arXiv:1705.02124v4 [math.CO] 15 Nov 2018

Terminal-Pairability in Complete Bipartite Graphs with Non-Bipartite Demands

Edge-disjoint paths in complete bipartite graphs

Lucas Colucci[2], Péter L. Erdős[1,†,‡],
Ervin Győri[1,2,†,‡], Tamás Róbert Mezei[1,2,†,§]

[1]Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences,
Reáltanoda u. 13–15, 1053 Budapest, Hungary
[2]Central European University, Department of Mathematics and its Applications,
Nádor u. 9, 1051 Budapest, Hungary

November 16, 2018

Abstract. We investigate the terminal-pairability problem in the case when the base graph is a complete bipartite graph, and the demand graph is a (not necessarily bipartite) multigraph on the same vertex set. In computer science, this problem is known as the edge-disjoint paths problem. We improve the lower bound on the maximum value of $\Delta(D)$ which still guarantees that the demand graph D has a realization in $K_{n,n}$. We also solve the extremal problem on the number of edges, i.e., we determine the maximum number of edges which guarantees that a demand graph is realizable in $K_{n,n}$.

Keywords: edge-disjoint paths; terminal-pairability; complete bipartite graph

1. Introduction and main results

The *terminal-pairability* problem has been introduced in [7]. The basic question is as follows: Let G be a simple graph — the *base graph*, and a let D be a loopless multigraph with the same vertex set $V(D) = V(G)$ — the *demand graph*. Can we find a path $P(e)$ for every edge $e \in E(D)$ such that $P(e)$ joins the end-vertices of e and these paths are pairwise edge-disjoint? If there is such a collection of paths then we say that D is *realizable* in G. The collection of paths is called a *realization* of D in G.

This problem has several names in the literature depending on motivation and background. In the terminal-pairability context, sufficient conditions (which guarantee the existence of a realization) and their extremum are sought after. In computer science, where the problem is referred to as the *edge-disjoint paths* problem (EDP problem for short), the complexity of constructing the set of edge-disjoint paths is studied (generally both D and G are part of the input).

In the following few paragraphs we take a short detour to survey the previous results about the complexity of the EDP problem.

The decision version of EDP was first shown to be NP-complete by Even, Itai, and Shamir [11]. Robertson and Seymour [30] proved that for a fixed number of paths the problem is solvable in polynomial time, and the running time was later improved by [22] (these results are about vertex-disjoint paths, but by moving to the line graph of G, edge-disjoint paths become vertex-disjoint). However, if the number of required paths is part of the input then the problem is NP-complete even for complete (see [24]) and series-parallel graphs (see [29]). The problem is NP-hard even if $G + D$ (the graph obtained by taking the disjoint union of the edge sets) is Eulerian and D consists of at most three set of parallel edges, as shown by Vygen [34]. If no restrictions are made on G, then the problem is NP-hard for one set of parallel edges which should be mapped to edge-disjoint paths of length exactly 3, see [3].

The edge-disjoint paths problem has many practical applications in telecommunications, VLSI design, network science, see [15], for example. Several random and deterministic methods have been developed to solve this problem for special classes of graphs when the number of demand edges is not too high. From a long series of papers we single out the paper of Alon and Capalbo [2], where the interested reader can also find a good survey of the earlier developments.

Theorem (Theorem 1.1 in [2]). *Let $G = (V, E)$ be a very strong d-regular expander on n-vertices. If D is a demand graph with respect to G with at most $\frac{nd}{150 \log n}$ edges and maximum degree $d/3$, then there is an online, deterministic, polynomial time algorithm that computes a realization of D in G.*

Online here means that the algorithm receives the demand edges one-by-one, and designs the paths immediately, without any information on the still forthcoming demand edges.

For the special case of complete bipartite base graphs, the upper bound on the number of edges in the demand graph is $\frac{n^2}{75 \log(2n)}$. Theorem 1 increases the number of possible demand edges asymptotically by a factor of $\log(2n)$ to $\frac{n^2}{4}$, at the cost of restricting the maximum degree of the demand graph to $(1 - o(1))\frac{n}{4}$.

The problem is also very closely related to the integer multicommodity flow problems and the theory of graph immersions, each with their own terminologies. In this paper from now on we use the terminology of terminal-pairability, as other papers [7, 13, 12, 14, 25, 18, 26, 27, 23, 28, 19] about sufficient conditions do.

The terminal-pairability problem arose as a theoretical framework for the practical problem of constructing high throughput packet switching networks. The problem was originally studied by Csaba, Faudree, Gyárfás, Lehel, and Schelp [7]. Their research served as a substrate for further theoretical studies by Gyárfás and Schelp [18] and Kubicka, Kubicki, and Lehel [26].

In general, it is hopeless to give a condition which is both necessary and sufficient for D to be realizable in G, because the problem is NP-hard. Instead, we group the instances of the problem according to the value of a parameter which corresponds to the complexity of it: the maximum degree, or the number of edges of D. Given a fixed value of one of these parameters, we are able to

give relatively tight conditions which guarantee the existence of a realization. Moreover, for an instance of the problem satisfying these conditions a solution can be constructed in polynomial time.

This paper is the latest piece in a series of papers about terminal-pairability [19, 20, 6]. Our previous paper in the series [6] also deals with terminal-pairability in complete bipartite base graphs, but with the not very natural restriction that D is bipartite with respect to the same vertex classes as the base graph. The novelty of this paper is that almost the same conditions are sufficient even if the bipartiteness condition on D is omitted.

We will refer to an instance of the edge-disjoint paths problem with a pair of graphs (G, D), which implicitly assumes that the underlying sets of vertices of the graphs are identical (or that there is a 1-to-1 correspondence between them).

For an edge $e \in E(D)$ with end-vertices x and y, we define the *lifting of e to a vertex* $z \in V(D)$, as an operation which transforms D by deleting e and adding a new edge joining the vertices x and z and another new one joining z and y; if $z = x$ or $z = y$, the operation does not do anything. We stress that we do not use any information about G to perform a lifting and that the graph obtained using a lifting operation is still a demand graph D'. Notice that D is realizable in G a realization if D' is realizable in G. Throughout the paper, the demand graphs will be denoted by D and its (indexed) derivatives.

