Local average of the hyperbolic circle problem for Fuchsian groups

by András BIRÓ

A. Rényi Institute of Mathematics, Hungarian Academy of Sciences

1053 Budapest, Reáltanoda u. 13-15., Hungary; e-mail: biroand@renyi.hu

Abstract. Let $\Gamma \subseteq PSL(2,\mathbf{R})$ be a finite volume Fuchsian group. The hyperbolic circle problem is the estimation of the number of elements of the Γ-orbit of z in a hyperbolic circle around w of radius R, where z and w are given points of the upper half plane and R is a large number. An estimate with error term $e^{\frac{2}{3}R}$ is known, and this has not been improved for any group. Recently Risager and Petridis proved that in the special case $\Gamma = PSL(2,\mathbf{Z})$ taking $z = w$ and averaging over z in a certain way the error term can be improved to $e^{\left(\frac{7}{12}+\epsilon\right)R}$. Here we show such an improvement for a general Γ, our error term is $e^{\left(\frac{5}{8}+\epsilon\right)R}$ (which is better that $e^{\frac{2}{3}R}$ but weaker than the estimate of Risager and Petridis in the case $\Gamma = PSL(2,\mathbf{Z})$). Our main tool is our generalization of the Selberg trace formula proved earlier.

1. Introduction

Let H be the open upper half plane. The elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the group $PSL(2,\mathbf{R})$ act on H by the rule $z \to (az+b)/(cz+d)$. Write

$$d\mu_z = \frac{dxdy}{y^2},$$

this is the $PSL(2,\mathbf{R})$-invariant measure on H.

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Let $\Gamma \subseteq PSL(2,\mathbf{R})$ be a finite volume Fuchsian group (see [I], p 40), i.e. Γ acts discontinuously on H and it has a fundamental domain of finite volume (with respect to the measure $d\mu_z$). Let F be a fixed fundamental domain of Γ in H (it contains exactly one point of each Γ-equivalence class of H).

For $z, w \in H$ let

$$u(z,w) = \frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w},$$

this is closely related to the hyperbolic distance $\rho(z,w)$ of z and w (see [I], (1.3)). For $X > 2$ define

$$N(z,w,X) := |\{\gamma \in \Gamma : \ 4u(\gamma z, w) + 2 \leq X\}|,$$

the condition here is equivalent to $\rho(z,w) \leq \cosh^{-1}(X/2)$, hence $N(z,w,X)$ is the number of points γz in the hyperbolic circle around w of radius $\cosh^{-1}(X/2)$. Therefore the estimation of $N(z,w,X)$ is called the hyperbolic circle (or lattice point) problem. This is a classical problem, see the Introduction of [R-P] for its history.

In order to give the main term in the asymptotic expansion of $N(z,w,X)$ as $X \to \infty$ we have to introduce Maass forms.

The hyperbolic Laplace operator is given by

$$\Delta := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

It is well-known that Δ commutes with the action of $PSL(2,\mathbf{R})$.

Let $\{u_j(z) : \ j \geq 0\}$ be a complete orthonormal system of Maass forms for Γ (the function $u_0(z)$ is constant), let $\Delta u_j = \lambda_j u_j$, where $\lambda_j = s_j(s_j - 1)$, $s_j = \frac{1}{2} + it_j$ and $\mathrm{Re}s_j = \frac{1}{2}$ or $\frac{1}{2} < s_j \leq 1$. Note that $s_j = 1$ if and only if $j = 0$, and $\frac{1}{2} < s_j < 1$ holds only for finitely many j.

We can now define

$$M(z,w,X) := \sqrt{\pi} \sum_{s_j \in \left(\frac{1}{2},1\right]} \frac{\Gamma\left(s_j - \frac{1}{2}\right)}{\Gamma(s_j+1)} u_j(z) \overline{u_j(w)} X^{s_j}.$$

It is well-known that

$$|N(z,w,X) - M(z,w,X)| = O_{z,w,\Gamma}\left(X^{\frac{2}{3}}\right),$$

see e.g. [I], Theorem 12.1. The error term here has never been improved for any group, but (as it is noted in [I]) it is conjectured that 2/3 might be lowered to any number greater than 1/2.

It was proved recently in [R-P] that in the case $\Gamma = PSL(2, \mathbf{Z})$ the error term $X^{\frac{2}{3}}$ can be improved taking a certain local average. More precisely, they proved that if f is a smooth nonnegative function which is compactly supported on F, then

$$\int_F f(z)\left(N(z,z,X) - M(z,z,X)\right) d\mu_z = O_{f,\epsilon}\left(X^{\frac{7}{12}+\epsilon}\right)$$

for any $\epsilon > 0$.

In the present paper we show that for this local average the error term $X^{\frac{2}{3}}$ can be improved in the case of any finite volume Fuchsian group Γ. In this generality we get the exponent $X^{\frac{5}{8}+\epsilon}$, which is better than $X^{\frac{2}{3}}$ but not as strong as the result of [R-P] in the special case $\Gamma = PSL(2, \mathbf{Z})$.

THEOREM 1.1. *Let f be a given smooth function on H such that it is compactly supported on F, and for $X > 2$ let*

$$N_f(X) := \int_F f(z)\, N(z,z,X)\, d\mu_z.$$

Then

$$N_f(X) = \int_F f(z)\left(\sqrt{\pi} \sum_{s_j \in \left(\frac{1}{2},1\right]} \frac{\Gamma\left(s_j - \frac{1}{2}\right)}{\Gamma(s_j+1)} X^{s_j} \left|u_j(z)\right|^2\right) d\mu_z + O_{f,\Gamma,\epsilon}\left(X^{\frac{5}{8}+\epsilon}\right)$$

for every given $\epsilon > 0$.

REMARK 1.1. As it is noted in Remark 1.3 of [R-P], the proof there is valid only for groups similar to $PSL(2, \mathbf{Z})$, as it requires strong arithmetic input not available for most groups. Our theorem is valid for any finite volume Fuchsian group. In particular, it is valid for cocompact groups.

The main tool of our proof is our generalization of the Selberg trace formula ([B1]), which is valid for every finite volume Fuchsian group Γ. Note that the operator whose trace is studied in [B1] has been used and analysed also in a series of papers by Zelditch (see [Z1], [Z2], [Z3]).

REMARK 1.2. As a very brief indication of the idea of our proof we mention that

$$N(z,z,X)=\sum_{\gamma\in\Gamma}k\left(u(z,\gamma z)\right), \tag{1.1}$$

where k is the characteristic function of the interval $[0,x]$ with a large real x (in fact $x=(X-2)/4$). We use the decomposition

$$k(v)=k^*(v)+\left(k(v)-k^*(v)\right), \tag{1.2}$$

where k^* is a certain smoothed version of k. We will estimate the contribution of k^* in (1.1) in the traditional way, using the spectral expansion of the automorphic kernel function given by k^* and estimating the Selberg-Harish-Chandra transform of k^*. However, the contribution of $k(v)-k^*(v)$ to $N_f(X)$ is estimated in a completely different way, using our generalization of the Selberg trace formula ([B1]).

REMARK 1.3. We have seen that $N(z,z,X)$ is the number of points in the Γ-orbit of z in a hyperbolic circle around z of large radius. Note that the analogous quantity in the euclidean case (if we choose in place of Γ the group of translations on the euclidean plane with vectors having integer coordinates in place of Γ) is independent of z, hence averaging in z does not help in the euclidean case, the problem there remains the same.

We mention that different kind of averages were considered by Chamizo in [C]. In particular, he proved a strong estimate for the integral with respect to z of the square of $N(z,w,X)$ for a fixed w over the whole fundamental domain in the case of a cocompact group, see Corollary 2.2.1 of [C].

REMARK 1.4. The structure of the paper is the following. In Section 2 we introduce the necessary notations. In Section 3 we give the two main lemmas (Lemmas 3.3 and 3.4) needed for the proof of the theorem. Lemma 3.3 is our main new tool, this is a consequence of our generalization of the Selberg trace formula in [B1]. Lemma 3.4 is the well-known spectral expansion of an automorphic kernel function. The proof of Theorem 1.1 is given in Section 4, using some results proved only later on special functions and automorphic functions in Sections 5 and 6, respectively.

2. Further notations

We fix a complete set A of inequivalent cusps of Γ, and we will denote its elements by a, b or c, so e.g. $\sum_a \sum_c$ or $\cup_a$ will mean that a and c run over A. We say that σ_a is a *scaling matrix* of a cusp a if $\sigma_a \infty = a$, $\sigma_a^{-1} \Gamma_a \sigma_a = B$, where Γ_a is the stability group of a in Γ, and B is the group of integer translations. The scaling matrix is determined up to composition with a translation from the right.

We also fix a complete set P of representatives of Γ-equivalence classes of the set

$$\{z \in H : \ \gamma z = z \text{ for some } id \neq \gamma \in \Gamma\} .$$

For a $p \in P$ let m_p be the order of the stability group of p in Γ.

