# Andronov-Hopf and Bautin bifurcation in a tritrophic food chain model with Holling functional response types IV and II 

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#### Abstract

The existence of an Andronov-Hopf and Bautin bifurcation of a given system of differential equations is shown. The system corresponds to a tritrophic food chain model with Holling functional responses type $I V$ and $I I$ for the predator and superpredator, respectively. The linear and logistic growth is considered for the prey. In the linear case, the existence of an equilibrium point in the positive octant is shown and this equilibrium exhibits a limit cycle. For the logistic case, the existence of three equilibrium points in the positive octant is proved and two of them exhibit a simultaneous Hopf bifurcation. Moreover the Bautin bifurcation on these points are shown.


Keywords: Andronov-Hopf bifurcation, Bautin bifurcation, limit cycle, food chain model.

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## 1 Introduction

In the task of understanding the complexity presented by the interactions among the different populations living in a habitat, the mathematical modeling has been a very important rule in ecology in the last decades. Some of the models which have been studied are the tritrophic systems (see Ref. [3] and references therein). In particular, in this work we analyzed a tritrophic model given by the following differential equation system,

$$
\begin{align*}
& \frac{d x}{d t}=h(x)-f(x) y, \\
& \frac{d y}{d t}=c_{1} y f(x)-g(y) z-c_{2} y,  \tag{1.1}\\
& \frac{d z}{d t}=c_{3} g(y) z-d_{2} z,
\end{align*}
$$

[^0]where $x$ represents the density of a prey that gets eaten by a predator of density $y$ (mesopredator), and the species $y$ feeds the top predator $z$ (superpredator). The function $h(x)$ represents the growth rate of the prey population in absence of the predators, and the functions $f(x)$ and $g(y)$ are the functional responses for the mesopredator and the superpredator, respectively. The parameters $c_{1}, c_{2}, c_{3}$ and $d_{2}$ are positive and we are interested to find the stable solutions in the positive octant $\Omega=\{x>0, y>0, z>0\}$. There are different proposals of functional responses in literature, among which are the Holling type (see Refs. [5,10,11]). In Ref. [3], is considered the case when $h(x)$ is a logistic map and the functional responses $f$ and $g$ are Holling type II. Using the averaging theory, they proved that the system (1.1) has an equilibrium point which exhibits a triple Andronov-Hopf bifurcation. It implies the existence of a stable periodic orbit contained in the domain of interest.

The local dynamics of the differential system (1.1) has been analyzed in Ref. [2], when $h(x)$ is a linear map, and the functional responses $f$ and $g$ are Holling type III. They proved the existence of two equilibrium points which exhibit simultaneously a zero-Hopf bifurcation in $\Omega$. In Ref. [1], the authors analyzed the case when $h(x)$ is a linear map, and the functional responses $f$ and $g$ are Holling type III and Holling type II, respectively. They proved that there is a domain in the parameter space where the system (1.1) has a stable periodic orbit which results from an Andronov-Hopf bifurcation.

In this paper we are interested in analyzed the dynamics of the differential system (1.1) when the functional responses $f(x)$ and $g(y)$ are Holling type $I V$ and $I I$, respectively. In particular, we are interested in stable equilibrium points or stable limit cycles inside the positive octant $\Omega$. We consider two cases, the linear case, taking $h(x)=\rho x$, and the logistic case taking $h(x)=\rho x\left(1-\frac{x}{R}\right)$. The functions $f$ and $g$ will be

$$
f(x)=\frac{a_{1} x}{b_{1}+x^{2}}, \quad g(y)=\frac{a_{2} y}{b_{2}+y^{\prime}},
$$

where $a_{1}, b_{1}, a_{2}, b_{2}$ are positive constants. Explicitly, we will study the differential system

$$
\begin{align*}
& \frac{d x}{d t}=h(x)-\frac{a_{1} x y}{x^{2}+b_{1}}, \\
& \frac{d y}{d t}=\frac{c_{1} a_{1} y x}{x^{2}+b_{1}}-\frac{a_{2} y z}{b_{2}+y}-c_{2} y,  \tag{1.2}\\
& \frac{d z}{d t}=\frac{c_{3} a_{2} y z}{b_{2}+y}-d_{2} z .
\end{align*}
$$

Along this manuscript the terms linear or logistic case will be used to refer cases when the prey has either linear or logistic growth rate, respectively.

The main results in this paper are contained in Sections 2 and 3.

## 2 Linear case

In this section we consider the differential system (1.2) with a linear growth for the prey, this means that the function $h(x)=\rho x$ and then the differential system becomes

$$
\begin{align*}
& \dot{x}=-\frac{a_{1} x y}{b_{1}+x^{2}}+\rho x, \\
& \dot{y}=-c_{2} y+\frac{a_{1} c_{1} x y}{b_{1}+x^{2}}-\frac{a_{2} y z}{b_{2}+y^{\prime}}  \tag{2.1}\\
& \dot{z}=\left(-d_{2}+\frac{a_{2} c_{3} y}{b_{2}+y}\right) z .
\end{align*}
$$

In next lemma we show the existence of an equilibrium point in the positive octant $\Omega$ under certain conditions on the parameters involved in the system of differential equations.
Lemma 2.1. The differential system (2.1) has only one equilibrium point $p_{0}=\left(x_{0}, y_{0}, z_{0}\right) \in \Omega$ if
(a) $a_{2} c_{3}-d_{2}>0$,
(b) $a_{1} y_{0}-\rho b_{1}>0$,
(c) $c_{2} y_{0}-c_{1} x_{0} \rho<0$.

Moreover, if ones of above condition does not hold, then the differential system (2.1) does not have any equilibrium point in $\Omega$.
Proof. The equilibrium points of the differential system (2.1) are the solutions of

$$
\begin{aligned}
-\frac{a_{1} x y}{b_{1}+x^{2}}+\rho x & =0 \\
-c_{2} y+\frac{a_{1} c_{1} x y}{b_{1}+x^{2}}-\frac{a_{2} y z}{b_{2}+y} & =0 \\
\left(-d_{2}+\frac{a_{2} c_{3} y}{b_{2}+y}\right) z & =0
\end{aligned}
$$

By multiplying the above equations by the term $\left(b_{1}+x^{2}\right)\left(b_{2}+y\right)$, (which is always positive in $\Omega$ ), we obtain that an equilibrium point in $\Omega$ must satisfy the system

$$
\begin{align*}
\rho\left(b_{1}+x^{2}\right)-a_{1} y & =0 \\
\left(b_{2}+y\right)\left(c_{2}\left(b_{1}+x^{2}\right)-a_{1} c_{1} x\right)+a_{2} z\left(b_{1}+x^{2}\right) & =0  \tag{2.2}\\
d_{2}\left(b_{2}+y\right)-a_{2} c_{3} y & =0
\end{align*}
$$

From the third equation in system (2.2), $y_{0}=\frac{d_{2} b_{2}}{a_{2} c_{3}-d_{2}}$ and it is positive by hypothesis (a).
Substituting $y=y_{0}$ in the first equation of (2.2), we obtain a unique positive solution $x=x_{0}$ by hypothesis (b). Now, substituting $x=x_{0}$ and $y=y_{0}$ in the second equation of system (2.2), we have that the unique solution $z=z_{0}$ of this equation is positive, if and only if, $\left(c_{2}\left(b_{1}+x_{0}^{2}\right)-a_{1} c_{1} x_{0}\right)<0$, but, from the first equation in system (2.2), we have that $b_{1}+x_{0}^{2}=a_{1} y_{0} / \rho$, then $\left(c_{2}\left(b_{1}+x_{0}^{2}\right)-a_{1} c_{1} x_{0}\right)=\frac{a_{1}}{\rho}\left(c_{2} y_{0}-c_{1} x_{0} \rho\right)$, and $z_{0}>0$ by hypothesis (c).

Clearly, if ones of the conditions $a_{2} c_{3}-d_{2}>0, a_{1} y_{0}-\rho b_{1}>0$ or $c_{2} y_{0}-c_{1} x_{0} \rho<0$ does not hold then the differential system (2.1) has no equilibrium points in $\Omega$.

In order to simplify the expression of the equilibrium point $p_{0}$ we introduce a new parameters given by the next result.
Lemma 2.2. If the parameters of the system (2.1) satisfy the conditions (a), (b) and (c) in Lemma 2.1, then there exist $k_{1}>0, k_{2}>0$ and $k_{3}>0$, such that the parameters $a_{1}, a_{2}$ and $b_{2}$ involved in the differential system (2.1) can be written as

$$
\begin{equation*}
a_{2}=\frac{d_{2} \rho+k_{1}^{2}}{c_{3} \rho}, \quad b_{2}=\frac{b_{1} k_{1}^{2}+k_{2}^{2}}{a_{1} d_{2}}, \quad a_{1}=\frac{b_{1} c_{2} k_{1}^{2}+c_{2} k_{2}^{2}+k_{3}}{c_{1} k_{1} k_{2}}, \tag{2.3}
\end{equation*}
$$

and the unique equilibrium point of the system (2.1) in $\Omega$, is

$$
p_{0}=\left(\frac{k_{2}}{k_{1}}, \frac{c_{1} k_{2} \rho\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)}{k_{1}\left(b_{1} c_{2} k_{1}^{2}+c_{2} k_{2}^{2}+k_{3}\right)}, \frac{c_{1} c_{3} k_{2} k_{3} \rho}{b_{1} c_{2} d_{2} k_{1}^{3}+c_{2} d_{2} k_{1} k_{2}^{2}+d_{2} k_{1} k_{3}}\right) .
$$

