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THE L_p-MINKOWSKI PROBLEM FOR $-n < p < 1$

GABRIELE BIANCHI, KÁROLY J. BÖRÖCZKY, ANDREA COLESANTI, DEANE YANG

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Abstract. Chou and Wang's existence result for the L_p-Minkowski problem on $\mathbb{S}^{n-1}$ for $p \in (-n, 1)$ and an absolutely continuous measure is discussed and extended to more general measures. In particular, we provide an almost optimal sufficient condition for the case $p \in (0, 1)$.

1. Introduction

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. A *convex body* K in $\mathbb{R}^n$ is a compact convex set that has non-empty interior. For any $x \in \partial K$, $\nu_K(x)$ ("the Gauß map") is the family of all unit exterior normal vectors at x; in particular $\nu_K(x)$ consists of a unique vector for $\mathcal{H}^{n-1}$ almost all $x \in \partial K$ (see, *e.g.*, Schneider [78]), where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.

The *surface area measure* S_K of K is a Borel measure on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, defined, for a Borel set $\omega \subset \mathbb{S}^{n-1}$ by

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right) = \mathcal{H}^{n-1}\left(\{x \in \partial K : \nu_K(x) \cap \omega \neq \emptyset\}\right)$$

(see, *e.g.*, Schneider [78]).

As one of the cornerstones of the classical Brunn-Minkowski theory, the Minkowski's existence theorem can be stated as follows (see, *e.g.*, Schneider [78]): If the Borel measure μ is not concentrated on a great subsphere of $\mathbb{S}^{n-1}$, then μ is the surface area measure of a convex body if and only if the following vector condition is verified

$$\int_{\mathbb{S}^{n-1}} u d\mu(u) = 0.$$

Moreover, the solution is unique up to translation. The regularity of the solution has been also well investigated, see *e.g.*, Lewy [54], Nirenberg [72], Cheng and Yau [20], Pogorelov [75], and Caffarelli [14, 15].

The surface area measure of a convex body has a clear geometric significance. In [59], Lutwak showed that there is an L_p analogue of the surface area measure (known as the L_p-surface area measure). For a convex compact set K in $\mathbb{R}^n$, let h_K be its support function:

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\} \quad \text{for } u \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product.

Let $\mathcal{K}_0^n$ denote the family of convex bodies in $\mathbb{R}^n$ containing the origin o. Note that if $K \in \mathcal{K}_0^n$, then $h_K \geq 0$. If $p \in \mathbb{R}$ and $K \in \mathcal{K}_0^n$, then the L_p-surface area measure is defined by

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

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where for $p > 1$ the right hand side is assumed to be a finite measure. In particular, if $p = 1$, then $S_{K,p} = S_K$, and if $p < 1$ and $\omega \subset \mathbb{S}^{n-1}$ is a Borel set, then

$$S_{K,p}(\omega) = \int_{x \in \nu_K^{-1}(\omega)} \langle x, \nu_K(x) \rangle^{1-p} d\mathcal{H}^{n-1}(x).$$

In recent years, the L_p-surface area measure appeared in, *e.g.*, [1,5,16,32,33,35,36,41,56–58,61–63,66,68,70,71,73,74,81]. In [59], Lutwak posed the associated L_p-Minkowski problem for $p \geq 1$ which extends the classical Minkowski problem. In addition, the L_p-Minkowski problem for $p < 1$ was publicized by a series of talks by Erwin Lutwak in the 1990's, and appeared in print in Chou and Wang [22] for the first time.

L_p-Minkowski problem: For $p \in \mathbb{R}$, what are the necessary and sufficient conditions on a finite Borel measure μ on $\mathbb{S}^{n-1}$ in order that μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$?

Besides discrete measures, an important special class is that of Borel measures μ on $\mathbb{S}^{n-1}$ which have a density with respect to $\mathcal{H}^{n-1}$:

$$d\mu = f \, d\mathcal{H}^{n-1} \tag{1}$$

for some non-negative measurable function f on $\mathbb{S}^{n-1}$. If (1) holds, then the L_p-Minkowski problem amounts to solving the Monge-Ampère type equation

$$h^{1-p} \det(\nabla^2 h + hI) = f \tag{2}$$

where h is the unknown non-negative (support) function on $\mathbb{S}^{n-1}$ to be found, $\nabla^2 h$ denotes the (covariant) Hessian matrix of h with respect to an orthonormal frame on $\mathbb{S}^{n-1}$, and I is the identity matrix. Recent extensions of the L_p-Minkowski problem are the L_p dual Minkowski problem proposed by Lutwak, Yang, Zhang [67], and the Orlicz Minkowski problem discussed by Haberl, Lutwak, Yang, Zhang [34] (extending the case $p > 1$, for even measures), Huang, He [44] (extending the case $p > 1$) and Jian, Lu [52] (extending the case $0 < p < 1$).

The case $p = 1$, namely the classical Minkowski problem, was solved by Minkowski [69] in the case of polytopes, and in the general case by Alexandrov [2], and Fenchel and Jessen [25]. The case $p > 1$ and $p \neq n$ was solved by Chou and Wang [22], Guan and Lin [31] and Hug, Lutwak, Yang, and Zhang [47]; Zhu [93] investigated the dependence of the solution on p for a given target measure. We note that the solution is unique if $p > 1$ and $p \neq n$, and unique up to translation if $p = 1$. In addition, if $p > n$, then the origin lies in the interior of the solution K; however, if $1 < p < n$, then possibly the origin lies on the boundary of the solution K even if (1) holds for a positive continuous f.

The goal of this paper is to discuss the L_p-Minkowski problem for $p < 1$. The case $p = 0$ is the so called logarithmic Minkowski problem see, *e.g.*, [9–12, 56–58, 70, 71, 73, 79–81, 89]. Additional references regarding the L_p Minkowski problem and Minkowski-type problems can be found in, *e.g.*, [19, 22, 30–34, 43, 45, 46, 51, 53, 55, 59, 60, 65, 69, 79, 80, 90, 91]. Applications of the solutions to the L_p Minkowski problem can be found in, *e.g.*, [3, 4, 21, 23, 26, 37–39, 48, 49, 64, 84, 85, 88].

We note that if $p < 1$, then non-congruent n-dimensional convex bodies may give rise to the same L_p-surface area measure, see Chen, Li, and Zhu [18] for examples when $0 < p < 1$, Chen, Li, and Zhu [17] for examples when $p = 0$ and Chou and Wang [22] for examples when $p < 0$.

If $0 < p < 1$, then the L_p-Minkowski problem is essentially solved by Chen, Li, and Zhu [18].

Theorem 1.1 (Chen, Li, and Zhu). *If $p \in (0, 1)$, and μ is a finite Borel measure on $\mathbb{S}^{n-1}$ not concentrated on a great subsphere, then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$.*

We believe that the following property characterizes L_p-surface area measures for $p \in (0,1)$.

Conjecture 1.2. *Let $p \in (0,1)$, and let μ be a non-trivial Borel measure on $\mathbb{S}^{n-1}$. Then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$ if and only if* $\operatorname{supp}\mu$ *is not a pair of antipodal points.*

Conjecture 1.2 is proved in the planar case $n = 2$ independently by Böröczky and Trinh [13] and Chen, Li,and Zhu [18]. Here we prove a slight extension of the result proved in [18]. We note that Lemma 11.1 of the present paper implies that $\operatorname{supp} S_{K,p}$ is not a pair of antipodal points for any convex body $K \in \mathcal{K}_0^n$ and $p < 1$. For $X \subset \mathbb{R}^n$, its positive hull is

$$\operatorname{pos} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \lambda_i \geq 0,\ x_i \in X \text{ and } k \geq 1 \text{ integer} \right\},$$

which is closed if $X \subset \mathbb{S}^{n-1}$ is compact. We prove the following result.

Theorem 1.3. *Let $p \in (0,1)$, let μ be a non-trivial finite Borel measure on $\mathbb{S}^{n-1}$, and let $L = \operatorname{lin}\operatorname{supp}\mu$. If either $\operatorname{supp}\mu$ spans $\mathbb{R}^n$, or $\dim L \leq n-1$ and $\operatorname{pos}\operatorname{supp}\mu \neq L$, then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$. In addition, if μ is invariant under a closed subgroup G of $O(n)$ acting as the identity on $L^\perp$, then K can be chosen to be invariant under G.*

The assumption in Theorem 1.3 can be equivalently stated in term of the subset $\operatorname{conv}(\{o\} \cup \operatorname{supp}\mu)$ in $\mathbb{R}^n$ (here $\operatorname{conv} A$ denotes the convex hull of the set A). We require that either $\operatorname{conv}(\{o\} \cup \operatorname{supp}\mu)$ has non-empty interior or, if this is not the case, that $\operatorname{conv}(\{o\} \cup \operatorname{supp}\mu)$ does not contain o in its relative interior.

The case $p = 0$ concerns the cone volume measure. We say that a Borel measure μ on $\mathbb{S}^{n-1}$ satisfies the subspace concentration condition if for any non-trivial linear subspace L we have

$$\mu(L \cap \mathbb{S}^{n-1}) \leq \frac{\dim L}{n}\, \mu(\mathbb{S}^{n-1}),$$

and equality holds if and only if there exists a complementary linear subspace L' such that $\operatorname{supp}\mu \subset L \cup L'$. Böröczky, Lutwak, Yang, and Zhang [10] proved that even cone volume measures are characterized by the subspace concentration condition. The sufficiency part has been extended to all Borel measures on $\mathbb{S}^{n-1}$ by Chen, Li, and Zhu [17]. The part of Theorem 1.4 concerning the action of a closed subgroup G of $O(n)$ is not actually in [17] but could be verified easily using the methods of our paper.

Theorem 1.4 (Chen, Li, Zhu)**.** *If μ is a Borel measure on $\mathbb{S}^{n-1}$ satisfying the subspace concentration condition, then μ is the L_0-surface area measure of a convex body $K \in \mathcal{K}_0^n$. In addition, if μ is invariant under a closed subgroup G of $O(n)$, then K can be chosen to be invariant under G.*

If $p = 0$, then not even a conjecture is known concerning which properties may characterize L_0-surface area measures. Note that Böröczky and Hegedűs [7] characterized the restriction of an L_0-surface area measure to a pair of antipodal points.

The main new result of this paper is the following statement regarding the case $p \in (-n, 0)$.

Theorem 1.5. *If $p \in (-n, 0)$, and μ is a non-trivial Borel measure on $\mathbb{S}^{n-1}$ satisfying (1) for a non-negative function f in $L_{\frac{n}{n+p}}(\mathbb{S}^{n-1})$, then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$. In addition, if μ is invariant under a closed subgroup G of $O(n)$, then K can be chosen to be invariant under G.*

It is not clear whether the analogue of Theorem 1.5 can be expected in the critical case $p = -n$. If ∂K is C_+^2 and $o \in \operatorname{int} K$, then L_{-n} surface area measure is

$$dS_{K,-n} = \frac{h_K(u)^{n+1}}{\kappa(u)}\, d\mathcal{H}^{n-1}, \tag{3}$$

where $\kappa(u)$ is the Gaussian curvature of ∂K at the point $x \in \partial K$ with $u \in \nu_K(x)$. Note that $\kappa_0(u) = \kappa(u)/h_K(u)^{n+1}$ is the so called centro-affine curvature (see Ludwig [57] or Stancu [81]), which is equi-affine invariant in the following sense. For any $A \in \mathrm{SL}(n)$, if $\tilde{A}(u) = \frac{Au}{\|Au\|}$ is the corresponding projective transformation of $\mathbb{S}^{n-1}$, and $\tilde{\kappa}_0$ is the centro-affine curvature function of $A^{-t}K$, then

$$\tilde{\kappa}_0(\tilde{A}(u)) = \kappa_0(u), \quad \forall\, u \in \mathbb{S}^{n-1}.$$

In particular, Chou and Wang [22] proved the following formula for the L_{-n} surface area measure.

Proposition 1.6 (Chou and Wang). *Let $K \in \mathcal{K}_0^n$ be such that $o \in \operatorname{int} K$ and ∂K is C_+^3, so that $dS_{K,-n} = f\, d\mathcal{H}^{n-1}$ for a C^1 function f according to (3). If $\mathcal{V}(\xi) = \xi_j A^{ij}\partial_i$ is a projective vector field on $\mathbb{S}^{n-1}$ for $A \in \mathrm{GL}(n)$, then*

$$\int_{\mathbb{S}^{n-1}} h^{-n}\, \mathcal{V} f\, d\mathcal{H}^{n-1} = 0.$$

For the sake of completeness, we provide a proof of Proposition 1.6 in Section 12.

We will prove Theorems 1.3 and 1.5 via an approximation argument based on Theorem 1.7, proved by Chou and Wang [22]. Of the latter, we will also provide a simplified and clarified argument. Again, the part of Theorem 1.7 concerning the action of a closed subgroup G of $O(n)$ is not actually in [17] but could be verified easily using the methods of our paper.

Theorem 1.7 (Chou and Wang). *If $p \in (-n, 1)$, and μ is a Borel measure on $\mathbb{S}^{n-1}$ satisfying (1) where f is bounded and $\inf_{u\in\mathbb{S}^{n-1}} f(u) > 0$, then μ is the L_p-surface area measure of a convex body $K \in \mathcal{K}_0^n$. In addition, if μ is invariant under the closed subgroup G of $O(n)$, then K can be chosen to be invariant under G, and $o \in \operatorname{int} K$ provided $p \in (-n, 2-n]$.*

Remark Theorems 1.3, 1.4 and 1.5 show that Theorem 1.7 holds for any $p \in (-n, 1)$ and non-negative bounded f with $\int_{\mathbb{S}^{n-1}} f\, d\mathcal{H}^{n-1} > 0$.

As already mentioned, if $p = 0$, then Böröczky and Hegedűs [7] provides some necessary condition on an L_0 surface area measure, more precisely, on the restriction of an L_0-surface area measure to pairs of antipodal points. Unfortunately, no necessary condition concerning L_p-surface area measures is known to us for the case $p < 0$.

We conclude by mentioning the related paper by G. Bianchi, K. J. Böröczky and A. Colesanti [6] which deals with the strict convexity and the C^1 smoothness of the solution to the L_p Minkowski problem when $p < 1$ and μ satisfies (1) for some function f which is bounded from above and from below by positive constants.

2. Preparation

Let κ_n be the volume of the n-dimensional unit Euclidean ball B^n, and let $\sigma(K)$ be the *centroid* of a convex body K.

Lemma 2.1. *For a convex body K in $\mathbb{R}^n$,*

- **(i):** $\frac{-1}{n}(x - \sigma(K)) + \sigma(K) \in K$ *for any $x \in K$;*
- **(ii):** *(Blaschke-Santaló inequality)*

$$\int_{\mathbb{S}^{n-1}} \frac{1}{n(h_K(u) - \langle \sigma(K), u\rangle)^n}\, d\mathcal{H}^{n-1}(u) \le \frac{\kappa_n^2}{V(K)}.$$

- **(iii):** *If $\varrho > 0$ is maximal and $R > 0$ is minimal such that $\sigma(K) + \varrho\, B^n \subset K$ and $K \subset \sigma(K) + R\, B^n$, then*

$$V(K) \le (n+1)\kappa_{n-1}\varrho R^{n-1}.$$

Proof. In the case of the Blaschke-Santaló inequality, we note that if the origin is the centroid of K, then the left hand side of (ii) is the volume of the polar body K^*, and the origin is the Santaló point of K^*. Therefore (i) and (ii) are well-known facts, see Lemma 2.3.3 and (10.28) in [78].

For (iii), we assume that $\sigma(K) = o$. Let $x_0 \in \varrho B^n \cap \partial K$, and let H be the common tangent hyperplane to K and ϱB^n at x_0. Since $-x/n \in K$ for any $x \in K$ as $\sigma(K) = o$, we deduce that K lies between the parallel hyperplanes H and $-nH$ whose distance is $(n+1)\varrho$. Note that x_0 is orthogonal to H. Now the projection of K into $x_0^\perp$ is contained in RB^n, we conclude (iii). Q.E.D.