Let

$$P(Y) = \{z = x + iy : \ 0 < x \leq 1, \ y > Y\} ,$$

and let Y_Γ be a constant (depending only on the group Γ) such that for any $Y \geq Y_\Gamma$ the cuspidal zones $F_a(Y) = \sigma_a P(Y)$ are disjoint, and the fixed fundamental domain F of Γ is partitioned into

$$F = F(Y) \cup \bigcup_a F_a(Y) ,$$

where $F(Y)$ is the central part,

$$F(Y) = F \setminus \bigcup_a F_a(Y) ,$$

and $F(Y)$ has compact closure.

For $j \geq 0$ and $a \in A$ we have the Fourier expansion

$$u_j(\sigma_a z) = \beta_{a,j}(0)\, y^{1-s_j} + \sum_{n \neq 0} \beta_{a,j}(n)\, W_{s_j}(nz) ,$$

where W is the Whittaker function.

The Fourier expansion of the Eisenstein series (as in [I], (3.20)) is given by

$$E_c(\sigma_a z, s) = \delta_{ca} y^s + \phi_{c,a}(s)\, y^{1-s} + \sum_{n \neq 0} \phi_{a,c}(n,s)\, W_s(nz) .$$

For $Y \geq Y_\Gamma$ let us define the truncated Eisenstein series (as in [I], pp 95-96) in the following way: for a given $c \in A$ and every $a \in A$ let

$$E_c^Y(z,s) = E_c(z,s) - \left(\delta_{ca}\left(\mathrm{Im}\sigma_a^{-1}z\right)^s + \phi_{c,a}(s)\left(\mathrm{Im}\sigma_a^{-1}z\right)^{1-s}\right) \text{ for } z \in F_a(Y),$$

let

$$E_c^Y(z,s) = E_c(z,s) \text{ for } z \in F(Y),$$

finally let $E_c^Y(\gamma z, s) = E_c^Y(z,s)$ for $\gamma \in \Gamma$ and $z \in F$.

For $Y \geq Y_\Gamma$ and $j \geq 0$ let us also define the truncation of u_j in the following way: for every $a \in A$ let

$$u_j^Y(z) = u_j(z) - \beta_{a,j}(0)\left(\mathrm{Im}\sigma_a^{-1}z\right)^{1-s_j} \text{ for } z \in F_a(Y),$$

let

$$u_j^Y(z) = u_j(z) \text{ for } z \in F(Y),$$

finally let $u_j^Y(\gamma z) = u_j^Y(z)$ for $\gamma \in \Gamma$ and $z \in F$.

Let $\{S_l : l \in L\}$ be the set of the poles in the half-plane $\mathrm{Re}s > \frac{1}{2}$ of the Eisenstein series for Γ. Then $\frac{1}{2} < S_l \leq 1$ for every $l \in L$, and L is a finite set. We have $\beta_{a,j}(0) = 0$ if $u_j(z)$ is not a linear combination of the residues of Eisenstein series, so if $j \geq 0$ is such that $\beta_{a,j}(0) \neq 0$ for some a, then $s_j = S_l$ for some $l \in L$. In particular, u_j^Y is the same as u_j for all but finitely many j.

The constants in the symbols O will depend on the group Γ. For a function g we will denote its jth derivative by $g^{(j)}$.

For $\lambda \leq 0$ define the special function $f_\lambda(\theta)$ in the following way: $f_\lambda(\theta)$ is the unique even solution of the differential equation

$$f^{(2)}(\theta) = \frac{\lambda}{\cos^2\theta} f(\theta), \qquad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \tag{2.1}$$

with $f_\lambda(0) = 1$. Note that this differential equation (which appeared in [B1] and also in [Hu], equations (10-(11)) is the Laplacian on functions depending only on the hyperbolic distance from the imaginary real axis, i.e. if for $z \in H$ we write $z = re^{i\left(\frac{\pi}{2}+\theta\right)}$ with $r > 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then for

$$F\left(z\right) := f_\lambda(\theta)$$

we have $\Delta F = \lambda F$.

For $\lambda \leq 0$ define also the special function $g_\lambda(r)$ $(r \in [0, \infty))$ as the unique solution of

$$g^{(2)}(r) + \frac{\cosh r}{\sinh r} g^{(1)}(r) = \lambda g(r)$$

with $g_\lambda(0) = 1$. Note that it is well-known (see e.g. [I], (1.20)) that this differential equation is the Laplacian on functions depending only on the hyperbolic distance $\rho(z, i)$ from the given point i, i.e. if for $z \in H$ we write

$$G\left(z\right) := g_\lambda\left(\rho(z, i)\right),$$

then we have $\Delta G = \lambda G$.

Note that $f_0(\theta)$ and $g_0(r)$ are the identically 1 functions.

If m is a compactly supported continuous function on $[0, \infty)$, let (see [I], (1.62))

$$g_m\left(a\right) = 2q_m\left(\frac{e^a + e^{-a} - 2}{4}\right), \text{ where } q_m(v) = \int_0^\infty \frac{m(v+\tau)}{\sqrt{\tau}} d\tau, \tag{2.2}$$

and let

$$h_m\left(r\right) = \int_{-\infty}^{\infty} g_m\left(a\right) e^{ira} da. \tag{2.3}$$

For $\gamma \in \Gamma$ denote by $[\gamma]$ the conjugacy class of γ in Γ, i.e.

$$[\gamma] = \left\{\tau^{-1}\gamma\tau : \tau \in \Gamma\right\}.$$

We use the general notation

$$(F, G) = \int_F F\left(z\right) \overline{G\left(z\right)} d\mu_z.$$

We write $F\begin{pmatrix} \alpha, \beta \\ \gamma \end{pmatrix}; z$ for the Gauss hypergeometric function. We use the notations $\Gamma\left(X \pm Y\right) = \Gamma\left(X + Y\right)\Gamma\left(X - Y\right)$.

3. Basic lemmas

Our two main results here are Lemmas 3.3 and 3.4, but first we have to prove two simple lemmas.

LEMMA 3.1. *Let $a \in A$. If Y is large enough (depending only on Γ), then for $z \in F_a(Y)$ and $\gamma \in \Gamma$ we have either*

$$u(\gamma z, z) \geq D_\Gamma Y^2 \tag{3.1}$$

with some constant $D_\Gamma > 0$ depending only on Γ, or we have $\gamma \in \Gamma_a$.

Proof. Let $z \in F_a(Y) = \sigma_a P(Y)$, then $z = \sigma_a w$ with some $w \in P(Y)$, and for $\gamma \in \Gamma$ we have

$$u(\gamma z, z) = u\left(\sigma_a^{-1}\gamma\sigma_a w, w\right). \tag{3.2}$$

Let $\sigma_a^{-1}\gamma\sigma_a = \begin{pmatrix} * & * \\ C & D \end{pmatrix}$. Assume $|C| > 0$, then $\mathrm{Im}\sigma_a^{-1}\gamma\sigma_a w = \frac{\mathrm{Im}w}{|Cw+D|^2} \leq \frac{1}{C^2\mathrm{Im}w}$. Since $\mathrm{Im}w > Y$, for large enough Y this implies

$$u\left(\sigma_a^{-1}\gamma\sigma_a w, w\right) \geq \frac{\left|Y - \frac{1}{C^2Y}\right|^2}{4Y\frac{1}{C^2Y}}. \tag{3.3}$$

Since

$$\min\left\{|C| > 0 : \begin{pmatrix} * & * \\ C & * \end{pmatrix} \in \sigma_a^{-1}\Gamma\sigma_a\right\}$$

exists (see [I], p 53), (3.2) and (3.3) imply (3.1).

Assume $C = 0$. Then $\sigma_a^{-1}\gamma\sigma_a\infty = \infty$, so $\gamma a = a$, because $\sigma_a\infty = a$, hence $\gamma \in \Gamma_a$, the lemma is proved.

LEMMA 3.2. *Assume that m is a compactly supported function with bounded variation on $[0,\infty)$. Let $a \in A$, and let Y be large enough depending on Γ and m. For $z, w \in H$ define*

$$M(z,w) := \sum_{\gamma\in\Gamma} m\left(\frac{|z-\gamma w|^2}{4\mathrm{Im}z\mathrm{Im}\gamma w}\right), \tag{3.4}$$

then for $z \in F_a(Y)$ we have

$$M(z,z) = 4\mathrm{Im}w\int_0^\infty m\left(y^2\right)dy + O_{\Gamma,m}(1),$$

where $z = \sigma_a w$.

Proof. It follows easily from Lemma 3.1 using (3.2) and $\sigma_a^{-1}\Gamma_a\sigma_a = B$ that if Y is large enough, then

$$M(z,z) = \sum_{l=-\infty}^{\infty} m\left(\frac{l^2}{4\mathrm{Im}^2 w}\right),$$

where $z = \sigma_a w$*.* Then using the inequality of Koksma (Theorem 5.1 of [K-N]) we get the lemma.