It is easy to see that the terminal-pairability problem defined by G and D is solvable if and only if there exists a series of liftings, which, applied successively to D, results in a (simple!) subgraph of G (which is a realization of D in G). The edge-disjoint paths can be recovered by assigning pairwise different labels to the edges of D, and performing the series of liftings so that new edges inherit the label of the edge they replace. Clearly, edges sharing the same label form a walk between the endpoints of the demand edge of the same label in D, and so there is also such a path.

Several attempts have been made to improve the general result for the case of complete base graphs. One of them is due to Kosowski [24], and an even better extremal bound is proven in [19].

In our previous paper [6] on terminal-pairability in complete bipartite graphs, we restricted ourselves to demand graphs that are bipartite with respect to the color classes of the base graph.

In this paper we explore the terminal-pairability problem when G is a complete symmetric bipartite graph, and the demand graph is assumed to be a loopless multigraph. Our approach is extremal in nature: we are searching for simple conditions on the maximum degree and the number of edges that guarantee realizability.

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. We define $K_{n,n}$ as the graph whose vertex set is $A \cup B$, and its edge set is the set of all unordered A, B pairs. For a (multi-)graph H, whose vertex set is $A \cup B$, let $H[A, B]$ be the bipartite (multi-)subgraph of H induced by A and B as the two color classes. For a (multi-)graph H, let $\Delta(H)$ be the maximum degree in H, and let $e(H)$ be the number of its edges (with multiplicity).

A motivation behind taking $K_{n,n}$ as the base graph is that from a bundle of parallel edges between the two color classes, each edge (except at most one) must be mapped to a path of at least 3 edges. If the base graph is a complete graph, multiple edges only need to be mapped to paths of length at least 2, so studying terminal-pairability in complete bipartite graphs is a logical next step for this reason, too.

The observation of the previous paragraph and the pigeonhole principle implies that a demand graph containing $\left\lceil \frac{n}{3} \right\rceil + 1$ copies of the edge $\{a_i, b_i\}$ for each $i = 1, \dots, n$ is not realizable in $K_{n,n}$:

$$n \cdot \left(3 \cdot \left\lceil \frac{n}{3} \right\rceil + 1 \right) > e(K_{n,n}).$$

Therefore, the extremal upper bound on the maximum degree of a demand graph realizable in $K_{n,n}$ is at most $\left\lceil \frac{n}{3} \right\rceil$. It is not clear if this is the extremal bound. Our attempts to adapt the proof of [17] (which improves on the trivial extremal upper bound on the demand degrees in complete graphs) to complete bipartite base graphs have been futile.

Although the following theorem is not likely to be sharp, in terms of maximum degree, it approaches the $\left\lceil \frac{n}{3} \right\rceil$ upper bound to a factor of $\frac{4}{3}$.

Theorem 1. *Let $(K_{n,n}, D)$ be an instance of the* EDP *(terminal-pairability) problem, where D is not necessarily bipartite. If*

$$\Delta(D) \leq (1 - o(1)) \cdot \frac{n}{4}$$

as $n \to \infty$, then D is realizable in $K_{n,n}$.

Such sufficient conditions are studied in the theory *immersions* as well. It is a relatively new and quickly growing subject of graph theory. In short, a loopless (multi)graph D has an immersion in a graph G if there exists a mapping of $V(D)$ into $V(G)$ so that D has a realization in G with respect to this vertex-map. The foundations have been laid down by Robertson and Seymour [31], Abu-Khzam and Langston [1], and Fellows and Langston [15, 16]. Recently, Collins and Heenehan [5], Devos, Dvořák, Fox, McDonald, Mohar, and Scheide [8], and Dvořák and Yepremyan [10] studied the problem of finding sufficient conditions on a simple graph so that it contains an immersion of K_n. Theorem 1 (and even more so, the main result in [19]) studies the converse of this problem: what is a sufficient condition on a loopless multigraph D so that it has an immersion into $K_{n,n}$ (respectively, K_n). In comparison, Theorem 2 in [6] is not invariant on the permutation of $V(D)$, but the technical novelty in the proofs of this paper (compared to [6]) allows us to reformulate Theorem 1 in the language of immersions.

Corollary 2. *If H is a loopless multigraph on at most $2n$ vertices with maximum degree at most $(1 - o(1)) \cdot \frac{n}{4}$, then there is an immersion of H in $K_{n,n}$.*

If a significant amount of the edges of D are inside the two color classes, Theorem 3 permits even higher degrees in D.

Theorem 3. *Let $(K_{n,n}, D)$ be an instance of the* EDP *problem. If*

$$\Delta(D) \leq (1 - o(1)) \cdot \left(\frac{2n}{7} - \frac{3}{7} \cdot \frac{e(D[A, B])}{n} \right)$$

as $n \to \infty$, then D is realizable in $K_{n,n}$.

Additionally, we prove a sharp bound on the maximum number of edges in a realizable demand graph:

Theorem 4. *Let $n \geq 2$ and $(K_{n,n}, D)$ be an instance of the* EDP *problem. If D has at most $2n - 3$ edges and $\Delta(D) \leq n$, then D is realizable in $K_{n,n}$.*

The assumption $\Delta(D) \leq n$ is trivially necessary: at any given vertex, there can be at most n edge-disjoint paths that terminate there. The result is sharp, as shown by the demand graph on $2n - 2$ edges consisting of two bundles of $n - 1$ edges, where one of the bundles joins an arbitrary pair of vertices in A, while the other bundle joins a pair in B.

The following NP-hardness is result probably well-known, but we have not been able to find a reference for it. For completeness' sake, we include a short reduction.