For $v \in \mathbb{S}^{n-1}$ and $\alpha \in (0, \frac{\pi}{2}]$, let $\Omega(v,\alpha)$ be the family of all $u \in \mathbb{S}^{n-1}$ with $\angle(u,v) \leq \alpha$, where $\angle(u,v)$ is the (smaller) angle formed by u and v, i.e. their geodesic distance on the unit sphere. The following lemma is needed to show that with modified "energy function" φ_ε (see next section), the optimal "center" is in the interior.

Lemma 2.2. *Let $\varepsilon \in (0, \frac{1}{3}]$, $R \geq 1$ and $q \geq n-1$; let $K \in \mathcal{K}_0^n$ with $o \in \partial K$ and* $\operatorname{diam} K \leq R$, *and let v be an exterior unit normal at o.*

(i): *For $\alpha = \arcsin \frac{\varepsilon}{2R}$, if $\xi \in \operatorname{int} K$ with $\|\xi\| < \varepsilon/2$ and $u \in \Omega(v,\alpha)$, then $h_K(u) - \langle \xi, u\rangle < \varepsilon$.*

(ii): *If $\delta \in (0, \sin\alpha)$ and $\xi \in \operatorname{int} K$ satisfies $\|\xi\| \leq R\delta$, then*

$$\int_{\Omega(v,\alpha)} (h_K(u) - \langle \xi, u\rangle)^{-q}\, d\mathcal{H}^{n-1}(u) \geq \frac{(n-2)\kappa_{n-2}}{2^q R^q} \log \frac{\sin\alpha}{\delta}.$$

Proof. We may assume that $K = \{x \in RB^n : \langle x, v\rangle \leq 0\}$, and hence $h_K(u) = R\|u|v^\perp\| = R \sin \angle(u,v)$ if $u \in \Omega(v, \frac{\pi}{2})$. In particular, $\alpha = \arcsin \frac{\varepsilon}{2R}$ works in (i).

For (ii), if $\delta \in (0, \sin\alpha)$, $u \in \Omega(v,\alpha)$ with $\|u|v^\perp\| > \delta$, and $\|\xi\| < R\delta$, then $h_K(u) - \langle \xi, u\rangle < 2R||u|v^\perp\|$. We deduce that if $\|\xi\| < R\delta$ for $\xi \in \operatorname{int} K$, then

$$\begin{aligned}\int_{\Omega(v,\alpha)} (h_K(u) - \langle \xi, u\rangle)^{-q}\, d\mathcal{H}^{n-1}(u) &\geq \int_{[(\sin\alpha\cdot B^n)\setminus(\delta B^n)]\cap v^\perp} \frac{1}{2^q R^q \|x\|^q}\, d\mathcal{H}^{n-1}(x) \\ &= \frac{(n-2)\kappa_{n-2}}{2^q R^q} \int_\delta^{\sin\alpha} t^{n-2-q}\, dt \geq \frac{(n-2)\kappa_{n-2}}{2^q R^q} \log \frac{\sin\alpha}{\delta},\end{aligned}$$

which in turn yields the lemma. Q.E.D.

Let K be a convex body in $\mathbb{R}^n$. A point p in its boundary is said to be *smooth* if there exists a unique hyperplane supporting K at p, and p is said to be *singular* if it is not smooth. We write $\partial' K$ and Ξ_K to denote the set of smooth and singular points of ∂K, respectively. It is well known that $\mathcal{H}^{n-1}(\Xi_K) = 0$. We call K *quasi-smooth* if $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}\setminus\nu_K(\partial' K)) = 0$; namely, the set of $u \in \mathbb{S}^{n-1}$ that are exterior normals only at singular points has $\mathcal{H}^{n-1}$-measure zero.

The following Lemma 2.3 will be used to prove first that the extremal convex body K^ε is quasi-smooth in Section 5, and secondly that it satisfies an Euler-Lagrange type equation in Section 6. Let K and C be convex bodies containing the origin in their interior such that $rC \subset K$ for some $r > 0$. For $t \in (-r, r)$, we consider the *Wulff shape*

$$K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u) + t h_C(u) \ \text{ for } u \in \mathbb{S}^{n-1}\},$$

and we denote by h_t the support function of K_t.

Lemma 2.3. *Using the notation above, let $u \in \mathbb{S}^{n-1}$.*

(i): *If $K \subset R\,B^n$ for $R > 0$ and $t \in (-r, r)$, then $|h_t(u) - h_K(u)| \leq \frac{R}{r}|t|$.*

(ii): *If u is the exterior normal at some smooth point $z \in \partial K$, then*

$$\lim_{t\to 0} \frac{h_t(u) - h_K(u)}{t} = h_C(u).$$

Proof. If $t \geq 0$ then $h_t = h_K + th_C$, therefore we may assume that $t < 0$.

For (i), we observe that

$$\left(1+\frac{t}{r}\right)K+|t|C\subset\left(1+\frac{t}{r}\right)K+\frac{|t|}{r}\cdot K=K.$$

In other words, $\widetilde{K}_t=(1+\frac{t}{r})K\subset K_t$, which in turn yields that if $u\in\mathbb{S}^{n-1}$, then

$$h_K(u)-h_t(u)\leq h_K(u)-h_{\widetilde{K}_t}(u)=\frac{|t|}{r}\cdot h_K(u)\leq\frac{R}{r}\cdot|t|.$$

We turn to (ii). For $u\in\mathbb{S}^{n-1}$, we have $h_K(u)-h_t(u)\geq|t|\,h_C(u)$, and hence it is sufficient to prove that if $\varepsilon>0$ then

$$h_K(u)-h_t(u)\leq(h_C(u)+\varepsilon)|t| \tag{4}$$

provided that $t<0$ has small absolute value. Let D be the diameter of C, and let $\delta=\frac{\varepsilon}{\sqrt{D^2+\varepsilon^2}}$. If u is an exterior normal to C at a point $q\in\partial C$, then $w=q+\varepsilon u$ satisfies

$$\langle u,w\rangle \;=\; h_C(u)+\varepsilon \tag{5}$$

$$\langle u,x-w\rangle \;\leq\; -\delta\|x-w\| \ \text{ for all } \ x\in C. \tag{6}$$

Since $z\in\partial K$ is a smooth point with exterior unit normal u, there exists $\varrho>0$ such that if $\|x-z\|\leq\varrho$ and $\langle u,x-z\rangle\leq-\delta\|x-z\|$, then $x\in K$. We deduce from (6) that if $(D+\varepsilon)|t|<\varrho$, then $y+|t|C\subset K$ for $y=z-|t|w$, and hence $y\in K_t$. Therefore

$$h_K(u)-h_t(u)\leq\langle u,z-y\rangle=(h_C(u)+\varepsilon)|t|,$$

proving (4). Q.E.D.

Remark. Results similar to those proved in the previous lemma are contained in [50, Section 3].

Using the notation of Lemma 2.3, if K is quasi-smooth, then

$$\lim_{t\to0}\frac{h_t(u)-h_K(u)}{t}=h_C(u)$$

holds for $\mathcal{H}^{n-1}$ almost all $u\in\mathbb{S}^{n-1}$. In particular, Lemma 3.5 below applies.

3. The energy function and optimal center

Let $p\in(-n,1)$. For $t>0$, we set

$$\varphi(t)=\begin{cases} t^p & \text{if } p\in(0,1),\\ \log t & \text{if } p=0,\\ -t^p & \text{if } p\in(-n,0).\end{cases}$$

The reasons behind this choice of φ are that if $t\in(0,\infty)$, then

$$\varphi'(t)=\begin{cases} |p|t^{p-1} & \text{if } p\in(-n,1)\backslash\{0\}\\ t^{p-1} & \text{if } p=0\end{cases} \tag{7}$$

is positive and decreasing, φ is strictly increasing and φ'' is negative and continuous, and hence φ is strictly concave. In addition,

$$\lim_{t\to\infty}\varphi(t)=\begin{cases}\infty & \text{if } p\in[0,1),\\ 0 & \text{if } p\in(-n,0).\end{cases} \tag{8}$$

Let $q=\max\{|p|,n-1\}$. In order to force the "optimal center" of a convex body K into its interior, we change $\varphi(t)$ into a function of order $-t^{-q}$ if t is small (see Proposition 3.2). For

$t \in (0,1)$, the equation $\psi(s) = -t^{-(n-1)} + (n-1)t^{-n}(s-t)$ of the tangent to the graph of $t \mapsto -t^{-(n-1)}$ satisfies $\psi(3t) \geq t^{-(n-1)} \geq 1$. Thus for any $\varepsilon \in (0, \frac{1}{3})$, there exists an increasing strictly concave function $\varphi_\varepsilon : (0,\infty) \to \mathbb{R}$, with continuous and negative second derivative, such that

$$\varphi_\varepsilon(t) = \begin{cases} \varphi(t) & \text{if } t \geq 3\varepsilon, \\ -t^{-q} & \text{if } 0 < t \leq \varepsilon, \end{cases} \tag{9}$$

and in addition

$$\varphi_\varepsilon(t) \geq -t^{-q} \quad \text{if } t \in (0,1). \tag{10}$$

Let us observe that if $p \in (-n, -(n-1)]$, we may choose $\varphi_\varepsilon = \varphi$.

Let f be a measurable function on $\mathbb{S}^{n-1}$ such that there exist $\tau_2 > \tau_1 > 0$ satisfying

$$\tau_1 < f(u) < \tau_2 \quad \text{for } u \in \mathbb{S}^{n-1}, \tag{11}$$

and let μ be the Borel measure defined by $d\mu = f\, d\mathcal{H}^{n-1}$. *We remark that, even when not explicitly stated, in all the results contained in Sections 3, 4, 5, 6 and 7 it is always assumed that* (11) *holds.*

For $\varepsilon \in (0, \frac{1}{3})$, a convex body K and $\xi \in \operatorname{int} K$, we define

$$\Phi_\varepsilon(K,\xi) = \int_{\mathbb{S}^{n-1}} \varphi_\varepsilon(h_K(u) - \langle u, \xi\rangle)\, d\mu(u).$$

The proofs of Proposition 3.2 and Lemma 3.4 depend on the concavity of φ_ε and the following Lemma 3.1. Here and throughout the paper, the convergence of sequence of convex bodies is always meant in the sense of the Hausdorff metric.

Lemma 3.1. *Let $\{K_m\}$ be a sequence of convex bodies tending to a convex body K in $\mathbb{R}^n$, and let $\xi_m \in \operatorname{int} K_m$ be such that $\lim_{m\to\infty} \xi_m = z_0 \in \partial K$. Then*

$$\lim_{m\to\infty} \Phi_\varepsilon(K_m, \xi_m) = -\infty.$$

Proof. Let $r_m > 0$ be maximal such that $\xi_m + r_m B^n \subset K_m$, and let $y_m \in (\xi_m + r_m B^n) \cap \partial K_m$. The condition $z_0 \in \partial K$ implies that $r_m = \|y_m - \xi_m\|$ tends to zero. Let $v_m \in \mathbb{S}^{n-1}$ be an exterior normal at y_m to K_m. For $R = 1 + \operatorname{diam} K$, we have $\operatorname{diam} K_m \leq R$ for large m; let $\alpha = \arcsin \frac{\varepsilon}{2R}$ be the constant of Lemma 2.2. It follows from Lemma 2.2 (i) that if $u \in \Omega(v_m, \alpha)$ (the geodesic ball on $\mathbb{S}^{n-1}$, centered at v_m with opening α), then $h_{K_m}(u) - \langle u, \xi_m\rangle < \varepsilon$ for all m, and hence

$$\varphi_\varepsilon(h_{K_m}(u) - \langle u, \xi_m\rangle) = -(h_{K_m}(u) - \langle u, \xi_m\rangle)^{-q}.$$

Therefore Lemma 2.2 (ii) and (11) yield that

$$\lim_{m\to\infty} \int_{\Omega(v_m,\alpha)} \varphi_\varepsilon(h_{K_m}(u) - \langle u, \xi_m\rangle)\, d\mu(u) = -\infty. \tag{12}$$

On the other hand, $\varphi_\varepsilon(h_{K_m}(u) - \langle u, \xi_m\rangle) \leq \varphi_\varepsilon(R)$ holds for all m and $u \in \mathbb{S}^{n-1}$. We deduce from (11) that

$$\int_{\mathbb{S}^{n-1}\setminus\Omega(v,\alpha)} \varphi_\varepsilon(h_{K_m}(u) - \langle u, \xi_m\rangle)\, d\mu(u) < \tau_2 n\kappa_n \varphi_\varepsilon(R) \tag{13}$$

for all m. Combining (12) and (13) we conclude the proof. Q.E.D.

Now we single out the optimal $\xi \in \operatorname{int} K$.

Proposition 3.2. *For $\varepsilon \in (0, \frac{1}{3})$ and a convex body K in $\mathbb{R}^n$, there exists a unique $\xi(K) \in \operatorname{int} K$ such that*

$$\Phi_\varepsilon(K, \xi(K)) = \max_{\xi \in \operatorname{int} K} \Phi_\varepsilon(K,\xi).$$

Proof. Let $\xi_1, \xi_2 \in \operatorname{int} K$, $\xi_1 \neq \xi_2$, and let $\lambda \in (0,1)$. If $u \in \mathbb{S}^{n-1} \backslash (\xi_1 - \xi_2)^\perp$, then $\langle u, \xi_1 \rangle \neq \langle u, \xi_2 \rangle$, and hence the strict concavity of φ_ε yields that

$$\varphi_\varepsilon(h_K(u) - \langle u, \lambda\xi_1 + (1-\lambda)\xi_2 \rangle) > \lambda\varphi_\varepsilon(h_K(u) - \langle u, \xi_1 \rangle) + (1-\lambda)\varphi_\varepsilon(h_K(u) - \langle u, \xi_2 \rangle).$$

We deduce from (11) that

$$\Phi_\varepsilon(K, \lambda\xi_1 + (1-\lambda)\xi_2) > \lambda\Phi_\varepsilon(K, \xi_1) + (1-\lambda)\Phi_\varepsilon(K, \xi_2),$$

thus $\Phi_\varepsilon(K, \xi)$ is a strictly concave function of $\xi \in \operatorname{int} K$.

Let $\xi_m \in \operatorname{int} K$ such that

$$\lim_{m \to \infty} \Phi_\varepsilon(K, \xi_m) = \sup_{\xi \in \operatorname{int} K} \Phi_\varepsilon(K, \xi).$$

We may assume that $\lim_{m \to \infty} \xi_m = z_0 \in K$, and Lemma 3.1 yields $z_0 \in \operatorname{int} K$. Since $\Phi_\varepsilon(K, \xi)$ is a strictly concave function of $\xi \in \operatorname{int} K$, we conclude Proposition 3.2. Q.E.D.

Since $\xi \mapsto \Phi_\varepsilon(K, \xi)$ is maximal at $\xi(K) \in \operatorname{int} K$, we deduce

Corollary 3.3. *For $\varepsilon \in (0, \frac{1}{3})$ and a convex body K in $\mathbb{R}^n$, we have*

$$\int_{\mathbb{S}^{n-1}} u \; \varphi_\varepsilon'\Big(h_K(u) - \langle u, \xi(K) \rangle\Big) \, d\mu(u) = o.$$

An essential property of $\xi(K)$ is its continuity with respect to K.

