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THE RECENT STATUS OF THE
VOLUME PRODUCT PROBLEM

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ABSTRACT. In this small survey we consider the volume product, and sketch some of the best upper and lower estimates known up to now, based on our paper [BMMR]. The author thanks the organizers of the conference in Jurata, March 2010, for their kind invitation, and the excellent atmosphere there. This paper is based on the talk of the author on that conference.

I. INTRODUCTION

For a finite dimensional Banach space X (we always consider only *real Banach spaces*), the *volume product of X* is the product of the volumes of the unit balls of the space X and its dual X'. (Of course, for this to make sense, we have to identify X and X', which can be done via the usual scalar product. However, the volume product is independent of the scalar product used.) This concept has a great importance in the local theory of Banach spaces, i.e., the asymptotical (functional analytical) investigation of Banach spaces of high finite dimension, cf. e.g., Pisier's book [Pi]. Moreover, it lies on the cross-road of several disciplines of mathematics, even seemingly unrelated, which points out its great importance and usefulness.

In more geometric terms, in $\mathbb{R}^n$ we have an 0-symmetric convex body K, and we consider its polar body K^*, and the product of their volumes $V(K)V(K^*)$.

Here a set $K \subset \mathbb{R}^n$ is a *convex body*, if it is a compact convex set with non-empty interior. We denote by $V(K)$ its volume.

Let us suppose that $0 \in \operatorname{int} K$. Then the *polar body K^* of K* is defined as $K^* := \{x \in \mathbb{R}^n \mid \forall k \in K \ \ \langle x, k \rangle \le 1\}$. This is also a convex body, with $0 \in \operatorname{int} K^*$. We have $(K^*)^* = K$. If K is 0-symmetric, hence is the unit ball of an n-dimensional Banach space X, then K^* is the unit ball of the dual space X'. (Of course, for this to make sense, we have to identify X and X', via the usual scalar product.)

We call in the above situation (i.e., when $0 \in \operatorname{int} K$) the quantity $V(K)V(K^*)$ the *volume product of K*. This is invariant under non-singular linear transformations.

The investigation of this quantity was originated by Blaschke [Bl], who used this concept for the affine geometry of convex bodies (i.e., investigation of properties of convex bodies that are invariant under — possibly volume preserving — *affine*

transformations, i.e., maps of the form $x \mapsto Tx + a$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a non-singular linear map, and $a \in \mathbb{R}^n$ is a vector), and Mahler [Ma38] and [Ma39], who used this concept in the geometry of numbers (i.e., a common part of geometry and number theory, dealing, among others, with relation of convex bodies, and also of their polars, to *lattices* in $\mathbb{R}^n$, i.e., non-singular linear images of $\mathbb{Z}^n$ in $\mathbb{R}^n$).

Later it became obvious, that the volume product is a very important quantity in the local theory of Banach spaces, it has relations to a number of other characteristics of these Banach spaces (cf., e.g., [Pi]). This fact drew the interest of functional analysts to the volume product problem, which later resulted in the solution of the problem of the lower estimate of the volume product for an n-dimensional Banach space in a way satisfactory for applications in the local theory of Banach spaces. (The problem of the sharp upper estimate had already been solved by this time.)

Several other mathematical disciplines also need the volume product. We mention stochastic geometry, i.e., geometric probability (in several different ways), and the geometry of Minkowski spaces (these are just finite dimensional Banach spaces, but the object is not their analytic properties, but their geometric properties, which are interesting already for the case $n = 2$). It is interesting enough, that the theory of functions of several complex variables is also connected to the volume product problem. Still one area is discrete geometry, that is the branch of geometry dealing with density estimates of systems of convex sets in $\mathbb{R}^d$ satisfying various hypotheses. The *density* of such a system is the "percentage of the volume of the whole space covered by the system, taken with multiplicity (i.e., doubly covered parts count doubly, etc.)".