Proof. The solutions of system (2.2) are

$$
\begin{aligned}
& p_{0}=\left(\frac{\sqrt{a_{1} b_{2} d_{2}+b_{1} \rho\left(d_{2}-a_{2} c_{3}\right)}}{\sqrt{\rho\left(a_{2} c_{3}-d_{2}\right)}}, \frac{b_{2} d_{2}}{a_{2} c_{3}-d_{2}}, \frac{c_{1} c_{3} \sqrt{\rho\left(a_{2} c_{3}-d_{2}\right)} \sqrt{\Delta_{1}}-b_{2} c_{2} c_{3} d_{2}}{d_{2}\left(a_{2} c_{3}-d_{2}\right)}\right), \\
& p_{1}=\left(-\frac{\sqrt{a_{1} b_{2} d_{2}+b_{1} \rho\left(d_{2}-a_{2} c_{3}\right)}}{\sqrt{\rho\left(a_{2} c_{3}-d_{2}\right)}}, \frac{b_{2} d_{2}}{a_{2} c_{3}-d_{2}},-\frac{c_{3}\left(c_{1} \sqrt{\rho\left(a_{2} c_{3}-d_{2}\right)} \sqrt{\Delta_{1}}+b_{2} c_{2} d_{2}\right)}{d_{2}\left(a_{2} c_{3}-d_{2}\right)}\right), \\
& \Delta_{1}=a_{1} b_{2} d_{2}+b_{1} \rho\left(d_{2}-a_{2} c_{3}\right) .
\end{aligned}
$$

Since $p_{1} \notin \Omega$, by Lemma $2.1 p_{0} \in \Omega$, and $\rho\left(a_{2} c_{3}-d_{2}\right)>0$, then there exists $k_{1}>0$ such that $a_{2}=\frac{d_{2} \rho+k_{1}^{2}}{c_{3} \rho}$. Hence

$$
p_{0}=\left(\frac{\sqrt{a_{1} b_{2} d_{2}-b_{1} k_{1}^{2}}}{k_{1}}, \frac{b_{2} d_{2} \rho}{k_{1}^{2}}, \frac{c_{3} \rho\left(c_{1} k_{1} \sqrt{a_{1} b_{2} d_{2}-b_{1} k_{1}^{2}}-b_{2} c_{2} d_{2}\right)}{d_{2} k_{1}^{2}}\right)
$$

Moreover, $a_{1} b_{2} d_{2}-b_{1} k_{1}^{2}>0$, then there exists $k_{2}>0$ such that $b_{2}=\frac{b_{1} k_{1}^{2}+k_{2}^{2}}{a_{1} d_{2}}$, then

$$
p_{0}=\left(\frac{k_{2}}{k_{1}}, \frac{\rho\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)}{a_{1} k_{1}^{2}}, \frac{c_{3} \rho\left(k_{2}\left(a_{1} c_{1} k_{1}-c_{2} k_{2}\right)-b_{1} c_{2} k_{1}^{2}\right)}{a_{1} d_{2} k_{1}^{2}}\right) .
$$

Since $k_{2}\left(a_{1} c_{1} k_{1}-c_{2} k_{2}\right)-b_{1} c_{2} k_{1}^{2}>0$, then there exists $k_{3}>0$ such that $a_{1}=\frac{b_{1} c_{2} k_{1}^{2}+c_{2} k_{2}{ }^{2}+k_{3}}{c_{1} k_{1} k_{2}}$, and

$$
p_{0}=\left(\frac{k_{2}}{k_{1}}, \frac{c_{1} k_{2} \rho\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)}{k_{1}\left(b_{1} c_{2} k_{1}^{2}+c_{2} k_{2}^{2}+k_{3}\right)}, \frac{c_{1} c_{3} k_{2} k_{3} \rho}{b_{1} c_{2} d_{2} k_{1}^{3}+c_{2} d_{2} k_{1} k_{2}^{2}+d_{2} k_{1} k_{3}}\right) .
$$

Lemma 2.3. Under the hypothesis of Lemma 2.2 and considering that the parameters $a_{1}, a_{2}$ and $b_{1}$ satisfy the conditions (2.3) and

$$
\begin{equation*}
k_{2}=\sqrt{2} \sqrt{b_{1}} k_{1}, \quad d_{2}=\frac{12 b_{1} k_{1}^{4}}{5 k_{3}}, \quad k_{3}=\frac{3}{2} b_{1} k_{1}^{2} \rho, \quad \text { and } \quad c_{2}=c_{20}(\rho):=\frac{9 k_{1}^{2}+38 \rho^{2}}{52 \rho} \tag{2.4}
\end{equation*}
$$

then the equilibrium point $p_{0}$ is given by

$$
p_{0}=\left(\sqrt{2} \sqrt{b_{1}}, \frac{52 \sqrt{2} \sqrt{b_{1}} c_{1} \rho^{2}}{9 k_{1}^{2}+64 \rho^{2}}, \frac{65 \sqrt{b_{1}} c_{1} c_{3} \rho^{4}}{\sqrt{2}\left(18 k_{1}^{4}+128 k_{1}^{2} \rho^{2}\right)}\right)
$$

and the eigenvalues of the linear approximation of system (2.1) at $p_{0}$ are

$$
\alpha=\frac{64 \rho}{39} \text { and } \pm i \omega,
$$

where

$$
\omega^{2}=\frac{k_{1}^{2}}{4}>0
$$

Proof. Taking into account the assignations of the parameters $a_{1}, a_{2}$ and $b_{1}$ given by (2.3), the characteristic polynomial of the linear approximation $M_{p_{0}}$ of differential system (2.1) at the equilibrium point $p_{0}$ is $P(\lambda)=\operatorname{det}\left(\lambda I-M_{p_{0}}\right)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}$, where,

$$
\begin{aligned}
& A_{1}=-\frac{\rho\left(2 k_{2}^{2}\left(d_{2} \rho+k_{1}^{2}\right)+d_{2} k_{3}\right)}{\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)\left(d_{2} \rho+k_{1}^{2}\right)}, \\
& A_{2}=\frac{d_{2} \rho^{2}\left(b_{1} k_{1}^{2}+k_{2}^{2}\right) h_{1}+k_{1}^{2} \rho\left(b_{1} k_{1}^{2}-k_{2}^{2}\right) h_{2}+d_{2} k_{1}^{2} k_{3}\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)}{\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)^{2}\left(d_{2} \rho+k_{1}^{2}\right)}, \\
& A_{3}=-\frac{2 d_{2} k_{1}^{2} k_{2}^{2} k_{3} \rho}{\left(b_{1} k_{1}^{2}+k_{2}^{2}\right)^{2}\left(d_{2} \rho+k_{1}^{2}\right)}, \\
& h_{1}=\left(b_{1} c_{2} k_{1}^{2}-c_{2} k_{2}^{2}+k_{3}\right), \\
& h_{2}=\left(b_{1} c_{2} k_{1}^{2}+c_{2} k_{2}^{2}+k_{3}\right) .
\end{aligned}
$$

If we consider the assignments for $k_{2}, k_{3}$ and $d_{2}$ given by (2.4), then $A_{1}, A_{2}$ and $A_{3}$ reduce to

$$
A_{1}=-\frac{64 \rho}{39}, \quad A_{2}=\frac{1}{78}\left(\rho\left(-26 c_{2}+19 \rho\right)+24 k_{1}^{2}\right) \quad \text { and } \quad A_{3}=-\frac{16 k_{1}^{2} \rho}{39} .
$$

The characteristic polynomial $P(\lambda)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$ if and only if $P(\lambda)=(\lambda-\alpha)\left(\lambda^{2}+\omega^{2}\right)=\lambda^{3}-\alpha \lambda^{2}+\omega^{2} \lambda-\alpha \omega^{2}$. Thus comparing coefficients, $P(\lambda)$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$ if and only if $A_{2}>0$ and

$$
\begin{equation*}
A_{1} A_{2}-A_{3}=0, \tag{2.5}
\end{equation*}
$$

where $\omega=\sqrt{A_{2}}$ and $\alpha=-A_{1}$. Since $A_{1} A_{2}-A_{3}=-\frac{16 \rho\left(-52 c_{2} \rho+9 k_{1}^{2}+38 \rho^{2}\right)}{1521}$, solving equation (2.5) for the parameter $c_{2}$, we have that if

$$
c_{2}=\frac{9 k_{1}^{2}+38 \rho^{2}}{52 \rho}
$$

then $A_{2}=\frac{k_{1}^{2}}{4}>0$. Thus, we conclude that the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$, where $\alpha=\frac{64 \rho}{39}$ and $\omega=\frac{k_{1}}{2}$. The equilibrium point $p_{0}$ becomes

$$
p_{0}=\left(\sqrt{2} \sqrt{b_{1}}, \frac{52 \sqrt{2} \sqrt{b_{1}} c_{1} \rho^{2}}{9 k_{1}^{2}+64 \rho^{2}}, \frac{65 \sqrt{b_{1}} c_{1} c_{3} \rho^{4}}{\sqrt{2}\left(18 k_{1}^{4}+128 k_{1}^{2} \rho^{2}\right)}\right) .
$$

In order to compute the Lyapunov coefficients and a regularity condition, from now in this section

$$
b_{1}=1, \quad k_{1}=1, \quad c_{1}=1 \quad \text { and } \quad c_{3}=1 .
$$

Remark 2.4. If the assumptions of Lemma 2.3 are satisfied, then the linear approximation of the differential system (2.1) at $p_{0}$ has the eigenvalues $\alpha=\frac{64 \rho}{39}$ and $\pm \frac{i}{2}$, when $c_{2}=c_{20}(\rho)$,
hence, by continuity on the eigenvalues, the linear approximation of the differential system at $p_{0}$ has a pair of complex eigenvalues,

$$
\lambda\left(p_{0}, c_{2}, \rho\right)=\xi\left(p_{0}, c_{2}, \rho\right) \pm i \omega\left(p_{0}, c_{2}, \rho\right)
$$

when $c_{2}$ is in a neighborhood of $c_{20}(\rho)$.
In order to compute the first Lyapunov coefficient $\ell_{1}$, we apply the Kuznetsov formula, (see Ref. [7]). Taking into account the assumptions of Lemma 2.3 and using the Mathematica software, we obtain the first Lyapunov coefficient of the differential system (2.1) at the equilibrium point $p_{0}$.