Proposition 5. *The terminal-pairability or edge-disjoint paths problem for $K_{n,n}$ is* NP-*hard.*

Proof. The EDP problem is NP-hard on complete base graphs [24]. Take an instance of the edge-disjoint paths problem (K_n, D) on the vertex set $\{a_1, \ldots, a_n\}$. Take $G = K_{\binom{n}{2}, \binom{n}{2}}$, such that its vertex classes are $\{a_1, \ldots, a_{\binom{n}{2}}\}$ and $\{b_1, \ldots, b_{\binom{n}{2}}\}$ Let (G, D') be an instance of the EDP problem, where

$$E(D') = E(D) \cup E(G) \setminus \left\{ (a_i, b_j), (b_j, a_k) \ : \ 1 \leq i < k \leq n, \ j = i - 1 + \sum_{l=0}^{k-1} l \right\}.$$

Obviously, if D has a realization in K_n then we may lift an edge $\{a_i, a_k\}$ of the realization of D to the vertex b_j such that $j = i - 1 + \sum_{l=0}^{k-1} l$. If D' has a realization in G then it also has such a realization where each edge

$$e \in E(G) \setminus \left\{ (a_i, b_j), (b_j, a_k) \ : \ 1 \leq i < k \leq n, \ j = i - 1 + \sum_{l=0}^{k-1} l \right\}$$

is mapped to a path of one edge, i.e., itself: this can clearly be done, if e is not in the realization of D'; if it is, we can simply modify a path $P(f) \ni e$ by replacing $P(f)$ with $P(f) \cup P(e) \setminus \{e\}$ (some cycles may have to be pruned). Such a solution trivially corresponds to a realization of D in K_n. The described reduction is polynomial. $\square$

2. Proofs of the degree versions (Theorem 1 and 3)

Before we proceed to prove the theorems, we state several definitions and four well-known results about edge colorings of multigraphs.

Let H be a loopless multigraph. Recall, that the *chromatic index* of H (also known as the *edge chromatic number*), denoted by $\chi'(H)$, is the minimum k

such that there is a proper k-coloring of the edges of H. An *equitable edge coloring* of H is a proper coloring of the edges $E(H)$ such that the sizes of the color classes differ by at most one. The *list chromatic index* of H (also known as the *list edge chromatic number*), denoted by $\mathrm{ch}'(H)$, is the smallest integer k such that if for each edge of H there is a list of k different colors given, then there exists a proper coloring of the edges of H where each edge gets its color from its list.

The *maximum multiplicity* $\mu(H)$ is the maximum number of edges joining the same pair of vertices in H. The number of edges joining a vertex $x \in V(H)$ to a subset $A \subseteq V(H)$ of vertices is denoted by $e_H(x, A)$. The *set of neighbors* of x in H is denoted by $N_H(x)$. For other notation the reader is referred to [9].

Proposition 6. *If H is a multigraph and $\chi'(H) \leq k$ for some integer k, then there is an equitable edge coloring of H with exactly k colors.*

Proof. Let $c : E(H) \to \{1, 2, \ldots, k\}$ be a proper edge coloring of H. Suppose there are two colors x and y for which $|c^{-1}(x)| \geq |c^{-1}(y)| + 2$. The connected components of $H_{x,y} = c^{-1}(x) \cup c^{-1}(y)$ are cycles (where two parallel edges are regarded as a 2-cycle) and paths. In any cycle of $H_{x,y}$, the number of edges of color x is equal to the number of edges of color y, therefore $H_{x,y}$ must contain a path component of odd length, with one more edge of color x than of color y. By switching the two colors in this path, the sum $\sum_{i=1}^{k} |c^{-1}(i)|^2$ decreases, and we end up with a coloring which is still proper. Thus, if we cannot repeat this procedure anymore, c must be an equitable coloring, as desired. $\square$

We will use the following well-known results about the edge colorings of multigraphs.

Proposition 7 (Greedy edge coloring). *For any multigraph H we have*
$$\chi'(H) \leq \mathrm{ch}'(H) \leq 2\Delta(H) - 1.$$

Theorem 8 (Vizing [33]). *For any multigraph H, its chromatic index*
$$\chi'(H) \leq \Delta(H) + \mu(H).$$

Theorem 9 (Shannon [32]). *For any multigraph H, its chromatic index*
$$\chi'(H) \leq \frac{3}{2}\Delta(H).$$

Theorem 10 (Kahn [21]). *For any multigraph H, its list chromatic index*
$$\mathrm{ch}'(H) \leq (1 + o(1))\chi'(H).$$

We need to prove a technical proposition before the proofs of Theorem 1 and 3.

Proposition 11. *If D is a demand graph on the vertex set $V(K_{n,n})$ and $\Delta(D) \leq n/4$, then there exists a proper edge $2\lfloor n/2 \rfloor$-coloring of $D[A, B] \cup D[B]$, which induces an equitable $2\lfloor n/2 \rfloor$-coloring on $D[B]$ and an almost equitable (the difference between the sizes of two color classes is ≤ 2) coloring on $D[A, B]$.*

Proof. Observe that (by Proposition 7)

$$\chi'(D[B]) \leq 2\Delta(D) \leq \frac{1}{2}n,$$

$$\chi'(D[A,B]) \leq 2\Delta(D) \leq \frac{1}{2}n.$$

By Proposition 6, there is a partition of $E(D[B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[B])/n \rfloor$ and $\lceil e(D[B])/n \rceil$, say $M_1, \ldots, M_{\lfloor n/2 \rfloor}$, so that $|M_i| \geq |M_j|$ for $i < j$. Similarly, there is a partition of $E(D[A,B])$ into $\lfloor n/2 \rfloor$ matchings of size $\lfloor e(D[A,B])/n \rfloor$ and $\lceil e(D[A,B])/n \rceil$, say $N_1, \ldots, N_{\lfloor n/2 \rfloor}$, so that $|N_i| \leq |N_j|$ for $i < j$. It is sufficient to prove now that for all $i = 1, \ldots, \lfloor n/2 \rfloor$, there exists a 2-coloring of $M_i \cup N_i$ which is induces an equitable 2-coloring on $D[B]$ and induces an almost equitable 2-coloring on $D[A,B]$.

Fix i. Observe, that $M_i \cup N_i$ is the vertex disjoint union of some edges and paths composed of two or three edges that alternate between elements of M_i and N_i. The paths of two and three edges contain one edge of M_i exactly. Let the number of components of $M_i \cup N_i$ containing k edges be c_k.

Color the M_i edge of $\lfloor c_3/2 \rfloor$ of the path components of length three with color 1, and color the M_i edges of the remaining $\lceil c_3/2 \rceil$ paths of length three with color 2. Similarly, color the M_i edge of $\lceil c_2/2 \rceil$ paths of length two with color 1, and color the remaining $\lfloor c_2/2 \rfloor$ uncolored M_i edges in paths of length two with color 2.