LEMMA 3.3. *Let* u *be a fixed* Γ*-automorphic eigenfunction of the Laplace operator with eigenvalue* $\lambda = s(s-1)$ *satisfying*

$$\int_F |u(z)|\, d\mu_z < \infty, \tag{3.5}$$

and let $\mathrm{Re}s = \frac{1}{2}$ *or* $\frac{1}{2} < s \le 1$*. Denote the Fourier expansion of* u *by*

$$u(\sigma_a z) = \beta_a(0)\, y^s + \tilde{\beta}_a(0)\, y^{1-s} + \sum_{n\neq 0} \beta_a(n)\, W_s(nz).$$

Introduce the notations

$$B_u = \sum_a \beta_a(0), \qquad \tilde{B}_u = \sum_a \tilde{\beta}_a(0).$$

Assume that m *is a compactly supported function with bounded variation on* $[0,\infty)$ *and*

$$\int_0^\infty \frac{m(v)}{\sqrt{v}} dv = 0. \tag{3.6}$$

Recalling the notation (3.4) we have that

$$\int_F M(z,z)u(z)d\mu_z = \Sigma_{hyp} + \Sigma_{ell} + \Sigma_{par}, \tag{3.7}$$

with the definitions

$$\Sigma_{hyp} := \sum_{\substack{[\gamma] \\ \gamma \text{ hyperbolic}}} \left(\int_{C_\gamma} u dS\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} m\left(\frac{N(\gamma) + N(\gamma)^{-1} - 2}{4\cos^2\theta}\right) f_\lambda(\theta) \frac{d\theta}{\cos^2\theta},$$

where the summation is over the hyperbolic conjugacy classes of Γ*,* $N(\gamma)$ *is the norm of (the conjugacy class of)* γ*,* C_γ *is the closed geodesic obtained by factorizing the noneuclidean* *line connecting the fixed points of* γ *by the action of the centralizer of* γ *in* Γ, $dS = \frac{|dz|}{y}$ *is the hyperbolic arc length,*

$$\Sigma_{ell} := \sum_{p\in P} \frac{2\pi}{m_p} u\left(p\right) \sum_{l=1}^{m_p-1} \int_0^{\infty} m\left(\sin^2 \frac{l\pi}{m_p} \sinh^2 r\right) g_{\lambda}(r) \sinh r dr,$$

and for $s \neq 1$ *we write*

$$\Sigma_{par} := B_u 2^{1-s} \zeta\left(1-s\right) \int_0^{\infty} \frac{m\left(v\right)}{v^{\frac{1+s}{2}}} dv + \tilde{B}_u 2^s \zeta\left(s\right) \int_0^{\infty} \frac{m\left(v\right)}{v^{\frac{2-s}{2}}} dv,$$

for $s = 1$ *we write*

$$\Sigma_{par} := \tilde{B}_u \int_0^{\infty} \frac{m\left(v\right)}{v^{\frac{1}{2}}} \log v dv$$

where ζ *is the Riemann zeta function. The left-hand side of (3.7) is absolutely convergent and* Σ_{hyp} *is a finite sum.*

Proof. This is essentially proved in [B1] in the case $s \neq 1$ and in [I] in the case $s = 1$ (since for $s = 1$ this follows from the classical Selberg trace formula), but it is not stated there exactly in this form, so we explain how it follows from [B1] and from [I].

It follows from Lemma 3.2 and condition (3.6) that $M(z, z)$ is bounded on H, hence by (3.5) the left-hand side of (3.7) is absolutely convergent. Let us write

$$M_{hyp}(z) := \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ hyperbolic}}} m\left(\frac{|z-\gamma z|^2}{4\text{Im}z\text{Im}\gamma z}\right),$$

$$M_{ell}(z) := \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ elliptic}}} m\left(\frac{|z-\gamma z|^2}{4\text{Im}z\text{Im}\gamma z}\right),$$

$$M_{par}(z) := \sum_{\substack{\gamma \in \Gamma \\ \gamma \text{ parabolic}}} m\left(\frac{|z-\gamma z|^2}{4\text{Im}z\text{Im}\gamma z}\right).$$

It is clear by Lemma 3.1 and the fact that Γ acts discontinuously on H (see p 40 of [I]) that there are only finitely many $\gamma \in \Gamma$ for which there is a $z \in H$ such that the contribution of γ to $M_{hyp}(z)$ or $M_{ell}(z)$ is nonzero. It follows then, on the one hand, that $M_{par}(z)$ is also bounded on H, and on the other hand that

$$\int_F M_{hyp}(z)u(z)d\mu_z = \sum_{\substack{[\gamma] \\ \gamma \text{ hyperbolic}}} T_\gamma, \qquad \int_F M_{ell}(z)u(z)d\mu_z = \sum_{\substack{[\gamma] \\ \gamma \text{ elliptic}}} T_\gamma$$

where the summation is over the hyperbolic and elliptic conjugacy classes of Γ, respectively, and

$$T_\gamma := \sum_{\delta\in[\gamma]} \int_F m\left(\frac{|z-\delta z|^2}{4\text{Im}z\text{Im}\delta z}\right) u(z)d\mu_z.$$

Then it follows from (3) and (4) of [B1] (and the reasoning there is valid also for the case $s=1$) that

$$\int_F M_{hyp}(z)u(z)d\mu_z = \Sigma_{hyp}, \qquad \int_F M_{ell}(z)u(z)d\mu_z = \Sigma_{ell}.$$

Since we have seen that $M_{par}(z)$ is bounded, so

$$\int_F M_{par}(z)u(z)d\mu_z = \Sigma_{par}$$

follows from Lemma 3 of [B1] in the case $s \neq 1$ and from (10.14) and (10.15) of [I] in the case $s=1$ (since in that case u is constant), taking into account that $g_m(0) = 0$ by our condition (3.6) (see (2.2)). The lemma is proved.

LEMMA 3.4. *Let m be a compactly supported continuous function on $[0,\infty)$, Assume that $h_m(r)$ (defined in (2.2) and (2.3)) is even, it is holomorphic in the strip $|\text{Im}r| \leq \frac{1}{2}+\epsilon$ and $h_m(r) = O\left((1+|r|)^{-2-\epsilon}\right)$ in this strip for some $\epsilon > 0$. Then for $z \in H$ we have (using definition (3.4)) that*

$$M(z,z) = \sum_{j=0}^{\infty} h_m(t_j)|u_j(z)|^2 + \sum_a \frac{1}{4\pi}\int_{-\infty}^{\infty} h_m(r)\left|E_a\left(z, \frac{1}{2}+ir\right)\right|^2 dr,$$

and this expression is absolutely and uniformly convergent on compact subsets of H.

Proof. This follows from Theorem 7.4 of [I].

4. Proof of the theorem

Let x be a large positive number and let $1 < d < \frac{x}{\log x}$ (say). We will choose later the parameter d optimally.

Let

$$k(y) = 1 \text{ for } 0 \le y \le x, \quad k(y) = 0 \text{ for } y > x, \tag{4.1}$$

and let k^* be a smoothed version of k, more precisely

$$k^*(y) := \int_{-\infty}^{\infty} \eta(\tau)\, k(ye^{\tau})\, d\tau, \tag{4.2}$$

where the function η will be a smooth even function satisfying $\eta(\tau) = 0$ for $|\tau| > d/x$. We define η precisely below.

But before defining η let us remark that our goal is to achieve

$$\int_0^{\infty} (k^*(u) - k(u))\, u^{-1/2} du = 0, \tag{4.3}$$

since we would like to apply Lemma 3.3 for this difference. By (4.2) we have

$$\int_0^{\infty} k^*(u) u^{-1/2} du = \int_{-\infty}^{\infty} \eta(\tau)\, e^{-\tau/2} d\tau \int_0^{\infty} k(v)\, v^{-1/2} dv, \tag{4.4}$$

and so we want to have

$$\int_{-\infty}^{\infty} \eta(\tau)\, e^{-\tau/2} d\tau = 1.$$

We will take

$$\eta(\tau) := \frac{x}{d} \eta_0 \left(\frac{x}{d} \tau \right), \tag{4.5}$$

where the function η_0 will be a smooth even function satisfying $\eta_0(\tau) = 0$ for $|\tau| > 1$. Then

$$\int_{-\infty}^{\infty} \eta(\tau)\, e^{-\tau/2} d\tau = \int_{-\infty}^{\infty} \eta_0(\tau)\, e^{-\tau \frac{d}{2x}} d\tau, \tag{4.6}$$

so we need

$$\int_{-\infty}^{\infty} \eta_0(\tau)\, e^{-\tau \frac{d}{2x}} d\tau = 1. \tag{4.7}$$

We now define η_0. First let ψ_0 be a given smooth even nonnegative function on the real line such that

$$\psi_0(\tau) = 0 \text{ for } |\tau| > 1$$

and

$$\int_{-\infty}^{\infty} \psi_0(\tau)\, d\tau = 1. \tag{4.8}$$

Then

$$I_{d,x} := \int_{-\infty}^{\infty} \psi_0(\tau)\, e^{-\tau \frac{d}{2x}}\, d\tau = 1 + O\left(\left(\frac{d}{x}\right)^2\right) \tag{4.9}$$

with implied absolute constant. If x is large enough, then clearly $1/2 < I_{d,x} < 2$. Let

$$\eta_0(\tau) = \frac{\psi_0(\tau)}{I_{d,x}} \tag{4.10}$$

for real τ, then we have (4.7). Note that η_0 slightly depends on d and x, but we do not denote it. Formulas (4.9), (4.10), (4.5) define η, and by (4.7), (4.4), (4.6) we get (4.3).
Note that by the definitions for any integer $j \geq 0$ we have that

$$\int_{-\infty}^{\infty} \left|\eta^{(j)}(\tau)\right| d\tau \ll_j \left(\frac{x}{d}\right)^j, \tag{4.11}$$

and we also have (by (4.8), (4.9), (4.10) and (4.5)) that

$$\int_{-\infty}^{\infty} \eta(\tau)\, d\tau = 1 + O\left(\left(\frac{d}{x}\right)^2\right). \tag{4.12}$$

So the smoothed version k^* of k is now defined. As it is mentioned in Remark 1.2, we will use the decomposition (1.2), and we will apply Lemma 3.4 for the first term there, we will apply Lemma 3.3 for the second term. To apply these lemmas we need estimates for the function transforms occurring in those lemmas. We give such estimates in the next three lemmas.
For simplicity introduce the abbreviations $q^* = q_{k^*}$, $g^* = g_{k^*}$ and $h^* = h_{k^*}$ (see (2.2) and (2.3)).

