THE VARIETY OF COSET RELATION ALGEBRAS

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ABSTRACT. Givant [6] generalized the notion of an atomic pair-dense relation algebra from Maddux [13] by defining the notion of a *measurable relation algebra*, that is to say, a relation algebra in which the identity element is a sum of atoms that can be measured in the sense that the "size" of each such atom can be defined in an intuitive and reasonable way (within the framework of the first-order theory of relation algebras). In Andréka-Givant [3], a large class of examples of such algebras is constructed from systems of groups, coordinated systems of isomorphisms between quotients of the groups, and systems of cosets that are used to "shift" the operation of relative multiplication. In Givant-Andréka [8], it is shown that the class of these *full coset relation algebras* is adequate to the task of describing all measurable relation algebras in the sense that every atomic and complete measurable relation algebra is isomorphic to a full coset relation algebra.

Call an algebra $\mathfrak{A}$ a *coset relation algebra* if $\mathfrak{A}$ is embeddable into some full coset relation algebra. In the present paper, it is shown that the class of coset relation algebras is equationally axiomatizable (that is to say, it is a variety), but that no finite set of sentences suffices to axiomatize the class (that is to say, the class is not finitely axiomatizable).

1. INTRODUCTION

In [6], a subidentity element x—that is to say, an element below the identity element—of a relation algebra is defined to be *measurable* if it is an atom and if the square $x;1;x$ is a sum of functional elements, that is to say, a set of abstract elements f satisfying the functional inequality $f^{\smile};f \leq 1$'. (A functional element is an abstract version of a function in that in a concrete algebra of binary relations an element is functional if and only if it is a function set theoretically, i.e., $(u,v) \in f$ and $(u,w) \in f$ imply $v = w$.) The number of non-zero functional elements below the square $x;1;x$ gives the *measure*, or the *size*, of the atom x. A relation algebra is said to be *measurable* if the identity element is the sum of measurable atoms, and *finitely measurable* if each of the measurable atoms has finite measure.

The group relation algebras constructed in [6] are examples of measurable relation algebras. Interestingly, the class GRA of algebras embeddable into the full group relation algebras coincides with the variety RRA of all representable relation algebras [6, section 5], in symbols

$$\mathsf{GRA} = \mathsf{RRA}.$$

It turns out that full group relation algebras are not the only examples of measurable relation algebras. In [3], a more general class of measurable relation algebras

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is constructed. The algebras are obtained from group relation algebras by "shifting" the relational composition operation by means of coset multiplication, using an auxiliary system of cosets. For that reason, they are called *full coset relation algebras*, and they are not too much of a distortion to representable algebras. They are a genuine generalization to group relation algebras, because among them are algebras that are not representable [3, Thm.5.2]. However, this class is adequate to the task of describing all atomic, complete measurable relation algebras in the sense that a relation algebra is atomic, complete and measurable if and only if it is isomorphic to a full coset relation algebra [8, Thm.7.2].

In the present paper, we show that the class CRA of algebras embeddable into full coset relation algebras is a variety. It is a generalization of the class RRA of representable relation algebras. Given the relationship between GRA and RRA, it is natural to ask whether CRA coincides with the class RA of all relation algebras. We prove that this is not the case, and in fact CRA is not finitely axiomatizable as RA is. Thus

$$\mathsf{GRA} = \mathsf{RRA} \subset \mathsf{CRA} \subset \mathsf{RA}.$$

Thus CRA shares the properties of RRA of being a variety and of being not finitely axiomatizable.

An extended abstract describing the above results and their interconnections was published by the authors in [7]. The reader may find the expository and motivational material of [7] helpful in connection with the present paper. Readers who wish to learn more about the subject of relation algebras and their connection to logic are recommended to look at one or more of the books Hirsch-Hodkinson [9], Maddux [14], Givant [4, 5], or Tarski-Givant [19].

2. Group and coset relation algebras

Here is a summary of the essential notions from [6, 3, 7] that will be needed in this paper. Fix a system

$$G = \langle\, G_x : x \in I \,\rangle$$

of groups that are pairwise disjoint, and an associated system

$$\varphi = \langle\, \varphi_{xy} : (x, y) \in \mathcal{E} \,\rangle$$

of isomorphisms between quotient groups. Specifically, we require that $\mathcal{E}$ be an equivalence relation on the index set I, and for each pair (x, y) in $\mathcal{E}$, the function φ_{xy} be an isomorphism from a quotient group of G_x to a quotient group of G_y. Call

$$\mathcal{F} = (G, \varphi)$$

a *group pair*. The set I is the *group index set*, and the equivalence relation $\mathcal{E}$ is the *(quotient) isomorphism index set* of $\mathcal{F}$. The normal subgroups of G_x and G_y from which the quotient groups are constructed are uniquely determined by φ_{xy}, and will be denoted by H_{xy} and K_{xy} respectively, so that φ_{xy} maps G_x/H_{xy} isomorphically onto G_y/K_{xy}.

Let κ_{xy} denote the cardinality of the quotient group G_x/H_{xy}. For a fixed enumeration $\langle H_{xy,\gamma} : \gamma < \kappa_{xy}\rangle$ (without repetitions) of the cosets of H_{xy} in G_x, the isomorphism φ_{xy} induces a *corresponding*, or *associated*, coset system of K_{xy} in G_y, determined by the rule

$$K_{xy,\gamma} = \varphi_{xy}(H_{xy,\gamma})$$

for each $\gamma < \kappa_{xy}$. In what follows, it is always assumed that the given coset systems for H_{xy} in G_x and for K_{xy} in G_y are associated in this manner. Further, we will always assume that the coset H_{xy} is the first one in the enumeration: $H_{xy,0} = H_{xy}$. In the following, $\circ$ will denote the group operations of the groups in question, we hope context will always tell which group we have in mind.

Definition 2.1. For each pair (x, y) in $\mathcal{E}$ and each $\alpha < \kappa_{xy}$, define a binary relation $R_{xy,\alpha}$ by

$$R_{xy,\alpha} = \bigcup_{\gamma < \kappa_{xy}} H_{xy,\gamma} \times \varphi_{xy}[H_{xy,\gamma} \circ H_{xy,\alpha}] = \bigcup_{\gamma < \kappa_{xy}} H_{xy,\gamma} \times (K_{xy,\gamma} \circ K_{xy,\alpha}).$$

$\square$

The set A of all possible unions of sets of such relations is a complete Boolean set algebra, but it may not contain the identity relation, nor need it be closed under the operations of relational converse and composition. The following theorems from [6] characterize when we do obtain such closure, so that A is the universe of a set relation algebra.