The already colored edges in M_i determine the colors of edges of N_i intersecting them (as we are looking for a proper edge 2-coloring). Let this proper partial edge 2-coloring be c. It trivially induces an equitable coloring on $D[B]$. On $D[A,B]$, we have $|c^{-1}(1) \cap D[A,B]| = 2\lceil c_3/2 \rceil + \lfloor c_2/2 \rfloor$ and $|c^{-1}(2) \cap D[A,B]| = 2\lfloor c_3/2 \rfloor + \lceil c_2/2 \rceil$, the difference of which is clearly at most 2. As the yet uncolored edges of $M_i \cup N_i$ are vertex disjoint, this partial coloring can be extended to a proper 2-coloring, which is equitable in M_i and almost equitable in N_i. □

Proof of Theorem 1. As D has an even number of vertices, we may assume that D is regular by adding edges, if necessary. Clearly, $e(D[A]) = e(D[B])$, $e(D) = e(D[A]) + e(D[A,B]) + e(D[B])$, and $e(D) = n \cdot \Delta(D)$.

Our proof consists of three steps. In the first step, we resolve the high multiplicity edges of $D[A]$, while leaving $D[A,B] \cup D[B]$ untouched. In the second step, we lift the edges of $D[B]$ to A, and resolve the multiplicities of $D[A,B]$. In the third step, we lift the edges induced by A to B, while preserving a simpleness of the bipartite subgraph induced by A and B, thus we end up with a graph which is a realization of D.

By Proposition 7, $\chi'(D[A]) \leq n$, so Proposition 6 implies the existence of an equitable edge n-coloring c_1 of $D[A]$. We construct D' from D by lifting the elements of $c_1^{-1}(i)$ to a_i for all $i = 1, \ldots, n$. As c_1 is a proper coloring, $\mu(D'[A]) \leq 2$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$e_{D'}(a, A) \leq e_D(a, A) + 2 \cdot \lceil e(D[A])/n \rceil,$$
$$e_{D'}(a, B) = e_D(a, B), \ e_{D'}(b, A) = e_D(b, A), \ e_{D'}(b, B) = e_D(b, B).$$

For the second step, we use Proposition 11 to take a proper edge n-coloring c_2 of $D[A,B] \cup D[B]$, which is an (almost) equitable n- or $(n-1)$-coloring if restricted

to both $D[A, B]$ and $D[B]$. We get D'' from D' by lifting the elements of $c_2^{-1}(i)$ to a_i for all $i = 1, \ldots, n$. As c_2 is a proper edge coloring, $D''[A, B]$ is simple, and $\mu(D''[A]) \leq \mu(D'[A]) + 2 \leq 4$. For any $a \in A$ and $b \in B$, we have the following estimates:

$$e_{D''}(a, A) \leq e_{D'}(a, A) + e_{D'}(a, B) + \lceil e(D[A, B])/(n - 1) \rceil + 1$$
$$e_{D''}(a, B) \leq \lceil e(D[A, B])/(n - 1) \rceil + 1 + 2 \cdot \lceil e(D[B])/(n - 1) \rceil$$
$$e_{D''}(b, A) = \Delta(D), \ e_{D''}(b, B) = 0.$$

To each edge $e \in E(D''[A])$ with end vertices a_i and a_j, we associate a list $L(e)$ of vertices of B, to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus (N_{D''}(a_i) \cup N_{D''}(a_j)),$$

whose size is bounded from below

$$|L(e)| \geq n - e_{D''}(a_i, B) - e_{D''}(a_j, B) \geq$$
$$\geq n - 2 \cdot \lceil e(D[A, B])/(n - 1) \rceil - 4 \cdot \lceil e(D[B])/(n - 1) \rceil - 2.$$

By Vizing's theorem (Theorem 8),

$$\chi'(D''[A]) \leq \Delta(D''[A]) + \mu(D''[A]) \leq$$
$$\leq \max_{a \in A} (e_{D'}(a, A) + e_{D'}(a, B) + \lceil e(D[A, B])/(n - 1) \rceil + 1) + 4 \leq$$
$$\leq \Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A, B])/(n - 1) \rceil + 5.$$

By Kahn's theorem (Theorem 10), $\mathrm{ch}'(D''[A]) \leq (1 + o(1))\chi'(D''[A])$. We have $\mathrm{ch}'(D''[A]) \leq |L(e)|$ for each edge e in $E(D''[A])$, if

$$(1 + o(1)) (\Delta(D) + 2 \cdot \lceil e(D[A])/n \rceil + \lceil e(D[A, B])/(n - 1) \rceil) \leq$$
$$\leq n - 2\lceil e(D[A, B])/(n - 1) \rceil - 4 \cdot \lceil e(D[B])/(n - 1) \rceil.$$

This inequality holds, if

$$(1 + o(1)) (\Delta(D) + 2 \cdot e(D[A])/n + 3 \cdot e(D[A, B])/n + 4 \cdot e(D[B])/n) \leq n.$$

Using our observations at the beginning of this proof, the previous inequality is a consequence of the regularity of D and

$$(1 + o(1)) \cdot 4 \cdot \Delta(D) \leq n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_3 which maps each $e \in E(D''[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D''[A])$ to $c_3(e)$. As we do not create multiple edges between A and B, the resulting graph is a realization of D. $\qquad\square$

Proof of Theorem 3. This proof is a slight variation on the previous proof. We do not lift edges of $D[A]$ to elements of A. Futhermore, instead of Vizing's theorem, Shannon's theorem (Theorem 9) will be used to bound the chromatic index of a graph induced by A.

We may assume that D is regular. For the first step, we use Proposition 11 to take a proper edge n-coloring c_1 of $D[A, B] \cup D[B]$, which is an (almost) equitable n- or $(n - 1)$-coloring if restricted to $D[A, B]$ and $D[B]$. Lift $c_1^{-1}(i)$ to a_i for all

$i = 1, \ldots, n$ to get D' from D. Now $D'[A, B]$ is simple and $D'[B]$ is an empty graph on n-vertices. For any $a \in A$ and $b \in B$, we have the following estimates:

$$e_{D'}(a, A) \leq e_D(a, A) + e_D(a, B) + \lceil e(D[A, B])/(n-1) \rceil + 1,$$
$$e_{D'}(a, B) \leq \lceil e(D[A, B])/(n-1) \rceil + 1 + 2 \cdot \lceil e(D[B])/(n-1) \rceil,$$
$$e_{D'}(b, A) = \Delta(D), \ e_{D'}(b, B) = 0.$$