LEMMA 4.1. *For every integer $j \geq 2$ we have for $r \geq 1$ that*

$$|h^*(r)| \ll_j \frac{d^{3/2}}{x} \left(\frac{x}{dr}\right)^j. \tag{4.13}$$

We also have for every complex r that

$$|h^*(r)| \ll x^{\frac{1}{2}+|\mathrm{Im} r|} \log x. \tag{4.14}$$

Proof. By (2.2) and (4.1) we have

$$q_k(y) = 2\sqrt{x-y} \text{ for } 0 \le y \le x, \quad q_k(y) = 0 \text{ for } y > x. \tag{4.15}$$

It is easy to see by (2.2) and (4.2) that we have

$$q^*(v) = \int_{-\infty}^{\infty} \eta(\tau) e^{-\frac{\tau}{2}} q_k(ve^{\tau}) \, d\tau, \tag{4.16}$$

and by the substitution $e^{\mu} = ve^{\tau}$ we can also write

$$q^*(v) = \sqrt{v} \int_{-\infty}^{\infty} \eta(\mu - \log v) e^{-\frac{\mu}{2}} q_k(e^{\mu}) \, d\mu.$$

Then for any integer $j \ge 0$ we have on the one hand that

$$(q^*(v))^{(j)} = \int_{-\infty}^{\infty} \eta(\tau) e^{\left(j-\frac{1}{2}\right)\tau} q_k^{(j)}(ve^{\tau}) \, d\tau, \tag{4.17}$$

and we have on the other hand that

$$(q^*(v))^{(j)} = \sqrt{v} \int_{-\infty}^{\infty} \frac{\sum_{l=0}^{j} c_{l,j} \eta^{(l)}(\mu - \log v)}{v^j} e^{-\frac{\mu}{2}} q_k(e^{\mu}) \, d\mu \tag{4.18}$$

with some constants $c_{l,j}$.

It is clear from (4.15) and (4.16) that for $v \ge xe^{d/x}$ we have

$$q^*(v) = 0. \tag{4.19}$$

It is also clear by the same formulas that for $0 \le v \le xe^{d/x}$ we have

$$0 \le q^*(v) \ll \sqrt{x}. \tag{4.20}$$

The estimate (4.14) follows at once from (4.19), (4.20), (2.2) and (2.3).

Assume that $0 \le v \le xe^{-2d/x}$. Then for $\eta(\tau) \ne 0$ we have

$$ve^{\tau} \le ve^{d/x} \le v + \frac{x-v}{2},$$

since this latter inequality is easily seen to be equivalent to

$$2e^{d/x} - 1 \le \frac{x}{v},$$

which is true, since $\frac{x}{v} \geq e^{2d/x}$. So for any integer $j \geq 1$ we have by (4.15) that

$$\left| q_k^{(j)} \left(ve^{\tau} \right) \right| \ll_j \left(x - ve^{\tau} \right)^{\frac{1}{2}-j} \ll_j \left(x - v \right)^{\frac{1}{2}-j},$$

hence by (4.17) we get that

$$\left| \left(q^* \left(v \right) \right)^{(j)} \right| \ll_j \left(x - v \right)^{\frac{1}{2}-j} \tag{4.21}$$

for $0 \leq v \leq xe^{-2d/x}$ and $j \geq 1$.

Now let $xe^{-2d/x} \leq v \leq xe^{d/x}$. Then we use (4.18). If the integrand here is nonzero, then we must have $|\mu - \log v| \leq d/x$, so

$$xe^{-3d/x} \leq e^{\mu} \leq xe^{2d/x},$$

hence by (4.15) one has

$$q_k \left(e^{\mu} \right) \ll \sqrt{d},$$

and by the upper and lower bounds for e^{μ} and v one also has

$$x \ll e^{\mu} \ll x, \qquad x \ll v \ll x$$

Using these estimates, by (4.18) and (4.11) we get for any integer $j \geq 1$ that

$$\left| \left(q^* \left(v \right) \right)^{(j)} \right| \ll_j d^{\frac{1}{2}-j} \tag{4.22}$$

for $xe^{-2d/x} \leq v \leq xe^{d/x}$.

We see by (2.2) for every $j \geq 1$ and real a that

$$\left| \left(g^* \left(a \right) \right)^{(j)} \right| \ll_j \sum_{l=1}^{j} \left| \left(q^* \left(\sinh^2 \frac{a}{2} \right) \right)^{(l)} \right| e^{l|a|}. \tag{4.23}$$

By (2.3) we have by repeated partial integration for every $j \geq 1$ and $r \geq 1$ that

$$\left| h^* \left(r \right) \right| \ll_j \frac{1}{r^j} \int_{-\infty}^{\infty} \left| \left(g^* \left(a \right) \right)^{(j)} \right| da.$$

By (4.19), (4.21), (4.22) and (4.23) we obtain (4.13). The lemma is proved.

LEMMA 4.2. *For $\frac{1}{100} \le it \le \frac{1}{2}$ (say) we have that*

$$h^*(t) = \sqrt{\pi}\frac{\Gamma(it)\, 2^{2it+1}}{\Gamma\left(\frac{3}{2}+it\right)} x^{\frac{1}{2}+it} + O\left(x\left(\frac{d}{x}\right)^2\right) + O\left(x^{\frac{1}{2}}\right).$$

Proof. Assume first $it < \frac{1}{2}$. By (1.62') of [I] (the function $F_s(u)$ is defined by the formulas on p.26., line 7, and (B.23) of [I]) and by [G-R], p 995, 9.113 we have that

$$h^*(t) = \frac{2}{i\Gamma\left(\frac{1}{2}\pm it\right)} \int_{(\sigma)} \frac{\Gamma\left(\frac{1}{2}\pm it+S\right)\Gamma(-S)}{\Gamma(1+S)} \left(\int_0^\infty k^*(u) u^S du\right) dS,$$

where $it - \frac{1}{2} < \sigma < 0$, so by (4.1) and (4.2) we get

$$h^*(t) = \frac{2}{i\Gamma\left(\frac{1}{2}\pm it\right)} \int_{-\infty}^{\infty} \eta(\tau) \int_{(\sigma)} \frac{\Gamma\left(\frac{1}{2}\pm it+S\right)\Gamma(-S)}{\Gamma(1+S)} \frac{(x/e^\tau)^{S+1}}{S+1} dS d\tau.$$

Shifting the integration to the left we get (and observe that it is also true for $it = \frac{1}{2}$) that

$$h^*(t) = \frac{4\pi}{\Gamma\left(\frac{1}{2}+it\right)} \frac{\Gamma(2it)}{\Gamma\left(\frac{3}{2}+it\right)} \int_{-\infty}^{\infty} \eta(\tau) (x/e^\tau)^{\frac{1}{2}+it} d\tau + O\left(x^{\frac{1}{2}}\right),$$

and taking into account the properties of η and the duplication formula for the Γ-function ([I], (B.5)) we obtain the lemma.

If m is a compactly supported continuous function on $[0,\infty)$, $\lambda \le 0$ and $T > 0$, introduce the notation

$$M_{m,\lambda}(T) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} m\left(\frac{T}{\cos^2\theta}\right) f_\lambda(\theta) \frac{d\theta}{\cos^2\theta}. \tag{4.24}$$

Note that this kind of transform appeared in [B1], equation (7) and also in [Hu], equation (31).

We obviously have

$$M_{k^*,\lambda}(T) = \int_{-\infty}^{\infty} \eta(\tau)\, M_{k,\lambda}(Te^\tau)\, d\tau. \tag{4.25}$$

LEMMA 4.3. *(i) For every $T > 0$ the functions*

$$M_{k,-\frac{1}{4}-t^2}(T), \qquad M_{k^*,-\frac{1}{4}-t^2}(T)$$

are entire functions of t.