Lemma 2.5. If the hypotheses of Lemma 2.3 hold, then the first Lyapunov coefficient of the differential system (2.1) at the equilibrium point $p_{0}$ is

$$
\ell_{1}\left(p_{0}, c_{20}(\rho), \rho\right)=\frac{\left(64 \rho^{2}+9\right)\left(30074175488 \rho^{8}+9866010240 \rho^{6}-2504091294 \rho^{4}-1131103197 \rho^{2}-80677701\right)}{169 \rho^{3}\left(4096 \rho^{2}+1521\right)\left(16384 \rho^{2}+1521\right)\left(100 \rho^{4}+1252 \rho^{2}+81\right)}
$$

Corollary 2.6. There exists a unique real number $\rho_{0}>0$ such that $\ell_{1}\left(p_{0}, c_{20}\left(\rho_{0}\right), \rho_{0}\right)=0$.
Proof. By Lemma 2.5, $\ell_{1}\left(p_{0}, c_{20}(\rho), \rho\right)=0$ if and only if

$$
\begin{equation*}
30074175488 \rho^{8}+9866010240 \rho^{6}-2504091294 \rho^{4}-1131103197 \rho^{2}-80677701=0 \tag{2.6}
\end{equation*}
$$

According to the Descartes rule, there is a unique real number $\rho_{0}>0$ such that $\ell_{1}\left(p_{0}, c_{20}\left(\rho_{0}\right), \rho_{0}\right)=0$. Indeed, solving numerically equation (2.6) for the parameter $\rho$, we have that $\rho_{0}(\approx 0.57721)$.

Since $\ell_{1}\left(p_{0}, c_{20}(\rho), \rho\right)$ takes positive and negative values, we will verify the transversality conditions to have Andronov-Hopf or Bautin bifurcation. At first we state the following result proposed as an exercise in Ref. [6], whose proof is straight forward and we omit the details.

Lemma 2.7. Let $M(\tau)$ be a parameter-dependent real $(n \times n)$-matrix which has a simple pair of complex eigenvalues $\xi(\tau) \pm i \omega(\tau)$ such that $\xi\left(\tau_{0}\right)=0$ and $\omega\left(\tau_{0}\right):=\omega_{0}>0$. Then, the derivative of the real part of the complex eigenvalues at $\tau_{0}$ is given by

$$
\frac{d \xi}{d \tau}\left(\tau_{0}\right)=\operatorname{Re}\left(\overline{\mathbf{p}}^{t r} \cdot\left(\frac{d M}{d \tau}\left(\tau_{0}\right) \cdot \mathbf{q}\right)\right)
$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{C}^{n}$ are eigenvectors satisfying the normalization conditions

$$
M\left(\tau_{0}\right) \mathbf{q}=i \omega_{0}, \quad M^{t r}\left(\tau_{0}\right) \mathbf{p}=-i \omega_{0}, \quad \overline{\mathbf{q}}^{t r} \cdot \mathbf{q}=1 \quad \text { and } \quad \overline{\mathbf{p}}^{t r} \cdot \mathbf{q}=1
$$

We know proceed to show the regularity condition in order to obtain a Bautin bifurcation.
Lemma 2.8 (Bautin regularity condition). If the parameters $a_{1}, a_{2}, b_{2}, k_{2}, k_{3}$ and $d_{2}$ satisfy the hypothesis of Lemma 2.3, then the map $\left(c_{2}, \rho\right) \mapsto\left(\xi\left(p_{0}, c_{2}, \rho\right), \ell_{1}\left(p_{0}, c_{2}, \rho\right)\right)$ is regular at $\left(c_{0}, \rho_{0}\right)$, where $\xi\left(p_{0}, c_{2}, \rho\right)$ is given in Remark 2.4 and $c_{0}:=c_{20}\left(\rho_{0}\right)$.

Proof. By hypothesis, the linear approximation of the differential system (2.1) at $p_{0}$ depends only on the parameters $c_{2}$ and $\rho$, let $M_{p_{0}}\left(c_{2}, \rho\right)$ be this linear approximation. By Lemma 2.5, the complex numbers $\pm \frac{i}{2}$ are eigenvalues of $M_{p_{0}}\left(c_{0}, \rho_{0}\right)$, hence, the real part of the complex eigenvalues of $M_{p_{0}}\left(c_{0}, \rho_{0}\right)$, are

$$
\begin{equation*}
\xi\left(p_{0}, c_{0}, \rho_{0}\right)=0 \tag{2.7}
\end{equation*}
$$

let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_{0}}\left(c_{0}, \rho_{0}\right)$ for the corresponding eigenvalues $-\frac{i}{2}$ and $\frac{i}{2}$, respectively, such that

$$
\begin{equation*}
\overline{\mathbf{q}}^{t r} \cdot \mathbf{q}=1 \quad \text { and } \quad \overline{\mathbf{p}}^{t r} \cdot \mathbf{q}=1 \tag{2.8}
\end{equation*}
$$

By (2.7) and (2.8), we can apply Lemma 2.7 to the linear approximation $M_{p_{0}}\left(c_{2}, \rho\right)$, then taking into account the values of $\overline{\mathbf{p}}, \mathbf{q}$, and $\frac{\partial M_{p_{0}}\left(c_{2}, \rho\right)}{\partial c_{2}}$, which we compute with the Mathematica software, we have that the partial derivative of the real part of the eigenvalues $\xi\left(c_{2}, \rho\right) \pm i \omega\left(c_{2}, \rho\right)$ of $M_{p_{0}}\left(c_{2}, \rho\right)$, with respect to the parameter $c_{2}$, at the point $\left(c_{0}, \rho_{0}\right)$, takes the value

$$
\begin{equation*}
\frac{\partial \xi}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)=\overline{\mathbf{p}}^{t r}\left(\frac{\partial M_{p_{0}}\left(c_{0}, \rho_{0}\right)}{\partial c_{2}} \cdot \mathbf{q}\right)=-\frac{1664 \rho_{0}^{2}}{16384 \rho_{0}^{2}+1521} \tag{2.9}
\end{equation*}
$$

Applying Lemma 2.7 one more time, and taking into account the values of $\overline{\mathbf{p}}, \mathbf{q}$ and $\frac{\partial M\left(c_{2}, \rho\right)}{\partial \rho}$ it follows that

$$
\begin{equation*}
\frac{\partial \xi}{\partial \rho}\left(c_{0}, \rho_{0}\right)=\overline{\mathbf{p}}^{\operatorname{tr}}\left(\frac{\partial M_{p_{0}}\left(c_{0}, \rho_{0}\right)}{\partial \rho} \cdot \mathbf{q}\right)=\frac{32\left(38 \rho_{0}^{2}-9\right)}{16384 \rho_{0}^{2}+1521} \tag{2.10}
\end{equation*}
$$

From the Kuznetsov formula (see Ref. [4]), the first Lyapunov coefficient at the equilibrium point $p_{0}$ is given by

$$
\begin{equation*}
\ell_{1}\left(p_{0}, c_{2}, \rho\right)=\frac{\operatorname{Re} \mathbf{C}_{1}\left(c_{2}, \rho\right)}{\omega\left(c_{2}, \rho\right)}-\xi\left(c_{2}, \rho\right) \frac{\operatorname{Im} \mathbf{C}_{1}\left(c_{2}, \rho\right)}{\omega^{2}\left(c_{2}, \rho\right)} \tag{2.11}
\end{equation*}
$$

where $\mathbf{C}_{1}\left(c_{2}, \rho\right)$ is a function that takes complex values as a differentiable function in the variables $\left(c_{2}, \rho\right)$. Notice that, from Corollary 2.6, (2.7), (2.11) and since $\omega\left(c_{0}, \rho_{0}\right)=1 / 2$,

$$
\begin{equation*}
\operatorname{Re} \mathbf{C}_{1}\left(c_{0}, \rho_{0}\right)=0 \tag{2.12}
\end{equation*}
$$