To each edge $e \in E(D'[A])$ with end vertices a_i and a_j, we associate a list $L(e)$ of vertices of B, to which we can lift e to without creating multiple edges:

$$L(e) = B \setminus (N_{D'}(a_i) \cup N_{D'}(a_j)),$$

whose size is bounded from below

$$|L(e)| \geq n - e_{D'}(a_i, B) - e_{D'}(a_j, B) \geq$$
$$\geq n - 2\lceil e(D[A, B])/(n-1) \rceil - 4 \cdot \lceil e(D[B])/(n-1) \rceil - 2 \geq$$
$$\geq n - (1 + o(1))2\Delta(D).$$

By Shannon's theorem (Theorem 9),

$$\chi'(D'[A]) \leq \frac{3}{2}\Delta(D'[A]) \leq$$
$$\leq \frac{3}{2} \cdot \max_{a \in A} \left(e_D(a, A) + e_D(a, B) + \lceil e(D[A, B])/(n-1) \rceil + 1 \right) \leq$$
$$\leq (1 + o(1)) \cdot \frac{3}{2} \cdot (\Delta(D) + e(D[A, B])/n).$$

Furthermore, by Kahn's theorem (Theorem 10), $\mathrm{ch}'(D'[A]) \leq (1 + o(1))\chi'(D'[A])$. We have $\mathrm{ch}'(D'[A]) \leq |L(e)|$ for each edge e in $E(D'[A])$, if

$$(1 + o(1)) \cdot \frac{3}{2} \cdot (\Delta(D) + e(D[A, B])/n) \leq n - 2\Delta(D).$$

This holds, if

$$(1 + o(1)) \cdot \left(\frac{7}{2} \cdot \Delta(D) + \frac{3}{2} \cdot \frac{e(D[A, B])}{n} \right) \leq n.$$

Thus, if the conditions of the statement of this theorem hold, there is a proper list edge coloring c_2 which maps each $e \in E(D'[A])$ to an element of $L(e)$. Finally, we lift every edge $e \in E(D'[A])$ to $c_2(e)$. As we do not create multiple edges between A and B, the resulting graph is a realization of D. $\qquad \square$

3. ALGORITHMIC VERSIONS OF THEOREMS 1 AND 3

3.1. **Analysis of the complexity of the EDP problem with degree conditions.** The proofs of Theorems 1 and 3 provide recipes for constructing realizations in $K_{n,n}$. In this section we derive randomized and deterministic polynomial algorithms from them.

Proposition 12. *Given a proper edge k-coloring $c : E(H) \rightarrow \{1, \ldots, k\}$, an equitable proper edge k-coloring of H can be computed in $\mathcal{O}(kn)$.*

Proof. Recall the proof of Proposition 6. Given two colors i and j such that $|c^{-1}(i)| > \lceil e(H)/k \rceil$ and $|c^{-1}(j)| < \lfloor e(H)/k \rfloor$, find the connected components of $c^{-1}(i) \cup c^{-1}(j)$. By switching colors on path components containing more edges of color i than j, we eventually reach a point where either $|c^{-1}(i)|$ has decreased to $\lceil e(H)/k \rceil$ or $|c^{-1}(j)|$ has increased to $\lfloor e(H)/k \rfloor$. This subroutine takes $\mathcal{O}(|c^{-1}(i)| + |c^{-1}(j)|)$ time.

Repeat the above procedure until every color class has cardinality $\leq \lceil e(H)/k \rceil$ or every color class has size $\geq \lfloor e(H)/k \rfloor$. The subroutine is called at most k times, so the algorithm took $\mathcal{O}(kn)$ time. To make the coloring equitable, at most k more color switches have to be performed, each taking at most $\mathcal{O}(n)$ time. □

Consequently, the algorithm in the proof of Proposition 11 also runs in $\mathcal{O}(kn)$ time.

The proof of Kahn's result [21] is probabilistic, and it can be emulated in polynomial time. It immediately follows that:

Theorem 13. *Given an instance $(K_{n,n}, D)$ of the* EDP *problem, where $\Delta(D) \leq (1 - o(1)) \cdot \frac{n}{4}$, there is a randomized polynomial time algorithm which computes a realization of D in $K_{n,n}$.*

By using greedy edge coloring algorithms, we get deterministic algorithms (albeit with tighter upper bounds on $\Delta(D)$).

Theorem 14. *Given an instance $(K_{n,n}, D)$ of the* EDP *problem, where*

$$\Delta(D) \leq \frac{1}{6}(n - 7), \; or$$

$$\Delta(D) \leq \frac{1}{4}\left(n - 2 \cdot \left\lceil \frac{e(D[A, B])}{n - 1} \right\rceil - 5\right),$$

there is a deterministic $\mathcal{O}(\Delta(D) \cdot n)$ time algorithm which computes a realization of D in $K_{n,n}$.

Proof. We make the demand graph regular in $\mathcal{O}(\Delta(D) \cdot n)$. The time complexity of the greedy edge coloring algorithm is linear, and emulating Proposition 11 requires $\mathcal{O}(\Delta(D) \cdot n)$ time (see Proposition 12). Constructing D' and D'' by lifting the appropriate edges only takes linear time.

In the proof of Theorem 1, replace Kahn's list edge coloring with the greedy algorithm (Proposition 7). To get a proper list edge coloring of $D''[A]$, it is sufficient to have

$$2\Delta(D''[A]) - 1 \leq n - 2 \cdot \left\lceil \frac{e(D[A, B])}{n - 1} \right\rceil - 4 \cdot \left\lceil \frac{e(D[B])}{n - 1} \right\rceil - 2$$

$$2\Delta(D) + 4 \cdot \left\lceil \frac{e(D[A])}{n} \right\rceil + 4 \cdot \left\lceil \frac{e(D[A, B])}{n - 1} \right\rceil + 4 \cdot \left\lceil \frac{e(D[B])}{n - 1} \right\rceil \leq n - 3$$

which is satisfied if $\Delta(D) \leq \frac{1}{6}(n - 7)$.

In the proof of Theorem 3, to get a proper list edge coloring of $D'[A]$ via greedy coloring, it is sufficient to have

$$2 \cdot \left(\Delta(D) + \left\lceil \frac{e(D[A,B])}{n-1} \right\rceil + 1 \right) - 1 \leq n - 2 \cdot \left\lceil \frac{e(D[A,B])}{n-1} \right\rceil - 4 \cdot \left\lceil \frac{e(D[B])}{n-1} \right\rceil - 2$$

$$2 \cdot \Delta(D) + 4 \cdot \left\lceil \frac{e(D[A,B])}{n-1} \right\rceil + 4 \cdot \left\lceil \frac{e(D[B])}{n-1} \right\rceil \leq n - 3$$

which is satisfied when the second inequality of the conditions of this theorem holds. □

These bounds are not as tight as our theoretical bounds, but are smaller only by a factor of $\frac{3}{2}$ and $\frac{8}{7}$, respectively.