(ii) Let $0 < \delta < \frac{1}{2}$ be given. There is a constant $A_\delta > 0$ depending only on δ such that for every t satisfying $\frac{1}{4} + t^2 \geq 0$ or $|\mathrm{Re}\,(it)| \leq \frac{1}{2} - \delta$ one has the following statements with the notation $\lambda = -\frac{1}{4} - t^2$:
For $T \geq xe^{d/x}$ we have that

$$M_{k^*,\lambda}(T) = M_{k,\lambda}(T) = 0, \tag{4.26}$$

for $xe^{-2d/x} \leq T \leq xe^{d/x}$ we have that

$$|M_{k^*,\lambda}(T)| + |M_{k,\lambda}(T)| \ll_\delta (1+|t|)^{A_\delta} \left(\frac{d}{x}\right)^{1/2}, \tag{4.27}$$

and for $0 < T \leq xe^{-2d/x}$ we have that

$$M_{k^*,\lambda}(T) - M_{k,\lambda}(T) \ll_\delta \frac{(1+|t|)^{A_\delta} d^2}{T^{1/2}(x-T)^{3/2}}.$$

Proof. Note first that we can give an explicit formula for f_λ, namely

$$f_\lambda(\theta) = F\left(\begin{matrix}\frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2}\\ \frac{1}{2}\end{matrix}; -\frac{\sin^2\theta}{\cos^2\theta}\right) \tag{4.28}$$

for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\lambda = -\frac{1}{4} - t^2$. This can be proved in the following way. One has

$$\frac{1}{\pi^{\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F\left(\begin{matrix}\frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2}\\ \frac{1}{2}\end{matrix}; -\frac{\sin^2\theta}{\cos^2\theta}\right) \cos^{2s}\theta \frac{d\theta}{\cos^2\theta} = \frac{\Gamma(s-\frac{1}{4}+\frac{it}{2})\Gamma(s-\frac{1}{4}-\frac{it}{2})}{\Gamma^2(s)}$$

for $\mathrm{Re}s > \frac{1}{2}$, as one can see by the substitution $y = \frac{\sin^2\theta}{\cos^2\theta}$ and by [G-R], p 807, 7.512.10. By Lemma 11 of [B1] it follows that

$$\int_0^{\frac{\pi}{2}} \left(F\left(\begin{matrix}\frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2}\\ \frac{1}{2}\end{matrix}; -\frac{\sin^2\theta}{\cos^2\theta}\right) - f_\lambda(\theta)\right) \cos^n\theta d\theta = 0$$

for every nonnegative integer n, which easily implies (4.28).
We can see part (i) at once.
To show part (ii) note that using (4.24), (4.28) and the substitution

$$Y = \log\frac{1}{\cos^2\theta}$$

one has for $T \leq x$ that

$$M_{k,\lambda}(T) = \int_0^{\log x - \log T} F\left(\begin{matrix} \frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2} \\ \frac{1}{2} \end{matrix}; 1-e^Y\right) \frac{e^Y dY}{\sqrt{e^Y-1}}, . \tag{4.29}$$

and for $T > x$ we have $M_{k,\lambda}(T) = 0$. Then (4.26) is obvious by (4.25). We also have by (4.29) and Lemma 5.1 for $T \leq x$ that

$$M_{k,\lambda}(T) \ll_\delta (1+|t|)^{A_\delta} \left(\frac{x}{T}-1\right)^{1/2}, \tag{4.30}$$

and then (using also (4.25)) (4.27) follows.

By (4.25), (4.12) and since η is even, we have that

$$M_{k^*,\lambda}(T) - M_{k,\lambda}(T) \tag{4.31}$$

equals

$$\int_0^\infty \eta(\tau)\left(M_{k,\lambda}(Te^\tau) + M_{k,\lambda}\left(Te^{-\tau}\right) - 2M_{k,\lambda}(T)\right) d\tau + O\left(\left(\frac{d}{x}\right)^2\right) M_{k,\lambda}(T). \tag{4.32}$$

Assuming $Te^\tau \leq x$ by (4.29) we have that

$$M_{k,\lambda}(Te^\tau) + M_{k,\lambda}\left(Te^{-\tau}\right) - 2M_{k,\lambda}(T) =$$

$$= \int_{\log\frac{x}{T}}^{\log\frac{x}{T}+\tau} \left(\frac{F\left(\begin{matrix} \frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2} \\ \frac{1}{2} \end{matrix}; 1-e^Y\right) e^Y}{\sqrt{e^Y-1}} - \frac{F\left(\begin{matrix} \frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2} \\ \frac{1}{2} \end{matrix}; 1-e^{Y-\tau}\right) e^{Y-\tau}}{\sqrt{e^{Y-\tau}-1}} \right) dY.$$

For $0 < T \leq xe^{-2d/x}$ and $|\tau| \leq d/x$ we then have by the mean-value theorem and by Lemma 5.1 that

$$M_{k,\lambda}(Te^\tau) + M_{k,\lambda}\left(Te^{-\tau}\right) - 2M_{k,\lambda}(T) \ll_\delta (1+|t|)^{A_\delta} \tau^2 \left(\frac{x}{T}\right)^{1/2} \left(1+\frac{1}{\log\frac{x}{T}}\right)^{3/2}.$$

So by (4.30), (4.11) with $j = 0$ and by (4.31), (4.32), using that $\eta(\tau) = 0$ for $|\tau| > d/x$ we get for $0 < T \leq xe^{-2d/x}$ that

$$M_{k^*,\lambda}(T) - M_{k,\lambda}(T) \ll_\delta (1+|t|)^{A_\delta} \left(\frac{d}{x}\right)^2 \left(\frac{x}{T}\right)^{1/2} \left(\frac{x}{x-T}\right)^{3/2}.$$

The lemma is proved.

REMARK 4.4. We now compare (4.28) to formulas (86) and (87) of [F]. Indeed, by the notation used there we see that

$$N^0_{\frac{1}{2}+it,0}\left(\theta+\frac{\pi}{2}\right) \text{ and } N^0_{\frac{1}{2}-it,0}\left(\theta+\frac{\pi}{2}\right)$$

are solutions of our differential equation (2.1) for $-\frac{\pi}{2}<\theta<0$. Using (87) of [F] and the quadratic transformation [G-R], p 999, 9.134.1 one sees that

$$N^0_{\frac{1}{2}+it,0}\left(\theta+\frac{\pi}{2}\right)=\left(\frac{\cos^2\theta}{\sin^2\theta}\right)^{\frac{1}{4}+\frac{it}{2}} F\left(\begin{matrix}\frac{1}{4}+\frac{it}{2},\frac{3}{4}+\frac{it}{2}\\ 1+it\end{matrix};-\frac{\cos^2\theta}{\sin^2\theta}\right)$$

for $-\frac{\pi}{2}<\theta<0$. Using the same relation with $-t$ in place of t we see by [G-R], p 999, 9.132.2 that the right-hand side of (4.28) is a linear combination of $N^0_{\frac{1}{2}+it,0}\left(\theta+\frac{\pi}{2}\right)$ and $N^0_{\frac{1}{2}-it,0}\left(\theta+\frac{\pi}{2}\right)$ on the interval $-\frac{\pi}{2}<\theta<0$, hence it is also a solution of (2.1). Since it is an even function, it gives a solution of (2.1) on the whole interval $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

Continuing the proof of Theorem 1.1 let

$$m_1(v)=k^*(v),\qquad m_2(v)=k(v)-k^*(v). \tag{4.33}$$

For $m_1(v)$ we will apply Lemma 3.4, and for $m_2(v)$ we will apply Lemma 3.3.

We can see e.g. by (1.62') of [I] (the function $F_s(u)$ is defined by the formulas on p.26., line 7, and (B.23) of [I]) and by Lemma 6.2 of [B2] that the conditions of Lemma 3.4 are satisfied writing m_1 in place of m, hence for $z\in H$ we have that

$$\sum_{\gamma\in\Gamma} m_1\left(\frac{|z-\gamma z|^2}{4\mathrm{Im}z\mathrm{Im}\gamma z}\right) \tag{4.34}$$

equals

$$\sum_{j=0}^{\infty} h_{m_1}(t_j)\,|u_j(z)|^2+\sum_a \frac{1}{4\pi}\int_{-\infty}^{\infty} h_{m_1}(r)\left|E_a\left(z,\frac{1}{2}+ir\right)\right|^2 dr.$$

Then applying Lemma 4.1 (we apply (4.13) with $j=2$ for $1\le r<\frac{x}{d}$, we apply (4.13) with $j=3$ for $r\ge\frac{x}{d}$, finally we apply (4.14) for $|\mathrm{Re}r|<1$, $|\mathrm{Im}r|<\frac{1}{100}$), Lemma 4.2 and [I], Proposition 7.2, for every $z\in H$ satisfying $f(z)\ne 0$ and for every $\epsilon>0$ we get that (4.34) equals

$$\sum_{j,\,it_j>0}\sqrt{\pi}\frac{\Gamma(it_j)\,2^{2it_j+1}}{\Gamma\left(\frac{3}{2}+it_j\right)}x^{\frac{1}{2}+it_j}\,|u_j(z)|^2+O_{f,\epsilon}\left(x^{\epsilon}\left(\frac{x}{\sqrt{d}}+x^{\frac{1}{2}+\frac{1}{100}}+\frac{d^2}{x}\right)\right), \tag{4.35}$$

where we took into account that f is compactly supported on F.

