k+N-1})\\
&=\sum_{\delta_k} p(\boldsymbol{r}_{k+N}^{K-1}|\boldsymbol{r}_{k+1}^{k+N-1}, \delta_{k+1})p(\boldsymbol{r}_0^{k-1}|\boldsymbol{r}_k^{k+N-2}, \delta_k)\\
&\cdot p(\boldsymbol{r}_k^{k+N-1}|\delta_k, u_{k+N-1})p(\delta_k)
\end{aligned}
$$

the classical forward, backward and transition kernel probabilities (denoted α, β and γ respectively) are identified in (6) as follows.

$$
\begin{aligned}
\gamma(\delta_k \rightarrow \delta_{k+1}, \boldsymbol{r}_k^{k+N-1}) &=p(\boldsymbol{r}_k^{k+N-1}|\delta_k, u_{k+N-1})\\
\alpha_k(\delta_k) &=p(\boldsymbol{r}_0^{k-1}|\boldsymbol{r}_k^{k+N-2}, \delta_k)p(\delta_k) \quad (7)\\
\beta_{k+1}(\delta_{k+1}) &=p(\boldsymbol{r}_{k+N}^{K-1}|\boldsymbol{r}_{k+1}^{k+N-1}, \delta_{k+1})
\end{aligned}
$$

As with the BCJR algorithm, α_k can be calculated as

$$\alpha_k(\delta_k) = p(\boldsymbol{r}_0^{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)p(\delta_k)$$

$$= p(\boldsymbol{r}_0^{k-2},\boldsymbol{r}_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)p(\delta_k)$$

$$= p(\delta_k)\cdot\sum_{\delta_{k-1}}p(\boldsymbol{r}_0^{k-2},\boldsymbol{r}_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k,\delta_{k-1})$$

$$\cdot p(\delta_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)$$

$$= \sum_{\delta_{k-1}}p(\boldsymbol{r}_0^{k-2}|\boldsymbol{r}_{k-1}^{k+N-3},\delta_{k-1})p(\boldsymbol{r}_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k,\delta_{k-1})$$

$$\cdot p(\delta_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)p(\delta_k)$$

$$(8)$$

where.

$$p(\boldsymbol{r}_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k,\delta_{k-1}) = p(\boldsymbol{r}_{k-1}|\boldsymbol{r}_k^{k+N-2},u_{k+N-2},\delta_{k-1})$$

$$= \frac{p(\boldsymbol{r}_{k-1}^{k+N-2}|u_{k+N-2},\delta_{k-1})}{p(\boldsymbol{r}_k^{k+N-2}|u_{k+N-2},\delta_{k-1})}$$

$$= \frac{\gamma(\delta_{k-1}\to\delta_k,r_{k-1}^{k+N-2})}{p(\boldsymbol{r}_k^{k+N-2}|u_{k+N-2},\delta_{k-1})}$$

$$(9)$$

(9) is derived from Bayes' theorem and (7). Moreover, we have

$$p(\delta_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)p(\delta_k) = p(u_{k+N-2})p(\delta_{k-1}) \qquad (10)$$

In (10), $p(\delta_{k-1}|\boldsymbol{r}_k^{k+N-2},\delta_k)$ is independent of observations $\boldsymbol{r}_k^{k+N-2}$. At last, after all terms have been collected, a recursion of α is obtained as follows

$$\alpha_k(\delta_k) = \sum_{\delta_{k-1}}\alpha_{k-1}(\delta_{k-1})\frac{\gamma(\delta_{k-1}\to\delta_k,r_{k-1}^{k+N-2})}{p(\boldsymbol{r}_k^{k+N-2}|u_{k+N-2},\delta_{k-1})}p(u_{k+N-2})$$

$$(11)$$

Similarly, β can be calculated using a backward recursion.