Hence, from (2.11), (2.7) and (2.12), the partial derivative of $\ell_{1}\left(c_{2}, \rho\right)$ with respect to $c_{2}$ at the point ( $c_{0}, \rho_{0}$ ) is given by

$$
\frac{\partial \ell_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)=\frac{1}{\omega^{2}\left(c_{0}, \rho_{0}\right)}\left(\omega\left(c_{0}, \rho_{0}\right) \operatorname{Re}\left(\frac{\partial \mathbf{C}_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)\right)-\operatorname{Im} \mathbf{C}_{1}\left(c_{0}, \rho_{0}\right) \frac{\partial \mathcal{\xi}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)\right)
$$

and the partial derivative of $\ell_{1}\left(c_{2}, \rho\right)$ with respect to $\rho$ at the point $\left(c_{0}, \rho_{0}\right)$ is given by

$$
\frac{\partial \ell_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)=\frac{1}{\omega^{2}\left(c_{0}, \rho_{0}\right)}\left(\omega\left(c_{0}, \rho_{0}\right) \operatorname{Re}\left(\frac{\partial \mathbf{C}_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)\right)-\operatorname{Im} \mathbf{C}_{1}\left(c_{0}, \rho_{0}\right) \frac{\partial \xi}{\partial \rho}\left(c_{0}, \rho_{0}\right)\right),
$$

thus, the determinant of interest reduces to

$$
\begin{align*}
\operatorname{det}\left(\begin{array}{ll}
\frac{\partial \tilde{\xi}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right) & \frac{\partial \tilde{\xi}}{\partial \rho}\left(c_{0}, \rho_{0}\right) \\
\frac{\partial \ell_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right) & \frac{\partial \ell_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)
\end{array}\right) & \\
& =\frac{\frac{\partial \tilde{\xi}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right) \operatorname{Re}\left(\frac{\partial c_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)\right)-\frac{\partial \tilde{\xi}}{\partial \rho}\left(c_{0}, \rho_{0}\right) \operatorname{Re}\left(\frac{\partial c_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)\right)}{\omega\left(c_{0}, \rho_{0}\right)} . \tag{2.13}
\end{align*}
$$

Numerically, one has that $\operatorname{Re}\left(\frac{\partial \mathrm{C}_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right)\right) \approx-0.9053$ and $\operatorname{Re}\left(\frac{\partial \mathrm{C}_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)\right) \approx 2.48325$, and by Corollary 2.6, $\rho_{0} \approx 0.57721$. Then by (2.9), (2.10) and (2.13)

$$
\operatorname{det}\left(\begin{array}{ll}
\frac{\partial \xi}{\partial c_{1}}\left(c_{0}, \rho_{0}\right) & \frac{\partial \xi}{\partial \rho}\left(c_{0}, \rho_{0}\right) \\
\frac{\partial \partial_{1}}{\partial c_{2}}\left(c_{0}, \rho_{0}\right) & \frac{\partial \ell_{1}}{\partial \rho}\left(c_{0}, \rho_{0}\right)
\end{array}\right) \approx-0.18205 .
$$

Hence, the map $\left(c_{2}, \rho\right) \mapsto\left(\xi\left(p_{0}, c_{2}, \rho\right), \ell_{1}\left(p_{0}, c_{2}, \rho\right)\right)$ is regular at $\left(c_{0}, \rho_{0}\right)$.

Theorem 2.9. If the parameters $a_{1}, a_{2}, b_{2}, k_{2}, k_{3}$ and $d_{2}$ satisfy the hypothesis given in Lemma 2.3, then the differential system (2.1) exhibits an Andronov-Hopf bifurcation at $p_{0}=\left(\sqrt{2}, \frac{52 \sqrt{2} \rho^{2}}{64 \rho^{2}+9}, \frac{65 \rho^{4}}{\sqrt{2}\left(128 \rho^{2}+18\right)}\right)$, with respect to the parameter $c_{2}$ and its bifurcation value is $c_{20}(\rho)$, where $\rho>0$ and $\rho \neq \rho_{0}$. Moreover, if $\rho>\rho_{0}$ the bifurcation is subcritical and if $\rho<\rho_{0}$ the bifurcation is supercritical.

Proof. From Lemma 2.3, the linearization $M_{p_{0}}\left(c_{2}, \rho\right)$ of differential system (2.1) at $p_{0}$ has a positive real eigenvalue and a conjugate pair of pure imaginary eigenvalues if $c_{2}=c_{20}(\rho)$. From Lemma 2.8, the derivative of the real part of the complex eigenvalues is

$$
\frac{\partial \xi}{\partial c_{2}}\left(c_{20}(\rho), \rho\right)=-\frac{1664 \rho^{2}}{16384 \rho^{2}+1521},
$$

which is negative for $\rho \neq 0$, and hence the transversality condition holds. The Lemma 2.5 and Corollary 2.6 give a negative first Lyapunov coefficient if $\rho<\rho_{0}$, and a positive first Lyapunov coefficient if $\rho>\rho_{0}$. Then the hypotheses of Andronov-Hopf bifurcation Theorem (see Refs. [7-9]) hold and we conclude the proof.

Lemma 2.10 (Second Lyapunov coefficient). If we have the assumptions given in Lemma 2.3, then the second Lyapunov coefficient of the differential system (2.1) at the equilibrium point $p_{0}$ is given by

$$
\begin{align*}
& \ell_{2}\left(p_{0}, c_{20}(\rho), \rho\right) \\
& \quad=-\frac{\left(64 \rho^{2}+9\right)^{2} s_{1}(\rho)}{65804544 \rho^{9}\left(4096 \rho^{2}+1521\right)^{3}\left(16384 \rho^{2}+1521\right)^{3}\left(16384 \rho^{2}+13689\right) s_{2}(\rho)^{2}}, \tag{2.14}
\end{align*}
$$

where

$$
\begin{aligned}
s_{1}(\rho)= & 1684088318371577781870044208361897984 \rho^{26}- \\
& 12159352425316235727712314958979006464 \rho^{24}+ \\
& 84451000135751630806296323148790890496 \rho^{22}+ \\
& 88370770237252221116361066360845893632 \rho^{20}- \\
& 109618776714701834747940641433301549056 \rho^{18}- \\
& 194370158327281073384907679985062379520 \rho^{16}- \\
& 113112389947859122362340150200812175360 \rho^{14}- \\
& 33189611310495737671149541682647842816 \rho^{12}- \\
& 5137221528028189621494819571679640576 \rho^{10}- \\
& 305998757885518907545964388766063032 \rho^{8}+ \\
& 30440310395959735846728047367564897 \rho^{6}+ \\
& 7031366298566120280440132136776088 \rho^{4}+ \\
& 492932708224495242328372625695584 \rho^{2}+ \\
& 12343578321586192504727388915456, \\
s_{2}(\rho)= & 100 \rho^{4}+1252 \rho^{2}+81 .
\end{aligned}
$$

Moreover, if $\rho=\rho_{0}$, then $\ell_{2}\left(p_{0}, c_{20}(\rho), \rho\right) \neq 0$, where $\rho_{0}$ is given in the Corollary 2.6.

Proof. In order to compute the second Lyapunov coefficient $\ell_{2}$, we apply the Kuznetsov formula, (see Ref. [4]). Taking into account the assumptions of this Lemma and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_{2}\left(p_{0}, c_{20}(\rho), \rho\right)$, of the differential system (2.1) at the equilibrium point $p_{0}$ is given by (2.14) and $\ell_{2}\left(p_{0}, c_{20}\left(\rho_{0}\right), \rho_{0}\right) \approx$ 7.40065 .

Corollary 2.6, Lemma 2.8 and Lemma 2.10 provide the validity of the necessary and sufficient conditions to apply the Bautin bifurcation theorem (see Ref. [4]). In summary we have the following result.

Theorem 2.11 (Bautin bifurcation in linear growth). If the parameters $a_{1}, a_{2}, b_{2}, k_{2}, k_{3}$ and $d_{2}$ satisfy the hypothesis given in Lemma 2.3, then the differential system (2.1) exhibits a Bautin bifurcation at $p_{0}$, with respect to the parameters $c_{2}$ and $\rho$ and its critical bifurcation value is $\left(c_{20}\left(\rho_{0}\right), \rho_{0}\right)$.

## 3 Logistic case

In this section we consider the differential system (1.2) with a logistic growth for the prey, this means that the function $h(x)=\rho x\left(1-\frac{x}{R}\right)$ and we will analyze the differential system

$$
\begin{align*}
& \dot{x}=\rho x\left(1-\frac{x}{R}\right)-\frac{a_{1} x y}{b_{1}+x^{2}}, \\
& \dot{y}=\frac{a_{1} c_{1} x y}{b_{1}+x^{2}}-\frac{a_{2} y z}{b_{2}+y}-c_{2} y,  \tag{3.1}\\
& \dot{z}=z\left(\frac{a_{2} c_{3} y}{b_{2}+y}-d_{2}\right) .
\end{align*}
$$

In order to make ecological sense we assume that all parameters of the system (3.1) are positive.