Lemma 2.2 (Partition Lemma)**.** *The relations $R_{xy,\alpha}$, for $\alpha < \kappa_{xy}$, are non-empty and partition the set $G_x \times G_y$.*

Theorem 2.3 (Boolean Reduct Theorem)**.** *The set A is the universe of a complete, atomic Boolean algebra of sets. The atoms are the relations $R_{xy,\alpha}$, and the elements in A are the unions of the various sets of atoms.*

In the following, e_x denotes the identity element of the group G_x, and $id_U = \{(u, u) : u \in U\}$ is the identity relation on the set U. Also, we often denote the domain of the group G_x also by G_x.

Theorem 2.4 (Identity Theorem)**.** *For each element x in I, the following conditions are equivalent.*

(i) *The identity relation id_{G_x} on G_x is in A.*
(ii) *$R_{xx,0} = id_{G_x}$.*
(iii) *φ_{xx} is the identity automorphism of $G_x/\{e_x\}$.*

Consequently, the set A contains the identity relation id_U on the base set U if and only if (iii) *holds for each x in I.*

Convention 2.5. Suppose that the identity relation is in A. Then $H_{xx} = \{e_x\}$ by (iii) of the Identity Theorem. Consequently, the cosets of H_{xx} are the singletons $\{g\}$ for $g \in G_x$. We will write simply $R_{xx,g}$ in place of $R_{xx,\gamma}$ for $\gamma = \{g\}$. Thus, for example, $\{R_{xx,g} : g \in H_{xy}\}$ means $\{R_{xx,\gamma} : \gamma = \{g\}$ for some $g \in H_{xy}\}$. Note that $\{R_{xx,g} : g \in G_x\}$ is the same as $\{R_{xx,\gamma} : \gamma < \kappa_{xx}\}$, and $\kappa_{xx} = |G_x|$.

In the following, $R^{-1} = \{(v, u) : (u, v) \in R\}$ denotes the inverse of the binary relation R. We also denote by a^{-1} the inverse of an element a in a group.

Theorem 2.6 (Converse Theorem)**.** *For each pair (x, y) in $\mathcal{E}$, the following conditions are equivalent.*

(i) *There are an $\alpha < \kappa_{xy}$ and a $\beta < \kappa_{yx}$ such that $R_{xy,\alpha}^{-1} = R_{yx,\beta}$.*
(ii) *For every $\alpha < \kappa_{xy}$ there is a $\beta < \kappa_{yx}$ such that $R_{xy,\alpha}^{-1} = R_{yx,\beta}$.*
(iii) *$\varphi_{xy}^{-1} = \varphi_{yx}$.*

Moreover, if one of these conditions holds, then we may assume that $\kappa_{yx} = \kappa_{xy}$, and the index β in (i) *and* (ii) *is uniquely determined by the equation $H_{xy,\alpha}^{-1} = H_{xy,\beta}$. The set A is closed under converse if and only if* (iii) *holds for all (x, y) in $\mathcal{E}$.*

Convention 2.7. Suppose A is closed under converse. If a pair (x, y) is in $\mathcal{E}$, then $H_{yx} = K_{xy}$, and therefore any coset system for H_{yx} is also a coset system for K_{xy}. Since the enumeration $\langle H_{yx,\gamma} : \gamma < \kappa_{yx} \rangle$ of the cosets of H_{yx} can be freely chosen, we can and always shall choose it so that $\kappa_{yx} = \kappa_{xy}$ and $H_{yx,\gamma} = K_{xy,\gamma}$ for $\gamma < \kappa_{xy}$. It then follows from the Converse Theorem that $K_{yx,\gamma} = H_{xy,\gamma}$ for $\gamma < \kappa_{xy}$.

In the following, $R \,|\, S = \{(u, w) : (u, v) \in R$ and $(v, w) \in S$ for some $v\}$ denotes the relational composition of the binary relations R and S.

Lemma 2.8. *If (x, y) and (w, z) are in $\mathcal{E}$, and if $y \neq w$, then*

$$R_{xy,\alpha} \,|\, R_{wz,\beta} = \varnothing$$

for all $\alpha < \kappa_{xy}$ and $\beta < \kappa_{wz}$.

The most important case regarding the composition of two atomic relations is when $y = w$.

Theorem 2.9 (Composition Theorem)**.** *For all pairs (x, y) and (y, z) in $\mathcal{E}$, the following conditions are equivalent.*

   (i) *The relation $R_{xy,0} \,|\, R_{yz,0}$ is in A.*
  (ii) *For each $\alpha < \kappa_{xy}$ and each $\beta < \kappa_{yz}$, the relation $R_{xy,\alpha} \,|\, R_{yz,\beta}$ is in A.*
 (iii) *For each $\alpha < \kappa_{xy}$ and each $\beta < \kappa_{yz}$,*

$$R_{xy,\alpha} \,|\, R_{yz,\beta} = \bigcup \{R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}]\}.$$

 (iv) *$H_{xz} \subseteq \varphi_{xy}^{-1}[K_{xy} \circ H_{yz}]$ and $\hat{\varphi}_{xy} \,|\, \hat{\varphi}_{yz} = \hat{\varphi}_{xz}$, where $\hat{\varphi}_{xy}$ and $\hat{\varphi}_{xz}$ are the mappings induced by φ_{xy} and φ_{xz} on the quotient of G_x modulo the normal subgroup $\varphi_{xy}^{-1}[K_{xy} \circ H_{yz}]$, while $\hat{\varphi}_{yz}$ is the isomorphism induced by φ_{yz} on the quotient of G_y modulo the normal subgroup $K_{xy} \circ H_{yz}$.*

Consequently, the set A is closed under relational composition if and only if (iv) *holds for all pairs (x, y) and (y, z) in $\mathcal{E}$.*

The next theorem clarifies the characters of the mappings induced by the quotient isomorphism.