$$\beta_{k+1}(\delta_{k+1}) = \sum_{\delta_{k+2}}\beta_{k+2}(\delta_{k+2})\frac{\gamma(\delta_{k+1}\to\delta_{k+2},r_{k+1}^{k+N})}{p(\boldsymbol{r}_{k+1}^{k+N-1}|u_{k+N},\delta_{k+1})}p(u_{k+N})$$

$$(12)$$

Finally gathering α, β and γ leads to

$$p(u_{k+N-1}|\boldsymbol{r}_0^{K-1}) \sim \sum_{\delta_k}\alpha_k(\delta_k)\beta_{k+1}(\delta_{k+1})$$

$$\cdot\gamma(\delta_k\to\delta_{k+1},\boldsymbol{r}_k^{k+N-1})p(u_{k+N-1})$$

$$(13)$$

4. MULTISYMBOL WITH MEMORY FOR A NONCOHERENT CPFSK MODULATION

Using sufficient statistics at the output of the filter bank of Fig. 1 we have [6])

$$p(\boldsymbol{r}_k|u_k,a,\psi_k) \sim e^{\frac{2a\sqrt{Es}}{N_0}\Re(e^{-j\psi_k}r_{u_k,k})} \qquad (14)$$

where $\Re(.)$ is the real part and $\psi_k = \phi_k + \theta$ (with $\psi_0 = \theta$). The branch
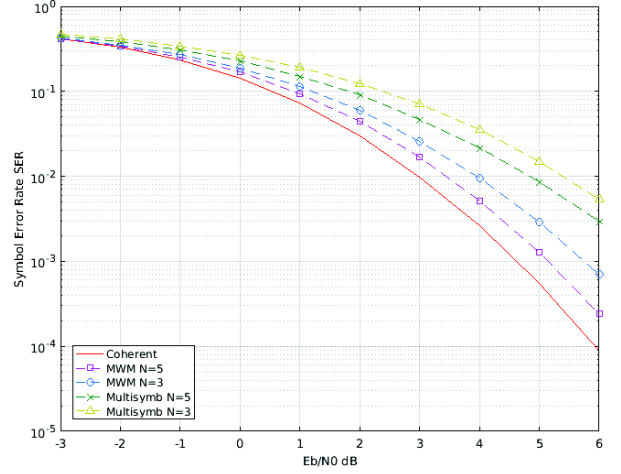

Fig. 3. 4-CPFSK SER with $h = \frac{5}{7}$ over an AWGN channel.

metric associated to the MWM requires the computation of the conditional probability related to γ and given in [6]

$$\gamma(\delta_k\to\delta_{k+1},\boldsymbol{r}_k^{k+N-1}) = p(\boldsymbol{r}_k^{k+N-1}|\delta_k,u_{k+N-1},a)$$

$$\sim e^{\frac{2a\sqrt{Es}}{N_0}\Re\left(e^{-j\psi_k}\mu(u_k^{k+N-1})\right)}$$

$$(15)$$

where,

$$\mu(u_k^{k+N-1}) = \sum_{i=k}^{k+N-1}r_{u_i,i}e^{-j2\pi h\sum_{n=k}^{i-1}u_n}$$

Averaging over the random phase ψ_0 yields the well known zero-order modified Bessel function

$$p(\boldsymbol{r}_k^{k+N-1}|\delta_k,u_{k+N-1},a) \sim I_0\left(\frac{2a\sqrt{Es}}{N_0}\Big|\mu(u_k^{k+N-1})\Big|\right) \quad (16)$$

Then equation (11) and (12) can be rewritten as

$$\alpha_k(\delta_k) \sim \sum_{\delta_{k-1}}\alpha_{k-1}(\delta_{k-1})\frac{I_0\left(\frac{2a\sqrt{Es}}{N_0}|\mu(u_{k-1}^{k+N-2})|\right)}{I_0\left(\frac{2a\sqrt{Es}}{N_0}|\mu(u_k^{k+N-2})|\right)}p(u_{k+N-2})$$

$$\beta_k(\delta_k) \sim \sum_{\delta_{k+1}}\beta_{k+1}(\delta_{k+1})\frac{I_0\left(\frac{2a\sqrt{Es}}{N_0}|\mu(u_k^{k+N-1})|\right)}{I_0\left(\frac{2a\sqrt{Es}}{N_0}|\mu(u_k^{k+N-2})|\right)}p(u_{k+N-1})$$

$$(17)$$

Symbol error rate (SER) of 4-CPFSK for a given modulation index using the method highlighted above is shown in Fig. 3. From this figure, it appears that noncoherent detection done with the MWM reaches better SER than multisymbol detector. The MWM's SER draws near to coherent detection when the number of symbols in states increases.