Lemma 3.1. If the parameters $a_{1}, a_{2}, b_{1}, c_{1}$ and $R$, satisfy

$$
\begin{array}{lll}
a_{1}=\frac{\rho\left(b_{1}+x_{0}^{2}\right)\left(R-x_{0}\right)}{R y_{0}}, & a_{2}=\frac{d_{2}\left(b_{2}+y_{0}\right)}{c_{3} y_{0}}, & R=k_{2}+x_{0}, \\
c_{1}=\frac{c_{2} c_{3} y_{0}\left(k_{2}+x_{0}\right)+k_{3}}{c_{3} k_{2} \rho x_{0}}, & b_{1}=k_{2} x_{0}+k_{4}, \tag{3.2}
\end{array}
$$

then the unique equilibrium points of the differential system (3.1) in the region $\Omega$ are

$$
\begin{aligned}
& p_{1}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2}\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{2}=\left(\frac{k_{7}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{7} y_{0}\left(k_{8}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+2 x_{0}\right)\left(k_{8}+4 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{3}=\left(\frac{k_{8}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{8} y_{0}\left(k_{7}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right) .
\end{aligned}
$$

Where, $x_{0}>0, y_{0}>0, k_{3}>0, k_{7} \geq 0, k_{8} \geq 0$ and

$$
\begin{equation*}
k_{2}=\frac{4 x_{0}+k_{7}+k_{8}}{2}, \quad k_{4}=\frac{1}{4} k_{5} k_{6}, \quad k_{5}=2 x_{0}+k_{7}, \quad k_{6}=2 x_{0}+k_{8} . \tag{3.3}
\end{equation*}
$$

Proof. The equilibrium points of the differential system (3.1) are the solutions of the system,

$$
\begin{aligned}
\rho x\left(1-\frac{x}{R}\right)-\frac{a_{1} x y}{b_{1}+x^{2}} & =0, \\
\frac{a_{1} c_{1} x y}{b_{1}+x^{2}}-\frac{a_{2} y z}{b_{2}+y}-c_{2} y & =0, \\
z\left(\frac{a_{2} c_{3} y}{b_{2}+y}-d_{2}\right) & =0 .
\end{aligned}
$$

Multiplying the above equations by $\left(b_{1}+x^{2}\right)\left(b_{2}+y\right)$, (which is always positive in the region $\Omega$ ), we obtain that the equilibrium point must satisfy (3.4). Correspondingly each solution of (3.4) must be an equilibrium point of the differential system (3.1).

$$
\begin{align*}
a_{1} R y-\rho\left(b_{1}+x^{2}\right)(R-x) & =0 \\
\left(b_{2}+y\right)\left(c_{2}\left(b_{1}+x^{2}\right)-a_{1} c_{1} x\right)+a_{2} z\left(b_{1}+x^{2}\right) & =0  \tag{3.4}\\
d_{2}\left(b_{2}+y\right)-a_{2} c_{3} y & =0
\end{align*}
$$

A point $\left(x_{0}, y_{0}, z_{0}\right) \in \Omega$ is an equilibrium point of the differential system (3.1) if

$$
\begin{align*}
a_{1} R y_{0}-\rho\left(b_{1}+x_{0}^{2}\right)\left(R-x_{0}\right) & =0 \\
\left(b_{2}+y_{0}\right)\left(c_{2}\left(b_{1}+x_{0}^{2}\right)-a_{1} c_{1} x_{0}\right)+a_{2} z_{0}\left(b_{1}+x_{0}^{2}\right) & =0  \tag{3.5}\\
d_{2}\left(b_{2}+y_{0}\right)-a_{2} c_{3} y_{0} & =0
\end{align*}
$$

We suppose $x_{0}>0, y_{0}>0$ and $z_{0}>0$. Note that the first equation of the system (3.5) is a linear equation in terms of $a_{1}$, and it has the unique solution,

$$
a_{1}=\frac{\rho\left(b_{1}+x_{0}^{2}\right)\left(R-x_{0}\right)}{R y_{0}} .
$$

Since $a_{1}>0, R-x_{0}$ must be positive, so there exists $k_{2}>0$ such that $R=x_{0}+k_{2}$. A similar argument using the third equation of system (3.5), we obtain that:

$$
a_{2}=\frac{d_{2}\left(b_{2}+y_{0}\right)}{c_{3} y_{0}}
$$

Using the values of $a_{1}, a_{2}$ and $R$, and solving the second equation of system (3.5) for $z_{0}$, we have that

$$
z_{0}=\frac{c_{1} c_{3} k_{2} \rho x_{0}-c_{2} c_{3} y_{0}\left(k_{2}+x_{0}\right)}{d_{2}\left(k_{2}+x_{0}\right)}
$$

Since $z_{0}>0$, there must exists $k_{3}>0$, such that $c_{1} c_{3} k_{2} \rho x_{0}-c_{2} c_{3} y_{0}\left(k_{2}+x_{0}\right)=k_{3}$. Then

$$
c_{1}=\frac{c_{2} c_{3} y_{0}\left(k_{2}+x_{0}\right)+k_{3}}{c_{3} k_{2} \rho x_{0}}
$$

Therefore, if $a_{1}, a_{2}, R$ and $c_{1}$ satisfy (3.2), then $\left(x_{0}, y_{0}, z_{0}\right)$ is a solution of system (3.4) in $\Omega$, where $z_{0}=\frac{k_{3}}{d_{2}\left(k_{2}+x_{0}\right)}$. Moreover, the system (3.4) takes the form

$$
\begin{align*}
\frac{k_{2} \rho y\left(b_{1}+x_{0}^{2}\right)}{y_{0}}-\rho\left(b_{1}+x^{2}\right)\left(k_{2}-x+x_{0}\right) & =0 \\
\left(b_{2}+y\right)\left(c_{2}\left(b_{1}+x^{2}\right)-Q\right)+\frac{d_{2} z\left(b_{1}+x^{2}\right)\left(b_{2}+y_{0}\right)}{c_{3} y_{0}} & =0  \tag{3.6}\\
\frac{b_{2} d_{2}\left(y_{0}-y\right)}{y_{0}} & =0
\end{align*}
$$

where $Q=\frac{x\left(b_{1}+x_{0}{ }^{2}\right)\left(c_{c} c_{3} y_{0}\left(k_{2}+x_{0}\right)+k_{3}\right)}{c_{3} x_{0} y_{0}\left(k_{2}+x_{0}\right)}$. Solving the third equation of system (3.6) for $y$, we have that $y=y_{0}$. Moreover, the first equation of system (3.6) reduce to:

$$
\rho\left(x-x_{0}\right)\left(x^{2}-k_{2} x+b_{1}-k_{2} x_{0}\right)=0 .
$$

Hence, the solutions of this equation are

$$
x_{0}, \quad x_{1}:=\frac{1}{2}\left(k_{2}-\sqrt{k_{2}\left(k_{2}+4 x_{0}\right)-4 b_{1}}\right), \quad x_{2}:=\frac{1}{2}\left(k_{2}+\sqrt{k_{2}\left(k_{2}+4 x_{0}\right)-4 b_{1}}\right) .
$$

Thus, a necessary condition to have at least two solutions of system (3.6) in $\Omega$ is that $k_{2}\left(k_{2}+\right.$ $\left.4 x_{0}\right)-4 b_{1} \geq 0$. On the other hand, $x_{1}>0$ if and only if $0<k_{2}^{2}-\left(k_{2}\left(k_{2}+4 x_{0}\right)-4 b_{1}\right)=$ $4\left(b_{1}-k_{2} x_{0}\right)$, then, $x_{1}>0$ if and only if there exists $k_{4}>0$ such that $b_{1}=k_{2} x_{0}+k_{4}$, which is a hypothesis in (3.2). Let $k_{5}=k_{2}-\sqrt{k_{2}^{2}-4 k_{4}}>0$, then $k_{4}=\frac{1}{4} k_{5} k_{6}$, where $k_{6}=2 k_{2}-k_{5}>0$. Hence, $x_{1}=\frac{k_{5}}{2}$ and $x_{2}=\frac{k_{6}}{2}$.

Substituting $b_{1}, k_{4}, k_{5}, k_{2}, y=y_{0}$ and $x=x_{1}$ in the second equation of system (3.6) and solving this equation for $z$, we have that

$$
z_{1}=\frac{c_{2} c_{3} k_{6} y_{0}\left(k_{5}-2 x_{0}\right)\left(k_{5}+k_{6}+2 x_{0}\right)+2 k_{3} k_{5}\left(k_{6}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{5}+k_{6}\right)\left(k_{5}+k_{6}+2 x_{0}\right)} .
$$

Moreover, if $k_{5}-2 x_{0} \geq 0$, then $z_{1}>0$. In the same way, replacing $y=y_{0}$ and $x=x_{2}$ in (3.6), we obtain

$$
z_{2}=\frac{c_{2} c_{3} k_{5} y_{0}\left(k_{6}-2 x_{0}\right)\left(k_{5}+k_{6}+2 x_{0}\right)+2 k_{3} k_{6}\left(k_{5}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{5}+k_{6}\right)\left(k_{5}+k_{6}+2 x_{0}\right)} .
$$

Also, if $k_{6}-2 x_{0} \geq 0$, then $z_{2}>0$.
Let $k_{7}=k_{5}-2 x_{0}$ and $k_{8}=k_{6}-2 x_{0}$, then $x_{1}=\frac{k_{7}}{2}+x_{0}, x_{2}=\frac{k_{8}}{2}+x_{0}$, and $z_{1}, z_{2}$ becomes

$$
\begin{aligned}
& z_{1}=\frac{c_{2} c_{3} k_{7} y_{0}\left(k_{8}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+2 x_{0}\right)\left(k_{8}+4 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)} \\
& z_{2}=\frac{c_{2} c_{3} k_{8} y_{0}\left(k_{7}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)} .
\end{aligned}
$$

Therefore, the unique equilibrium points of the differential system (3.1) in the region $\Omega$, are:

$$
p_{1}=\left(x_{0}, y_{0}, z_{0}\right), \quad p_{2}=\left(x_{1}, y_{0}, z_{1}\right) \quad \text { and } \quad p_{3}=\left(x_{2}, y_{0}, z_{2}\right),
$$

which completes the proof.
Remark 3.2. Choosing the values $k_{7}, k_{8}$ adequately, we obtain one, two or three equillibria.