Theorem 2.10 (Image Theorem)**.** *If the set A is closed under converse and composition, then*

$$\varphi_{xy}[H_{xy} \circ H_{xz}] = K_{xy} \circ H_{yz}, \qquad \varphi_{yz}[K_{xy} \circ H_{yz}] = K_{xz} \circ K_{yz},$$
$$\varphi_{xz}[H_{xy} \circ H_{xz}] = K_{xz} \circ K_{yz}$$

for all (x, y) and (y, z) in $\mathcal{E}$.

Full group relation algebras by themselves are not sufficient to represent all atomic, measurable relation algebras, because the operation of relative multiplication need not coincide with that of relational composition in the most natural candidate for a representable copy of a measurable relation algebra [3, Thm.5.2]. The operation in an arbitrary measurable relation algebra may be a kind of "shifted" relational composition. It is therefore necessary to add one more ingredient to a group pair $\mathcal{F} = (G, \varphi)$, namely a system of cosets

$$\langle C_{xyz} : (x, y, z) \in \mathcal{E}_3 \rangle,$$

where $\mathcal{E}_3$ is the set of all triples (x, y, z) such that the pairs (x, y) and (y, z) are in $\mathcal{E}$, and for each such triple, the set C_{xyz} is a coset of the normal subgroup $H_{xy} \circ H_{xz}$ in G_x. Call the resulting triple

$$\mathcal{F} = (G, \varphi, C)$$

a *group triple*.

Define a new binary multiplication operation $\otimes$ on the pairs of atomic relations in the Boolean algebra A of Theorem 2.3 as follows.

Definition 2.11. For pairs (x, y) and (y, z) in $\mathcal{E}$, put

$$R_{xy,\alpha} \otimes R_{yz,\beta} = \bigcup \{R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}] \circ C_{xyz}\}$$

for all $\alpha < \kappa_{xy}$ and all $\beta < \kappa_{yz}$, and for all other pairs (x, y) and (w, z) in $\mathcal{E}$ with $y \neq w$, put

$$R_{xy,\alpha} \otimes R_{wz,\beta} = \varnothing$$

for all $\alpha < \kappa_{xy}$ and $\beta < \kappa_{wz}$. Extend $\otimes$ to all of A by requiring it to distribute over arbitrary unions. □

Comparing the formula defining $R_{xy,\alpha} \otimes R_{yz,\beta}$ in Definition 2.11 with the value of the relational composition $R_{xy,\alpha} \mid R_{yz,\beta}$ given in Composition Theorem 2.9(iii), it is clear that they are very similar in form. In the first case, however, the coset $\varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}]$ of the composite group $H_{xy} \circ H_{xz}$ has been shifted, through coset multiplication by C_{xyz}, to another coset of $H_{xy} \circ H_{xz}$, so that in general the value of the $\otimes$-product and the value of relational composition on a given pair of atomic relations will be different, except in certain cases, for example, the case in which the value is the empty set.

Lemma 2.12. $R_{xy,\alpha} \otimes R_{xy,\alpha}^{-1} = R_{xy,\alpha} \mid R_{xy,\alpha}^{-1} = \bigcup \{R_{xx,g} : g \in H_{xy}\}.$

Proof. The relation $R_{xy,\alpha}^{-1}$ is equal to $R_{yx,\beta}$ for β such that

$$(1) \qquad H_{xy,\beta} = H_{xy,\alpha}^{-1},$$

by Converse Theorem 2.6. Note in passing that (1) and the isomorphism properties of φ_{xy} imply that

$$K_{xy,\beta} = K_{xy,\alpha}^{-1},$$

and hence that

$$(2) \qquad H_{yx,\beta} = H_{yx,\alpha}^{-1},$$

by Convention 2.7.

Lemma 6.5 in [3] implies that the first equality in (1) holds with $R_{yx,\beta}$ in place of $R_{xy,\alpha}^{-1}$ if and only if $C_{xyx} = H_{xy} \circ H_{xx} = H_{xy}$. This last equality does hold, by the coset conditions listed in Theorem 7.6(v) of [3], so the first equality of the lemma holds.

As regards the second equality of the lemma, we have

$$(3) \qquad R_{xy,\alpha} \mid R_{yx,\beta} = \bigcup \{R_{xx,g} : g \in \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yx,\beta}]\},$$

by the Composition Theorem 2.9. Now $K_{xy,\alpha} = H_{yx,\alpha}$, by Convention 2.7, so

$$(4) \qquad K_{xy,\alpha} \circ H_{yx,\beta} = H_{yx,\alpha} \circ H_{yx,\beta} = H_{yx,\alpha} \circ H_{yx,\alpha}^{-1} = H_{yx},$$

by (2) and the group inverse property. Consequently,

$$(5) \qquad \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yx,\beta}] = \varphi_{xy}^{-1}(H_{yx}) = \varphi_{xy}^{-1}(K_{xy}) = H_{xy},$$

by (4), Convention 2.7, and the definition of φ_{xy}. Replace the left side of (5) in (1) by the right side of (5) to arrive at the second equality of the lemma. □

Lemma 2.13. $(G_x \times G_y) \otimes (G_y \times G_z) = (G_x \times G_y) \,|\, (G_y \times G_z) = G_x \times G_z$.

Proof. The second equality is obviously true. To derive the first equality, it is helpful to derive the second equality in a more roundabout way. Use Partition Lemma 2.2, the distributivity of relational composition over unions, and Composition Theorem 2.9, to obtain

$$
\begin{aligned}
(G_x \times G_y)\,|\,(G_y \times G_z) &= (\textstyle\bigcup\{R_{xy,\alpha} : \alpha < \kappa_{xy}\})\,|\,(\textstyle\bigcup\{R_{yz,\beta} : \beta < \kappa_{yz}\}) \\
&= \textstyle\bigcup\{R_{xy,\alpha}\,|\,R_{yz,\beta} : \alpha < \kappa_{xy} \text{ and } \beta < \kappa_{yz}\} \\
&= \textstyle\bigcup\{R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}], \alpha < \kappa_{xy}, \beta < \kappa_{yz}\}.
\end{aligned}
$$