Lemma 3.4. *For $\varepsilon \in (0, \frac{1}{3})$, both $\xi(K)$ and $\Phi_\varepsilon(K, \xi(K))$ are continuous functions of the convex body K in $\mathbb{R}^n$.*

Proof. Let $\{K_m\}$ be a sequence convex bodies tending to a convex body K in $\mathbb{R}^n$. We may assume that $\lim_{m \to \infty} \xi(K_m) = z_0 \in K$. There exists $r > 0$ such that $\xi(K) + 2r\, B^n \subset K$, and hence we may also assume that $\xi(K) + r\, B^n \subset K_m$ for all m. Thus

$$\Phi_\varepsilon(K_m, \xi(K_m)) \geq \Phi_\varepsilon(K_m, \xi(K)) \geq \Phi_\varepsilon(\xi(K) + r\, B^n, \xi(K)),$$

and in turn Lemma 3.1 yields that $z_0 \in \operatorname{int} K$. It follows that $\varphi_\varepsilon(h_{K_m}(u) - \langle u, \xi(K_m) \rangle)$ tends uniformly to $\varphi_\varepsilon(h_K(u) - \langle u, z_0 \rangle)$. In particular,

$$\Phi_\varepsilon(K, z_0) = \lim_{m \to \infty} \Phi_\varepsilon(K_m, \xi(K_m)) \geq \limsup_{m \to \infty} \Phi_\varepsilon(K_m, \xi(K)) = \Phi_\varepsilon(K, \xi(K)).$$

Since $\xi(K)$ is the unique maximum point of $\xi \mapsto \Phi_\varepsilon(K, \xi)$ on $\operatorname{int} K$ according to Proposition 3.2, we have $z_0 = \xi(K)$. In turn, we conclude Lemma 3.4. Q.E.D.

The next lemma shows that if we perturb a convex body K in a differentiable way, then $\xi(K)$ changes also in a differentiable way.

Lemma 3.5. *For $\varepsilon \in (0, \frac{1}{3})$, let $c > 0$ and $t_0 > 0$, and let K_t be a family of convex bodies with support function h_t for $t \in [0, t_0)$. Assume that*

(1) $|h_t(u) - h_0(u)| \leq ct$ for each $u \in \mathbb{S}^{n-1}$ and $t \in [0, t_0)$,

(2) $\lim_{t \to 0^+} \frac{h_t(u) - h_0(u)}{t}$ exists for $\mathcal{H}^{n-1}$-almost all $u \in \mathbb{S}^{n-1}$.

Then $\lim_{t \to 0^+} \frac{\xi(K_t) - \xi(K_0)}{t}$ exists.

Proof. We may assume that $\xi(K_0) = o$. Since $\xi(K) \in \operatorname{int} K$ is the unique maximizer of $\xi \mapsto \Phi_\varepsilon(K, \xi)$, we deduce that

$$\lim_{t \to 0^+} \xi(K_t) = o.$$

Let $g(t,u)=h_t(u)-h_0(u)$ for $u\in\mathbb{S}^{n-1}$ and $t\in[0,t_0)$. In particular, there exists constant $\gamma>0$ such that if $u\in\mathbb{S}^{n-1}$ and $t\in[0,t_0)$, then

$$\varphi_\varepsilon'(h_t(u)-\langle u,\xi(K_t)\rangle)=\varphi_\varepsilon'(h_0(u))+\varphi_\varepsilon''(h_0(u))\,\big(g(t,u)-\langle u,\xi(K_t)\rangle\big)+e(t,u)$$

where, setting $\gamma_1=2\gamma c^2$ and $\gamma_2=2\gamma$, we have

$$|e(t,u)|\le\gamma(g(t,u)-\langle u,\xi(K_t)\rangle)^2\le\gamma(ct+\|\xi(K_t)\|)^2\le\gamma_1t^2+\gamma_2\|\xi(K_t)\|^2.$$

In particular, $e(t,u)=e_1(t,u)+e_2(t,u)$ where

$$|e_1(t,u)|\le\gamma_1t^2 \text{ and } |e_2(t,u)|\le\gamma_2\|\xi(K_t)\|^2. \tag{14}$$

It follows from applying Corollary 3.3 to K_t and K_0 that

$$\int_{\mathbb{S}^{n-1}}u\,\Big(\varphi_\varepsilon''(h_0(u))\,\big(g(t,u)-\langle u,\xi(K_t)\rangle\big)+e(t,u)\Big)\,d\mu(u)=o,$$

which can be written as

$$\int_{\mathbb{S}^{n-1}}u\,\big(\varphi_\varepsilon''(h_0(u))\,g(t,u)+e_1(t,u)\big)\,d\mu(u)=\int_{\mathbb{S}^{n-1}}u\,\langle u,\xi(K_t)\rangle\varphi_\varepsilon''(h_0(u))\,d\mu(u)-\int_{\mathbb{S}^{n-1}}u\,e_2(t,u)\,d\mu(u).$$

Since $\varphi_\varepsilon''(s)<0$ for all $s>0$, the symmetric matrix

$$A=\int_{\mathbb{S}^{n-1}}u\otimes u\;\varphi_\varepsilon''(h_0(u))\,d\mu(u)$$

is negative definite because for any $v\in\mathbb{S}^{n-1}$, we have

$$v^TAv=\int_{\mathbb{S}^{n-1}}\langle u,v\rangle^2\;\varphi_\varepsilon''(h_0(u))\;f(u)\,d\mathcal{H}^{n-1}(u)<0.$$

In addition, A satisfies that

$$\int_{\mathbb{S}^{n-1}}u\;\langle u,\xi(K_t)\rangle\;\varphi_\varepsilon''(h_0(u))\,d\mu(u)=A\,\xi(K_t).$$

It follows from (14) that if t is small, then

$$A^{-1}\int_{\mathbb{S}^{n-1}}u\;\varphi_\varepsilon''(h_0(u))\;g(t,u)\,d\mu(u)+\psi_1(t)=\xi(K_t)-\psi_2(t), \tag{15}$$

where $\|\psi_1(t)\|\le\alpha_1t^2$ and $\|\psi_2(t)\|\le\alpha_2\|\xi(K_t)\|^2$ for constants $\alpha_1,\alpha_2>0$. Since $\xi(K_t)$ tends to o, if t is small, then $\|\xi(K_t)-\psi_2(t)\|\ge\frac12\|\xi(K_t)\|$, thus $\|\xi(K_t)\|\le\beta\,t$ for a constant $\beta>0$ by $g(t,u)\le ct$. In particular, $\|\psi_2(t)\|\le\alpha_2\beta^2t^2$. Since there exists $\lim_{t\to0^+}\frac{g(t,u)-g(0,u)}{t}=\partial_1g(0,u)$ for μ almost all $u\in\mathbb{S}^{n-1}$, and $\frac{g(t,u)-g(0,u)}{t}<c$ for all $u\in\mathbb{S}^{n-1}$ and $t>0$, we conclude that

$$\frac{d}{dt}\,\xi(K_t)\Big|_{t=0}=A^{-1}\int_{\mathbb{S}^{n-1}}u\;\varphi_\varepsilon''(h_0(u))\;\partial_1g(0,u)\,d\mu(u).$$

Q.E.D.

Corollary 3.6. *Under the conditions of Lemma 3.5, and denoting K_0 by K, we have*

$$\frac{d}{dt}\,\Phi_\varepsilon(K_t,\xi(K_t))\Big|_{t=0}=\int_{\mathbb{S}^{n-1}}\frac{\partial}{\partial t}h_{K_t}(u)\Big|_{t=0}\;\varphi_\varepsilon'\big(h_K(u)-\langle u,\xi(K)\rangle\big)\,d\mu(u).$$

Proof. We write $h(t,u)=h_{K_t}(u)$ and $\xi(t)=\xi(K_t)$; Corollary 3.3 and Lemma 3.5 yield

$$\begin{aligned}\frac{d}{dt}\,\Phi_\varepsilon(K_t,\xi(K_t))\Big|_{t=0} &= \frac{d}{dt}\int_{\mathbb{S}^{n-1}}\varphi_\varepsilon\big(h(t,u)-\langle u,\xi(t)\rangle\big)\,d\mu(u)\Big|_{t=0}\\ &= \int_{\mathbb{S}^{n-1}}\partial_1 h(0,u)\;\varphi_\varepsilon'\big(h_K(u)-\langle u,\xi(K)\rangle\big)\,d\mu(u)-\\ &\quad\int_{\mathbb{S}^{n-1}}\langle u,\xi'(0)\rangle\;\varphi_\varepsilon'\big(h_K(u)-\langle u,\xi(K)\rangle\big)\,d\mu(u)\\ &= \int_{\mathbb{S}^{n-1}}\partial_1 h(0,u)\;\varphi_\varepsilon'\big(h_K(u)-\langle u,\xi(K)\rangle\big)\,d\mu(u).\end{aligned}$$

Q.E.D.

4. The existence of the minimum convex body K^ε

Let $p\in(-n,1)$, and let $\mathcal{K}_1\subset\mathcal{K}_0^n$ be the set of convex bodies with volume one and containing the origin.

We observe that $\kappa_n^{-1/n}>\frac{1}{2}$, $\kappa_n^{-1/n}B^n\in\mathcal{K}_1$ and the diameter of $\kappa_n^{-1/n}B^n$ is $2\kappa_n^{-1/n}$. It follows from $\varphi_\varepsilon\le\varphi$ and the monotonicity of φ, that if $\varepsilon\in(0,\frac{1}{6})$, then

$$\text{(16)}\qquad \begin{aligned}\Phi_\varepsilon(\kappa_n^{-1/n}B^n,\xi(\kappa_n^{-1/n}B^n)) &\le \int_{\mathbb{S}^{n-1}}\varphi(2\kappa_n^{-1/n})d\mu=\varphi(2\kappa_n^{-1/n})\mu(\mathbb{S}^{n-1})\\ &\le \begin{cases}2^p\kappa_n^{\frac{-p}{n}}\,n\kappa_n\cdot\tau_2 & \text{if } p\in(0,1),\\ \log\big(2\kappa_n^{\frac{-1}{n}}\big)\,n\kappa_n\cdot\tau_2 & \text{if } p=0,\\ -2^p\kappa_n^{\frac{-p}{n}}\,n\kappa_n\cdot\tau_1 & \text{if } p\in(-n,0).\end{cases}\end{aligned}$$

For $K\in\mathcal{K}_1$, let $R(K)=\max\{\|x-\sigma(K)\|:\,x\in K\}$. We define the measure of the empty set to be zero. We note that if $\alpha\in(0,\frac{\pi}{2})$ and $v\in\mathbb{S}^{n-1}$, then

$$\text{(17)}\qquad \mathcal{H}^{n-1}\left(\{u\in\mathbb{S}^{n-1}:\,\langle u,v\rangle\ge\cos\alpha\}\right)\ge(\sin\alpha)^{n-1}\kappa_{n-1}.$$

Lemma 4.1. *Let $p\in[0,1)$. There exists $R_0>1$, depending on n, p, τ_1 and τ_2, such that if $K\in\mathcal{K}_1$, $R(K)>R_0$ and $\varepsilon\in(0,\frac{1}{6})$, then*

$$\Phi_\varepsilon(K,\xi(K))>\Phi_\varepsilon(\kappa_n^{-1/n}B^n,\xi(\kappa_n^{-1/n}B^n)).$$

Proof. Let $K\in\mathcal{K}_1$. We may assume $\sigma(K)=o$ and $R=R(K)>2n$. Let $v\in\mathbb{S}^{n-1}$ satisfy $Rv\in K$. It follows from Lemma 2.1 (i) that $(-R/n)v\in K$, as well.

We write c_0,c_1 to denote positive constants depending on n,p,τ_1,τ_2. We consider

$$\Xi_0=\{u\in\mathbb{S}^{n-1}:\,h_K(u)<1\},$$

and $\Xi_1=\mathbb{S}^{n-1}\backslash\Xi_0$. We observe that if $u\in\Omega(v,\frac{\pi}{3})$, then $h_K(u)\ge\langle u,Rv\rangle\ge R/2$, and in turn $\Omega(v,\frac{\pi}{3})\subset\Xi_1$. Since $\mu(\Omega(v,\frac{\pi}{3}))\ge\tau_1(\frac{\sqrt{3}}{2})^{n-1}\kappa_{n-1}$ by (17) and $\varphi_\varepsilon(t)=\varphi(t)>0$ for $t>1$, we have

$$\text{(18)}\qquad \int_\Xi\varphi_\varepsilon\circ h_K\,d\mu\ge\int_{\Omega(v,\frac{\pi}{3})}\varphi_\varepsilon\circ h_K\,d\mu\ge\tau_1\left(\frac{\sqrt{3}}{2}\right)^{n-1}\kappa_{n-1}\varphi(R/2)=c_1\varphi(R/2).$$

However, if $u\in\Xi_0$, then $|\langle u,v\rangle|<n/R$ as $1>h_K(u)\ge|\langle(R/n)v,u\rangle|$. It follows that

$$\text{(19)}\qquad \mathcal{H}^{n-1}(\Xi_0)\le(n-1)\kappa_{n-1}\cdot\frac{2n}{R}<(n-1)\kappa_{n-1}.$$

We deduce from (10), the Hölder inequality, the Blaschke-Santaló inequality Lemma 2.1 (ii) and (19) that

$$\begin{aligned}\int_{\Xi_0} \varphi_\varepsilon \circ h_K\, d\mu &\geq -\tau_2 \int_{\Xi_0} h_K^{-(n-1)}\, d\mathcal{H}^{n-1}\\ &\geq -\tau_2 \left(\int_{\Xi_0} h_K^{-n}\, d\mathcal{H}^{n-1}\right)^{\frac{n-1}{n}} \mathcal{H}^{n-1}(\Xi_0)^{\frac{1}{n}}\\ &\geq -\tau_2 (n\kappa_n^2)^{\frac{n-1}{n}}((n-1)\kappa_{n-1})^{\frac{1}{n}} = -c_0. \qquad (20)\end{aligned}$$

Writing $c(n, p, \tau_1, \tau_2)$ to denote the constant on the right hand side of (16), comparing (16), (18) and (20) yields

$$c_1\varphi(R/2) - c_0 \leq c(n, p, \tau_1, \tau_2),$$

and, in turn, the existence of R_0 as $\lim_{R\to\infty} \varphi(R/2) = \infty$ by (8). Q.E.D.

The argument in the case $p \in (-n, 0)$ is similar to the previous one, but it needs to be refined as now $\lim_{t\to\infty} \varphi(t) = 0$.

Lemma 4.2. *Let $p \in (-n, 0)$. There exists $R_0 > 1$, depending on n, p, τ_1 and τ_2, such that if $K \in \mathcal{K}_1$, $R(K) > R_0$, and $\varepsilon \in (0, \frac{1}{6})$, then*

$$\Phi_\varepsilon(K, \xi(K)) > \Phi_\varepsilon(\kappa_n^{-1/n}B^n, \xi(\kappa_n^{-1/n}B^n)).$$

Proof. Let $K \in \mathcal{K}_1$. We may assume $\sigma(K) = o$ and $R = R(K) > 4n^2$. Let $v \in \mathbb{S}^{n-1}$ satisfy $Rv \in K$. It follows from Lemma 2.1 (i) that $(-R/n)v \in K$, as well.