Let $f(z)$ be as in the theorem, and consider the integral

$$\int_F f(z)\left(\sum_{\gamma\in\Gamma} m_2\left(\frac{|z-\gamma z|^2}{4\mathrm{Im}z\mathrm{Im}\gamma z}\right)\right)d\mu_z. \tag{4.36}$$

By (4.3) and Lemma 3.2 we see that the function in the bracket here is bounded. Then it follows from the Spectral Theorem ([I], Theorems 4.7 and 7.3) that (4.36) equals

$$\sum_{j=0}^{\infty}(f,u_j)\int_F u_j(z)\left(\sum_{\gamma\in\Gamma} m_2\left(\frac{|z-\gamma z|^2}{4\mathrm{Im}z\mathrm{Im}\gamma z}\right)\right)d\mu_z+ \tag{4.37}$$

$$+\sum_a\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(f,E_a\left(*,\frac{1}{2}+ir\right)\right)\int_F E_a\left(z,\frac{1}{2}+ir\right)\left(\sum_{\gamma\in\Gamma} m_2\left(\frac{|z-\gamma z|^2}{4\mathrm{Im}z\mathrm{Im}\gamma z}\right)\right)d\mu_z dr.$$

By (4.3) we see that the conditions of Lemma 3.3 are satisfied writing m_2 in place of m and writing $u=u_j$ or $u=E_a\left(*,\frac{1}{2}+ir\right)$. By Lemma 6.3, Lemma 5.2, (4.2), (4.11) with $j=0$, using also (6.28) of [I] and a convexity bound for the Riemann zeta function we get that after applying Lemma 3.3 the contribution of Σ_{ell} and Σ_{par} to (4.37) is $O_{f,\epsilon}\left(x^{\frac{1}{2}+\epsilon}\right)$ for every $\epsilon>0$.

Therefore, for a hyperbolic $\gamma\in\Gamma$ introducing the notation

$$T(\gamma)=\frac{N(\gamma)+N(\gamma)^{-1}-2}{4}$$

and recalling (4.24) we get that (4.36) equals

$$O_{f,\epsilon}\left(x^{\frac{1}{2}+\epsilon}\right)+\sum_{\substack{[\gamma]\\ \gamma\text{ hyperbolic}}}\left(\Sigma_1(\gamma)+\Sigma_2(\gamma)\right) \tag{4.38}$$

with the notations

$$\Sigma_1(\gamma):=\sum_{j=0}^{\infty}(f,u_j)\left(\int_{C_\gamma}u_j^{Y_\Gamma}dS\right)M_{m_2,-\frac{1}{4}-t_j^2}(T(\gamma))+$$

$$+\sum_a\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(f,E_a\left(*,\frac{1}{2}+ir\right)\right)\left(\int_{C_\gamma}E_a^{Y_\Gamma}\left(*,\frac{1}{2}+ir\right)dS\right)M_{m_2,-\frac{1}{4}-r^2}(T(\gamma))\,dr, \tag{4.39}$$

$$\Sigma_2(\gamma) := \sum_{j=0}^{\infty} (f, u_j) \left(\int_{C_\gamma} \left(u_j - u_j^{Y_\Gamma} \right) dS \right) M_{m_2, -\frac{1}{4} - t_j^2} (T(\gamma)) +$$

$$+ \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(f, E_a \left(*, \frac{1}{2} + ir \right) \right) \left(\int_{C_\gamma} D_a^{Y_\Gamma} \left(*, \frac{1}{2} + ir \right) dS \right) M_{m_2, -\frac{1}{4} - r^2} (T(\gamma))\, dr,$$

where for simplicity we wrote

$$D_a^{Y_\Gamma} \left(*, \frac{1}{2} + ir \right) := E_a \left(*, \frac{1}{2} + ir \right) - E_a^{Y_\Gamma} \left(*, \frac{1}{2} + ir \right).$$

Observe that for any $a \in A$ and any $y > 0$ we have by [I], (6.22) and (6.27) that

$$\sum_c \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(f, E_c \left(*, \frac{1}{2} + ir \right) \right) \left(\delta_{ca} y^{\frac{1}{2} + ir} + \phi_{c,a} \left(\frac{1}{2} + ir \right) y^{\frac{1}{2} - ir} \right) M_{m_2, -\frac{1}{4} - r^2} (T(\gamma))\, dr$$

equals

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(f, E_a \left(*, \frac{1}{2} - ir \right) \right) y^{\frac{1}{2} - ir} M_{m_2, -\frac{1}{4} - r^2} (T(\gamma))\, dr.$$

We then see using the notations of Lemma 6.2 (taking into account also that $E_a\left(*, \frac{1}{2} - ir\right)$ and $E_a\left(*, \frac{1}{2} + ir\right)$ are conjugates of each other for real r) that

$$\Sigma_2(\gamma) = \sum_{j=0}^{\infty} (f, u_j) \left(\int_{C_\gamma} \left(u_j - u_j^{Y_\Gamma} \right) dS \right) M_{m_2, -\frac{1}{4} - t_j^2} (T(\gamma)) +$$

$$+ \sum_a \frac{1}{2\pi i} \int_{\left(\frac{1}{2}\right)} \left(\int_F f(z) E_a(z, s)\, d\mu_z \right) \left(\int_{C_\gamma} A_a(*, s)\, dS \right) M_{m_2, s(s-1)} (T(\gamma))\, ds.$$

Since $M_{m_2, s(s-1)}(T(\gamma))$ is analytic in s by Lemma 4.3, so shifting the line of integration to the right, using Lemma 6.4 to see that the residues cancel out and using also Lemma 6.3 (iii) and Lemma 4.3 we get that

$$\Sigma_2(\gamma) = (f, u_0) \left(\int_{C_\gamma} \left(u_0 - u_0^{Y_\Gamma} \right) dS \right) M_{m_2, 0} (T(\gamma)) +$$

$$+ \sum_a \frac{1}{2\pi i} \int_{(1-\delta)} \left(\int_F f(z) E_a(z, s)\, d\mu_z \right) \left(\int_{C_\gamma} A_a(*, s)\, dS \right) M_{m_2, s(s-1)} (T(\gamma))\, ds, \tag{4.40}$$

where $\delta > 0$ is a small number chosen in such a way that $1 - \delta > S_l$ for every $l \in L$ satisfying $S_l < 1$.

Choosing δ small enough in terms of ϵ, applying Lemma 4.3, Lemma 6.3, Lemma 6.2, using (4.38), (4.39) and (4.40) we get that (4.36) equals

$$O_{f,\epsilon}\left(x^{\frac{1}{2}+\epsilon}\right)+$$

$$O_{f,\epsilon}\left(x^{\epsilon}\sum_{\substack{[\gamma]\\ \gamma \text{ hyperbolic, } T(\gamma)\leq xe^{-2d/x}}}\frac{d^2\log N(\gamma)}{T(\gamma)^{1/2}(x-T(\gamma))^{3/2}}\right)+$$

$$O_{f,\epsilon}\left(x^{\epsilon}\left(\frac{d}{x}\right)^{1/2}\sum_{\substack{[\gamma]\\ \gamma \text{ hyperbolic, } xe^{-2d/x}\leq T(\gamma)\leq xe^{d/x}}}\log N(\gamma)\right).$$

From the prime geodesic theorem (Theorem 10.5 of [I]) we then get assuming

$$d\geq x^{3/4}$$

that (4.36) equals

$$O_{f,\epsilon}\left((x^{\epsilon})\left(x^{\frac{1}{2}}+\frac{d^2}{x}+\frac{d^{3/2}}{\sqrt{x}}\right)\right). \tag{4.41}$$

Then by (4.33), (4.34), (4.36), (4.41) and (4.35) we get for $x^{3/4}\leq d\leq \frac{x}{\log x}$ that

$$\int_F f(z)\left(\sum_{\gamma\in\Gamma}k\left(\frac{|z-\gamma z|^2}{4\mathrm{Im}z\mathrm{Im}\gamma z}\right)\right)d\mu_z$$

equals

$$\int_F f(z)\left(\sum_{j,\, it_j>0}\sqrt{\pi}\frac{\Gamma(it_j)\,2^{2it_j+1}}{\Gamma\left(\frac{3}{2}+it_j\right)}x^{\frac{1}{2}+it_j}\left|u_j(z)\right|^2\right)d\mu_z+O_{f,\epsilon}\left((x^{\epsilon})\left(\frac{x}{\sqrt{d}}+\frac{d^{3/2}}{\sqrt{x}}\right)\right).$$

Choosing $d=x^{3/4}$ and $x=\frac{X-2}{4}$ we get the theorem.