As α and β vary over their index sets, the cosets $K_{xy,\alpha} \circ H_{yz,\beta}$ of $K_{xy} \circ H_{yz}$ in G_y vary over all of the cosets of $K_{xy} \circ H_{yz}$, the union of which is just G_y. Continue with the preceding string of equalities to arrive at

$$
\begin{aligned}
(6) \qquad (G_x \times G_y)\,|\,(G_y \times G_z) &= \textstyle\bigcup\{R_{xz,\gamma} : H_{xz,\gamma} \subseteq \varphi_{xy}^{-1}[G_y]\} \\
&= \textstyle\bigcup\{R_{xz,\gamma} : H_{xz,\gamma} \subseteq G_x\} \\
&= \textstyle\bigcup\{R_{xz,\gamma} : \gamma < \kappa_{xz}\} \\
&= G_x \times G_z.
\end{aligned}
$$

The computation with $\otimes$ in place of $|$ is nearly the same, but the composition with C_{xyz} must be adjoined on the right to each of the terms

$$
\varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}], \qquad \varphi_{xy}^{-1}[G_y], \qquad G_x,
$$

that is to say, these three terms must be replaced by

$$
\varphi_{xy}^{-1}[K_{xy,\alpha} \circ H_{yz,\beta}] \circ C_{xyz}, \qquad \varphi_{xy}^{-1}[G_y] \circ C_{xyz}, \qquad G_x \circ C_{xyz}
$$

respectively. Note that $G_x = G_x \circ C_{xyz}$, so we arrive at the same final equality. Combine these observations to obtain the first equality of the lemma. □

3. The variety generated by the class of full coset relation algebras

Call an algebra $\mathfrak{A}$ a *coset relation algebra* if $\mathfrak{A}$ is embeddable into a full coset relation algebra, and let CRA be the class of all coset relation algebras. The class CRA is an analogue of RRA. A rather surprising consequence of the Representation Theorem for measurable relation algebras [8, Theorem 7.4] is that the class CRA is equationally axiomatizable, or a *variety*, as such classes are usually called. The proof of this theorem is analogous to the proof of Tarski's theorem in [18] that the class of representable relation algebras forms a variety.

Theorem 3.1. *The class of coset relation algebras is a variety.*

Proof. Let K be the class of all atomic, measurable relation algebras, and denote by S(K) the class of algebras that are embeddable into some algebra in K. The first step in proving the theorem is to show that the class K is first-order axiomatizable. In other words, there is a set Γ of first-order sentences such that an algebra $\mathfrak{A}$ is in K just in case $\mathfrak{A}$ is a model of Γ, that is to say, just in case all the sentences of Γ are true of $\mathfrak{A}$, where everything is taken in the signature of relation algebras.

First, put the relation algebraic axioms into Γ. Next, observe that the property of being an atom is expressible in first-order logic: an atom is a minimal non-zero element. Consequently, the property of being an atomic algebra is expressible by a first-order sentence φ saying that below every non-zero element there is an atom. Put φ into Γ. The property of being a measurable atom is also first-order expressible as follows: an element x is a measurable atom just in case x is a subidentity atom (an atom below 1'), and every non-zero element below $x; 1; x$ is above some non-zero functional element (an element f satisfying the functional inequality $f^{\smile}; f \leq 1$'). The first-order sentence ψ stating that below every non-zero subidentity element there is a measurable atom expresses the property of an algebra being measurable. Put ψ into Γ. Clearly, Γ is a set of axioms for K, in symbols,

$$(1) \qquad\qquad\qquad \mathsf{Mo}(\Gamma) = \mathsf{K},$$

where $\mathsf{Mo}(\Gamma)$ is the class of all models of Γ. Let Θ be the set of universal sentences true in K. A well-known theorem of Tarski [17] says that, for any first-order axiomatizable class L of algebras, the class $\mathsf{S}(\mathsf{L})$ of algebras embeddable into algebras of L is axiomatizable by a set of (first-order) universal sentences. In particular,

$$(2) \qquad\qquad\qquad \mathsf{Mo}(\Theta) = \mathsf{S}(\mathsf{K}).$$

The next step is to prove that

$$(3) \qquad\qquad\qquad \mathsf{CRA} = \mathsf{S}(\mathsf{K}).$$

Every full coset relation algebra is in the class K (this is proved in [3]). Consequently, every coset relation algebra is in $\mathsf{S}(\mathsf{K})$, because this class is closed under subalgebras and isomorphic images. This establishes the inclusion from left to right in (3). To establish the reverse inclusion, use the representation theorem for measurable relation algebras [8, Theorem 7.4]. This theorem says that every algebra in K is embeddable into a full coset relation algebra, and consequently belongs to the class CRA of all coset relation algebras. It follows that every algebra in $\mathsf{S}(\mathsf{K})$ is embeddable into an algebra in the class CRA and therefore belongs to this class, because the class is closed under subalgebras and isomorphic images. This completes the proof of (3).

The remarks after the proof of Theorem 6.1 in [3, p.51] imply that the direct product of a system of full coset relation algebras is isomorphic to a full coset relation algebra. It follows that the direct product of a system of coset relation algebras is embeddable into a full coset relation algebra. Thus, $\mathsf{S}(\mathsf{K})$ is closed under direct products, and consequently under subdirect products.

Consider again the set Θ of universal sentences that axiomatizes $\mathsf{S}(\mathsf{K})$. It is a well-known theorem in the theory of relation algebras (due to Tarski—see Theorem 9.5 in [4]) that for every universal sentence θ in the language of relation algebras, there is an (effectively constructible) equation ε_θ in the language of relation algebras such that θ and ε_θ are equivalent in all simple relation algebras, that is to say, θ is valid in a simple relation algebra $\mathfrak{A}$ just in case ε_θ is valid in $\mathfrak{A}$. Let Δ be the set of equations corresponding to universal sentences in Θ,

$$\{\varepsilon_\theta : \theta \in \Theta\},$$

together with the axioms of the theory of relation algebras.

$$(4) \qquad\qquad\qquad \mathsf{Mo}(\Delta) = \mathsf{S}(\mathsf{K}).$$

To prove (4), consider any model $\mathfrak{A}$ of Δ. Certainly, $\mathfrak{A}$ is a relation algebra, because the relation algebraic axioms are all in Δ. Every relation algebra is isomorphic to a subdirect product of simple relation algebras (see Theorem 12.10 in [4]). Let $\mathfrak{B}$ be a simple subdirect factor of $\mathfrak{A}$. Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, every equation true of $\mathfrak{A}$ is true of $\mathfrak{B}$. (Recall that equations are preserved under the passage to homomorphic images.) It follows that each equation in Δ is valid in $\mathfrak{B}$. Now $\mathfrak{B}$ is simple, by assumption, so each sentence in Θ is valid in $\mathfrak{B}$. Consequently, $\mathfrak{B}$ belongs to $\mathsf{S}(\mathsf{K})$, by (2). This shows that every simple, subdirect factor of $\mathfrak{A}$ is in $\mathsf{S}(\mathsf{K})$. Since $\mathsf{S}(\mathsf{K})$ is closed under subalgebras and direct products, it follows that $\mathfrak{A}$ is in $\mathsf{S}(\mathsf{K})$. In other words, every model of Δ is in $\mathsf{S}(\mathsf{K})$.