In this case, we divide $\mathbb{S}^{n-1}$ into three parts:

$$\begin{aligned}\Xi_0 &= \{u \in \mathbb{S}^{n-1} : h_K(u) < 1\},\\ \Xi_1 &= \{u \in \mathbb{S}^{n-1} : 1 \leq h_K(u) < \sqrt{R}\},\\ \Xi_2 &= \{u \in \mathbb{S}^{n-1} : h_K(u) \geq \sqrt{R}\}.\end{aligned}$$

If $u \in \Xi_0 \cup \Xi_1$, then

$$\sqrt{R} > h_K(u) \geq \max\{\langle u, Rv\rangle, \langle u, (-R/n)v\rangle\} \geq (R/n)|\langle u, v\rangle|.$$

Thus $|\langle u, v\rangle| \leq n/\sqrt{R}$, which in turn yields that

$$\mathcal{H}^{n-1}(\Xi_0 \cup \Xi_1) \leq \frac{4n(n-1)\kappa_{n-1}}{\sqrt{R}}. \qquad (21)$$

We write c_0, c_1, c_2 to denote positive constants depending on n, p, τ_1, τ_2. If $u \in \Xi_0$, then $\varphi_\varepsilon(h_K(u)) \geq -h_K(u)^{-q}$ according to (10), and hence we deduce from the Hölder inequality, the Blaschke-Santaló inequality Lemma 2.1 (ii) and (21) that

$$\begin{aligned}\int_{\Xi_0} \varphi_\varepsilon \circ h_K\, d\mu &\geq -\tau_2 \int_{\Xi_0} h_K^{-q}\, d\mathcal{H}^{n-1}\\ &\geq -\tau_2 \left(\int_{\Xi_0} h_K^{-n}\, d\mathcal{H}^{n-1}\right)^{\frac{q}{n}} \mathcal{H}^{n-1}(\Xi_0)^{\frac{n-q}{n}}\\ &\geq -\tau_2 (n\kappa_n^2)^{\frac{q}{n}} \left(\frac{4n(n-1)\kappa_{n-1}}{\sqrt{R}}\right)^{\frac{n-q}{n}} = -c_0 R^{-\frac{n-q}{2n}}. \qquad (22)\end{aligned}$$

Next if $u \in \Xi_1$, then $\varphi_\varepsilon(h_K(u)) = -h_K(u)^{-|p|}$, and hence we deduce from the Hölder inequality, the Blaschke-Santaló inequality Lemma 2.1 (ii) and (21) that

$$\begin{aligned}\int_{\Xi_1} \varphi_\varepsilon \circ h_K \, d\mu &\geq -\tau_2 \int_{\Xi_1} h_K^{-|p|} \, d\mathcal{H}^{n-1} \\ &\geq -\tau_2 \left(\int_{\Xi_1} h_K^{-n} \, d\mathcal{H}^{n-1} \right)^{\frac{|p|}{n}} \mathcal{H}^{n-1}(\Xi_1)^{\frac{n-|p|}{n}} \\ &\geq -\tau_2 (n\kappa_n^2)^{\frac{|p|}{n}} \left(\frac{4n(n-1)\kappa_{n-1}}{\sqrt{R}} \right)^{\frac{n-|p|}{n}} = -c_1 R^{-\frac{n-|p|}{2n}}. \end{aligned} \tag{23}$$

Finally, if $u \in \Xi_2$, then $\varphi_\varepsilon(h_K(u)) \geq \varphi_\varepsilon(\sqrt{R})$, and hence

$$\int_{\Xi_2} \varphi_\varepsilon \circ h_K \, d\mu \geq \tau_2 n \kappa_n \cdot \varphi_\varepsilon(\sqrt{R}) = c_2 \varphi_\varepsilon(\sqrt{R}). \tag{24}$$

Writing $c(n, p, \tau_1, \tau_2) < 0$ to denote the constant on the right hand side of (16) in the case $p \in (-n, 0)$, comparing (16), (22), (23) and (24) yields

$$-c_0 R^{-\frac{n-q}{2n}} - c_1 R^{-\frac{n-|p|}{2n}} + c_2 \varphi_\varepsilon(\sqrt{R}) \leq c(n, p, \tau_1, \tau_2) < 0,$$

and in turn the existence of R_0 as $\lim_{R\to\infty} \varphi(\sqrt{R}) = 0$ by (8). Q.E.D.

We deduce from the Blaschke selection theorem and the continuity of $\Phi_\varepsilon(K, \xi(K))$ (see Lemma 3.4) the existence of the extremal body K^ε.

Corollary 4.3. *For every $\varepsilon \in (0, \frac{1}{6})$, if $R_0 > 0$ is the number depending on n, p, τ_1 and τ_2 of Lemma 4.1 and Lemma 4.2, there exists $K^\varepsilon \in \mathcal{K}_1$ with $R(K^\varepsilon) \leq R_0$, such that*

$$\Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) = \min_{K \in \mathcal{K}_1} \Phi_\varepsilon(K, \xi(K)).$$

5. K^ε is quasi-smooth

Lemma 5.1 below is essential in order to apply Lemma 3.5. For any convex body K and $\omega \subset \mathbb{S}^{n-1}$, we define

$$\nu_K^{-1}(\omega) = \{x \in \partial K : \nu_K(x) \cap \omega \neq \emptyset\}.$$

For $u \in \mathbb{S}^{n-1}$, we write $F(K, u)$ to denote the face of K with exterior unit normal u; in other words,

$$F(K, u) = \{x \in \partial K : \langle x, u \rangle = h_K(u)\}.$$

Lemma 5.1. *Let K be a convex body with $rB^n \subset \operatorname{int} K$ for $r > 0$, let $\omega \subset \mathbb{S}^{n-1}$ be closed, and let*

$$K_t = \{x \in K : \langle x, v \rangle \leq h_K(v) - t \quad \text{for every } v \in \omega\}$$

for $t \in (0, r)$. If h_t is the support function of K_t, then $\lim_{t\to 0^+} \frac{h_t(u) - h_K(u)}{t}$ exists for all $u \in \mathbb{S}^{n-1}$.

Remark Readily, $\lim_{t\to 0^+} \frac{h_t(u) - h_K(u)}{t} \leq -1$ if $u \in \omega$.

Proof. We set $X = \nu_K^{-1}(\omega)$; this is a compact set. We consider two cases: either u is an exterior unit normal at some $y \notin X$, or $F(K, u) \subset X$.

In the first case $h_t(u) = h_K(u)$ for sufficiently small t, and hence $\lim_{t\to 0} \frac{h_t(u) - h_K(u)}{t} = 0$.

Next let $F(K, u) \subset X$ for $u \in \mathbb{S}^{n-1}$, and let $z \in \operatorname{relint} F(K, u)$. We define Σ to be the support cone at z; namely,

$$\Sigma = \operatorname{cl}\{\alpha(y - z) : y \in K \text{ and } \alpha \geq 0\} = \{y \in \mathbb{R}^n : \langle y, v \rangle \leq 0 \text{ for } v \in \nu_K(z)\}.$$

For small $t > 0$, let

$$C_t = \{x \in \Sigma : \langle x, v\rangle \leq -t \ \text{ for } \ v \in \omega \cap \nu_K(z)\};$$

note that C_t is a closed convex set satisfying $K_t - z \subset C_t$, and $C_t = tC_1$. We define

$$\aleph = \sup\{\langle x, u\rangle : x \in C_1\} \leq 0,$$

and claim that for any $\tau > 0$ there exists $t_0 > 0$ depending on z, K and τ such that if $t \in (0, t_0)$, then

$$(\aleph - \tau)t \leq h_t(u) - h_K(u) \leq \aleph t. \tag{25}$$

To prove (25), we may assume that $z = o$, and hence $h_K(v) = 0$ for all $v \in \nu_K(z)$. For the upper bound in (25), we observe that $K_t \subset C_t$, and hence

$$h_t(u) - h_K(u) = h_t(u) \leq \sup\{\langle x, u\rangle : x \in C_t\} = \aleph t.$$

For the lower bound, let $y_\tau \in \operatorname{int} C_1$ be such that

$$\langle y_\tau, u\rangle > \aleph - \tau.$$

Since $\omega \cap \nu_K(o)$ is compact, there exists $\delta > 0$ such that

$$\langle y_\tau, v\rangle \leq -1 - \delta \ \text{ for } v \in \omega \cap \nu_K(o).$$

Moreover, $y_\tau \in \operatorname{int} \Sigma$ yields the existence of $t_1 > 0$ such that $ty_\tau \in K$ if $t \in (0, t_1]$.

We also need one more constant reflecting the boundary structure of K near o. Recall that $h_K(w) \geq 0$ for all $w \in \mathbb{S}^{n-1}$, and $h_K(w) = 0$ if and only if $w \in \nu_K(o)$. Since ω is compact, there exists $\gamma > 0$ such that

$$\text{if } w \in \omega \text{ and } \|w - v\| \geq \delta/\|y_\tau\| \text{ for all } v \in \omega \cap \nu_K(o), \text{ then } h_K(w) \geq \gamma.$$

We finally define $t_0 \in (0, t_1]$ by the condition $t_0\|y_\tau\| + t_0 < \gamma$.

Let $t \in (0, t_0)$, and hence $ty_\tau \in K$. If $w \in \omega$ satisfies $\|w - v\| \geq \delta/\|y_\tau\|$ for all $v \in \omega \cap \nu_K(o)$, then

$$\langle ty_\tau, w\rangle \leq t_0\|y_\tau\| < \gamma - t_0 < h_K(w) - t.$$

However, if $w \in \omega$ and there exists $v \in \omega \cap \nu_K(o)$ satisfying $\|w - v\| < \delta/\|y_\tau\|$, then

$$\langle ty_\tau, w\rangle = \langle ty_\tau, w - v\rangle + \langle ty_\tau, v\rangle \leq t\delta + t(-1 - \delta) = -t \leq h_K(w) - t.$$

We deduce that $ty_\tau \in K_t$, thus

$$h_t(u) - h_K(u) \geq \langle ty_\tau, u\rangle \geq (\aleph - \tau)t,$$

concluding the proof of (25).

In turn, (25) yields that $\lim_{t\to 0^+} \frac{h_t(u) - h_K(u)}{t} = \aleph$. Q.E.D.

A crucial fact for us is Alexandrov's Lemma 5.2 (see Lemma 7.5.3 in [78]). To state this, let $g : (-r, r) \times \mathbb{S}^{n-1} \to \mathbb{R}$, $r > 0$, verify

- $g(0, u) = h_K(u)$ for a convex body K;
- for every $u \in \mathbb{S}^{n-1}$ the limit $\lim_{t\to 0} \frac{g(t,u) - g(0,u)}{t} = \partial_1 g(0, u)$ exists (finite) and the convergence is uniform with respect to $u \in \mathbb{S}^{n-1}$; moreover $\partial_1 g(0, u)$ is continuous with respect to $u \in \mathbb{S}^{n-1}$;
- $K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq g(t, u) \ \text{ for any } u \in \mathbb{S}^{n-1}\}$ is a convex body for $t \in (-r, r)$.

Lemma 5.2 (Alexandrov)**.** *In the notation introduced above, we have*

$$\lim_{t\to 0} \frac{V(K_t) - V(K)}{t} = \int_{\mathbb{S}^{n-1}} \partial_1 g(0, u)\, dS_K(u).$$

Next we present a way to improve on $\Phi_\varepsilon(K, \xi(K))$ while staying in the family $\mathcal{K}_1$.

Proposition 5.3. *If for $K\in\mathcal{K}_1$ there exists a closed set $\omega\subset\mathbb{S}^{n-1}$ with $\mathcal{H}^{n-1}(\omega)>0$, such that $S_K(\omega)=0$, then there exists a convex body $\widetilde{K}\in\mathcal{K}_1$ such that $\Phi_\varepsilon(\widetilde{K},\xi(\widetilde{K}))<\Phi_\varepsilon(K,\xi(K))$.*

Proof. For small $t\geq 0$, we consider

$$K_t=\{x\in K:\,\langle x,u\rangle\leq h_K(u)-t\quad\text{for }u\in\omega\},$$

and

$$\widetilde{K}_t=V(K_t)^{-1/n}K_t\in\mathcal{K}_1.$$

We define $\alpha(t)=V(K_t)^{-1/n}$, so that in particular $\alpha(0)=1$. We claim that

$$\alpha'(0)=0. \tag{26}$$

Since α is monotone decreasing, it is equivalent to prove that if $\eta\in(0,1)$, then

$$\liminf_{t\to 0^+}\frac{V(K_t)-V(K)}{t}\geq-\eta. \tag{27}$$

Since $S_K(\omega)=0$ and ω is closed, we can choose a continuous function $\psi:\,\mathbb{S}^{n-1}\to[0,1]$ such that $\psi(u)=1$ if $u\in\omega$, and

$$\int_{\mathbb{S}^{n-1}}\psi\,dS_K\leq\eta.$$

For small $t>0$, we consider $\gamma_t=h_K-t\psi$ and

$$K_{\psi,t}=\{x\in K:\,\langle x,u\rangle\leq\gamma_t(u)\quad\text{for }u\in\omega\},$$

and hence $K_{\psi,t}\subset K_t$. Using Lemma 5.2, we deduce that

$$\liminf_{t\to 0^+}\frac{V(K_t)-V(K)}{t}\geq\frac{d}{dt}V(K_{\psi,t})\Big|_{t=0^+}=-\int_{\mathbb{S}^{n-1}}\psi\,dS_K\geq-\eta.$$

We conclude (27), and in turn (26).

We set $h(t,u)=h_{K_t}(u)$. As

$$K_{0,t}=\{x\in K:\,x+tB^n\subset K\}\subset K_t,$$

Lemma 2.3 (i), with $C=B^n$, yields that there is $c>0$ such that if $t>0$ is small, then

$$-ct\leq h_{K_{0,t}}(u)-h_K(u)\leq h(t,u)-h(0,u)\leq 0$$

for any $u\in\mathbb{S}^{n-1}$. In addition, we deduce from Lemma 5.1 that $\lim_{t\to 0^+}\frac{h(t,u)-h(0,u)}{t}=\partial_1 h(0,u)\leq 0$ exists for any $u\in\mathbb{S}^{n-1}$ where $\partial_1 h(0,u)\leq -1$ for $u\in\omega$ by definition. Next let $\tilde{h}(t,u)=\alpha(t)h(t,u)=h_{\widetilde{K}_t}(u)$ for $u\in\mathbb{S}^{n-1}$ and small $t>0$. Therefore there exists $\tilde{c}>0$ such that if $t>0$ is small, then $|\tilde{h}(t,u)-\tilde{h}(0,u)|\leq\tilde{c}t$ for any $u\in\mathbb{S}^{n-1}$, and $\alpha(0)=1$ and (26) implies that

$$\lim_{t\to 0^+}\frac{\tilde{h}(t,u)-\tilde{h}(0,u)}{t}=\partial_1\tilde{h}(0,u)=\partial_1 h(0,u)\leq 0$$

exists for any $u\in\mathbb{S}^{n-1}$, where $\partial_1\tilde{h}(0,u)\leq -1$ for $u\in\omega$. We may assume that $\xi(K)=o$ and $K\subset RB^n$ for $R>0$ where $K=\tilde{K}_0$. As φ'_ε is positive and monotone decreasing, $\mathcal{H}^{n-1}(\omega)>0$ and Corollary 3.6 imply

$$\frac{d}{dt}\Phi_\varepsilon(\widetilde{K}_t,\xi(\widetilde{K}_t))\Big|_{t=0}=\int_{\mathbb{S}^{n-1}}\partial_1\tilde{h}(0,u)\cdot\varphi'_\varepsilon(h_K(u))\,d\mu(u)\leq\int_\omega(-1)\varphi'_\varepsilon(R)\,d\mu(u)<0.$$

Therefore $\Phi_\varepsilon(\widetilde{K}_t,\xi(\widetilde{K}_t))<\Phi_\varepsilon(K,\xi(K))$ for small $t>0$, which proves Lemma 5.3. Q.E.D.

Corollary 5.4. *K^ε is quasi-smooth.*

Proof. Let $\partial' K$ and Ξ_K be as in the definition of quasi-smooth body, immediately after the proof of Lemma 2.2. If $K \in \mathcal{K}_1$ is not quasi-smooth, then $\mathcal{H}^{n-1}(\mathbb{S}^{n-1} \backslash \nu_K(\partial' K)) > 0$. Now there exists a closed set $\omega \subset \mathbb{S}^{n-1} \backslash \nu_K(\partial' K)$ such that $\mathcal{H}^{n-1}(\omega) > 0$. If an exterior normal at $x \in \partial K$ lies in ω, then $x \in \Xi_K$, and hence $S_K(\omega) \leq \mathcal{H}^{n-1}(\Xi_K) = 0$. Thus Proposition 5.3 yields the existence of a convex body $\widetilde{K} \in \mathcal{K}_1$ such that $\Phi(\widetilde{K}, \xi(\widetilde{K})) < \Phi(K, \xi(K))$. We conclude that K^ε is quasi-smooth by its extremality property. Q.E.D.