3.2. **Approximate solutions to the MaxEDP problem.** Given an instance (G, D) of the EDP problem, the MaxEDP problem asks for the subgraph $D' \subseteq D$ with the maximum number of edges such that (G, D') is realizable.

The algorithms for EDP in the previous section can be turned into constant factor approximation algorithms for the MaxEDP problem at the cost of a sensible additional term to their running time.

Theorem 15. *Let $(K_{n,n}, D)$ be an instance of the MaxEDP problem. Let $\mathcal{A}$ be an algorithm that can solve an instance $(K_{n,n}, D^*)$ of the EDP problem given a maximum degree condition $\Delta(D^*) \leq t$. Then there is an algorithm $\mathcal{B}$ which gives a $(3n/2t)$-approximation solution to the MaxEDP problem. The running time of algorithm $\mathcal{B}$ is at most $\mathcal{O}(e(D) \cdot n)$ plus the running time of algorithm $\mathcal{A}$ on the instance $(K_{n,n}, D^*)$, where D^* is some subgraph of D.*

Proof. Let D_{opt} be a subgraph of D that has the maximum number of edges among those that are realizable in $K_{n,n}$. Trivially, $\Delta(D_{\mathrm{opt}}) \leq n$. Since D is bipartite, using a folklore reduction to a maximum flow algorithm, the subgraph $D^* \subseteq D$ with the maximum number of edges such that $\Delta(D') \leq t$ can be found in $\mathcal{O}(e(D) \cdot n)$. (For regular D, the time complexity can be improved to $\mathcal{O}(e(D) \cdot \log \Delta(D))$ using the algorithm of [4].)

Partition $E(D_{\mathrm{opt}})$ into $\lfloor \frac{3}{2} \Delta(D) \rfloor$ matchings (see Theorem 9). Order the matchings in decreasing order of their cardinality. Let the spanning subgraph of D_{opt} formed by the union of the first t of its largest matchings be D'. Since $\Delta(D') \leq t$, we have $e(D') \leq e(D^*)$. Furthermore, because we take the largest matchings,

$$|E(D^*)| \geq |E(D')| \geq \frac{2t}{3n} \cdot |E(D^{\mathrm{opt}})|.$$

Lastly, Algorithm $\mathcal{A}$ can be used to compute a realization of D^* in $K_{n,n}$. □

Using Theorems 13 and 14, we have the following corollaries.

Corollary 16. *Given an instance $(K_{n,n}, D)$ of MaxEDP, there is a randomized polynomial time algorithm which produces a $(6 + \mathcal{O}(1/n))$-approximation solution.*

Corollary 17. *Given an instance $(K_{n,n}, D)$ of MaxEDP, there is a deterministic $\mathcal{O}(e(D) \cdot n)$ time algorithm which produces a $(9 + \mathcal{O}(1/n))$-approximation solution.*

4. Proof of the edge version (Theorem 4)

We apply induction on n. It is easy to check the result for $n \leq 3$, so let us assume from now on that $n \geq 4$. Since a subgraph of a realizable graph is realizable as well, it is enough to prove the result for demand graphs D on exactly $2n - 3$ edges. Recall that A and B be are the color classes of $K_{n,n}$, and let

$$S = \{v \in A \cup B : d_D(v) \geq n - 1\}.$$

Since D has $2n - 3$ edges, it is clear that $|S| \leq 3$ and that for every pair of vertices in S there is at least one edge joining them.

For a vertex $v \in V(D)$, we denote by $d(v)$ its degree and by $\gamma_A(v)$, $\gamma_B(v)$ the number of neighbors of v in class A and B, respectively. Let $d'(v), \gamma'_A(v), \gamma'_B(v)$ denote the value of these quantities after resolution of a vertex in D; similarly, $d''(v)$, $\gamma''_A(v)$, $\gamma''_B(v)$ denotes the values after the resolution of a second vertex, and so on. We denote the multiplicity of an edge uv by $\mu(uv)$, and we call it *monochromatic* if u and v are in the same color class of D, and *crossing*, otherwise.

Notice that, for a vertex $v \in A$, we need precisely $d(v) - \gamma_B(v)$ vertices in $B \backslash N_B(v)$ (which can be freely chosen in this set) to lift all the multiple edges and monochromatic edges incident to v. After these liftings, which increased the number of edges of the graph by $d(v) - \gamma_B(v)$, all the edges incident to v have their other endpoint in B and are simple. Clearly, we have the same for a vertex in B, exchanging all the occurrences of A and B. We say in this case that v is *resolved*.

For the induction step, we will resolve $t = 1$ or 3 vertices in each color class of D (possibly making some liftings before), remove them from the graph, getting a smaller graph D', and apply the induction hypothesis on D'. It is clear that D is realizable if D' is. By the inductive hypothesis, D' is realizable if the following conditions hold:

  (1) $\Delta(D') \leq n - t$,
  (2) D' has at most $2(n-t)-3$ edges, i.e., there were at least $2t$ edges incident to the $2t$ removed vertices after their resolution.

Assume first that there are 3 vertices of degree n in D lying on the same color class (this can only happen if $n \geq 6$, since we must have $3n \leq \sum_{v \in D} d(v) = 4n-6$). In this case, all other vertices in D have degree at most $4n-6-3n = n-6$.

Let $x, y, z \in A$ be the vertices of degree n. As $e(D) = 2n - 3$, it is clear that we have $\mu(xy) + \mu(xz) + \mu(yz) \geq n + 3$ and that there are at least 6 isolated vertices in B. We choose three from them, say, a, b, c. Without loss of generality, we may assume that $\mu(xy) + \mu(xz) \geq 2/3 \cdot (n + 3) \geq 6$.