To establish the reverse inclusion, consider first an arbitrary full coset relation algebra $\mathfrak{C}[\mathcal{F}]$. Certainly, $\mathfrak{C}[\mathcal{F}]$ is in $\mathsf{S}(\mathsf{K})$, by (3), and hence is a model of Θ, by (2). If $\mathcal{F}$ is simple in the sense that the quotient isomorphism system index set $\mathcal{E}$ coincides with $I \times I$ (the universal relation on the group system index set I), then $\mathfrak{C}[\mathcal{F}]$ is simple in the algebraic sense of the word that it has exactly two ideals, by Theorem 6.1 in [3]. Each equation corresponding to a sentence in Θ is therefore true of $\mathfrak{C}[\mathcal{F}]$, so $\mathfrak{C}[\mathcal{F}]$ is a model of Δ.

Next, consider the case when $\mathcal{F}$ is not simple. By Decomposition Theorem 6.2 in [3], the algebra $\mathfrak{C}[\mathcal{F}]$ is isomorphic to a direct product of coset relation algebras $\mathfrak{C}[\mathcal{F}^{(\xi)}]$, where each $\mathcal{F}^{(\xi)}$ is *simple* in the sense that it is a maximal connected component of $\mathcal{E}$. Each algebra $\mathfrak{C}[\mathcal{F}^{(\xi)}]$ must be a model of Δ, by the observations of the preceding paragraph. Since equations are preserved under the passage to direct products, it follows that $\mathfrak{C}[\mathcal{F}]$ is a model of Δ. In other words, every full coset relation algebra is a model of Δ.

Finally, equations are also preserved under that passage to subalgebras, so any coset relation algebra—that is to say, any algebra embeddable into a full coset relation algebra—will be a model of Δ. This proves (4). Combine (3) and (4) to arrive at the desired conclusion of the theorem. $\square$

4. CRA IS NOT FINITELY AXIOMATIZABLE

We shall need the notion of the Lyndon algebra $\mathfrak{B}$ of a (projective) line ℓ (of order at least three) with at least two points. Let ℓ be any finite set, that is to say, any finite projective line, with at least two elements, and take 1' to be a new element not occurring in ℓ. The Boolean part of $\mathfrak{B}$ is the Boolean algebra of all subsets of the set $\ell^+ = \ell \cup \{1'\}$. Singletons $\{p\}$ are identified with the points p themselves. The identity element is taken to be 1', and converse is defined to be the identity function on the universe. Define the relative product of any two points p and q in ℓ^+ as follows:

$$
p\,;q = \begin{cases}
\ell \sim \{p, q\} & \text{if} \quad p \neq q, \\
p + 1' & \text{if} \quad p = q, \\
p & \text{if} \quad q = 1' \\
q & \text{if} \quad p = 1'.
\end{cases}
$$

Extend ; to a binary operation on the universe by making it distributive over arbitrary unions. The resulting algebra $\mathfrak{B}$ is well known to be a simple relation algebra (see Lyndon [12]).

Fix a Lyndon algebra $\mathfrak{B}$ on a finite line ℓ with at least two points. Assume, from now till Theorem 4.7, that $\mathfrak{B}$ is embeddable via a mapping ϑ into a full coset relation algebra $\mathfrak{C}[\mathcal{F}]$. Since $\mathfrak{B}$ is simple, it may be assumed that the triple $\mathcal{F}$ is simple in the sense that its quotient isomorphism index set $\mathcal{E}$ coincides with the universal relation $I \times I$ on the group index set I. In more detail, if $\vartheta(1)$ includes an atom of the form $R_{xy,\alpha}$ for some pair (x, y) in $\mathcal{E}$, take J to be the equivalence class of x in $\mathcal{E}$, and let $\mathcal{F}'$ be the restriction of $\mathcal{F}$ to J:

$$\mathcal{F}' = (G', \varphi', C'),$$

where G' is the system of groups G_x with x in J, and φ' is the system of quotient isomorphisms φ_{xy} with x, y in J, and similarly for the coset system C'. The projection π of $\mathfrak{C}[\mathcal{F}]$ to $\mathfrak{C}[\mathcal{F}']$ is a non-trivial homomorphism, since it maps the atom $R_{xy,\alpha}$ to itself, so the composition $\pi \circ \vartheta$ is a non-trivial homomorphism, and therefore an embedding, of the simple algebra $\mathfrak{B}$ into $\mathfrak{C}[\mathcal{F}']$.

The strategy of the proof is to show that all the subgroups H_{xy} are trivial, and $\mathfrak{C}[\mathcal{F}]$ is representable. Hence $\mathfrak{B}$ has to be representable since it is embeddable into $\mathfrak{C}[\mathcal{F}]$. Thus no non-representable Lyndon algebra can be in CRA. We then adapt Monk's proof in [15] that RRA is not finitely axiomatizable to show that the same applies to CRA.