6. The variational formula (to get λ_ε)

We define

$$\lambda_\varepsilon = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K^\varepsilon - \xi(K^\varepsilon)}(u) \cdot \varphi'_\varepsilon(h_{K^\varepsilon - \xi(K^\varepsilon)}(u))\, d\mu(u). \tag{28}$$

Proposition 6.1. *$\varphi'_\varepsilon(h_{K^\varepsilon}(u) - \langle \xi(K^\varepsilon), u\rangle)\, d\mu(u) = \lambda_\varepsilon\, dS_{K^\varepsilon}$ as measures on $\mathbb{S}^{n-1}$.*

Proof. To simplify the argument, we write $K = K^\varepsilon$, and assume that $\xi(K) = o$. First we claim that if C is any convex body with $o \in \mathrm{int} C$, then

$$\int_{\mathbb{S}^{n-1}} h_C \lambda_\varepsilon\, dS_K = \int_{\mathbb{S}^{n-1}} h_C(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u). \tag{29}$$

Assuming $rC \subset K$ for $r > 0$, if $t \in (-r, r)$, then we consider

$$K_t = \{x \in K : \langle x, u\rangle \leq h_K(u) + t h_C(u) \quad \text{for } u \in \mathbb{S}^{n-1}\},$$

and

$$\widetilde{K}_t = V(K_t)^{-1/n} K_t \in \mathcal{K}_1.$$

We define $\alpha(t) = V(K_t)^{-1/n}$, so that in particular $\alpha(0) = 1$. Lemma 5.2 yields that

$$\frac{d}{dt} V(K_t)\Big|_{t=0} = \int_{\mathbb{S}^{n-1}} h_C\, dS_K,$$

and hence

$$\alpha'(0) = \frac{-1}{n} \int_{\mathbb{S}^{n-1}} h_C\, dS_K. \tag{30}$$

We write $h(t, u) = h_{K_t}(u)$. Since K is quasi-smooth, Lemma 2.3 (i) and (ii) imply that there exists $c > 0$ such that if $t \in (-r, r)$, then $|h(t, u) - h(0, u)| \leq c|t|$ for any $u \in \mathbb{S}^{n-1}$, and $\lim_{t\to 0} \frac{h(t,u) - h(0,u)}{t} = h_C(u)$ exists for $\mathcal{H}^{n-1}$-a.e. $u \in \mathbb{S}^{n-1}$. Next let $\tilde{h}(t, u) = \alpha(t) h(t, u) = h_{\widetilde{K}_t}(u)$ for $u \in \mathbb{S}^{n-1}$ and $t \in (-r, r)$. From the properties of $h(t, u)$ mentioned above and (30) it follows the existence of $\tilde{c} > 0$ such that if $t \in (-r, r)$, then $|\tilde{h}(t, u) - \tilde{h}(0, u)| \leq \tilde{c}|t|$ for any $u \in \mathbb{S}^{n-1}$, and

$$\lim_{t\to 0} \frac{\tilde{h}(t, u) - \tilde{h}(0, u)}{t} = \partial_1 \tilde{h}(0, u) = \alpha'(0) h_K(u) + h_C(u)$$

for any $u \in \mathbb{S}^{n-1}$. As $\Phi(\widetilde{K}_t, \xi(\widetilde{K}_t))$ has a minimum at $t = 0$ by the extremal property of $K^\varepsilon = \widetilde{K}_0 = K$, Corollary 3.6 implies

$$\begin{aligned} 0 &= \frac{d}{dt} \Phi(\widetilde{K}_t, \xi(\widetilde{K}_t))\Big|_{t=0} = \int_{\mathbb{S}^{n-1}} \partial_1 \tilde{h}(0, u) \cdot \varphi'_\varepsilon(h_K(u))\, d\mu(u) \\ &= \int_{\mathbb{S}^{n-1}} (\alpha'(0) h_K(u) + h_C(u)) \varphi'_\varepsilon(h_K(u))\, d\mu(u) \\ &= \int_{\mathbb{S}^{n-1}} h_C(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u) - \int_{\mathbb{S}^{n-1}} h_C \lambda_\varepsilon\, dS_K, \end{aligned}$$

and in turn we deduce (29).

Since differences of support functions are dense among continuous functions on $\mathbb{S}^{n-1}$ (see e.g. [78]), we have

$$\int_{\mathbb{S}^{n-1}} g\lambda_\varepsilon \, dS_K = \int_{\mathbb{S}^{n-1}} g(u)\varphi'_\varepsilon(h_K(u))\, d\mu(u)$$

for any continuous function g on $\mathbb{S}^{n-1}$. Therefore $\lambda_\varepsilon \, dS_K = \varphi'_\varepsilon \circ h_K \, d\mu$. Q.E.D.

7. Proof of Theorem 1.7

We start recalling that, by Corollary 4.3, $K^\varepsilon \subset \sigma(K^\varepsilon) + R_0 B^n$ where $\sigma(K^\varepsilon)$ is the centroid and $R_0 > 1$ depends on n, p, τ_1 and τ_2. The following lemma is a simple consequence of Lemma 2.1 (iii) and $V(K^\varepsilon) = 1$.

Lemma 7.1. *For $r_0 = \frac{1}{(n+1)R_0^{n-1}\kappa_{n-1}}$, we have $\sigma(K^\varepsilon) + r_0 B^n \subset K^\varepsilon$.*

Next we show that λ_ε is bounded and bounded away from zero.

Lemma 7.2. *There exist $\tilde\tau_2 > \tilde\tau_1 > 0$ depending on n, p, τ_1 and τ_2 such that $\tilde\tau_1 \le \lambda_\varepsilon \le \tilde\tau_2$ if $\varepsilon < \min\{\frac{r_0}{6}, \frac{1}{6}\}$.*

Proof. We assume $\xi(K^\varepsilon) = o$. To simplify the notation, we set $K = K^\varepsilon$ and $\sigma = \sigma(K)$. Let $w \in \mathbb{S}^{n-1}$ and $\varrho \ge 0$ be such that $\sigma = \varrho w$. Since $r_0 w \in K$, if $u \in \mathbb{S}^{n-1}$ and $\langle u, w\rangle \ge \frac{1}{2}$, then $h_K(u) \ge r_0/2$. Moreover, since φ'_ε is monotone decreasing, we have $\varphi'_\varepsilon(h_K(u)) \ge \varphi'_\varepsilon(2R_0) = \varphi'(2R_0)$ for all $u \in \mathbb{S}^{n-1}$, and hence (17) yields

$$\int_{\mathbb{S}^{n-1}} h_K(u)\cdot\varphi'_\varepsilon(h_K(u))\, d\mu(u) \ge \int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle \ge \frac{1}{2}}} (r_0/2)\cdot\varphi'(2R_0)\, d\mu(u) \ge (r_0/2)\cdot\varphi'(2R_0)\tau_1\cdot(\sqrt{3}/2)^{n-1}\kappa_{n-1},$$

which in turn yields the required lower bound on λ_ε.

To have a suitable upper bound on λ_ε, the key observation is that using $\varrho \le R_0$, we deduce that if $u \in \mathbb{S}^{n-1}$ with $\langle u, w\rangle \ge -\frac{r_0}{2R_0}$ and $\varepsilon < \frac{r_0}{6}$ then

$$h_K(u) \ge \langle u, \varrho w + r_0 u\rangle \ge r_0 - \frac{r_0\varrho}{2R_0} \ge r_0/2,$$

therefore

$$\varphi'_\varepsilon(h_K(u)) \le \varphi'_\varepsilon(r_0/2) = \varphi'(r_0/2). \tag{31}$$

Another observation is that $K \subset 2R_0 B^n$ implies

$$h_K(u) < 2R_0 \quad \text{for any } u \in S^{n-1}. \tag{32}$$

It follows directly from (31) and (32) that

$$\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle \ge \frac{-r_0}{2R_0}}} h_K(u)\varphi'_\varepsilon(h_K(u))\, d\mu(u) \le (2R_0)\varphi'(r_0/2)\tau_2 n\kappa_n. \tag{33}$$

However, if $\langle u, w\rangle < \frac{-r_0}{2R_0}$ for $u \in S^{n-1}$, then $\varphi'_\varepsilon(h_{K^\varepsilon}(u))$ can be arbitrary large as $\xi(K^\varepsilon)$ can be arbitrary close to ∂K^ε if $\varepsilon > 0$ is small, and hence we transfer the problem to the case $\langle u, w\rangle \ge \frac{-r_0}{2R_0}$ using Corollary 3.3. First we claim that

$$\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle < \frac{-r_0}{2R_0}}} \varphi'_\varepsilon(h_K(u))\, d\mu(u) \le \frac{2R_0}{r_0}\cdot\varphi'(r_0/2)\tau_2 n\kappa_n. \tag{34}$$

On the one hand, first applying Corollary 3.3, and after that $\mu(\mathbb{S}^{n-1}) \le \tau_2 n\kappa_n$ and (31) imply

$$\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle < \frac{-r_0}{2R_0}}} \langle u, -w\rangle\varphi'_\varepsilon(h_K(u))\, d\mu(u) = \int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle \ge \frac{-r_0}{2R_0}}} \langle u, w\rangle\varphi'_\varepsilon(h_K(u))\, d\mu(u) \le \varphi'(r_0/2)\tau_2 n\kappa_n.$$

On the other hand, as $\langle u,w\rangle < \frac{-r_0}{2R_0}$ is equivalent to $\langle u,-w\rangle > \frac{r_0}{2R_0}$, we have

$$\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle<\frac{-r_0}{2R_0}}} \langle u,-w\rangle \varphi'_\varepsilon(h_K(u))\,d\mu(u) \geq \frac{r_0}{2R_0}\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle<\frac{-r_0}{2R_0}}} \varphi'_\varepsilon(h_K(u))\,d\mu(u),$$

and in turn deduce (34).

Now (32) and (34) yield

$$\int_{\substack{u\in\mathbb{S}^{n-1}\\ \langle u,w\rangle<\frac{-r_0}{2R_0}}} h_K(u)\varphi'_\varepsilon(h_K(u))\,d\mu(u) \leq \frac{(2R_0)^2}{r_0}\cdot\varphi'(r_0/2)\tau_2 n\kappa_n,$$

which estimate combined with (33) leads to $\lambda_\varepsilon < \frac{(2R_0)^2+2R_0}{r_0}\varphi'(r_0/2)\tau_2 n\kappa_n$. In turn, we conclude Lemma 7.2. Q.E.D.

Proof of Theorem 1.7 We assume that $\xi(K^\varepsilon) = o$ for all $\varepsilon \in (0, \min\{\frac{1}{6}, \frac{r_0}{6}\})$. It follows from Lemma 6.1 that

$$\varphi'_\varepsilon(h_{K^\varepsilon}(u))\,d\mu(u) = \lambda_\varepsilon\,dS_{K^\varepsilon} \tag{35}$$

as measures on $\mathbb{S}^{n-1}$.

Using the constants r_0, R_0 of Lemma 7.1, if ε is small then $K^\varepsilon \subset 2R_0B^n$ and K^ε contains a ball of radius r_0. According to the Blaschke selection Theorem and Lemma 7.2, there exists a sequence $\{\varepsilon_m\}$ tending to zero, $\varepsilon_m > 0$, such that K^{ε_m} tends to a convex body K_0, and $\lim_{m\to\infty}\lambda_{\varepsilon_m} = \lambda_0 > 0$. In particular, the surface area measure of K^{ε_n} tends weakly to S_{K_0}, and we may assume that

$$\lambda_{\varepsilon_m}S(K^{\varepsilon_m}) \leq (\lambda_0+1)S(K) \tag{36}$$

for all m. Here, for a convex body K, $S(K)$ denotes its surface area: $S(K) = S_K(\mathbb{S}^{n-1})$.

We claim that the closed set $X = \{u\in\mathbb{S}^{n-1} : h_{K_0}(u) = 0\}$ satisfies

$$\mu(X) = 0. \tag{37}$$

We may assume that $X \neq \emptyset$. It follows from (7) that: setting $c = |p|$ if $p \in (-n,1)\backslash\{0\}$ and $c = 1$ if $p = 0$, we have

$$\varphi'(t) = c\,t^{p-1} \quad \text{if } t \in (0,1).$$

Let $\tau \in (0,1)$. We can choose m sufficiently large such that $3\varepsilon_m < \tau$ and $|h_{K^{\varepsilon_m}}(u) - h_{K_0}(u)| < \tau$ for $u \in \mathbb{S}^{n-1}$; thus, if $0 < t < \tau$, then

$$\varphi'_{\varepsilon_m}(t) \geq \varphi'_{\varepsilon_m}(\tau) = \varphi'(\tau) = c\,\tau^{p-1}.$$

In particular, $\varphi'_{\varepsilon_m}(h_{K^{\varepsilon_m}}(u)) \geq c\,\tau^{p-1}$ holds for $u \in X$. It follows from (35) and (36) that

$$\mu(X) \leq \frac{(\lambda_0+1)S(K)}{c\,\tau^{p-1}} = \frac{(\lambda_0+1)S(K)}{c}\cdot\tau^{1-p}$$

holds for any $\tau \in (0,1)$, and in turn we conclude (37) as $1-p > 0$.

Next, for $\delta \in (0,1)$, we define the closed set

$$\Xi_\delta = \{u\in\mathbb{S}^{n-1} : h_{K_0}(u) \geq \delta\},$$

so that $\mathbb{S}^{n-1}\backslash X = \cup_{\delta\in(0,1)}\Xi_\delta$. For large m, we have $\varphi'_{\varepsilon_m}\circ h_{K^{\varepsilon_m}} = \varphi'\circ h_{K^{\varepsilon_m}}$ on Ξ_δ, and the latter sequence tends uniformly to $\varphi'\circ h_{K_0}$ on Ξ_δ. Therefore, if $g : \mathbb{S}^{n-1}\to\mathbb{R}$ is a continuous function, then (35) and the convergence of K_{ε_m} to K_0 imply

$$\int_{\Xi_\delta} g(u)\varphi'(h_{K_0}(u))\,d\mu(u) = \lambda_0\int_{\Xi_\delta} g(u)\,dS_{K_0}(u).$$

We define

$$\lambda = \begin{cases} (\lambda_0/|p|)^{\frac{1}{n-p}} & \text{if } p \in (-n,1)\backslash\{0\}, \\ \lambda_0^{\frac{1}{n-p}} & \text{if } p = 0, \end{cases}$$

and hence (7) yields

$$\int_{\Xi_\delta} g(u) h_{K_0}(u)^{p-1}\, d\mu(u) = \lambda^{n-p} \int_{\Xi_\delta} g(u)\, dS_{K_0}(u). \tag{38}$$

For any continuous $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$, $\psi(u)/h_{K_0}(u)^{p-1}$ is a continuous function on Ξ_δ that can be extended to a continuous function on $\mathbb{S}^{n-1}$. Using this function in place of g in (38), we deduce that

$$\int_{\Xi_\delta} \psi(u)\, d\mu(u) = \lambda^{n-p} \int_{\Xi_\delta} \psi(u) h_{K_0}(u)^{1-p}\, dS_{K_0}(u).$$

As this holds for all $\delta \in (0,1)$, it follows that

$$\int_{\mathbb{S}^{n-1}\backslash X} \psi(u)\, d\mu(u) = \int_{\mathbb{S}^{n-1}\backslash X} \psi(u) h_{\lambda K_0}(u)^{1-p}\, dS_{\lambda K_0}(u). \tag{39}$$

Combining (37) and (39) implies that

$$\int_{\mathbb{S}^{n-1}} \psi(u)\, d\mu(u) = \int_{\Xi_\delta} \psi(u) h_{\lambda K_0}(u)^{1-p}\, dS_{\lambda K_0}(u),$$

for any continuous function $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$, and hence $d\mu = h_M(u)^{1-p}\, dS_M(u)$ for $M = \lambda K_0$. Q.E.D.

We still need to address the case when μ is invariant under certain closed subgroup G of $O(n)$. Here the main additional difficulty is that we always have to deform the involved bodies in a G-invariant way.