So the question is: what is the minimum, and the maximum of the volume product $V(K)V(K^*)$. Seemingly a bit more generally (but in fact equivalently): for $x \in \mathrm{int}\, K$ we consider $V(K)V((K - x)^*)$, and investigate this quantity. Here, for the upper estimate one has to be a bit careful, since for $\mathrm{dist}\,(x,\ \mathrm{bd}\, K) \to 0$ we have $V(K)V((K - x)^*) \to \infty$. Therefore we have to take rather $\min_{x \in \mathrm{int}\, K} V(K)V((K - x)^*)$ (for the 0-symmetric case this is just $V(K)V(K^*)$). This quantity is affine invariant.

We give some examples. For K the Euclidean unit ball, whose volume is denoted by κ_n, we have

$$V(K)V(K^*) = \kappa_n^2 = n^{-n} \left(2e\pi + o(1)\right)^n = n^{-n} \left(17.0794... + o(1)\right)^n .$$

If K is the cube $[-1, 1]^n$, or its polar, the *regular cross-polytope* conv $\{\pm e_i\}$ (with e_i the standard unit vectors), then

$$V(K)V(K^*) = 4^n/n! = n^{-n} \left(4e + o(1)\right)^n = n^{-n} \left(10.8731... + o(1)\right)^n .$$

For $x \in \mathrm{int}\, K$ one has the simple formula

$$V((K - x)^*) = \frac{1}{n} \int_{S^{n-1}} (h_K(u) - \langle x, u \rangle)^{-n} du ,$$

where $h_K(u) := \max\{\langle k, u\rangle \mid k \in K\}$ is the *support function of K*, defined for $u \in S^{n-1}$. This readily implies that the second differential of $V\left((K-x)^*\right)$ with respect to $x \in \operatorname{int} K$ is a positive definite quadratic form, hence $V\left((K-x)^*\right)$ is a strictly convex function of $x \in \operatorname{int} K$. Hence there is a unique point $x \in \operatorname{int} K$, such that $V\left((K-x)^*\right)$ is minimal: this point x is called the *Santaló point of K*, and is denoted by $s(K)$. If K is 0-symmetric, we have $s(K) = 0$.

As a further example, for K a simplex, we have that $s(K)$ is the barycentre of K, and

$$\begin{cases} V(K)V[(K-s(K))^*] = (n+1)^{n+1}/(n!)^2 = n^{-n}\left(e^2 + o(1)\right)^n = \\ n^{-n}\left(7.3890... + o(1)\right)^n . \end{cases}$$

In geometry it is unnatural to restrict ourselves to 0-symmetric bodies, which *is* natural in the local theory of Banach spaces. Actually, the importance of the Santaló point shows up only in the asymmetric case, and also leads to proofs, even in the 0-symmetric case, which would be hardly guessed if we would restrict ourselves to the 0-symmetric case only.

So, the correct question is: what is the minimum, and the maximum of $V(K) \times V[(K-s(K))^*]$. This quantity is invariant under affinities.

In all our theorems $K \subset \mathbb{R}^n$ will be a convex body.

II. Upper bound

Theorem 1. *([Bl], [San], [SR], [Pe], [Ba], [MP]) We have*

$$V(K)V[(K-s(K))^*] \leq \kappa_n^2,$$

with equality if and only if K is an ellipsoid.

III. Lower bound

Conjecture A. (Mahler-Guggenheimer-Saint Raymond, [Ma39], [G], [SR]) If K is 0-symmetric, we have

$$V(K)V[(K-s(K))^*] = V(K)V(K^*) \geq 4^n/n!,$$

with equality exactly for the unit balls of the Hanner-Hansen-Lima spaces.