Lemma 4.1. *If $H_{xy} \neq \{e_x\}$, then there is a unique point p in ℓ such that $(G_x \times G_y) \cap \vartheta(p) \neq \varnothing$. For this point p, we have*

$$(G_x \times G_y) \cup (G_y \times G_x) \subseteq \vartheta(p)$$

and

$$(G_x \cup G_y) \times (G_x \cup G_y) \subseteq \vartheta(p + 1').$$

Proof. Observe first that the hypothesis on H_{xy} implies that $x \neq y$, since $H_{xx} = \{e_x\}$. The set $U = \bigcup_{x \in I} G_x$ is the base set of $\mathfrak{C}[\mathcal{F}]$. The unit 1 of $\mathfrak{B}$ is the sum of the singletons, so it is the set ℓ^+, that is to say, it is the line ℓ with the identity element 1' adjoined. Use this observation, use that ℓ is finite and the fact that ϑ is an embedding of $\mathfrak{B}$ into $\mathfrak{C}[\mathcal{F}]$ to obtain

$$(1) \quad \bigcup\{G_u \times G_v : u, v \in I\} = \left(\bigcup_{u \in I} G_u\right) \times \left(\bigcup_{v \in I} G_v\right) = U \times U$$
$$= \vartheta(1) = \vartheta(\{1'\} \cup \sum\{p : p \in \ell\}) = \{\vartheta(1')\} \cup \bigcup\{\vartheta(p) : p \in \ell\}.$$

It is clear from (1) that

$$(2) \qquad\qquad (G_x \times G_y) \cap \vartheta(p) \neq \varnothing$$

for some p in ℓ^+. It is equally clear that $p \neq 1'$, since $G_x \times G_y$ is disjoint from the identity relation id_U, because the groups G_x and G_y are assumed to be disjoint, and id_U is the image of 1' under ϑ. The set $G_x \times G_y$ is the union of the relations $R_{xy,\alpha}$ for various α, so there must be an index α for which

$$R_{xy,\alpha} \cap \vartheta(p) \neq \varnothing,$$

by (2). The relation $R_{xy,\alpha}$ is an atom in $\mathfrak{C}[\mathcal{F}]$, and the image $\vartheta(p)$ is an element in $\mathfrak{C}[\mathcal{F}]$, so

$$(3) \qquad\qquad R_{xy,\alpha} \subseteq \vartheta(p),$$

by the definition of an atom.

Form the converse of both sides of (3), and use monotony, the embedding properties of ϑ, and the fact that converse is the identity function in $\mathfrak{B}$ to obtain

$$(4) \qquad\qquad R_{xy,\alpha}^{-1} \subseteq \vartheta(p)^{-1} = \vartheta(p^{\smile}) = \vartheta(p).$$

Apply Lemma 2.12, and then use (4), monotony, the embedding properties of ϑ, and the definition of relative multiplication in $\mathfrak{B}$ to arrive at

$$(5) \quad \bigcup\{R_{xx,g} : g \in H_{xy}\} = R_{xy,\alpha} \otimes R_{xy,\alpha}^{-1} \subseteq \vartheta(p) \otimes \vartheta(p) = \vartheta(p;p) = \vartheta(p+1\text{'}).$$

Use (5) and the fact that $R_{xx,g}$ is disjoint from the identity relation $id_U = \vartheta(1\text{'})$ when $g \neq e_x, g \in H_{xy}$, to conclude that

$$(6) \qquad\qquad R_{xx,g} \subseteq \vartheta(p)$$

for $g \neq e_x, g \in H_{xy}$.

Assume now for a contradiction that $R_{xy,\gamma}$ is not included in $\vartheta(p)$ for some γ. The first part of the proof shows that there must be a point q different from p such that

$$R_{xy,\gamma} \subseteq \vartheta(q).$$

The argument of the preceding paragraphs, with q in place of p, shows that

$$(7) \qquad\qquad R_{xx,g} \subseteq \vartheta(q)$$

for all $g \neq e_x, g \in H_{xy}$. Choose such a g, which certainly exists by the assumption that $H_{xy} \neq \{e_x\}$. We then have

$$R_{xx,g} \subseteq \vartheta(p) \cap \vartheta(q) = \vartheta(p \cdot q) = \vartheta(0) = \varnothing,$$

by (6), (7), and the embedding properties of ϑ. The desired contradiction has arrived, because the relation $R_{xx,g}$ is not empty. Conclusion:

$$R_{xy,\gamma} \subseteq \vartheta(p)$$

for all γ, that is to say,

$$(8) \qquad\qquad G_x \times G_y \subseteq \vartheta(p),$$

by Partition Lemma 2.2.

There cannot be another point

$$(G_x \times G_y) \cap \vartheta(q) \neq \varnothing,$$

for the preceding argument with q in place of p would give

$$(9) \qquad\qquad G_x \times G_y \subseteq \vartheta(q),$$

and therefore

$$G_x \times G_y \subseteq \vartheta(p) \cap \vartheta(q) = \vartheta(p \cdot q) = \vartheta(0) = \varnothing,$$

by (8), (9), and the embedding properties of ϑ. This is a clear absurdity.

Finally,

$$(10) \qquad G_y \times G_x = (G_x \times G_y)^{-1} \subseteq \vartheta(p)^{-1} = \vartheta(p^{\smile}) = \vartheta(p),$$

and

$$G_x \times G_x = (G_x \times G_y)\,|\,(G_y \times G_x) = (G_x \times G_y) \otimes (G_y \times G_x)$$
$$\subseteq \vartheta(p) \otimes \vartheta(p) = \vartheta(p;p) = \vartheta(p+1\text{'}),$$

by Lemma 2.13, (8), (10), the embedding properties of ϑ, and the definition of relative multiplication in $\mathfrak{B}$. Interchange x and y in this last computation to arrive at $G_y \times G_y \subseteq \vartheta(p + 1')$. This completes the proof of the lemma. $\square$

Definition 4.2. For a given point p in ℓ, define a binary relation $\sim_p$ on I by $x \sim_p y$ if and only if $G_x \times G_y \subseteq \vartheta(p + 1')$.

Lemma 4.3. $\sim_p$ *is an equivalence relation on its domain.*

Proof. If x is in the domain of $\sim_p$, then $x \sim_p y$ for some y, and consequently $G_x \times G_y$ is included in $\vartheta(p + 1')$, by Definition 4.2. Apply Lemma 4.1 to see that $G_x \times G_x$ and $G_y \times G_x$ are both included in $\vartheta(p + 1')$, so that $x \sim_p x$ and $y \sim_p x$. Thus, $\sim_p$ is reflexive on its domain, and also symmetric. If $x \sim_p y$ and $y \sim_p z$, then both $G_x \times G_y$ and $G_y \times G_z$ are included in $\vartheta(p)$. It follows from Lemma 2.13, the preceding inclusions, monotony, the embedding properties of ϑ, and the definition of relative multiplication in $\mathfrak{B}$ that

$$G_x \times G_z = (G_x \times G_y) \otimes (G_y \times G_z) \subseteq \vartheta(p) \otimes \vartheta(p) = \vartheta(p; p) = \vartheta(p + 1'),$$

so that $x \sim_p z$. Thus, $\sim_p$ is transitive. $\square$

Lemma 4.4. *For every x in I, there is a y in I such that $x \nsim_p y$.*

Proof. Suppose, for a contradiction, that $x \sim_p y$ for all y in I. In particular, for each z in I, we have $x \sim_p z$. Use symmetry and transitivity to obtain $y \sim_p z$ for all y and z in I. This means that $G_y \times G_z$ is included in $\vartheta(p + 1')$ for all y and z in I, by Definition 4.2, and therefore

$$U \times U = \bigcup \{G_y \times G_z : y, z \in I\} \subseteq \vartheta(p + 1').$$