1. If $k_{7}=k_{8}=0$, then $p_{1}=p_{2}=p_{3}=\left(x_{0}, y_{0}, \frac{k_{3}}{3 d_{2} x_{0}}\right)$, is the unique equilibrium point of the differential system (3.1) in $\Omega$.
2. If $k_{7}=0$, and $k_{8}>0$ then $p_{1}=p_{2}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2} k_{8}+6 d_{2} x_{0}}\right)$, and

$$
p_{3}=\left(\frac{k_{8}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{8} y_{0}\left(k_{8}+6 x_{0}\right)+4 k_{3}\left(k_{8}+2 x_{0}\right)}{d_{2}\left(k_{8}+4 x_{0}\right)\left(k_{8}+6 x_{0}\right)}\right),
$$

hence, there are two equilibrium points of the differential system (3.1) in $\Omega$.
3. If $k_{7}>0, k_{8}>0$ and $k_{7} \neq k_{8}$ then

$$
\begin{aligned}
& p_{1}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2}\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{2}=\left(\frac{k_{7}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{7} y_{0}\left(k_{8}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+2 x_{0}\right)\left(k_{8}+4 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{3}=\left(\frac{k_{8}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{8} y_{0}\left(k_{7}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right)
\end{aligned}
$$

are three different equilibrium points of system (3.1) in $\Omega$.

### 3.1 One equilibrium point of the differential system

In this subsection, we assume that the parameters $a_{1}, a_{2}, b_{1}, c_{1}, R, k_{2}, k_{4}, k_{5}$ and $k_{6}$ satisfy the conditions (3.2) and (3.3) of Lemma 3.1, and $k_{7}=k_{8}=0$. Then according to Remark 3.2, $p_{1}=\left(x_{0}, y_{0}, \frac{k_{3}}{3 d_{2} x_{0}}\right)$ is the unique equilibrium point of the differential system (3.1) in $\Omega$.

Proposition 3.3. The equilibrium point $p_{1}$ is not hyperbolic and it has a local unstable manifold of dimension 2.

Proof. Under the hypothesis of this subsection, the characteristic polynomial of the linear approximation $M_{p_{1}}$ of differential system (3.1) at $p_{1}$ is

$$
P(\lambda)=-\lambda^{3}+\frac{k_{3}}{3 b_{2} c_{3} x_{0}+3 c_{3} x_{0} y_{0}} \lambda^{2}-\frac{\left(\rho\left(b_{2}+y_{0}\right)\left(3 c_{2} c_{3} x_{0} y_{0}+k_{3}\right)+3 b_{2} d_{2} k_{3}\right)}{9 c_{3} x_{0} y_{0}\left(b_{2}+y_{0}\right)} \lambda .
$$

The eigenvalues of $M_{p_{1}}$ are

$$
\begin{aligned}
& \lambda_{1}=0, \\
& \lambda_{2}=\frac{k_{3} y_{0}-\sqrt{y_{0}\left(4 c_{3} x_{0}\left(b_{2}+y_{0}\right)\left(-\rho\left(b_{2}+y_{0}\right)\left(3 c_{2} c_{3} x_{0} y_{0}+k_{3}\right)-3 b_{2} d_{2} k_{3}\right)+k_{3}^{2} y_{0}\right)}}{6 c_{3} x_{0} y_{0}\left(b_{2}+y_{0}\right)},
\end{aligned}
$$

and

$$
\lambda_{3}=\frac{k_{3} y_{0}+\sqrt{y_{0}\left(4 c_{3} x_{0}\left(b_{2}+y_{0}\right)\left(-\rho\left(b_{2}+y_{0}\right)\left(3 c_{2} c_{3} x_{0} y_{0}+k_{3}\right)-3 b_{2} d_{2} k_{3}\right)+k_{3}^{2} y_{0}\right)}}{6 c_{3} x_{0} y_{0}\left(b_{2}+y_{0}\right)} .
$$

Since $\lambda_{1}=0$, the equilibrium point $p_{1}$ of differential system (3.1) is not hyperbolic. Moreover, if $\lambda_{2}$ and $\lambda_{3}$ are complex then

$$
\operatorname{Re}\left(\lambda_{2}\right)=\operatorname{Re}\left(\lambda_{3}\right)=\frac{k_{3} y_{0}}{6 c_{3} x_{0} y_{0}\left(b_{2}+y_{0}\right)}>0
$$

It is not difficult to see that if $\lambda_{2}$ and $\lambda_{3}$ are real, then $\lambda_{2}>0$ and $\lambda_{3}>0$. Therefore the equilibrium point $p_{1}$ of differential system (3.1) has a local unstable manifold of dimension 2.

Corollary 3.4. The differential system (3.1) does not exhibit an Andronov-Hopf bifurcation at the equilibrium point $p_{1}=\left(x_{0}, y_{0}, \frac{k_{3}}{3 d_{2} x_{0}}\right)$.

### 3.2 Two equilibrium points of the differential system

From now on this subsection, we assume that the parameters $a_{1}, a_{2}, b_{1}, c_{1}, R, k_{2}, k_{4}, k_{5}$ and $k_{6}$ satisfy the conditions (3.2) and (3.3) of Lemma 3.1, $k_{7}=0$ and $k_{8}>0$. By Remark 3.2, $p_{1}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2} k_{8}+6 d_{2} x_{0}}\right)$, and $p_{2}=\left(\frac{k_{8}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{8} y_{0}\left(k_{8}+6 x_{0}\right)+4 k_{3}\left(k_{8}+2 x_{0}\right)}{d_{2}\left(k_{8}+4 x_{0}\right)\left(k_{8}+6 x_{0}\right)}\right)$ are the unique two equilibrium points of differential system (3.1) in $\Omega$.
Proposition 3.5. The equilibrium point $p_{1}$ is not hyperbolic and it has a local unstable manifold of dimension 2.

Proof. Considering the assignations for the parameters $a_{1}, a_{2}, b_{1}, c_{1}, R, k_{2}, k_{4}, k_{5}, k_{6}$ and $k_{7}$ given in this subsection, the characteristic polynomial of the linear approximation $M_{p_{1}}$ of differential system (3.1) at $p_{1}$ is

$$
\begin{aligned}
P(\lambda)= & -\lambda^{3}+\frac{2 k_{3}}{c_{3}\left(b_{2}+y_{0}\right)\left(k_{8}+6 x_{0}\right)} \lambda^{2} \\
& -\frac{\left(\rho\left(b_{2}+y_{0}\right)\left(k_{8}+2 x_{0}\right)\left(c_{2} c_{3} y_{0}\left(k_{8}+6 x_{0}\right)+2 k_{3}\right)+2 b_{2} d_{2} k_{3}\left(k_{8}+6 x_{0}\right)\right)}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{8}+6 x_{0}\right)^{2}} \lambda,
\end{aligned}
$$

and the eigenvalues of $M_{p_{1}}$ are

$$
\begin{aligned}
\lambda_{1}= & 0 \\
\lambda_{2}= & \frac{k_{3}}{c_{3}\left(b_{2}+y_{0}\right)\left(k_{8}+6 x_{0}\right)}-\sqrt{\frac{Q 1}{c_{3}^{2} y_{0}\left(b_{2}+y_{0}\right)^{2}\left(k_{8}+6 x_{0}\right)^{2}}}, \\
\lambda_{3}= & \frac{k_{3}}{c_{3}\left(b_{2}+y_{0}\right)\left(k_{8}+6 x_{0}\right)}+\sqrt{\frac{Q 1}{c_{3}^{2} y_{0}\left(b_{2}+y_{0}\right)^{2}\left(k_{8}+6 x_{0}\right)^{2}}}, \\
Q_{1}= & -2 b_{2}^{2} c_{3} d_{2} k_{3}\left(k_{8}+6 x_{0}\right)-c_{3} \rho\left(b_{2}+y_{0}\right)^{2}\left(k_{8}+2 x_{0}\right)\left(c_{2} c_{3} y_{0}\left(k_{8}+6 x_{0}\right)+2 k_{3}\right) \\
& \quad+k_{3} y_{0}\left(k_{3}-2 b_{2} c_{3} d_{2}\left(k_{8}+6 x_{0}\right)\right) .
\end{aligned}
$$

Then the equilibrium point $p_{1}$ of differential system (3.1) is not hyperbolic. Moreover, if $\lambda_{2}$ and $\lambda_{3}$ are complex then

$$
\operatorname{Re}\left(\lambda_{2}\right)=\operatorname{Re}\left(\lambda_{3}\right)=\frac{k_{3}}{c_{3}\left(b_{2}+y_{0}\right)\left(k_{8}+6 x_{0}\right)}>0 .
$$

And it can be verify that if $\lambda_{2}$ and $\lambda_{3}$ are real then $\lambda_{2}>0$ and $\lambda_{3}>0$. Therefore the equilibrium point $p_{1}$ of differential system (3.1) is not hyperbolic and has a local unstable manifold of dimension 2.

Corollary 3.6. The differential system (3.1) does not exhibit an Andronov-Hopf bifurcation at the equilibrium point $p_{1}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2} k_{8}+6 d_{2} x_{0}}\right)$.