We resolve x, y and z in this order. After resolving x, we have $\gamma'_B(y) \geq \mu(xy)$, and after resolving y, we have $\gamma''_B(z) \geq \mu(xz)$. In total, we add $d(x) - \gamma_B(x) + d'(y) - \gamma'_B(y) + d''(z) - \gamma''_B(z)$ edges to D, and we delete at least $d(x) + d'(y) + d''(z)$ edges when we remove x, y, z, a, b, c from D, so $e(D') \leq e(D) - (\gamma_B(x) + \gamma'_B(y) + \gamma''_B(z)) \leq e(d) - (\mu(xy) + \mu(xz)) \leq e(D) - 6$ edges, and we can apply induction since $\Delta(D') \leq n - 3$.

From now on, we may assume that there are at most two vertices of degree n in a class.

Let u be a maximum degree vertex in D. We may assume that $u \in A$. We distinguish some cases based on the value of $\gamma_B(u)$:

Case 1. $\gamma_B(u) \geq 2$, or $\gamma_B(u) = 1$ and $N_A(u) \neq \emptyset$.

We resolve the demands of u first. Then, if $N_A(u) \neq \emptyset$, let $u' \in N_A(u)$ be a vertex of maximum degree in this set, and $v \in B$ be a vertex that was used for a lifting of an edge uu'. Otherwise, if $N_A(u) = \emptyset$, let v be an arbitrary neighbor of u in B. We resolve the vertex v using the available vertices in A for the lifts in increasing order of degree, and then delete u and v.

We claim that the remaining graph D' satisfies $\Delta(D') \leq n - 1$. Indeed, the procedure above increases the degree of a vertex by one if it was used for a lift of either u or v, does not increase the degree of any other vertex in the graph, and decreases the degree of the neighbors of u in B by at least one. Since the vertices used for a lift of u are not joined to it, and hence have degree at most $n - 2$ (recall that every vertex of degree at least $n - 1$ is joined to a maximum degree vertex), no vertex in B has degree more than $n - 1$ after the procedure. On the other hand, we could have a non-neighbor of v, $x \in A$, distinct from u and u', which has degree at least $n - 1$ originally. This vertex would end up with degree at least n after the procedure in case it is used for a lift of v. The way we chose the vertices in A for lifts of v would imply, however, that $d(v) \leq n - 2$, and so $4n - 6 = \sum_{v \in D} d(v) \geq d(u) + d(u') + d(x) + d(v) = 4n - 5$, a contradiction.

The liftings added $d(u) - \gamma_B(u) + d'(v) - \gamma'_A(v)$ edges to D, and we deleted $d(u) + d'(v) - 1$ edges when we remove u and v, so $e(D') \leq e(D) - (\gamma_B(u) + \gamma'_A(v) - 1) \leq e(D) - 2$, so we can apply the induction hypothesis on D'.

Case 2. $\gamma_B(u) = 1$ and $N_A(u) = \emptyset$.

Let u' be the neighbor of u in B. If u' has another neighbor distinct from u, we would have $d(u') > d(u)$, a contradiction. So uu' forms a bundle. Also, if there is any crossing edge vv' not belonging to this bundle, we resolve u first and then $v' \in B$ without using u in a lift (which is possible since we need $d'(v') - \gamma'_A(v') \leq n - 2 - 1 = n - 3$ vertices of A for the lifts). We are done by induction again after we deleting u and v, since $e(D') \leq e(D) - 2$ and $\Delta(D') \leq n - 1$.

Assume now that $E(D)$ consists of the bundle uu' and monochromatic edges not incident to u or u'. In this case, we take $a \neq u$ in A, $b \neq u'$ in B with smallest degree (by the number of edges, it is at most 3). Let e be an edge, say, in A, which is not incident to a. We lift e to b, and replace one copy of the edge uu' by the path $ubau'$. Then we resolve the multiple edges of a and b, and delete both of them. The remaining graph D' has $\Delta(D') \leq n - 1$ and two less edges than D, so we may apply the induction hypothesis on D'.

Case 3. $\gamma_B(u) = 0$.

Among the neighbors of u, let u' be one with the largest degree. Let us consider two cases:

Case 3.1. There is an edge e independent of uu'.

If e is a crossing edge, let $e = ab$. If not, let a and b be vertices in A and B, respectively, distinct from u, u' and the endpoints of e. In the first case, we lift uu' to b, and in the second, we also lift e to the vertex a or b which is in the opposite class of e. Then, we resolve the vertices a and b and delete them. In both cases, it is clear that the remaining graph D' satisfies $e(D') \le e(D) - 2$ and $\Delta(D') \le n - 1$, so the result follows from induction applied in D'.

Case 3.2. There is no edge independent from uu'.

As $e(D) = 2n - 3$ and $d(u), d(u') \le n$, it follows that uu' is an edge of multiplicity at most 3. So, it is clear that there are two independent edges e and f such that u and u' are incident to e and f, respectively. Again, we let a, b be vertices in A and B not incident to e or f, and we lift both edges to b. After resolving and deleting a and b, we are left with D' with $\Delta(D') \le n - 1$ and $e(D') \le e(D) - 2$, so we are done by the induction hypothesis on D'.

REFERENCES

[1] Faisal N. Abu-Khzam and Michael A. Langston. "Graph coloring and the immersion order". In: *Computing and combinatorics*. Vol. 2697. Lecture Notes in Comput. Sci. Springer, Berlin, 2003, pp. 394–403.

[2] Noga Alon and Michael Capalbo. "Finding disjoint paths in expanders deterministically and online". In: *Foundations of Computer Science, 2007. FOCS'07. 48th Annual IEEE Symposium on*. IEEE. 2007, pp. 518–524.

[3] Hannah Alpert and Jennifer Iglesias. "Length 3 edge-disjoint paths is NP-hard". In: *Comput. Complexity* 21.3 (2012), pp. 511–513. ISSN: 1016-3328.

[4] Richard Cole, Kirstin Ost, and Stefan Schirra. "Edge-coloring bipartite multigraphs in $O(E \log D)$ time". In: *Combinatorica* 21.1 (2001), pp. 5–12. ISSN: 0209-9683.

[5] Karen L. Collins and Megan E. Heenehan. "Constructing graphs with no immersion of large complete graphs". In: *J. Graph Theory* 77.1 (2014), pp. 1–18. ISSN: 0364-9024.

[6] Lucas Colucci, Péter L. Erdős, Ervin Győri, and Tamás Róbert Mezei. "Terminal-pairability in complete bipartite graphs". In: *Discrete Applied Mathematics* (2017). ISSN: 0166-218X. arXiv: 1702.04313.