Proposition 7.3. *If $-n < p < 1$ and the Borel measure μ satisfies $d\mu = f\, d\mathcal{H}^{n-1}$ where f is bounded, $\inf_{u \in \mathbb{S}^{n-1}} f(u) > 0$ and f is invariant under the closed subgroup G of $O(n)$, then there exists $M \in \mathcal{K}_0^n$ invariant under G such that $\mu = S_{M,p}$.*

To indicate the proof of Proposition 7.3, we only sketch the necessary changes in the argument leading to Theorem 1.7.

In this case, we consider the family $\mathcal{K}_1^G$ of convex bodies $K \in \mathcal{K}_1$ satisfying $AK = K$ for any $A \in G$. It follows from the uniqueness of $\xi(K)$ (see Proposition 3.2) that if $K \in \mathcal{K}_1^G$ and $A \in G$, then $A\xi(K) = \xi(K)$.

The argument for Corollary 4.3 carries over to yield the following analogue statement. For the $R_0 > 0$ depending on n, p, τ_1 and τ_2 of Lemma 4.1 and Lemma 4.2, there exists $K^\varepsilon \in \mathcal{K}_1^G$ with $R(K^\varepsilon) \le R_0$ for any $\varepsilon \in (0, \frac{1}{6})$ such that

$$\Phi_\varepsilon(K^\varepsilon, \xi(K^\varepsilon)) = \min_{K \in \mathcal{K}_1^G} \Phi_\varepsilon(K, \xi(K)).$$

Let us discuss how to prove a G invariant version of Corollary 5.4; namely, that K^ε is quasi-smooth. In this case, a more subtle modification is needed.

Lemma 7.4. *$K^\varepsilon \in \mathcal{K}_1^G$ is quasi-smooth.*

Proof. We suppose that $K = K^\varepsilon \in \mathcal{K}_1^G$ is not quasi-smooth, and seek a contradiction. We have $\mathcal{H}^{n-1}(\mathbb{S}^{n-1}\backslash \nu_K(\partial' K)) > 0$, therefore there exists a closed set $\tilde{\omega} \subset \mathbb{S}^{n-1}\backslash \nu_K(\partial' K)$ with $\mathcal{H}^{n-1}(\tilde{\omega}) > 0$. We define

$$\omega = \cup_{A \in G} A\tilde{\omega},$$

which is compact as both G and $\tilde{\omega}$ are compact. Readily, $\mathcal{H}^{n-1}(\tilde{\omega}) > 0$ and ω is G invariant. Since K is G invariant, we deduce that even $\omega \subset \mathbb{S}^{n-1}\backslash \nu_K(\partial' K)$, and hence $S_K(\omega) = 0$. Thus we can apply Lemma 5.3. We observe that the set K_t defined in Lemma 5.3 is now G invariant, and hence there exists a convex body $\widetilde{K} \in \mathcal{K}_1^G$ such that $\Phi_\varepsilon(\widetilde{K}, \xi(\widetilde{K})) < \Phi_\varepsilon(K, \xi(K))$. This contradiction with the extremality of $K = K^\varepsilon$ proves Lemma 7.4. Q.E.D.

Let us turn to the G-invariant version of Proposition 6.1.

Proposition 7.5. *$\varphi'_\varepsilon(h_{K^\varepsilon}(u) - \langle \xi(K^\varepsilon), u\rangle)\, d\mu(u) = \lambda_\varepsilon\, dS_{K^\varepsilon}$ as measures on $\mathbb{S}^{n-1}$.*

Proof. The key statement in the proof of Proposition 6.1 is (29), claiming that, if we assume $K = K^\varepsilon$ and $\xi(K) = o$, for any convex body C with $o \in \mathrm{int} C$ we have

$$\int_{\mathbb{S}^{n-1}} h_C \lambda_\varepsilon\, dS_K = \int_{\mathbb{S}^{n-1}} h_C(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u). \tag{40}$$

To prove (40), we write ϑ_G to denote the G-invariant Haar probability measure on $\mathbb{S}^{n-1}$. We define the G-invariant convex body C_0 by

$$h_{C_0} = \int_G h_{AC}\, d\vartheta_G(A).$$

Running the proof of (29), using C_0 in place of C, and observing that

$$K_t = \{x \in K : \langle x, u\rangle \le h_K(u) + t h_{C_0}(u) \quad \text{for } u \in \mathbb{S}^{n-1}\}$$

is G-invariant, we deduce that

$$\int_{\mathbb{S}^{n-1}} h_{C_0} \lambda_\varepsilon\, dS_K = \int_{\mathbb{S}^{n-1}} h_{C_0}(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u). \tag{41}$$

Therefore the G-invariance of K and μ, the Fubini theorem and (41) imply that

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} h_C \lambda_\varepsilon\, dS_K &= \int_G \int_{\mathbb{S}^{n-1}} h_{AC} \lambda_\varepsilon\, dS_K\, d\vartheta_G(A) \\
&= \int_{\mathbb{S}^{n-1}} h_{C_0} \lambda_\varepsilon\, dS_K = \int_{\mathbb{S}^{n-1}} h_{C_0}(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u) \\
&= \int_G \int_{\mathbb{S}^{n-1}} h_{AC}(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u)\, d\vartheta_G(A) \\
&= \int_{\mathbb{S}^{n-1}} h_C(u) \varphi'_\varepsilon(h_K(u))\, d\mu(u),
\end{aligned}$$

yielding (40). The rest of the proof of Proposition 6.1 carries over without any change. Q.E.D.

Having these tailored statements, the rest of the proof of Theorem 1.7 yields Proposition 7.3. The only part we do not prove here is that $o \in \mathrm{int}\, K$ when $p \le -n + 2$, which fact is verified using a simple argument by Chou and Wang [22], and is also proved as Lemma 4.1 in [6]. Q.E.D.

8. Some more simple facts needed to prove Theorems 1.3 and 1.5

In order to prove Theorems 1.3 and 1.5, we continue our study using the same notation. However we now drop the assumption (11) on f, unless explicitly stated. The following is a simple consequence of the proof of Theorem 1.7.

Lemma 8.1. *Let $p \in (-n,1)$ and μ be a measure on $\mathbb{S}^{n-1}$ with a bounded density function f with respect to $\mathcal{H}^{n-1}$, such that* $\inf f > 0$*; then there exists a convex body M with $o \in M$, $S_{M,p} = \mu$ and*

$$\int_{\mathbb{S}^{n-1}} \varphi\left(V(M)^{\frac{-1}{n}} h_{M-\sigma(M)}(u)\right) d\mu \leq \varphi(2\kappa_n^{-1/n})\mu(\mathbb{S}^{n-1}).$$

In addition, if μ is invariant under a closed subgroup G of $O(n)$, then M can be chosen to be invariant under G.

Proof. We recall that for any small $\varepsilon > 0$, $K^\varepsilon \in \mathcal{K}_1$ satisfies

$$\int_{\mathbb{S}^{n-1}} \varphi_\varepsilon \circ h_{K^\varepsilon - \xi(K^\varepsilon)}\, d\mu = \min_{K \in \mathcal{K}_1} \max_{\xi \in \operatorname{int} K} \int_{\mathbb{S}^{n-1}} \varphi_\varepsilon \circ h_{K-\xi}\, d\mu$$

where $\xi(K^\varepsilon) \in \operatorname{int} K^\varepsilon$. In addition, if μ is invariant under the closed subgroup G of $O(n)$, then K^ε can be chosen to be invariant under G, and hence $\sigma(K^\varepsilon)$ is invariant under G, as well. We deduce that (16) yields

$$\int_{\mathbb{S}^{n-1}} \varphi_\varepsilon \circ h_{K^\varepsilon - \sigma(K^\varepsilon)}\, d\mu \leq \int_{\mathbb{S}^{n-1}} \varphi_\varepsilon \circ h_{K^\varepsilon - \xi(K^\varepsilon)}\, d\mu \leq \varphi(2\kappa_n^{-1/n})\mu(\mathbb{S}^{n-1}) \tag{42}$$

for any small $\varepsilon > 0$. In the proof of Theorems 1.7 in Section 7, we have proved that there exist a sequence ε_m with $\lim_{m\to\infty} \varepsilon_m = 0$ and convex body M with $o \in M$ and $S_{M,p} = \mu$ such that K^{ε_m} tends to some $\widetilde{K} \in \mathcal{K}_1$ where $\widetilde{K} = V(M)^{\frac{-1}{n}} M$. As $\sigma(K^{\varepsilon_m})$ tends to $\sigma(\widetilde{K})$, we have that $K^{\varepsilon_m} - \sigma(K^{\varepsilon_m})$ tends to $\widetilde{K} - \sigma(\widetilde{K})$. Therefore we conclude Lemma 8.1 from $\sigma(\widetilde{K}) \in \operatorname{int} \widetilde{K}$ and (42). Q.E.D.

The following lemma bounds the inradius in terms of the L_p-surface area.

Lemma 8.2. *Let $p < 1$, and let K be a convex body in $\mathbb{R}^n$ which contains o and a ball of radius r, then*

$$S_{K,p}(\mathbb{S}^{n-1}) \geq \kappa_{n-1} r^{n-p}.$$

Proof. Let $x_0 \in \mathbb{R}^n$ be such that $x_0 + rB^n \subset K$. If $x_0 \neq o$ let $x_0 = \theta v$ for $\theta > 0$ and $v \in \mathbb{S}^{n-1}$, otherwise let v be any unit vector and let $\theta = 0$. We define a subset of ∂K as follows:

$$\Xi = \{x \in \partial K : x = y + sv \text{ for } y \in r(\operatorname{int} B^n) \cap v^\perp \text{ and } s > \theta\}.$$

Let $x \in \Xi$, with $x = y + sv$ for some $y \in r(\operatorname{int} B^n) \cap v^\perp$ and $s > \theta$, and let $\nu_K(x)$ be an outer unit normal of K at x. Since $x_0 + r\nu_K(x) \in K$ and $x_0 + y \in K$ we have

$$\langle \nu_K(x), x_0 + r\nu_K(x) - x\rangle \leq 0, \tag{43}$$

$$\langle \nu_K(x), x_0 + y - x\rangle \leq 0. \tag{44}$$

Formula (44) implies $\langle \nu_K(x), v\rangle \geq 0$, and, as a consequence,

$$\langle \nu_K(x), x_0\rangle \geq 0. \tag{45}$$

Formula (43) implies $\langle \nu_K(x), x\rangle \geq \langle \nu_K(x), x_0\rangle + r$, and, in view of (45),

$$\langle \nu_K(x), x\rangle \geq r.$$

It follows from $\mathcal{H}^{n-1}(\Xi) \geq \kappa_{n-1} r^{n-1}$ that

$$S_{K,p}(\mathbb{S}^{n-1}) \geq \int_\Xi \langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x) \geq r^{1-p}\kappa_{n-1} r^{n-1},$$

which proves Lemma 8.2. Q.E.D.

9. Proof of Theorem 1.5

We have a non-trivial measure μ on $\mathbb{S}^{n-1}$ satisfying that $d\mu = f\, d\mathcal{H}^{n-1}$ for a non-negative $L_{\frac{n}{n+p}}$ function f. For any integer $m \geq 2$, we define f_m on $\mathbb{S}^{n-1}$ as follows

$$f_m(u) = \begin{cases} m & \text{if } f(u) \geq m, \\ f(u) & \text{if } \frac{1}{m} < f(u) < m, \\ \frac{1}{m} & \text{if } f(u) \leq \frac{1}{m} \end{cases}$$

and define the measure μ_m on $\mathbb{S}^{n-1}$ by $d\mu_m = f_m\, d\mathcal{H}^{n-1}$. Since f is also in L_1 by Hölder's inequality, it follows from Lebesgue's Dominated Convergence theorem that μ_m tends weakly to μ. We choose m_0 such that

$$\frac{\mu(\mathbb{S}^{n-1})}{2} < \mu_m(\mathbb{S}^{n-1}) < 2\mu(\mathbb{S}^{n-1}) \quad \text{for } m \geq m_0. \tag{46}$$

According to Lemma 8.1, there exists a convex body K_m with $o \in K$, $S_{K_m,p} = \mu_m$ and

$$-V(K_m)^{\frac{|p|}{n}} \int_{\mathbb{S}^{n-1}} h^p_{K_m-\sigma(K_m)}\, d\mu_m = \int_{\mathbb{S}^{n-1}} -\left(V(K_m)^{\frac{-1}{n}} h_{K_m-\sigma(K_m)}\right)^p d\mu_m \tag{47}$$

$$\leq -(2\kappa_n^{-1/n})^p \mu_m(\mathbb{S}^{n-1}) \leq -\frac{(2\kappa_n^{-1/n})^p}{2} \cdot \mu(\mathbb{S}^{n-1}). \tag{48}$$

In addition, if μ is invariant under the closed subgroup G of $O(n)$, then each μ_m is invariant under G, and hence K_m can be chosen to be invariant under G.

Lemma 9.1. *$\{K_m\}$ is bounded.*

Proof. We set

$$\begin{aligned} \varrho_m &= \max\{\varrho :\ \sigma(K_m) + \varrho\, B^n \subset K_m\} \\ R_m &= \min\{\|x - \sigma(K_m)\| :\ x \in K_m\} \\ t_m &= \min\left\{\frac{1}{2}, R_m^{\frac{-1}{2n}}\right\}, \end{aligned}$$

choose $v_m \in \mathbb{S}^{n-1}$ such that $\sigma(K_m) + R_m v_m \in \partial K_m$, and define

$$\Xi_m = \{u \in \mathbb{S}^{n-1} :\ |\langle u, v_m\rangle| \leq t_m\}.$$

Lemma 8.2 and (46) imply

$$\varrho_m \leq \left(\frac{S_{K,p}(\mathbb{S}^{n-1})}{\kappa_{n-1}}\right)^{\frac{1}{n-p}} \leq \left(\frac{2\mu(\mathbb{S}^{n-1})}{\kappa_{n-1}}\right)^{\frac{1}{n-p}}.$$

Thus, by Lemma 2.1 (iii), we have

$$V(K_m) \leq (n+1)\kappa_{n-1}\varrho_m R_m^{n-1} \leq (n+1)\kappa_{n-1}\left(\frac{2\mu(\mathbb{S}^{n-1})}{\kappa_{n-1}}\right)^{\frac{1}{n-p}} R_m^{n-1} \leq c_0 R_m^{n-1} \tag{49}$$

for a $c_0 > 0$ depending on μ, n, p.

We suppose that $\{K_m\}$ is unbounded, thus there exists a subsequence $\{R_{m'}\}$ of $\{R_m\}$ tending to infinity, and seek a contradiction. We may assume that $\{v_{m'}\}$ tends to $v \in \mathbb{S}^{n-1}$. In addition, the definition of t_m yields

$$\lim_{m'\to\infty} t_{m'} = 0. \tag{50}$$

We claim that

$$\lim_{m'\to\infty}\int_{\Xi_{m'}} f^{\frac{n}{n-|p|}}\,d\mathcal{H}^{n-1}=0, \tag{51}$$

which is equivalent to show that the left hand side in (51) is at most τ for any small $\tau>0$. For $s\in(0,1)$, we set

$$\widetilde{\Xi}(s)=\{u\in\mathbb{S}^{n-1}:\,\langle u,v\rangle\le s\}.$$

Since f is in $L_{\frac{n}{n+p}}$ with respect to $\mathcal{H}^{n-1}$, there exists $\delta\in(0,\frac12)$ such that

$$\int_{\widetilde{\Xi}(2\delta)} f^{\frac{n}{n-|p|}}\,d\mathcal{H}^{n-1}<\tau. \tag{52}$$

Now if m' is large, then $t_{m'}<\delta$ by (50), and hence $\Xi_{m'}\subset\widetilde{\Xi}(2\delta)$ as $v_{m'}$ tends to v. Therefore (52) implies (51).