5. ASYMPTOTIC PERFORMANCE ANALYSIS

5.1. Mutual Information Rate

The mutual information rate of finite-state machine channels have been studied in [10] and [11].The mutual information rate between
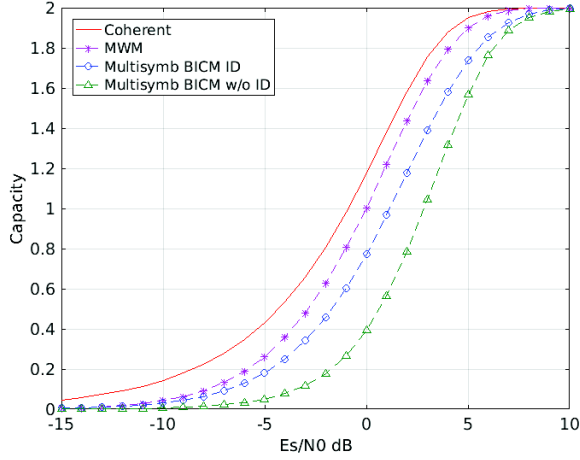
Fig. 4. MWM and Multisymbol noncoherent detection of 4-CPFSK with $h = \frac{5}{7}$ and $N = 5$.

the channel input source $\mathcal{U}$ and the channel output $\mathcal{R}$ can be described as follows [12].

$$I(\mathcal{U}, \mathcal{R}) = \lim_{K \to \infty} \frac{1}{K} I(u_0^{K-1}, \boldsymbol{r}_0^{K-1} | \delta_0) \qquad (18)$$

Where $I(u_0^{K-1}, \boldsymbol{r}_0^{K-1} | \delta_0)$ is the mutual information between the input process u_0^{K-1} and the output process $\boldsymbol{r}_0^{K-1}$ conditioned on the initial state δ_0. The expression of $I(u_0^{K-1}, \boldsymbol{r}_0^{K-1} | \delta_0)$ can be derived as follows [13]

$$
\begin{aligned}
I(u_0^{K-1}, \boldsymbol{r}_0^{K-1} | \delta_0) &= E\left[log\left(\frac{p(u_0^{K-1} | \boldsymbol{r}_0^{K-1}, \delta_0)}{p(u_0^{K-1} | \delta_0)}\right)\right] \\
&= \left(K - (N-1)\right) log(M) \\
&\quad + E\left[log\left(p(u_0^{K-1} | \boldsymbol{r}_0^{K-1}, \delta_0)\right)\right]
\end{aligned}
\qquad (19)
$$

To express (19) in terms of bits per channel used, use a base-2 logarithm. The probability function $p(u_0^{K-1} | \boldsymbol{r}_0^{K-1}, \delta_0)$ which appears in (19) may be reduced to $p(u_{N-1}^{K-1} | \boldsymbol{r}_0^{K-1}, \delta_0)$ since $\delta_0 = \{\phi_0, u_0; ...; u_{N-2}\}$. It may be evaluated as follows.

$$p(u_{N-1}^{K-1} | \boldsymbol{r}_0^{K-1}, \delta_0) = \prod_{k=0}^{K-N} p(u_{k+N-1} | \delta_0^k, \boldsymbol{r}_0^{K-1}) \qquad (20)$$

Computation of $p(u_{k+N-1} | \delta_0^k, \boldsymbol{r}_0^{K-1})$ in (20) has been done in [14]. The idea is to compute the probability of a symbol knowing perfectly all the previous states from the beginning of the transmission. This amounts to perform the BCJR algorithm as usual but taking into account the complete knowledge of the forward recursion. Meaning α is fixed to 1 for the correct state and 0 to all other states. γ and β remained unchanged beside the traditional BCJR. Computing $p(u_{k+N-1} | \delta_0^k, \boldsymbol{r}_0^{K-1})$ leads to full determination of the mutual information as well as the mutual information rate.