[7] László Csaba, Ralph J. Faudree, András Gyárfás, Jenő Lehel, and Richard H. Schelp. "Networks communicating for each pairing of terminals". In: *Networks* 22.7 (1992), pp. 615–626. ISSN: 0028-3045.

[8] Matt Devos, Zdeněk Dvořák, Jacob Fox, Jessica McDonald, Bojan Mohar, and Diego Scheide. "A minimum degree condition forcing complete graph immersion". In: *Combinatorica* 34.3 (2014), pp. 279–298. ISSN: 0209-9683.

[9] Reinhard Diestel. *Graph theory*. Fourth Edition. Vol. 173. Graduate Texts in Mathematics. Springer, Heidelberg, 2010, pp. xviii+437. ISBN: 978-3-642-14278-9.

[10] Zdeněk Dvořák and Liana Yepremyan. "Complete graph immersions and minimum degree". In: *arXiv preprint* (2015). arXiv: 1512.00513v1.

[11] S. Even, A. Itai, and A. Shamir. "On the complexity of timetable and multicommodity flow problems". In: *SIAM J. Comput.* 5.4 (1976), pp. 691–703. ISSN: 0097-5397.

[12] Ralph J. Faudree, András Gyárfás, and Jenő Lehel. "Minimal path pairable graphs". In: *Proceedings of the Twenty-third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1992)*. Vol. 88. 1992, pp. 111–128.

[13] Ralph J. Faudree, András Gyárfás, and Jenő Lehel. "Three-regular path pairable graphs". In: *Graphs Combin.* 8.1 (1992), pp. 45–52. ISSN: 0911-0119.

[14] Ralph J. Faudree, András Gyárfás, and Jenő Lehel. "Path-pairable graphs". In: *J. Combin. Math. Combin. Comput.* 29 (1999), pp. 145–157. ISSN: 0835-3026.

[15] Michael R. Fellows and Michael A. Langston. "On well-partial-order theory and its application to combinatorial problems of VLSI design". In: *SIAM J. Discrete Math.* 5.1 (1992), pp. 117–126. ISSN: 0895-4801.

[16] Michael R. Fellows and Michael A. Langston. "On search, decision, and the efficiency of polynomial-time algorithms". In: *J. Comput. System Sci.* 49.3 (1994), pp. 769–779. ISSN: 0022-0000.

[17] António Girão and Gábor Mészáros. "An improved upper bound on the maximum degree of terminal-pairable complete graphs". In: *Discrete Math.* 341.9 (2018), pp. 2606–2607. ISSN: 0012-365X.

[18] András Gyárfás and R. H. Schelp. "A communication problem and directed triple systems". In: *Discrete Appl. Math.* 85.2 (1998), pp. 139–147. ISSN: 0166-218X.

[19] Ervin Győri, Tamás Róbert Mezei, and Gábor Mészáros. "Terminal-Pairability in Complete Graphs". In: *Journal of Comb. Math.* (2016), accepted for publication. arXiv: 1605.05857.

[20] Ervin Győri, Tamás Róbert Mezei, and Gábor Mészáros. "Note on terminal-pairability in complete grid graphs". In: *Discrete Mathematics* 340.5 (2017), pp. 988–990. arXiv: 1606.06826.

[21] Jeff Kahn. "Asymptotics of the list-chromatic index for multigraphs". In: *Random Structures Algorithms* 17.2 (2000), pp. 117–156. ISSN: 1042-9832.

[22] Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce Reed. "The disjoint paths problem in quadratic time". In: *J. Combin. Theory Ser. B* 102.2 (2012), pp. 424–435. ISSN: 0095-8956.

[23] Teeradej Kittipassorn and Gábor Mészáros. "Frustrated triangles". In: *Discrete Math.* 338.12 (2015), pp. 2363–2373. ISSN: 0012-365X.

[24] Adrian Kosowski. "The maximum edge-disjoint paths problem in complete graphs". In: *Theoret. Comput. Sci.* 399.1-2 (2008), pp. 128–140. ISSN: 0304-3975.

[25] Ewa Kubicka, Grzegorz Kubicki, and Jenő Lehel. "Path-pairable property for complete grids". In: *Combinatorics, graph theory, and algorithms, Vol. I, II (Kalamazoo, MI, 1996)*. New Issues Press, Kalamazoo, MI, 1999, pp. 577–586.

[26] Ewa Kubicka, Grzegorz Kubicki, and Jenő Lehel. "Path-pairable property for complete grids". In: *Combinatorics, graph theory, and algorithms, Vol. I, II (Kalamazoo, MI, 1996)*. New Issues Press, Kalamazoo, MI, 1999, pp. 577–586.

[27] Gábor Mészáros. "Note on the diameter of path-pairable graphs". In: *Discrete Math.* 337 (2014), pp. 83–86. ISSN: 0012-365X.

[28] Gábor Mészáros. "On path-pairability in the Cartesian product of graphs". In: *Discuss. Math. Graph Theory* 36.3 (2016), pp. 743–758. ISSN: 1234-3099.

[29] Takao Nishizeki, Jens Vygen, and Xiao Zhou. "The edge-disjoint paths problem is NP-complete for series-parallel graphs". In: *Discrete Appl. Math.* 115.1-3 (2001). 1st Japanese-Hungarian Symposium for Discrete Mathematics and its Applications (Kyoto, 1999), pp. 177–186. ISSN: 0166-218X.

[30] Neil Robertson and Paul D. Seymour. "Graph minors. XIII. The disjoint paths problem". In: *J. Combin. Theory Ser. B* 63.1 (1995), pp. 65–110. ISSN: 0095-8956.

[31] Neil Robertson and Paul D. Seymour. "Graph minors XXIII. Nash-Williams' immersion conjecture". In: *J. Combin. Theory Ser. B* 100.2 (2010), pp. 181–205. ISSN: 0095-8956.

[32] Claude E. Shannon. "A theorem on coloring the lines of a network". In: *Studies in Applied Mathematics* 28.1-4 (1949), pp. 148–152.

[33] Vadim Georgievich Vizing. "The chromatic class of a multigraph". In: *Kibernetika (Kiev)* 1965.3 (1965), pp. 29–39. ISSN: 0023-1274.

[34] Jens Vygen. "NP-completeness of some edge-disjoint paths problems". In: *Discrete Appl. Math.* 61.1 (1995), pp. 83–90. ISSN: 0166-218X.