Next we claim

$$\lim_{m'\to\infty}V(K_{m'})^{\frac{|p|}{n}}\int_{\Xi_{m'}} h^p_{K_{m'}-\sigma(K_{m'})}\,d\mu=0. \tag{53}$$

We deduce from the Hölder inequality and the form of the Blaschke-Santaló inequality given in Lemma 2.1 (ii)

$$\begin{aligned}\int_{\Xi_{m'}} h^p_{K_{m'}-\sigma(K_{m'})}\,d\mu &= \int_{\Xi_{m'}} h^{-|p|}_{K_{m'}-\sigma(K_{m'})}f\,d\mathcal{H}^{n-1}\\ &\le \left(\int_{\Xi_{m'}} h^{-n}_{K_{m'}-\sigma(K_{m'})}\,d\mathcal{H}^{n-1}\right)^{\frac{|p|}{n}}\left(\int_{\Xi_{m'}} f^{\frac{n}{n-|p|}}\,d\mathcal{H}^{n-1}\right)^{\frac{n-|p|}{n}}\\ &\le \kappa_n^{\frac{2|p|}{n}}\,n^{\frac{|p|}{n}}\,V(K_{m'})^{\frac{-|p|}{n}}\left(\int_{\Xi_{m'}} f^{\frac{n}{n-|p|}}\,d\mathcal{H}^{n-1}\right)^{\frac{n-|p|}{n}}.\end{aligned}$$

In turn, (51) yields (53).

We also prove

$$\lim_{m'\to\infty}V(K_{m'})^{\frac{|p|}{n}}\int_{\mathbb{S}^{n-1}\backslash\Xi_{m'}} h^p_{K_{m'}-\sigma(K_{m'})}\,d\mu=0. \tag{54}$$

We observe that if $u\in\mathbb{S}^{n-1}\backslash\Xi_{m'}$, then $|\langle u,v_{m'}\rangle|>t_{m'}$. Since $\sigma(K_{m'})-\frac{R_{m'}}{n}\,v_{m'}\in K$ according to Lemma 2.1 (i), we deduce that

$$h_{K_{m'}-\sigma(K_{m'})}(u)\ge\max\left\{\left\langle u,-\frac{R_{m'}}{n}\,v_{m'}\right\rangle,\langle u,R_{m'}\,v_{m'}\rangle\right\}\ge\frac{R_{m'}t_{m'}}{n}.$$

It follows, by (49) and the definition of $t_{m'}$, that

$$V(K_{m'})^{\frac{|p|}{n}}\int_{\mathbb{S}^{n-1}\backslash\Xi_{m'}} h^p_{K_{m'}-\sigma(K_{m'})}\,d\mu\le n^{|p|}c_0^{\frac{|p|}{n}}R_{m'}^{\frac{|p|(n-1)}{n}}(R_{m'}t_{m'})^{-|p|}\mu(\mathbb{S}^{n-1})=n^{|p|}c_0^{\frac{|p|}{n}}\mu(\mathbb{S}^{n-1})R_{m'}^{\frac{-|p|}{2n}}$$

proving (54).

We deduce from (53) and (54) that

$$\lim_{m'\to\infty}V(K_{m'})^{\frac{|p|}{n}}\int_{\mathbb{S}^{n-1}} h^p_{K_{m'}-\sigma(K_{m'})}\,d\mu=0,$$

contradicting (47), and proving Lemma 9.1. Q.E.D.

Proof of Theorem 1.5. It follows from Lemma 9.1 that there is a subsequence $\{K_{m'}\}$ of $\{K_m\}$ that tends to a compact convex set K_0. Since $S_{K_{m'},p}$ tends weakly to $S_{K_0,p}$, we deduce that $\mu = S_{K_0,p}$. Since $S_{K,p}$ is the null measure when $p<1$ and K has empty interior, we deduce that $\operatorname{int} K_0 \neq \emptyset$. We note that if μ is invariant under the closed subgroup G of $O(n)$, then K_0 is invariant under G. Q.E.D.

10. Proof of Theorem 1.3 when any open hemisphere has positive measure

Let $p\in(0,1)$, and let μ be a non-trivial measure on $\mathbb{S}^{n-1}$ such that that any open hemisphere of $\mathbb{S}^{n-1}$ has positive measure. In addition, we assume that μ is invariant under the closed subgroup G of $O(n)$ (possibly G is a trivial subgroup). For a finite set Z, we write $\#Z$ to denote its cardinality.

First we construct a sequence $\{\mu_m\}$ of G invariant Borel measures weakly approximating μ. For any $u\in\mathbb{S}^{n-1}$, we write $\Gamma_u=\{Au:\ A\in G\}$ to denote its orbit. The space of orbits is $X=\mathbb{S}^{n-1}/\sim$ where $u\sim v$ if and only if $v=Au$ for some $A\in G$; let $\psi:\mathbb{S}^{n-1}\to X$ be the quotient map. Since G is compact, X is a metric space with the metric

$$d(\psi(u),\psi(v))=\min\{\angle(y,z):\ y\in\Gamma_u \ \text{ and } \ z\in\Gamma_v\}.$$

For $m\geq 2$, let $x_1,\ldots,x_k\in X$ be an $1/m$-net; namely, for any $x\in X$, there exists x_i with $d(x,x_i)\leq 1/m$. For any x_i, $i=1,\ldots,k$, we consider its Dirichlet-Voronoi cell

$$D_i=\{x\in X:\ d(x,x_i)\leq d(x,x_j) \ \text{ for } j=1,\ldots,k\},$$

and hence $d(x,x_i)\leq 1/m$ for $x\in D_i$. We set $U_0=\emptyset$ and, for $i=1,\ldots,k-1$, we define

$$U_i=\bigcup\{\psi^{-1}(D_j):\ j=1,\ldots,i\}.$$

We subdivide $\mathbb{S}^{n-1}$ into the pairwise disjoint Borel sets

$$\mathcal{D}_m=\{\psi^{-1}(D_i)\backslash U_{i-1}:\ i=1,\ldots,k\}$$

where each $\Pi\in\mathcal{D}_m$ satisfies that Π is G invariant, $\mathcal{H}^{n-1}(\Pi)>0$ and for any $u\in\Pi$, there exists $A\in G$ with $\angle(Au,z(\Pi))\leq 1/m$ for a fixed $z(\Pi)\in\Pi$ with $\psi(z(\Pi))\in\{x_1,\ldots,x_k\}$.

It is time to define the density function for μ_m by

$$f_m(u)=\frac{\mu(\Pi)}{\mathcal{H}^{n-1}(\Pi)}+\frac{1}{(\#\mathcal{D}_m)^2} \qquad \text{if } u\in\Pi \text{ and } \Pi\in\mathcal{D}_m,$$

in other words, $d\mu_m=f_m\,d\mathcal{H}^{n-1}$. It follows that each μ_m is invariant under G, each f_m is bounded with $\inf_{u\in\mathbb{S}^{n-1}}f_m(u)>0$.

Let us show that the sequence $\{\mu_m\}$ tends weakly to μ. For any continuous $g:\mathbb{S}^{n-1}\to\mathbb{R}$, we define the G invariant function $g_0:\mathbb{S}^{n-1}\to\mathbb{R}$ by

$$g_0(u)=\int_G g(Au)\,d\vartheta_G(A)$$

where ϑ_G is the invariant Haar probability measure on G. Since μ is G invariant, the Fubini theorem yields

$$\int_{\mathbb{S}^{n-1}} g\,d\mu=\int_{\mathbb{S}^{n-1}} g_0\,d\mu \ \text{ and } \ \int_{\mathbb{S}^{n-1}} g\,d\mu_m=\int_{\mathbb{S}^{n-1}} g_0\,d\mu_m$$

for $m\geq 2$. The construction of $\mathcal{D}_m$ implies that $\lim_{m\to\infty}\int_{\mathbb{S}^{n-1}} g_0\,d\mu_m=\int_{\mathbb{S}^{n-1}} g_0\,d\mu$, and hence $\{\mu_m\}$ tends weakly to μ.

We may assume that m_0 is large enough to ensure that

$$\mu_m(\mathbb{S}^{n-1})<2\mu(\mathbb{S}^{n-1}) \ \text{ for } m\geq m_0. \tag{55}$$

According to Lemma 8.1, there exists a convex body K_m with $o \in K_m$, $S_{K_m,p} = \mu_m$ and

$$V(K_m)^{\frac{-p}{n}} \int_{\mathbb{S}^{n-1}} h^p_{K_m-\sigma(K_m)}\, d\mu_m = \int_{\mathbb{S}^{n-1}} \left(V(K_m)^{\frac{-1}{n}} h_{K_m-\sigma(K_m)}\right)^p d\mu_m \leq (2\kappa_n^{-1/n})^p \mu_m(\mathbb{S}^{n-1}) \leq 2(2\kappa_n^{-1/n})^p \mu(\mathbb{S}^{n-1}). \tag{56}$$

In addition, each K_m can be chosen to be invariant under G.

Lemma 10.1. *$\{K_m\}$ is bounded.*

Proof. For $m \geq m_0$, we set

$$\begin{aligned} \varrho_m &= \max\{\varrho : \sigma(K_m) + \varrho B^n \subset K_m\} \\ R_m &= \min\{\|x - \sigma(K_m)\| : x \in K_m\}, \end{aligned}$$

and choose $v_m \in \mathbb{S}^{n-1}$ such that $\sigma(K_m)+R_m v_m \in \partial K_m$. It follows from Lemma 2.1 (iii), Lemma 8.2 and (55) that

$$V(K_m) \leq (n+1)\kappa_{n-1}\varrho_m R_m^{n-1} \leq (n+1)\kappa_{n-1}\left(\frac{2\mu(\mathbb{S}^{n-1})}{\kappa_{n-1}}\right)^{\frac{1}{n-p}} R_m^{n-1} \leq c_0 R_m^{n-1} \tag{57}$$

for a $c_0 > 0$ depending on μ, n, p.

We suppose that $\{K_m\}$ is unbounded, thus there exists a subsequence $\{R_{m'}\}$ of $\{R_m\}$ tending to infinity, and seek a contradiction. We may assume that $\{v_{m'}\}$ tends to $v \in \mathbb{S}^{n-1}$.

For $w \in \mathbb{S}^{n-1}$ and $\alpha \in (0, \frac{\pi}{2}]$, we recall that $\Omega(w, \alpha)$ is the family of all $u \in \mathbb{S}^{n-1}$ with $\angle(u, w) \leq \alpha$. Since the μ measure of the open hemisphere centered at v is positive, there exists $\delta > 0$ and $\gamma \in (0, \frac{\pi}{6})$ such that $\mu(\Omega(v, \frac{\pi}{2} - 3\gamma)) > 2\delta$. As μ_m tends to μ weakly, there exists $m_1 \geq m_0$ such that if $m' \geq m_1$, then $\mu_{m'}(\Omega(v, \frac{\pi}{2} - 2\gamma)) > \delta$ and $\angle(v_{m'}, v) < \gamma$. Therefore if $m' \geq m_1$, then

$$\mu_{m'}\left(\Omega\left(v_{m'}, \frac{\pi}{2} - \gamma\right)\right) > \delta.$$

If $u \in \Omega(v_m, \frac{\pi}{2} - \gamma)$ then $\langle u, v_m\rangle \geq \sin\gamma$. Therefore $h_{K_{m'}-\sigma(K_{m'})}(u) \geq R_{m'} \sin\gamma$ and

$$\int_{\Omega(v_{m'}, \frac{\pi}{2}-\gamma)} h^p_{K_{m'}-\sigma(K_{m'})}\, d\mu_{m'} \geq (R_{m'} \sin\gamma)^p \delta.$$

Inequality (57) yields

$$\lim_{m'\to\infty} V(K_{m'})^{\frac{-p}{n}} \int_{\mathbb{S}^{n-1}} h^p_{K_{m'}-\sigma(K_{m'})}\, d\mu_{m'} \geq \lim_{m'\to\infty} c_0^{\frac{-p}{n}} R_{m'}^{\frac{-p(n-1)}{n}} \cdot (R_{m'}\sin\gamma)^p \delta = \infty.$$

This contradicts (56), and proves Lemma 10.1. Q.E.D.

Proof of Theorem 1.3 under the assumption that $\mu(\Sigma) > 0$, for each open hemisphere Σ of $\mathbb{S}^{n-1}$. It follows from Lemma 10.1 that there is a subsequence $\{K_{m'}\}$ of $\{K_m\}$ that tends to a compact convex set K_0. Since $S_{K_{m'},p}$ tends weakly to $S_{K_0,p}$, we deduce that $\mu = S_{K_0,p}$ and $\operatorname{int} K_0 \neq \emptyset$. We note that if μ is invariant under the closed subgroup G of $O(n)$, then K_0 is invariant under G. Q.E.D.

11. Proof of Theorem 1.3 when the measure is concentrated on a closed hemisphere

Let $p \in (0, 1)$. First we show that the assumption required in Conjecture 1.2 is necessary.

Lemma 11.1. *If $p < 1$ and $K \in \mathcal{K}_0$, then* $\operatorname{supp} S_{K,p}$ *is not a pair of antipodal points.*

Proof. We suppose that $\operatorname{supp} S_{K,p} = \{w, -w\}$ for some $w \in \mathbb{S}^{n-1}$, and seek a contradiction. Since the surface area measure of any open hemi-sphere is positive, we have $o \in \partial K$. Let σ be the exterior normal cone at o; namely,

$$\sigma = \{y \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \ \forall x \in K\} = \{y \in \mathbb{R}^n : h_K(y) = 0\}.$$

It follows that $w, -w \notin \sigma$ by $p < 1$, therefore the orthogonal projection σ' of σ into $w^\perp$ does not contain the origin in its interior. We deduce from the Hanh-Banach theorem the existence of a $(n-2)$-dimensional linear subspace $L_0 \subset w^\perp$ supporting σ'. Therefore the $(n-1)$-dimensional linear subspace $L = L_0 + \mathbb{R}w$ is a supporting hyperplane to σ at o. We write L^+ to denote the open halfspace determined by L not containing σ. We have $S_K(L^+ \cap \mathbb{S}^{n-1}) > 0$ on the one hand, and $h_K(u) > 0$ if $u \in L^+ \cap \mathbb{S}^{n-1}$ on the other hand. We deduce that

$$S_{K,p}(L^+ \cap \mathbb{S}^{n-1}) = \int_{L^+ \cap \mathbb{S}^{n-1}} h_K^{1-p}\, dS_K > 0.$$

In particular, $\operatorname{supp} S_{K,p} \cap (L^+ \cap \mathbb{S}^{n-1}) \neq \emptyset$, contradicting $\operatorname{supp} S_{K,p} = \{w, -w\}$. Q.E.D.

We remark that $\operatorname{supp} S_{K,p}$ can consist of a single point, as the example of a pyramid with apex at o shows.

Now we prove a sufficient condition ensuring that a measure μ on $\mathbb{S}^{n-1}$ is an L_p-surface area measure. For any closed convex set $X \subset \mathbb{R}^n$, we write $\operatorname{relint} X$ to denote the interior of X with respect to $\operatorname{aff} X$.

Completion of the proof of Theorem 1.3. The idea is that we associate a measure μ_0 on $\mathbb{S}^{n-1}$ to μ such that the μ_0 measure of any open hemisphere is positive, construct a convex body K_0 whose L_p-surface area measure is μ_0, and then take a suitable section of K_0.

Let $C = \operatorname{pos} \operatorname{supp} \mu$ and $L = \operatorname{lin} \operatorname{supp} \mu$, and let $v_0 \in \operatorname{relint} C \cap \mathbb{S}^{n-1}$. For

$$\sigma = \{y \in L \, \langle y, v\rangle \leq 0 \text{ for } v \in C\},$$

the condition $L \neq C$ yields that $\sigma \cap L \neq \{o\}$.