The channel capacity $\mathcal{C}$ is defined as the supremum of the mutual information rate in [12].

$$\mathcal{C} = \lim_{K \to \infty} \sup\left[\frac{1}{K} \cdot I\left(u_0^{K-1}, \boldsymbol{r}_0^{K-1} | \delta_0\right)\right] \qquad (21)$$

MWM capacity for CPFSK is shown in Fig. 4 in noncoherent detection. Those curves are compared to the coherent case ($\theta = 0$), as well as to two classical receivers based on a simple multisymbol receivers. The first is based on a BICM (bit-interleaved coded modulation) scheme using iterative decoding (ID) between soft multisymbol receiver and an outer blockcode. The second one is a receiver based on serial concatenation of the multisymbol receiver and an outer channel code without iterative decoding. It appears that MWM reaches higher capacity than multisymbol detector. Multisymbol BICM is added to Fig. 4. The MWM capacity draws near to the coherent case when states cardinality increases.

6. CONCLUSION

Multisymbol receiver is usually considered to cope with noncoherent channel, trading complexity versus performance. However traditional approaches do not take into account for the possible memory that can be used at the receiver side. In this paper, we proposed to extend these approaches to the case of a multisymbol receiver including some memory in the decoding process. It leads to improved performances but with the expense of an increasing complexity.

7. APPENDIX

A short example is given Fig. 5 to clarify the mathematical reduction of (22) and (23) essential to section 3.
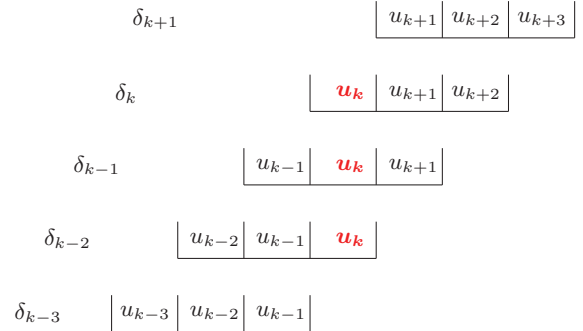
Fig. 5. Multisymbol with Memory State Model for N=4

For ease of reading, in the representation illustrated Fig. 5, states have been depicted without their phase. We can notice u_k is used in the following transition $\{\delta_{k-3} \to \delta_{k-2}, \delta_{k-2} \to \delta_{k-1}, \delta_{k-1} \to \delta_k, \delta_k \to \delta_{k+1}\}$. Thus, in a more generic way, u_k intervenes only on states transition involving $\delta_{k-(N-1)}^{k+1}$. It comes out from the previous example,

$\forall k \in \{0, ..., K-1\}$,

$$
\begin{aligned}
p(\boldsymbol{r}_k | \boldsymbol{r}_0^{K-1}, \delta_0^{K-(N-1)}) &= p(\boldsymbol{r}_k | \boldsymbol{r}_{k-(N-1)}^{k+N-1}, \delta_{k-(N-1)}^{k+1}) \\
&= p(\boldsymbol{r}_k | \boldsymbol{r}_{k-(N-1)}^{k+N-1}, u_k^{k+N-1}, \delta_{k-(N-1)})
\end{aligned}
\qquad (22)
$$

A much more generic expression of the hypothesis is given,

$\forall (i, j) \in \{0, ..., K-1\}^2$ such as $i \leq j$

$$
\begin{aligned}
p(\boldsymbol{r}_i^j | \boldsymbol{r}_0^{K-1}, \delta_0^{K-(N-1)}) &= p(\boldsymbol{r}_i^j | \boldsymbol{r}_{i-(N-1)}^{j+N-1}, \delta_{i-(N-1)}^{j+1}) \\
&= p(\boldsymbol{r}_i^j | \boldsymbol{r}_{i-(N-1)}^{j+N-1}, u_i^{j+N-1}, \delta_{i-(N-1)})
\end{aligned}
\qquad (23)
$$

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