Thus,

$$\vartheta(1) = U \times U = \vartheta(p + 1'),$$

and therefore $1 = p + 1'$, because ϑ is an embedding. But then the line ℓ has just one point, namely p, in contradiction to the assumption that it has at least two points. $\square$

We are close to our goal of proving that all subgroups H_{xy} must be trivial. We need one more lemma.

Lemma 4.5. *If $x \sim_p y$ and $H_{xy} \neq \{e_x\}$, then*

$$H_{xv} = \{e_x\}, \quad H_{yv} = \{e_y\}, \quad \text{and} \quad H_{vx} = H_{vy} = \{e_v\}$$

for all v in I such that $x \nsim_p v$.

Proof. Consider an element v in I such that $x \nsim_p v$, and suppose that $H_{xv} \neq \{e_x\}$ or $H_{vx} \neq \{e_v\}$. Apply Lemma 4.1 to obtain a unique q such that

$$(1) \qquad\qquad (G_x \cup G_v) \times (G_x \cup G_v) \subseteq \vartheta(q + 1'),$$

and therefore $x \sim_q v$. Observe that $q \neq p$, since $x \nsim_p v$. The assumption that $x \sim_p y$ implies that $G_x \times G_y$ is included in $\vartheta(p + 1')$, by Definition 4.2, so

$$(2) \qquad\qquad (G_x \cup G_y) \times (G_x \cup G_y) \subseteq \vartheta(p + 1'),$$

by Lemma 4.1. In particular, combine (1) and (2), and use the embedding properties of ϑ, and Boolean algebra, to see that

$$G_x \times G_x \subseteq \vartheta(p + 1') \cap \vartheta(q + 1') = \vartheta((p + 1') \cdot (q + 1'))$$
$$= \vartheta(p \cdot q + p \cdot 1' + q \cdot 1' + 1' \cdot 1') = \vartheta(1') = id_U.$$

This inclusion can hold only if G_x has just one element, that is to say, it can hold only if $G_x = \{e_x\}$, which would force $H_{xy} = \{e_x\}$. The desired contradiction has arrived, because it was assumed that $H_{xy} \neq \{e_x\}$, so we must have $H_{xv} = \{e_x\}$ and $H_{vx} = \{e_v\}$.

Next, suppose that $H_{yv} \neq \{e_y\}$ or $H_{vy} \neq \{e_v\}$. We must have $y \nsim_p v$, by transitivity, since $x \sim_p y$ and $x \nsim_p v$. Apply the preceding argument with x and y interchanged to arrive at a contradiction, and therefore to conclude that $H_{yv} = \{e_y\}$ and $H_{vy} = \{e_v\}$.   □

Theorem 4.6. $H_{xy} = \{e_x\}$ *for all x and y in I.*

Proof. Assume, for a contradiction, that $H_{xy} \neq \{e_x\}$, and observe as before that this forces $x \neq y$. By Lemma 4.1, there is a unique point p such that $G_x \times G_y$ is included in $\vartheta(p + 1')$, and consequently $x \sim_p y$. There is also a point v such that $x \nsim_p v$, by Lemma 4.4. Apply Lemma 4.5 to obtain

$$(1) \qquad H_{xv} = \{e_x\}, \qquad H_{yv} = \{e_y\}, \qquad H_{vx} = H_{vy} = \{e_v\}.$$

The quotient isomorphism φ_{xv} maps G_x/H_{xv} isomorphically to G_v/H_{vx} (recall that $K_{xv} = H_{vx}$, by Convention 2.7), so it maps $G_x/\{e_x\}$ isomorphically to $G_v/\{e_v\}$, by (1), that is to say, it maps distinct cosets of $\{e_x\}$ to distinct cosets of $\{e_v\}$. Image Theorem 2.10, together with Convention 2.7 and (1), implies that

$$(2) \qquad \varphi_{xv}[H_{xv} \circ H_{xy}] = K_{xv} \circ H_{vy} = H_{vx} \circ H_{vy} = \{e_v\}.$$

The composite subgroup $H_{xv} \circ H_{xy}$ is a union of cosets of H_{xv}, and φ_{xv} maps distinct cosets of H_{xv} to distinct cosets of H_{vx}, so (2) and the isomorphism properties of φ_{xv} imply that $H_{xv} \circ H_{xy}$ must be a coset of H_{xv}, and in fact it must be the identity coset $\{e_x\}$. Thus, $H_{xy} = \{e_x\}$, in contradiction to the assumption that these two subgroups are distinct.   □

A relation algebra is called *completely representable* if it has a representation in which all existing suprema are taken to set theoretic unions.

Theorem 4.7. *If a Lyndon algebra $\mathfrak{B}$ of a finite line with at least two points is embeddable into a full coset relation algebra $\mathfrak{C}[\mathcal{F}]$, then $\mathfrak{C}[\mathcal{F}]$ is completely representable and in fact it is isomorphic to a full group relation algebra. Hence, $\mathfrak{B}$ is representable.*