The *Hanner-Hansen-Lima spaces* are inductively defined from lower dimensions, by taking, for some decomposition $n_1 + n_2 = n$ (where $n_i \geq 1$), either the l^1, or l^∞-sums of the already defined Hanner-Hansen-Lima spaces in n_1 and n_2 dimensions (in other words, we take in $\mathbb{R}^n = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}$ for the unit ball either the convex hull, or the sum of the unit balls of $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$). The basis of induction is $n = 1$, when there is a unique 1-dimensional Banach space, which is by definition a Hanner-Hansen-Lima space. The unit balls of these spaces include, among others, the cube $[-1, 1]^n$, the regular

cross-polytope conv $\{\pm e_i\}$, but also many other bodies. A *Hanner-Hansen-Lima body* is the unit ball of a Hanner-Hansen-Lima space. For all these bodies K we have $V(K)V(K^*) = 4^n/n!$.

Conjecture B. (Mahler, [Ma39]) We have

$$V(K)V[(K - s(K))^*] \geq (n+1)^{n+1}/(n!)^2 \,,$$

with equality exactly for the simplex.

III.1. GENERAL THEOREMS

Theorem 2A. *([BMi], [K], [N]) If K is 0-symmetric, we have*

$$V(K)V(K^*) \geq \kappa_n^2/2^n = n^{-n}\left(e\pi + o(1)\right)^n = n^{-n}\left(8.5397... + o(1)\right)^n \,.$$

Theorem 2B. *([BMi], [K]) We have*

$$V(K)V[(K - s(K))^*] \geq an^{-n}(e\pi/2)^n = n^{-n}\left(4.2698... + o(1)\right)^n \,,$$

for some constant $a > 0$.

We remark that [BMi] proved only κ_n^2/A^n, for a non-explicit constant $A > 0$, in both theorems. They have used for this aim the so called *quotient of subspace theorem* of V. D. Milman (cf., e.g., [Pi]) that is another very important theorem in the local theory of Banach spaces. This says the following. Any n-dimensional Banach space X has a subspace X_1, and X_1 has a quotient X_2, such that $\dim X_2 \geq cn$, and such that the Banach-Mazur distance of X_2 and the Hilbert space of dimension $\dim X_2$ is at most C. Here $c \in (0,1)$, and $C \in (1,\infty)$ are some constants. More exactly: for any $c \in (0,1)$ there exists a $C = C(c) \in (1,\infty)$, such that the previous statement is true, and conversely, for any $C \in (1,\infty)$ there exists a $c = c(C) \in (0,1)$, such that the previous statement is true. Moreover, [N] proved only $(4^n/n!)(\pi/4)^{3n}$, and for this aim he used the theory of functions of several complex variables.

III.2. SHARP THEOREMS FOR SPECIAL BODIES

III.2.A. BODIES WITH HIGH SYMMETRY

Theorem 3A. *([SR], [Me86], [R87]) If K is symmetric with respect to all coordinate hyperplanes (thus, in particular, is 0-symmetric), then*

$$V(K)V(K^*) \geq 4^n/n! \,,$$

with equality exactly for the unit balls of the Hanner-Hansen-Lima spaces.

Bodies with the property in the hypothesis of this theorem, and also the corresponding norms, are called *unconditional*.

Theorem 3B. *([BF]) If K has all the symmetries (i.e., congruences) of a regular simplex, then*

$$V(K)V[(K - s(K))^*] \geq (n+1)^{n+1}/(n!)^2 \,,$$

with equality exactly for a regular simplex.

III.2.B. ZONOIDS

A *zonoid* is a limit, in the Hausdorff-metric, of some sequence of finite sums of segments.

Theorem 4. *([R85], [R86], [GMR]) Let K be a convex body, that is also an 0-symmetric zonoid. Then*

$$V(K)V(K^*) \geq 4^n/n! \,,$$

with equality if and only if K is a parallelepiped, with centre at 0.

Of course, this also means that the other Hanner-Hansen-Lima bodies are not zonoids.

III.2.C. PLANAR CASE

Theorem 5A. *([Ma38], [R86]) Let $n = 2$ and let K be 0-symmetric. Then*

$$V(K)V(K^*) \geq 8 \,,$$

with equality if and only if K is a parallelogram with centre at 0.