We claim that $(-\sigma) \cap \operatorname{relint} C \neq \emptyset$. If it didn't hold, then the Hahn-Banach theorem applied to C and σ yields a $w \in S^{n-1} \cap L$ such that $\langle w, x\rangle \leq 0$ for $x \in C$, and $\langle w, y\rangle \geq 0$ for $y \in -\sigma$. In particular, $w \in \sigma$, and as $y = -w \in \sigma$, we have

$$-1 = \langle w, y\rangle \geq 0.$$

This contradiction proves that there exists a $v_0 \in (-\sigma) \cap \operatorname{relint} C \cap S^{n-1}$. In particular, we have

$$\langle u, v_0\rangle \geq 0 \text{ for all } u \in \operatorname{supp} \mu. \tag{58}$$

We write $\widetilde{L} = L \cap v_0^\perp$, and set $d = n - \dim \widetilde{L}$ where $1 \leq d \leq n$. We observe that $\operatorname{supp} \mu$ is contained in the half space of L bounded by $\widetilde{L}$ and containing v_0 by (58). We consider a d-dimensional regular simplex S_0 in $\widetilde{L}^\perp$ with vertices $v_0, \ldots, v_d \in \mathbb{S}^{n-1} \cap \widetilde{L}^\perp$, and the $A \in O(n)$ that acts as the identity map on $\widetilde{L}$, and satisfies $Av_i = v_{i+1}$ for $i = 0, \ldots, d-1$. We consider the cyclic group G_0 of the isometries of S_0 of order $d+1$ generated by A, and the subgroup $\widetilde{G}$ of $O(n)$ generated by G and G_0. We define the Borel measure μ_0 invariant under $\widetilde{G}$ in a way such that if $\omega \subset \mathbb{S}^{n-1}$ is Borel, then

$$\mu_0(\omega) = \sum_{i=0}^{d} \mu(A^i \omega).$$

In particular, $\operatorname{supp} \mu_0 = \cup_{i=0}^d A^i \operatorname{supp} \mu$.

We prove that for any $w \in \mathbb{S}^{n-1}$, there exists

$$u \in \operatorname{supp} \mu_0 \ \text{ such that } \ \langle w, u\rangle > 0. \tag{59}$$

Since $v_0+\ldots+v_d=0$, either there exists $i\in\{0,\ldots,d\}$ such that $\langle w,v_i\rangle>0$, or $w\in\widetilde{L}$, and hence $\langle w,v_i\rangle=0$. For $L_i=\mathrm{lin}\{v_i,\widetilde{L}\}=A^iL$, we write $w=w_i+\tilde{w}_i$ where $w_i\in L_i$ and $\tilde{w}_i\in L_i^\perp$, and hence either $\langle w_i,v_i\rangle>0$, or $w_i=w\in\widetilde{L}$, which in turn also yield that $w_i\neq 0$. Since $v_i\in\mathrm{relint}\,A^iC$, there exists $u\in A^i\mathrm{supp}\,\mu$ with $\langle w_i,u\rangle>0$, and hence $\langle w_i,u\rangle>0$. In turn, we conclude (59), therefore the μ_0 measure of any open hemisphere of $\mathbb{S}^{n-1}$ is positive.

Now the argument in Section 10 provides a convex body $K_0\in\mathcal{K}_0^n$ whose L_p-surface area is μ_0 and is invariant under $\widetilde{G}$. For $i=0,\ldots,d$, the Dirichlet-Voronoi cell of v_i is defined by

$$D(v_i)=\{x\in\mathbb{R}^n:\ \langle x,v_i\rangle\geq\langle x,v_j\rangle\ \text{ for } j=0,\ldots,d\},$$

which is a polyhedral cone with $v_i\in\mathrm{int}\,D(v_i)$. Readily, $AD(v_i)=D(v_{i+1})$ for $i=0,\ldots,d-1$ and $\mathbb{R}^n=\bigcup_{j=0}^d A^jD(v_0)$, where the sets in the union have disjoint interiors.

We define

$$K=K_0\cap D(v_0)$$

and prove that $S_p(K,\omega)=\mu(\omega)$ for each Borel set $\omega\subset\mathbb{S}^{n-1}$. Let

$$N=\bigcup_{x\in\mathrm{int}D(v_0)}\nu_K(x)=\bigcup_{x\in\mathrm{int}D(v_0)}\nu_{K_0}(x).$$

First we observe that

$$S_p(K,\omega)=S_p(K,\omega\cap N). \tag{60}$$

Indeed, if $u\notin N$ then either $u\in\nu_K(o)$ and, as a consequence, $h_K(u)=0$, or $u\in\nu_K(x)$ for some x in the intersection of $\partial D(v_0)$ and of the closure of $(\partial K)\cap\mathrm{int}D(v_0)$, an intersection whose $(n-1)$-dimensional Hausdorff measure is zero. These facts imply $S_p(K,\omega\setminus N)=0$ and (60).

Then we prove that if $u\in\mathrm{supp}\,\mu_0\setminus\tilde{L}$ and $u\in\nu_{K_0}(x)$ for some $x\in\partial K_0\setminus D(v_j)$ then

$$u\notin A^j\mathrm{supp}\,\mu. \tag{61}$$

We prove (61) for $j=0$ arguing by contradiction; the other cases can be proved similarly. Assume that $u\in\mathrm{supp}\,\mu$. Since $x\notin D(v_0)$ we have that $x\in D(v_i)\setminus D(v_0)$, for some $i\in\{1,\ldots,d\}$, that is $\langle x,v_0\rangle<\langle x,v_i\rangle$. The symmetries of K_0 imply that $x=A^iy$ for some $y\in K_0$. The inclusion $\mathrm{supp}\,\mu\subset C$ and (58) imply $u=\alpha v_0+p$ for some $\alpha>0$ and $p\in\widetilde{L}$. It follows that

$$\begin{aligned}\langle y,u\rangle &= \alpha\langle y,v_0\rangle+\langle y,p\rangle=\alpha\langle A^iy,A^iv_0\rangle+\langle A^iy,A^ip\rangle=\alpha\langle x,v_i\rangle+\langle x,p\rangle\\ &> \alpha\langle x,v_0\rangle+\langle x,p\rangle=\langle x,u\rangle.\end{aligned}$$

This contradicts the fact that u is an exterior unit normal at x to ∂K_0 and conclude the proof of (61). The previous claim easily implies

$$N\cap\mathrm{supp}\,\mu_0\subset\mathrm{supp}\,\mu\quad\text{ and }\quad\nu_{K_0}^{-1}(N\cap\mathrm{supp}\,\mu_0\setminus\tilde{L})\subset D(v_0). \tag{62}$$

Formulas (62) imply

$$S_p(K,\omega\cap N\setminus\tilde{L})=S_p(K_0,\omega\cap N\setminus\tilde{L})=\mu(\omega\cap N\setminus\tilde{L}). \tag{63}$$

On the other hand, if $u\in\tilde{L}$ then $A^i\nu_{K_0}^{-1}(u)=\nu_{K_0}^{-1}(u)$, for each i, and

$$\nu_{K_0}^{-1}(u)=\bigcup_{i=0}^d\nu_{K_0}^{-1}(u)\cap A^iD(v_0)=\bigcup_{i=0}^d A^i\Big(\nu_{K_0}^{-1}(u)\cap D(v_0)\Big)=\bigcup_{i=0}^d A^i\Big(\nu_K^{-1}(u)\Big),$$

where the sets in the last union have disjoint relative interiors. Moreover $h_{K_0}(u) = h_K(u)$. Thus

$$\begin{aligned} S_p(K, \omega \cap N \cap \tilde{L}) =& \int_{\nu_K^{-1}(\omega\cap N\cap\tilde{L})} \langle x, \nu_K(x)\rangle^{1-p} d\mathcal{H}^{n-1}(x) \\ =& \frac{1}{d+1} \int_{\nu_{K_0}^{-1}(\omega\cap N\cap\tilde{L})} \langle x, \nu_{K_0}(x)\rangle^{1-p} d\mathcal{H}^{n-1}(x) \\ =& \frac{1}{d+1}\mu_0\big(\omega \cap N \cap \tilde{L}\big) \\ =& \mu\big(\omega \cap N \cap \tilde{L}\big) \end{aligned} \tag{64}$$

Formulas (60), (63) and (64) imply that $S_p(K, \omega) = \mu(\omega)$, or in other words, that μ is the L_p-surface area measure of K. Q.E.D.

Example 11.2. *If $L \subset \mathbb{R}^n$ is a linear d-subspace with $2 \le d \le n-1$, then there exists a convex body K such that $L = \operatorname{pos}\operatorname{supp}\mu$ for the L_p-surface area measure of K. To construct such a K, we take a d-ball $B \subset L$ such that $o \in \partial B$, and the exterior unit normal v to B at o. We also consider an $(n-d+1)$-dimensional convex cone $\sigma \subset \operatorname{lin}\{L^\perp, v\}$ with $v \in \operatorname{relint}\sigma$ and $\langle v, w\rangle > 0$ for $w \in \sigma\backslash\{o\}$. We define K with the formula*

$$K = \{x \in B + L^\perp : \langle x, y\rangle \le 0 \text{ for } y \in \sigma\}.$$

12. The critical case $p = -n$

Let $K \in \mathcal{K}_0^n$ with $o \in \operatorname{int} K$ and ∂K is C^3_+, and hence

$$dS_{K,-n} = f\, d\mathcal{H}^{n-1}$$

for a C^1 function $f(u) = h_K(u)^{n+1}/\kappa(u)$ on $\mathbb{S}^{n-1}$ (see (3)), where $\kappa(u)$ is the Gaussian curvature at $x \in \partial K$ with $\nu_K(x) = u$. For basic notions in this section, we refer to Schneider [78] and Yang [87].

Let $h = h_K$, and let $\tilde{h} = h_{K^*}$ be the support function of the polar body K^*, defined as follows:

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1\, \forall y \in K\}.$$

In particular, $h_{K*}(u)^{-1}u \in \partial K$ for $u \in \mathbb{S}^{n-1}$, and both h and $\tilde{h}$ are C^2 on $\mathbb{R}^n\backslash\{o\}$. We write $\tilde{f}$ to denote the curvature function on $\mathbb{R}^n$, that is the $(-n-1)$ homogeneous function satisfying $\tilde{f}(u) = \kappa(u)^{-1}$ for $u \in \mathbb{S}^{n-1}$.

We also recall some definitions and results from [87]. Given a function $\phi : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}$, let $\nabla\phi : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}^n$ denote its gradient and $\nabla^2\phi : \mathbb{R}^n\backslash\{0\} \to S^2\mathbb{R}^n$ its Hessian, where $S^2\mathbb{R}^n$ stands for symmetric 2 tensors. Let

$$H = \frac{1}{2}h^2 : \mathbb{R}^n \to (0, \infty). \tag{65}$$

Under the assumptions above, the gradient map, $\nabla H = h\nabla h : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}^n\backslash\{o\}$, is a C^1 diffeomorphism, and, by Lemma 5.5 in [87], the following relations hold for any $\xi \in \mathbb{R}^n\backslash\{o\}$ and $x = \nabla H$:

$$h(\xi) = \tilde{h}(\nabla H(\xi)) \tag{66}$$

$$h(\xi)\nabla h(\xi) = x \tag{67}$$

$$\xi = h(\xi)\nabla\tilde{h}(\nabla H(\xi)) \tag{68}$$

$$\det \nabla^2 H(\xi) = h^{n+1}(\xi)\tilde{f}(\xi). \tag{69}$$

The homogeneous contour integral of a function $\phi : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$, with homogeneity degree $-n$, is defined as

$$\oint \phi(x)\,dx = \int_{\mathbb{S}^{n-1}} \phi(u)\,d\mathcal{H}^{n-1}(u). \tag{70}$$

The volume of K is given by

$$V(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} \tilde{h}(u)^{-n}\,du = \frac{1}{n}\oint \tilde{h}(x)^{-n}\,dx = \frac{1}{n}\oint h(\xi)f(\xi)\,d\xi. \tag{71}$$

We also use the following integration by parts and change of variables lemmas.

Lemma 12.1. *(Corollary 6.6, [87]) Given a C^1 function $\phi : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$, homogeneous of degree $-n+1$, we have, for every $j \in \{1, \dots, n\}$,*

$$\oint \partial_j\phi(x)\,dx = 0.$$

Lemma 12.2. *(Corollary 6.8, [87]) Given a C^1 function $\phi : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}$ homogeneous of degree $-n$ and a C^1 diffeomorphism $\Phi : \mathbb{R}^n\backslash\{o\} \to \mathbb{R}^n\backslash\{o\}$ homogeneous of degree 1, we have*

$$\oint \phi(x)\,dx = \oint \phi(\Phi(\xi))\,\det\nabla\Phi(\xi)\,d\xi.$$

The following is the core result leading to Proposition 1.6 where δ_{ij} stands for the usual Kronecker symbols δ.

Lemma 12.3. *Given $1 \le i, j \le n$ and $p \ne 0$,*

$$\int_{\mathbb{S}^{n-1}} u_i h^p(u)\partial_j f_p(u)\,du = -(n+p)V(K)\delta_{ij}, \tag{72}$$

where $f_p = h^{1-p}f$.

Proof. By (70) and Lemma 12.1,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} u_i\partial_j\tilde{h}(u)(\tilde{h}(u))^{-n-1}\,du &= \oint x_i\partial_j\tilde{h}(x)(\tilde{h}(x))^{-n-1}\,dx \\
&= -\frac{1}{n}\oint x_i\partial_j(\tilde{h}(x))^{-n}\,dx \\
&= \frac{1}{n}\oint \partial_j(x_i)(\tilde{h}(x))^{-n}\,dx \\
&= \frac{1}{n}\oint \delta_{ij}(\tilde{h}(x))^{-n}\,dx \\
&= V(K)\delta^i_j.
\end{aligned} \tag{73}$$

On the other hand, using the change of variable $x = \nabla H(\xi)$, it follows by Lemma 12.2, (67), (68), (69), Lemma 12.1, and (71) that

$$
\begin{aligned}
\oint x_i \partial_j \tilde{h}(x)(\tilde{h}(x))^{-n-1}\, dx &= \oint (h(\xi)\partial_i h(\xi))\xi_j h^{-n-2}(\xi) \det \nabla^2 H(\xi)\, d\xi \\
&= \oint \partial_i h(\xi)\xi_j \tilde{f}(\xi)\, d\xi \\
&= \oint (h^{p-1}\partial_i h)\xi_j h^{1-p}\tilde{f}\, d\xi \\
&= \frac{1}{p}\oint \partial_i(h^p(\xi))\xi_j(h^{1-p}\tilde{f})\, d\xi \\
&= -\frac{1}{p}\oint h^p(\xi)\partial_i(\xi_j h^{1-p}\tilde{f})\, d\xi \\
&= -\frac{1}{p}\oint \delta_{ij}h(\xi)\tilde{f}(\xi) + \xi_j h^p(\xi)\partial_i f_p(\xi)\, d\xi \\
&= -\frac{n}{p}V(K)\delta_{ij} - \frac{1}{p}\oint \xi_j h^p(\xi)\partial_i f_p(\xi)\, d\xi \\
&= -\frac{n}{p}V(K)\delta_{ij} - \frac{1}{p}\int_{\mathbb{S}^{n-1}} u_j h^p(u)\partial_i f_p(u)\, du
\end{aligned} \tag{74}
$$

The lemma now follows by (73) and (74). □

Setting $p = -n$ in Lemma 12.3, we get Proposition 1.6.

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Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, Firenze, Italy I-50134
E-mail address: gabriele.bianchi@unifi.it

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reltanoda u. 13-15, H-1053 Budapest, Hungary, and Department of Mathematics, Central European University, Nador u 9, H-1051, Budapest, Hungary
E-mail address: boroczky.karoly.j@renyi.mta.hu

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze, Viale Morgagni 67/A, Firenze, Italy I-50134
E-mail address: andrea.colesanti@unifi.it

Department of Mathematics, New York University Tandon School of Engineering, 6 Metrotech Center, Brooklyn, NY 11201 U.S.A.
E-mail address: deane.yang@nyu.edu