Whereas the equilibrium point $p_{1}$ does not have an Andronov-Hopf bifurcation, we will show that the equilibrium point $p_{2}$ can have a pair of purely imaginary eigenvalues and consequently it can exhibit an Andronov-Hopf bifurcation.
Lemma 3.7. If the parameters $k_{8}, b_{2}, k_{3}, c_{2}, \rho$ and $d_{2}$ satisfy the conditions

$$
\begin{align*}
& k_{8}=x_{0}, \quad b_{2}=\frac{c_{3} x_{0} y_{0}\left(35 c_{2}+\rho\right)+60 k_{3}}{3 c_{3} \rho x_{0}}, \quad k_{3}=c_{2} c_{3} x_{0} y_{0}, \quad c_{2}=\frac{581875-5877 \rho^{2}}{143640 \rho}, \\
& \rho<\sqrt{\frac{581875}{5877}}, \quad d_{2}=d_{20}(\rho):=\frac{320060160 \rho^{3}}{\left(116375-873 \rho^{2}\right)\left(581875-5877 \rho^{2}\right)^{\prime}} \tag{3.7}
\end{align*}
$$

then the equilibrium point $p_{2}$ of differential system (3.1) is given by

$$
p_{2}=\left(\frac{3 x_{0}}{2}, y_{0}, \frac{c_{3}\left(581875-5877 \rho^{2}\right)^{2}\left(116375-873 \rho^{2}\right) y_{0}}{84687918336000 \rho^{4}}\right)
$$

and the eigenvalues of the linear approximation of system (3.1) at $p_{2}$ are

$$
\alpha=-\frac{2592 \rho^{3}}{475\left(9 \rho^{2}+30625\right)} \text { and } \pm i .
$$

Proof. Let $M_{p_{2}}$ be the Jacobian matrix of the differential system (3.1) evaluated at the equilibrium point $p_{2}$, then the characteristic polynomial $P(\lambda)=\operatorname{det}\left(\lambda I-M_{p_{2}}\right)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+$ $A_{3}$, where

$$
\begin{aligned}
A_{1}= & -\frac{\lambda^{2}\left(\left(k_{8}+4 x_{0}\right)\left(c_{2} c_{3} k_{8} y_{0}\left(k_{8}+6 x_{0}\right)+4 k_{3}\left(k_{8}+2 x_{0}\right)\right)-c_{3} k_{8}^{2} \rho\left(b_{2}+y_{0}\right)\left(k_{8}+2 x_{0}\right)\right)}{c_{3}\left(b_{2}+y_{0}\right)\left(k_{8}+4 x_{0}\right)^{2}\left(k_{8}+6 x_{0}\right)}, \\
A_{2}= & -\frac{-\frac{-b_{2} d_{2}\left(k_{8}+6 x_{0}\right)\left(k_{8}+4 x_{0}\right)^{2}\left(c_{2} c_{3} k_{8} y_{0}\left(k_{8}+\left(x_{0}\right)\right)+4 k_{3}\left(k_{8}+2 x_{0}\right)\right)-B_{1}}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{8}+4 x_{0}\right)^{3}\left(k_{8}+6 x_{0}\right)^{2}},}{} \\
& -\frac{\rho y_{0}\left(k_{8}+2 x_{0}\right)\left(k_{8}+4 x_{0}\right)\left(c_{2} c_{3} y_{0}\left(k_{8}+6 x_{0}\right)\left(k_{8}^{2}+4 k_{8} x_{0}-16 x_{0}{ }^{2}\right)+4 k_{3}\left(k_{8}+4 x_{0}\right)\left(k_{8}-2 x_{0}\right)\right)}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{8}+4 x_{0}\right)^{3}\left(k_{8}+6 x_{0}\right)^{2}}, \\
A_{3}= & \frac{b_{2} d_{2} k_{8}^{2} \rho\left(k_{8}+2 x_{0}\right)\left(c_{2} c_{3} k_{8 y} y_{0}\left(k_{8}+6 x_{0}\right)+4 k_{3}\left(k_{8}+2 x_{0}\right)\right),}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{8}+4 x_{0}\right)^{3}\left(k_{8}+6 x_{0}\right)^{2}}, \\
B_{1}= & 8 b_{2} \rho x_{0}\left(k_{8}+2 x_{0}\right)\left(8 x_{0}^{2}-k_{8}^{2}\right)\left(c_{2} c_{3} y_{0}\left(k_{8}+6 x_{0}\right)+k_{3}\right) .
\end{aligned}
$$

Taking $k_{8}, b_{2}$ and $k_{3}$ satisfying (3.7), then $A_{1}, A_{2}$ and $A_{3}$ are reduced to

$$
\begin{aligned}
& A_{1}=\frac{12 \rho^{2}}{175\left(95 c_{2}+4 \rho\right)}, \quad A_{2}=\frac{c_{2}\left(665 c_{2}\left(475 d_{2}+216 \rho\right)+\rho\left(3325 d_{2}+5877 \rho\right)\right)}{6125\left(95 c_{2}+4 \rho\right)} \\
& A_{3}=\frac{57 c_{2} d_{2} \rho\left(95 c_{2}+\rho\right)}{6125\left(95 c_{2}+4 \rho\right)} .
\end{aligned}
$$

Following the proof of Lemma 2.3, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$ if and only if $A_{2}>0$ and

$$
\begin{equation*}
A_{1} A_{2}-A_{3}=0, \tag{3.8}
\end{equation*}
$$

where $\omega=\sqrt{A_{2}}$ and $\alpha=-A_{1}$.
Since $A_{1} A_{2}-A_{3}=-\frac{3 c_{2} \rho\left(30008125 c_{2}{ }^{2} d_{2}+665 c_{2} \rho\left(475 d_{2}-864 \rho\right)-23508 \rho^{3}\right)}{1071875\left(95 c_{2}+4 \rho\right)^{2}}$, solving equation (3.8) for the parameter $d_{2}$, we have that

$$
d_{2}=\frac{36 \rho^{2}\left(15960 c_{2}+653 \rho\right)}{315875 c_{2}\left(95 c_{2}+\rho\right)} .
$$

Taking into account this assignment for $d_{2}$, the coefficient $A_{2}=\frac{9 \rho\left(15960 c_{2}+653 \rho\right)}{581875}>0$, which is equal to 1 , when $c_{2}=\frac{581875-5877 \rho^{2}}{143640 \rho}$. Moreover, $c_{2}>0$, if

$$
\rho<\sqrt{\frac{581875}{5877}} .
$$

Therefore, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i$ and a real root $\alpha=-\frac{2592 \rho^{3}}{475\left(9 \rho^{2}+30625\right)}$. The equilibrium point $p_{2}$ of system (3.1) becomes

$$
p_{2}=\left(\frac{3 x_{0}}{2}, y_{0}, \frac{c_{3}\left(581875-5877 \rho^{2}\right)^{2}\left(116375-873 \rho^{2}\right) y_{0}}{84687918336000 \rho^{4}}\right) .
$$

Remark 3.8. If the assumptions of Lemma 3.7 are satisfied, then the linear approximation of the differential system (3.1) at $p_{2}$ has two eigenvalues purely imaginary when $d_{2}=d_{20}(\rho)$. Hence, by continuity on the eigenvalues, the linear approximation of the differential system at $p_{2}$ has a pair of complex eigenvalues,

$$
\lambda\left(p_{2}, d_{2}, \rho\right)=\xi\left(p_{2}, d_{2}, \rho\right) \pm i \omega\left(p_{2}, d_{2}, \rho\right),
$$

when $d_{2}$ is in a neighborhood of $d_{20}(\rho)$.

In order to compute the first Lyapunov coefficient $\ell_{1}$, we make the following assignations.

$$
c_{3}=1, \quad y_{0}=1, \quad \text { and } \quad x_{0}=1 .
$$

Applying the Kuznetsov formula, (see Ref. [7]) and using the Mathematica software, we obtain the first Lyapunov coefficient $\ell_{1}\left(p_{2}, d_{20}(\rho), \rho\right)$, of the differential system (3.1) at the equilibrium point $p_{2}$.

Lemma 3.9. If we have the assumptions given in Lemma 3.7, then the eigenvalues of the linear approximation of system (3.1) at the equilibrium point $p_{2}$ are $\alpha=-\frac{2592 \rho^{3}}{475\left(9 \rho^{2}+30625\right)}$ and $\pm i$, and the first Lyapunov coefficient

$$
\ell_{1}\left(p_{2}, d_{20}(\rho), \rho\right)=\frac{37791360 \rho^{5}\left(9 \rho^{2}+30625\right) s_{3}(\rho)}{49\left(5877 \rho^{2}-581875\right) s_{4}(\rho) s_{6}(\rho) s_{5}(\rho)}
$$

where

$$
\begin{align*}
s_{3}(\rho)= & 667911733488169984885774464 \rho^{14} \\
& +617401358762851620995638930875 \rho^{12} \\
& -44174467587992130895876439343500 \rho^{10} \\
& +233287934641973329538653798974609375 \rho^{8} \\
& +177275906705124540208423981933593750000 \rho^{6}  \tag{3.9}\\
& -84149358184504925595752439022064208984375 \rho^{4} \\
& +10783804142077921019941784620285034179687500 \rho^{2} \\
& -433192260734995606409440033137798309326171875
\end{align*}
$$

and

$$
\begin{aligned}
s_{4}(\rho)= & 211611572265625+9 \rho^{2}\left(13819531250+2030625 \rho^{2}+186624 \rho^{4}\right) \\
s_{5}(\rho)= & 211611572265625+9 \rho^{2}\left(13819531250+2030625 \rho^{2}+746496 \rho^{4}\right) \\
s_{6}(\rho)= & 4585416449713134765625+9 \rho^{2}(87419516846757812500 \\
& \left.+27 \rho^{2}\left(2877811231676281250+792497391678300 \rho^{2}+108326221587 \rho^{4}\right)\right) .
\end{aligned}
$$

Corollary 3.10. If we have the assumptions given in Lemma 3.7, then there exists a unique real number $0<\rho_{0}<\sqrt{\frac{581875}{5877}}$, such that the first Lyapunov coefficient $\ell_{1}\left(p_{2}, d_{20}\left(\rho_{0}\right), \rho_{0}\right)=0$.

Proof. By Lemma 3.9 and equation (3.9), the first Lyapunov coefficient $\ell_{1}\left(p_{2}, d_{20}(\rho), \rho\right)=0$ if and only if $s_{3}(\rho)=0$. By Descartes rule of signs, there are 1,3 or 5 positive real numbers $\rho$ such that $s_{3}(\rho)=0$. Indeed, numerically this equation has three positive solutions, but only $\rho_{0}(\approx 9.76907)$ is less than $\sqrt{\frac{581875}{5877}}$.