Theorem 5B. *([Ma38], [Me91]) Let $n = 2$. Then*

$$V(K)V[(K - s(K))^*] \geq 27/4 \,,$$

with equality if and only if K is a triangle.

III.2.D. LOCAL MINIMA

Theorem 6A. *([NPRZ]) Among 0-symmetric bodies K, the volume product $V(K)$ $\times V(K^*)$ has a strict local minimum for 0-symmetric parallelepipeds.*

Theorem 6B. *([KR]) The volume product $V(K)V[(K - s(K))^*]$ has a strict local minimum for simplices.*

III.2.E. POLYHEDRA WITH SMALL NUMBERS OF VERTICES, OR OF $(n-1)$-FACES

Theorem 7A. *([LR]) Let $n \le 8$, and let K be an 0-symmetric convex polyhedron, with at most $n+1$ opposite (i.e., symmetric w.r.t. 0) pairs of vertices, or of $(n-1)$-faces. Then*

$$V(K)V(K^*) \ge 4^n/n! \,,$$

with equality if and only if K is a Hanner-Hansen-Lima body, with at most $n+1$ opposite pairs of vertices, or of $(n-1)$-faces.

Theorem 7B. *([MR]) Let K be a convex polyhedron, with at most $n+3$ vertices, or $(n-1)$-faces. Then*

$$V(K)V[(K-s(K))^*] \ge (n+1)^{n+1}/(n!)^2 \,,$$

with equality if and only if K is a simplex.

Polyhedra (parallelepipeds) in higher dimensional Euclidean spaces are usually called *polytopes (parallelotopes)* in geometry.

IV. STABILITY VARIANTS

If one proves an inequality, and also determines the cases of equality, that is not yet the end of the story. One can further ask the following. If a convex body has a volume product $V(K)V[(K-s(K))^*]$ (or $V(K)V(K^*)$ for the 0-symmetric case) ε-close to the extremal value, does it follow, that K is $(1+f(\varepsilon))$-close, in the Banach-Mazur distance, to some of the extremal bodies. Here f is some positive function, with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. (Of course, in case of compactness, the existence of such a function f is evident, but the aim is to give an explicit such function.) Typically, at such theorems, $f(\varepsilon)$ is some constant times some power of ε, sometimes also with some logarithmic factor, also on some power. At such theorems it is considered as satisfactory, if the sharp order of magnitude of the function f is determined.

The *Banach-Mazur distance* $\delta_{BM}(K,L)$ of two 0-symmetric convex bodies K and L is the Banach-Mazur distance of the Banach spaces with these unit balls. In geometrical terms, this can be described as follows.

$$\left\{ \begin{array}{l} \delta_{BM}(K,L) = \min\{\lambda_2/\lambda_1 \mid \lambda_1, \lambda_2 > 0, \ \exists\, T \text{ non-singular linear map,} \\ \text{such that } \lambda_1 K \subset TL \subset \lambda_2 K\} \,. \end{array} \right.$$

One can extend this naturally to the not 0-symmetric case as follows. *For K and L convex bodies,*

$$\left\{ \begin{array}{l} \delta_{BM}(K,L) := \min\{\lambda_2/\lambda_1 \mid \lambda_1, \lambda_2 > 0, \ \exists\, T \text{ non-singular linear map,} \\ \exists\, a_1, a_2 \text{ vectors, such that } \lambda_1 K + a_1 \subset TL \subset \lambda_2 K + a_2\} \,. \end{array} \right.$$

For K and L both 0-symmetric, this reduces to the previous formula. (We recall that actually $\log \delta_{BM}(K,L)$ is a metric.)

Then, for Theorem 1, we have stability, but probably not with the sharp order ([Bo], [BB], [BMa]). Concerning the theorems with lower estimates, we have stability of Theorem 4 ([BH]), Theorem 5A ([BH],[BMMR]), of Theorem 5B ([BMMR]), of Theorem 6A ([NPRZ]), and of Theorem 6B ([KR]), at each of these with the sharp order, namely, with linear order.