5. Lemmas on special functions

LEMMA 5.1. *Let $0 < \delta < \frac{1}{2}$ be given. There is a constant $A_\delta > 0$ depending only on δ such that for every t satisfying $\frac{1}{4} + t^2 \geq 0$ or $|\mathrm{Re}\,(it)| \leq \frac{1}{2} - \delta$ and for every $X \geq 0$ one has that*

$$\left| F\left({\frac{1}{4} + \frac{it}{2}, \frac{1}{4} - \frac{it}{2} \atop \frac{1}{2}} ; -X \right) \right| + \left| (1+X) \frac{d}{dX} F\left({\frac{1}{4} + \frac{it}{2}, \frac{1}{4} - \frac{it}{2} \atop \frac{1}{2}} ; -X \right) \right| \leq A_\delta \left(1 + |t|\right)^{A_\delta}.$$

Proof. Note that

$$\frac{d}{dX} F\left({\frac{1}{4} + \frac{it}{2}, \frac{1}{4} - \frac{it}{2} \atop \frac{1}{2}} ; -X \right) = -2\left(\frac{1}{16} + \frac{t^2}{4}\right) F\left({\frac{5}{4} + \frac{it}{2}, \frac{5}{4} - \frac{it}{2} \atop \frac{3}{2}} ; -X \right)$$

and so

$$(1+X) \frac{d}{dX} F\left({\frac{1}{4} + \frac{it}{2}, \frac{1}{4} - \frac{it}{2} \atop \frac{1}{2}} ; -X \right) = -2\left(\frac{1}{16} + \frac{t^2}{4}\right) F\left({\frac{1}{4} + \frac{it}{2}, \frac{1}{4} - \frac{it}{2} \atop \frac{3}{2}} ; -X \right),$$

where we used the third line of [G-R], p 998, 9.131.1. Then it is trivial by [G-R], p 995, 9.111 that the statement of the lemma is true for $|t| < 1/10$ (say). The satement of the lemma is also trivial for every t and for $|X| < \frac{1}{10(1+|t|)^2}$ (say) by estimating trivially the series definition of the hypergeometric function.

For $j = 0, 1$ and $X > 0$ one has that

$$F\left({\frac{1}{4} + j + \frac{it}{2}, \frac{1}{4} + j - \frac{it}{2} \atop \frac{1}{2} + j} ; -X \right)$$

equals the sum of

$$\frac{\Gamma\left(\frac{1}{2} + j\right) \Gamma(it) \Gamma(1 - it)}{\Gamma\left(\frac{1}{4} \pm \frac{it}{2}\right) \Gamma\left(\frac{3}{4} - \frac{it}{2}\right) \Gamma\left(\frac{1}{4} + j + \frac{it}{2}\right)} \int_0^1 y^{-\frac{1}{4} - \frac{it}{2}} (1 - y)^{-\frac{3}{4} - \frac{it}{2}} (X + y)^{-\frac{1}{4} - j + \frac{it}{2}}\, dy$$

and the same expression writing $-t$ in place of t, this follows from [G-R], p 999, 9.132.2 and [G-R], p 995, 9.111. Then the statement follows for the case $|t| \geq 1/10$, $|X| \geq \frac{1}{10(1+|t|)^2}$.

The lemma is proved.

LEMMA 5.2. *Let $x > 0$ and let the function k be defined by (4.1). Let $0 < a < 1$. There is an absolute constant $A_0 > 0$ such that for every t satisfying $\frac{1}{4} + t^2 \geq 0$ one has, writing $\lambda = -\frac{1}{4} - t^2$ that*

$$\left| \int_0^\infty k\left(a \sinh^2 r\right) g_\lambda(r) \sinh r dr \right| \leq A_0 \left(1 + |t|\right)^{A_0} \left(1 + \frac{x}{a}\right)^{1/2}.$$

Proof. Note first that we can give an explicit formula for g_λ, namely

$$g_\lambda(r) = F\left(\begin{matrix}\frac{3}{4}+\frac{it}{2}, \frac{3}{4}-\frac{it}{2} \\ 1\end{matrix}; -\sinh^2 r\right)\cosh r \tag{5.1}$$

for $r \geq 0$, where $\lambda = -\frac{1}{4} - t^2$. This can be proved in the following way. One has

$$\int_0^\infty F\left(\begin{matrix}\frac{3}{4}+\frac{it}{2}, \frac{3}{4}-\frac{it}{2} \\ 1\end{matrix}; -\sinh^2 r\right)\cosh r \sinh^{1-2s} r dr = \frac{\Gamma(s-\frac{1}{4}\pm\frac{it}{2})\Gamma(1-s)}{2\Gamma(\frac{3}{4}\pm\frac{it}{2})\Gamma(s)}$$

for $\frac{1}{2} < \mathrm{Re}s < 1$, as one can see by the substitution $y = \sinh^2 r$ and by [G-R], p 806, 7.511. By Lemma 11 of [B1] it follows that

$$\int_0^\infty \left(g_\lambda(r) - F\left(\begin{matrix}\frac{3}{4}+\frac{it}{2}, \frac{3}{4}-\frac{it}{2} \\ 1\end{matrix}; -\sinh^2 r\right)\cosh r\right)\sinh^{1-2s} r dr = 0$$

for every $\frac{1}{2} < \mathrm{Re}s < 1$, which easily implies (5.1).

Then by the substitution $y = \sinh^2 r$ and by the particular shape of k we see that

$$\int_0^\infty k\left(a\sinh^2 r\right) g_\lambda(r)\sinh r dr$$

equals

$$\frac{1}{2}\int_0^{x/a} F\left(\begin{matrix}\frac{3}{4}+\frac{it}{2}, \frac{3}{4}-\frac{it}{2} \\ 1\end{matrix}; -y\right) dy.$$

Since the integrand here equals

$$(1+y)^{-1/2}\frac{1}{\pi}\int_0^1 q^{-1/2}(1-q)^{-1/2} F\left(\begin{matrix}\frac{1}{4}+\frac{it}{2}, \frac{1}{4}-\frac{it}{2} \\ 1/2\end{matrix}; -qy\right) dq$$

by [G-R], p 807, 7.512.11 and the third line of [G-R], p 998, 9.131.1, so using Lemma 5.1 the present lemma is proved.

6. Lemmas on automorphic functions

LEMMA 6.1. *There is a constant $A_\Gamma > 0$ depending only on Γ such that if $\gamma \in \Gamma$ is hyperbolic and $z \in H$ is a point on the noneuclidean line connecting the fixed points of γ, then for every $a \in A$ one has*

$$\sigma_a^{-1}\delta z \notin P\left(A_\Gamma\sqrt{N(\gamma)}\right)$$

for every $\delta \in \Gamma$.

Proof. By the conditions on γ and z we have (see [I], p 19) that

$$\rho(z, \gamma z) = \log N(\gamma),$$

where ρ is the distance function on H defined in (1.2) of [I]. Then by (1.3) of [I] we have that

$$u(z, \gamma z) = \frac{N(\gamma) + N(\gamma)^{-1} - 2}{4},$$

and so for every $\delta \in \Gamma$ one has

$$u\left(\delta z, \left(\delta\gamma\delta^{-1}\right)\delta z\right) = \frac{N(\gamma) + N(\gamma)^{-1} - 2}{4}. \tag{6.1}$$

Now, $\delta\gamma\delta^{-1}$ is hyperbolic, so it is not an element of Γ_a. Assume that A_Γ is large enough and $\sigma_a^{-1}\delta z \in P\left(A_\Gamma\sqrt{N(\gamma)}\right)$, then by Lemma 3.1 we have

$$u\left(\delta z, \left(\delta\gamma\delta^{-1}\right)\delta z\right) \geq D_\Gamma A_\Gamma^2 N(\gamma).$$

If A_Γ is large enough, then this contradicts (6.1), the lemma is proved.

LEMMA 6.2. *For $a \in A$ and any complex s write*

$$A_a(z, s) := \left(\mathrm{Im}\sigma_a^{-1}z\right)^{1-s} \text{ for } z \in F_a(Y_\Gamma),$$

$$A_a(z, s) := 0 \text{ for } z \in F \setminus F_a(Y_\Gamma),$$

finally let $A_a(\gamma z, s) = A_a(z, s)$ for $\gamma \in \Gamma$ and $z \in F$. Let $\gamma \in \Gamma$ be hyperbolic, then for $0 \leq \sigma := \mathrm{Re}s \leq 1$ one has (using the notations of Lemma 3.3) that

$$\int_{C_\gamma}\left(1 + \sum_{a\in A}|A_a(*, s)|\right)dS \ll (N(\gamma))^{\frac{1-\sigma}{2}}\log N(\gamma).$$

Proof. It is well-known that $\int_{C_\gamma} 1dS \leq \log N(\gamma)$, so it is enough to show that

$$|A_a(z, s)| \ll (N(\gamma))^{\frac{1-\sigma}{2}}$$

for every $a \in A$ and every $z \in H$ lying on the noneuclidean line connecting the fixed points of γ, and this follows from Lemma 6.1. The lemma is proved.

LEMMA 6.3. *Let the function f be as in Theorem 1.1.*

(i) For integers $j \geq 0$ one has that

$$\sup_{z \in F} \left| u_j^{Y_\Gamma}(z) \right| \ll (1 + |t_j|)^C$$

and for any $a \in A$ and $R > 0$ one has that

$$\sup_{z \in F} \int_{-R}^{R} \left| E_a^{Y_\Gamma}\left(z, \frac{1}{2} + ir\right) \right|^2 dR \ll (1 + R)^C$$

with some absolute constant C.