Proof. Because $\mathfrak{B}$ is simple, it may be assumed that the group triple $\mathcal{F}$ is simple as well, that is to say, its quotient isomorphism index set is the universal relation on the group index set I (see the remarks at the beginning of the section). The normal subgroups H_{xy} are all trivial, by Theorem 4.6. The definition of the atomic relations $R_{xy,\alpha}$ therefore implies that

$$R_{xy,\alpha} = \bigcup \{H_{xy,\gamma} \times (K_{xy,\gamma} \circ K_{xy,\alpha}) : \gamma < \kappa_{xy}\}$$
$$= \bigcup \{\{g\} \times \{\bar{g} \circ \bar{f}\} : g \in G_x\} = \{(g, \bar{g} \circ \bar{f}) : g \in G_x\},$$

where $\alpha = \{f\}$ and the quotient isomorphism φ_{xy} maps each element $\{g\}$ in $G_x/\{e_x\}$ to the corresponding element $\{\bar{g}\}$ in $G_y/\{e_y\}$. Such an atom is clearly a function, so $\mathfrak{C}[\mathcal{F}]$ is an atomic relation algebra with functional atoms, by Boolean Reduct Theorem 2.3. The Jónsson-Tarski [11] Representation Theorem for atomic relation algebras with functional atoms, in the form given by Andréka-Givant [2], implies that $\mathfrak{C}[\mathcal{F}]$ is completely representable. An atomic measurable relation algebra is completely representable if and only if it has a scaffold, which in turn happens if and only if its completion is isomorphic to a full group relation algebra, by Scaffold Representation Theorem 7.6, Corollary 7.7, and Theorem 7.8 in [8]. Thus, $\mathfrak{C}[\mathcal{F}]$ (which, being complete, is its own completion) is isomorphic to a full group relation algebra, and consequently $\mathfrak{B}$ is representable since it is isomorphic to a subalgebra of $\mathfrak{C}[\mathcal{F}]$. $\square$

Corollary 4.8. *No finite non-representable Lyndon algebra of a line with at least two points is in* CRA.

The only properties of $\mathfrak{B}$ that are used in the proofs leading up to Theorem 4.7 are that the unit 1 of $\mathfrak{B}$ is the sum of finitely many equivalence elements $e_i = p_i + 1$' for $1 \leq i \leq n$ and some $n \geq 2$, and these equivalence elements satisfy the equation $e_i \cdot e_j = 1$' for $i \neq j$.

Corollary 4.9. *Let $\mathfrak{C}[\mathcal{F}]$ be a full coset relation algebra on a simple group triple $\mathcal{F}$. If in $\mathfrak{C}[\mathcal{F}]$ the unit is the sum of finitely many reflexive equivalence elements for which the pairwise distinct meets are always the identity element, then $\mathfrak{C}[\mathcal{F}]$ is completely representable and is isomorphic to a full group relation algebra.*

Theorem 4.10. CRA *is not finitely axiomatizable. Moreover, if K is any class such that* RRA $\subseteq K \subseteq$ CRA*, then K is not finitely axiomatizable.*

Proof. The proof is a modified version of Monk's proof that the class RRA of representable relation algebras is not finitely axiomatizable. Assume RRA $\subseteq K \subseteq$ CRA. Let $\langle \mathfrak{B}_n : n \in \mathbb{N} \rangle$ be an infinite sequence of finite non-representable Lyndon algebras of lines with at least $n + 2$ points, indexed by the set $\mathbb{N}$ of natural numbers. Such a sequence exists by the Bruck-Ryser Theorem (for more details, see Monk [15]). None of the algebras in this sequence can belong to CRA, by Corollary 4.8. Let D be a non-principal ultrafilter in the Boolean algebra of subsets of $\mathbb{N}$, and form the ultraproduct

$$\mathfrak{A} = (\textstyle\prod_{n \in \mathbb{N}} \mathfrak{B}_n)/D.$$

Monk [15] proved that $\mathfrak{A}$ is representable. Consequently, $\mathfrak{A}$ belongs to RRA, which is a subclass of K by our assumption. Hence, the complement of K is not closed under ultraproducts, and so K cannot be finitely axiomatized by a well-known theorem of model theory (again, see Monk [15] for details). Since RRA coincides with GRA which is a subclass of CRA, we have RRA $\subseteq$ CRA $\subseteq$ CRA, hence CRA is not finitely axiomatizable. $\square$

We can also use Corollary 4.8 to prove an analogue of Jónsson's theorem [10, Theorem 3.5.6].

Theorem 4.11. *Any equational axiom system for* CRA *must use infinitely many variables. Moreover, if K is any class such that* RRA $\subseteq K \subseteq$ CRA*, then K is not axiomatizable by any set of universal formulas that contains only finitely many variables.*

Proof. The proof is a modified version of Jónsson's proof that the class RRA of representable relation algebras is not axiomatizable by any set of equations containing finitely many variables. Assume RRA $\subseteq$ K $\subseteq$ CRA. In the proof of [10, Theorem 3.5.6], Jónsson shows that for any natural number k there is a finite non-representable Lyndon algebra $\mathfrak{B}$ of a finite line with more than 2 points such that each k-generated subalgebra of $\mathfrak{B}$ is representable. By Corollary 4.8, this algebra $\mathfrak{B}$ is not in K, but all k-generated subalgebra of $\mathfrak{B}$ does belong to K, by our assumption RRA $\subseteq$ K $\subseteq$ CRA. This proves that K cannot be axiomatized by any set Σ of universal formulas such that Σ contains at most k variables. Since k can be chosen to be any natural number, we get that K cannot be axiomatized with any set of universal formulas that contains only finitely many variables. Since equations are universal formulas and RRA $\subseteq$ CRA, we get that CRA cannot be axiomatized with any set of equations that contains only finitely many variables. $\square$

It is shown in [1] that there are as many varieties between RRA and CRA as possible, i.e., continuum many. By our theorems above, none of these continuum many varieties can be axiomatized by a set of equations containing finitely many variables only, in particular, none of them is finitely axiomatizable.

We use infinitely many non-representable coset relation algebras when constructing the above continuum many varieties. However, any ultraproduct of these is also non-representable, because the "cause" of the non-representability in these algebras is expressible by a common first-order formula. This leaves open the following.

Problem 4.12. Is RRA finitely axiomatizable over CRA, i.e., is there a finite set Σ of equations such that RRA is exactly those coset relation agebras that satisfy Σ?

In the proof of the present Theorem 3.1, we also prove that CRA is the variety generated by the atomic measurable relation algebras. Problem 8.5 in [3] asks whether each measurable relation algebra can be embedded into an atomic measurable relation algebra. In the light of Theorem 3.1, this problem is equivalent with asking whether there is an equation that holds in all atomic measurable relation algebras but not in all measurable relation algebras.

Problem 4.13. Is CRA the variety generated by the class of measurable relation algebras?

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