V. FUNCTIONAL VARIANTS

Recently there have appeared several functional variants of the volume product problem. We discuss here only one of these. The objects of investigation are the *log-concave functions*, i.e., functions $f : \mathbb{R}^n \to [0, \infty)$, whose logarithm is concave. To a convex body K, with $0 \in \operatorname{int} K$, we associate the function $f(x) := \exp\left(-\|x\|_K^2/2\right)$, where $\| \cdot \|_K$ means the asymmetric norm (i.e., $\| - x\|_K \neq \|x\|_K$) associated to the "unit ball" K (*i.e.,* $\|x\|_K := \min\{\lambda \geq 0 \mid x \in \lambda K\}$). Then $V(K) = \operatorname{const}_n \cdot \int_{\mathbb{R}^n} f(x)\, dx$, so here the right hand side is the proper generalization of $V(K)$.

Moreover, polarity of K and K^* goes over to the following. The corresponding functions f and f^* have their negative logarithms, with values in $(-\infty, \infty]$, which are the Legendre transforms of each other. The *Legendre transform* of a function $\varphi : \mathbb{R}^n \to [-\infty, \infty]$ is the function $\mathcal{L}\varphi : \mathbb{R}^n \to [-\infty, \infty]$, defined by

$$(\mathcal{L}\varphi)(y) := \sup\{\langle x, y \rangle - \varphi(x) \mid x \in \mathbb{R}^n\}\,.$$

Thus, the object of investigation is

$$\int_{\mathbb{R}^n} f(x)\, dx \cdot \int_{\mathbb{R}^n} f^*(x)\, dx\,,$$

where one supposes

$$\int_{\mathbb{R}^n} f(x)\, dx \in (0, \infty)\,.$$

Cf. the nice exposition in [A-AKM].

Unfortunately, translations of convex bodies have no (good) generalizations to log-concave functions. Thus, in place of a translation $K \mapsto K - x$, where $x \in \operatorname{int} K$, one considers an arbitrary translate of the function f (i.e., $x \mapsto f(x - x_0)$), and proves the sharp upper bound for a suitable translate of the original function f. Here, for even functions f, one may choose $x_0 = 0$ (as for 0-symmetric bodies one may choose $x = s(K) = 0$), cf. [A-AKM]. Of course, the problem of translations does not concern the question of the lower bound (as it is a minimum problem), but, in case of the upper bound, only the 0-symmetric case of the volume product problem generalizes this way to even log-concave functions. For the upper bound, in the even case, the sharp upper bound for the functional variant (cf. [Ba], [A-AKM], [FM07]) immediately implies the sharp upper bound for the 0-symmetric case of the volume product problem. Namely, the extremal even functions (up to constant factors) are ones associated to 0-symmetric convex bodies,

more exactly, to 0-symmetric ellipsoids. For the lower bound, the best, known lower bound for the functional variant (cf., [FM08a], Theorem 7) implies the best, known lower bound for the volume product problem, apart from the actual value of the base of the exponential (when the second lower bound is written in the form $n^{-n} \left(\operatorname{const} + o(1)\right)^n$). In the case of *unconditional functions* (i.e., $f(x_1, ..., x_n) = f(|x_1|, ..., |x_n|)$), with $f > 0$, the sharp lower bound is known (cf., [FM08a], Theorem 6).

Still we note that the conjecture in $\mathbb{R}^n$ about the lower bound for the functional variant, for the even, or the general case (cf., [FM08b]), would imply Conjecture A or Conjecture B, about the lower bound for the volume product, in the 0-symmetric, or the general case, for $\mathbb{R}^n$, or $\mathbb{R}^{n-1}$, respectively (cf., [FM08b]). However, the conjecture about the lower bound for the functional variant, for the even, or the general case, *for all n*, is equivalent to Conjecture A, or Conjecture B, *for all n*, respectively (cf. [FM08b], Propositions 1, 2).

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