(ii) For every positive integer K one has for $j \geq 0$ that

$$|(f, u_j)| \ll_{f,K} (1 + |t_j|)^{-K}.$$

(iii) For any $a \in A$ the function

$$\int_F f(z) E_a(z, s) \, d\mu_z$$

is meromorphic for $\frac{1}{2} \leq \operatorname{Re} s \leq 2$ having poles only at the points $\{S_l : l \in L\}$, and for every positive integer K and every $\frac{1}{2} \leq \sigma \leq 2$ one has

$$\int_{-\infty}^{-1} \left| \int_F f(z) E_a(z, \sigma + ir) \, d\mu_z \right|^2 r^{2K} dr \ll_{f,K} 1$$

and

$$\int_{1}^{\infty} \left| \int_F f(z) E_a(z, \sigma + ir) \, d\mu_z \right|^2 r^{2K} dr \ll_{f,K} 1.$$

Proof. Part (i) follows e.g. from [I], Proposition 7.2 and formulas (9.13), (8.1), (8.2), (8.5), (8.6). For parts (ii) and (iii) we use that the Laplace operator is self-adjoint (see (4.2) of [I]), and we apply repeatededly (4.2) of [I]. Part (ii) follows at once in this way. Part (iii) also follows in this way if we can show that there is an absolute constant $K_0 > 0$ such that

$$\int_{-\infty}^{-1} \frac{\int_F \left| E_a^Y(z, \sigma + ir) \right|^2 d\mu_z}{r^{2K_0}} dr + \int_{1}^{\infty} \frac{\int_F \left| E_a^Y(z, \sigma + ir) \right|^2 d\mu_z}{r^{2K_0}} dr \ll_Y 1 \qquad (6.2)$$

for $\frac{1}{2} < \sigma \leq 2$ and $Y \geq 1$. (We can assume indeed $\sigma > \frac{1}{2}$, since the case $\sigma = \frac{1}{2}$ of the last statement of (iii) follows from the $\frac{1}{2} < \sigma \leq 2$ case of that statement by continuity, since the upper bound is uniform in σ.)

Statement (6.2) can be deduced from the Maass-Selberg relations in the following way. One first shows by (6.31) of [I] (using the notations of that book) that

$$|\phi_{a,a}(\sigma+ir)| \ll 1$$

for $\frac{1}{2} < \sigma \leq 2$, $|r| \geq 1$ and for any cusp a, then the same estimate follows for $\phi_{a,b}(\sigma+ir)$ for any two cusps a, b. Finally, still by (6.31) of [I], we get

$$\sum_{b} |\phi_{a,b}(\sigma+ir)|^2 \leq 1 + O\left(\sigma - \frac{1}{2}\right)$$

for $\frac{1}{2} < \sigma \leq 2$, $|r| \geq 1$ and for any cusp a. By the Hadamard inequality (see e.g. Corollary 7.8.2 of [H-J]) we then see that for the determinant $\phi = \det(\phi_{a,b})$ we have that

$$|\phi(\sigma+ir)|^2 \leq \prod_{a} \left(\sum_{b} |\phi_{a,b}(\sigma+ir)|^2\right) \leq \left(1 + O\left(\sigma - \frac{1}{2}\right)\right) \sum_{b} |\phi_{a_0,b}(\sigma+ir)|^2$$

for any fixed cusp a_0, and combining it with Propositions 12.7 and 12.8 of [He] we get that

$$\sum_{b} |\phi_{a_0,b}(\sigma+ir)|^2 \geq 1 - O\left(\left(\sigma - \frac{1}{2}\right)\omega(r)\right)$$

for $\frac{1}{2} < \sigma \leq 2$, $|r| \geq 1$ and for any cusp a_0 with ω defined in Proposition 12.7 of [He]. Then using again (6.31) of [I] and Proposition 12.7 of [He] we get (6.2). The lemma is proved.

LEMMA 6.4. *For any $a \in A$ and $l \in L$ such that $\frac{1}{2} < S_l < 1$ we have that*

$$\sum_{j \geq 0,\, s_j = S_l} (f, u_j)\, \beta_{a,j}(0) = \int_F f(z)\, \mathrm{res}_{s=S_l} E_a(z,s)\, d\mu_z. \tag{6.3}$$

REMARK 6.5. Note that since f could be any function satisfying the conditions of Theorem 1.1, so we could easily get

$$\sum_{j \geq 0,\, s_j = S_l} \overline{u_j(z)} \beta_{a,j}(0) = \mathrm{res}_{s=S_l} E_a(z,s)$$

for every $z \in H$, but we will use (6.3) during the proof of Theorem 1.1, so it is enough for our purposes.

Proof. For $z \in H$ we have that

$$f(z) = \sum_{j=0}^{\infty} (f, u_j)\, \mathrm{u}_j(z) + \sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(f, E_a\left(*, \frac{1}{2} + ir \right) \right) \mathrm{E}_a\left(z, \frac{1}{2} + ir \right) dr, \qquad (6.4)$$

and this expression is uniformly and absolutely convergent on compact subsets of H. Formula (6.4) follows from the Spectral Theorem ([I], Theorems 4.7 and 7.3).

Using Lemma 6.3 and that $f(\sigma_a z)$ is also bounded, (6.4) implies that for any $a \in A$ the sum

$$\sum_{j=0}^{\infty} (f, u_j)\, \beta_{a,j}(0)\, y^{1-s_j} +$$

$$+ \sum_c \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(f, E_c\left(*, \frac{1}{2} + ir \right) \right) \left(\delta_{ca} y^{\frac{1}{2}+ir} + \phi_{c,a}\left(\frac{1}{2} + ir \right) y^{\frac{1}{2}-ir} \right) dr$$

is bounded for $z \in P(Y_\Gamma)$. Since by [I], (6.22) and (6.27) we have

$$\sum_c \left(f, E_c\left(*, \frac{1}{2} + ir \right) \right) \phi_{c,a}\left(\frac{1}{2} + ir \right) = \left(f, E_a\left(*, \frac{1}{2} - ir \right) \right)$$

for any real r, so for any $a \in A$ the sum

$$\sum_{j=0}^{\infty} (f, u_j)\, \beta_{a,j}(0)\, y^{1-s_j} +$$

$$+ \frac{1}{4\pi} \int_{-\infty}^{\infty} \left(\left(f, E_a\left(*, \frac{1}{2} + ir \right) \right) y^{\frac{1}{2}+ir} + \left(f, E_a\left(*, \frac{1}{2} - ir \right) \right) y^{\frac{1}{2}-ir} \right) dr$$

is bounded for $z \in P(Y_\Gamma)$, i.e.

$$\sum_{j=0}^{\infty} (f, u_j)\, \beta_{a,j}(0)\, y^{1-s_j} + \frac{1}{2\pi i} \int_{\left(\frac{1}{2}\right)} \left(\int_F f(z)\, E_a(z, s)\, d\mu_z \right) y^{1-s} ds$$

is bounded for $z \in P(Y_\Gamma)$. We now shift the integration to the right, to $\mathrm{Re} s = 1 - \delta$ with a small $\delta > 0$, and we see that every residue must be 0 because of the boundedness. The lemma follows.

References

[B1] A. Biró, *On a generalization of the Selberg trace formula*, Acta Arithmetica, 87 (4) (1999), 319-338.

[B2] A. Biró, *A relation between triple products of weight* 0 *and weight* $\frac{1}{2}$ *cusp forms,* Israel J. of Math., 182 (2011), no. 1, 61-101.

[C] F. Chamizo, *Some applications of large sieve in Riemann surfaces*, Acta Arithmetica, 77 (4) (1996), 315-337.

[F] J.D. Fay, *Fourier coefficients of the resolvent for a Fuchsian group*, J. Reine Angew. Math., 294 (1977), 143-203.

[G-R] I.S. Gradshteyn, I.M. Ryzhik, Table of integrals, series and products, 6th edition, *Academic Press*, 2000

[He] D.A. Hejhal, The Selberg Trace Formula for $PSL(2,\mathbf{R})$, vol. 2, Lecture Notes in Mathematics 1001 *Springer, New York*, 1983

[Hu] H. Huber, *Ein Gitterpunktproblem in der hyperbolischen Ebene,* J. Reine Angew. Math., 496 (1998), 15-53.

[H-J] R.A. Horn, C.R. Johnson, Matrix Analysis, *Cambridge University Press Press*, 1985

[I] H. Iwaniec, Introduction to the spectral theory of automorphic forms, *Rev. Mat. Iberoamericana*, 1995

[K-N] L. Kuipers, H. Niederreiter Uniform distribution of sequences, *New York, Wiley*, 1974

[R-P] M. Risager, Y. Petridis, *Local average in hyperbolic lattice point counting, with an appendix by Niko Laaksonen*, Math. Z. (2016), doi:10.1007/s00209-016-1749-z, available online at https://arxiv.org/abs/1408.5743

[Z1] S. Zelditch, *Selberg trace formulae, pseudodifferential operators, and geodesic periods of automorphic forms,* Duke Math. J., 56 (1988), no. 2, 295-344.

[Z2] S. Zelditch, *Trace formula for compact* $\Gamma \setminus PSL(2,\mathbf{R})$ *and the equidistribution theory of closed geodesics,* Duke Math. J., 59 (1989), no. 1, 27-81.

[Z3] S. Zelditch, *Selberg trace formulae and equidistribution theorems for closed geodesics and Laplace eigenfunctions: finite area surfaces.,* Mem. Amer. Math. Soc. 96 (1992), no. 465, vi+102 pp