Lemma 3.11 (Bautin regularity condition). If the parameters $k_{8}, b_{2}, k_{3}, c_{2}$ and $\rho$ satisfy the relations (3.7) of Lemma 3.7, then the map $\left(d_{2}, \rho\right) \mapsto\left(\xi\left(p_{2}, d_{2}, \rho\right), \ell_{1}\left(p_{2}, d_{2}(\rho), \rho\right)\right)$ is regular at $\left(d_{0}, \rho_{0}\right)$, where $\xi\left(p_{2}, d_{2}, \rho\right)$ is given in Remark 3.8 and $d_{0}:=d_{20}\left(\rho_{0}\right)$.

Proof. By hypothesis, the linear approximation of the differential system (3.1) at $p_{2}$ depends only on the parameters $d_{2}$ and $\rho$, let $M_{p_{2}}\left(d_{2}, \rho\right)$ be this linear approximation. By Lemma 3.7, the real part of the complex eigenvalues of $M_{p_{2}}\left(d_{0}, \rho_{0}\right)$ are

$$
\begin{equation*}
\xi\left(p_{2}, d_{0}, \rho_{0}\right)=0 \tag{3.10}
\end{equation*}
$$

Let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_{2}}\left(d_{0}, \rho_{0}\right)$ for the corresponding eigenvalues $-i$ and $i$, respectively, such that

$$
\begin{equation*}
\overline{\mathbf{q}}^{t r} \cdot \mathbf{q}=1 \quad \text { and } \quad \overline{\mathbf{p}}^{t r} \cdot \mathbf{q}=1 \tag{3.11}
\end{equation*}
$$

By (3.10) and (3.11), we can apply Lemma 2.7 to the linear approximation $M_{p_{2}}\left(d_{2}, \rho\right)$. Taking into account the values of $\mathbf{q}, \overline{\mathbf{p}}$ and $\frac{\partial M_{p_{0}}\left(d_{2}, \rho\right)}{\partial d_{2}}$, and using the Mathematica software, we obtain the partial derivative of the real part of the eigenvalues $\xi\left(d_{2}, \rho\right) \pm i \omega\left(d_{2}, \rho\right)$ of $M_{p_{2}}\left(d_{2}, \rho\right)$,

$$
\begin{gather*}
\frac{\partial \xi}{\partial d_{2}}\left(d_{0}, \rho_{0}\right)=\frac{5\left(581875-5877 \rho_{0}^{2}\right)^{2}\left(116375-873 \rho_{0}^{2}\right)}{Q_{2}}  \tag{3.12}\\
Q_{2}=49392\left(9\left(746496 \rho_{0}^{4}+2030625 \rho_{0}^{2}+13819531250\right) \rho_{0}^{2}+211611572265625\right)
\end{gather*}
$$

Applying Lemma 2.7 one more time, it follows from the values of $\mathbf{q}, \overline{\mathbf{p}}$ and $\frac{\partial M_{p_{2}}\left(d_{2}, \rho\right)}{\partial \rho}$ that

$$
\begin{gather*}
\frac{\partial \xi}{\partial \rho}\left(d_{0}, \rho_{0}\right)=-\frac{97200 \rho_{0}^{2}\left(1710207 \rho_{0}^{4}+397304250 \rho_{0}^{2}-67715703125\right)}{Q_{3}}  \tag{3.13}\\
Q_{3}=\left(873 \rho_{0}^{2}-116375\right)\left(9\left(746496 \rho_{0}^{4}+2030625 \rho_{0}^{2}+13819531250\right) \rho_{0}^{2}+211611572265625\right)
\end{gather*}
$$

Using the Wolfram Mathematica software, we have that $\operatorname{Re}\left(\frac{\partial \mathrm{C}_{1}}{\partial d_{2}}\left(d_{0}, \rho_{0}\right)\right) \approx-0.22637$ and $\operatorname{Re}\left(\frac{\partial \mathbf{C}_{1}}{\partial \rho}\left(d_{0}, \rho_{0}\right)\right) \approx 158.86065$, where, $\mathbf{C}_{1}\left(d_{2}, \rho\right)$ is the function given in the proof of Lemma 2.8, and by Corollary 3.10, $\rho_{0} \approx 9.76907$. Then by (3.12), (3.13) and the analogous of formula (2.13) given in the proof of Lemma 2.8, we have that

$$
\operatorname{det}\left(\begin{array}{ll}
\frac{\partial \xi}{\partial d_{2}}\left(d_{0}, \rho_{0}\right) & \frac{\partial \xi}{\partial \rho}\left(d_{0}, \rho_{0}\right) \\
\frac{\partial \ell_{1}}{\partial d_{2}}\left(d_{0}, \rho_{0}\right) & \frac{\partial \ell_{1}}{\partial \rho}\left(d_{0}, \rho_{0}\right)
\end{array}\right) \approx-0.00291456
$$

Hence, the map $\left(d_{2}, \rho\right) \mapsto\left(\xi\left(p_{2}, d_{2}, \rho\right), \ell_{1}\left(p_{2}, d_{2}, \rho\right)\right)$ is regular at $\left(d_{0}, \rho_{0}\right)$.

Theorem 3.12. If the parameters $k_{8}, b_{2}, k_{3}, c_{2}$ and $\rho$ satisfy the relations (3.7) of Lemma 3.7, then the differential system (3.1) exhibits an Andronov-Hopf bifurcation at

$$
p_{2}=\left(\frac{3}{2}, 1, \frac{\left(581875-5877 \rho^{2}\right)^{2}\left(116375-873 \rho^{2}\right)}{84687918336000 \rho^{4}}\right)
$$

with respect to the parameter $d_{2}$ and its critical bifurcation value is $d_{20}(\rho)$, where $\rho \in\left(0, \sqrt{\frac{581875}{5877}}\right)$ and $\rho \neq \rho_{0}$. Moreover, if $\rho<\rho_{0}$ the bifurcation is subcritical and if $\rho>\rho_{0}$ the bifurcation is supercritical.

Proof. From Lemma 3.7, the linear approximation $M_{p_{2}}\left(d_{2}, \rho\right)$ of differential system (3.1) at $p_{2}$ has a negative real eigenvalue and a pair of purely imaginary eigenvalues if $d_{2}=d_{20}(\rho)$. From Lemma 3.11, the derivative of the real part of the complex eigenvalues

$$
\begin{gathered}
\frac{\partial \xi}{\partial d_{2}}\left(d_{20}(\rho), \rho\right)=\frac{5\left(581875-5877 \rho^{2}\right)^{2}\left(116375-873 \rho^{2}\right)}{Q_{4}}, \\
Q_{4}=49392\left(9\left(746496 \rho^{4}+2030625 \rho^{2}+13819531250\right) \rho^{2}+211611572265625\right),
\end{gathered}
$$

which is positive if $\rho \in\left(0, \sqrt{\frac{581875}{5877}}\right)$, and hence the transversality condition holds. By Corollary 3.10 the first Lyapunov coefficient is negative if $\rho>\rho_{0}$, and is positive if $\rho<\rho_{0}$. Then the hypotheses of Andronov-Hopf bifurcation Theorem hold and we conclude the proof (see Refs. [7-9]).

Lemma 3.13 (Second Lyapunov coefficient). If we have the assumptions given in Lemma 3.7 and $\rho=\rho_{0}$, then the second Lyapunov coefficient of differential system (3.1) at the equilibrium point $p_{2}$, $\ell_{2}\left(p_{2}, d_{20}\left(\rho_{0}\right), \rho_{0}\right) \neq 0$.

Proof. In order to compute the second Lyapunov coefficient $\ell_{2}$, we apply the Kuznetsov formula, (see Ref. [4]). Taking into account the assumptions of this Lemma and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_{2}\left(p_{2}, d_{20}(\rho), \rho\right)$, of the differential system (3.1) at the equilibrium point $p_{2}$ takes the value $\ell_{2}\left(p_{2}, d_{20}\left(\rho_{0}\right), \rho_{0}\right) \approx 8894.15$, if $\rho=\rho_{0}$.

Corollary 3.10, Lemma 3.11 and Lemma 3.13 provide the validity of the necessary and sufficient conditions to apply the Bautin bifurcation theorem (see Ref. [4]). Then we have obtained the following.

Theorem 3.14. If the parameters $k_{8}, b_{2}, k_{3}, c_{2}$ and $\rho$ satisfy the relations (3.7) of Lemma 3.7, then the differential system (3.1) exhibits a Bautin bifurcation at $p_{2}$, with respect to the parameters $d_{2}$ and $\rho$ and its critical bifurcation value is $\left(d_{20}\left(\rho_{0}\right), \rho_{0}\right)$.

### 3.3 Three equilibrium points of the differential system

From now on in this subsection, we assume that the parameters $a_{1}, a_{2}, b_{1}, c_{1}, R, k_{2}, k_{4}, k_{5}$ and $k_{6}$ satisfy the conditions (3.2) and (3.3) of Lemma 3.1, $k_{7}>0, k_{8}>0$ and $k_{7} \neq k_{8}$.

Then by Remark 3.2,

$$
\begin{aligned}
& p_{1}=\left(x_{0}, y_{0}, \frac{2 k_{3}}{d_{2}\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{2}=\left(\frac{k_{7}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{7} y_{0}\left(k_{8}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+2 x_{0}\right)\left(k_{8}+4 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right), \\
& p_{3}=\left(\frac{k_{8}}{2}+x_{0}, y_{0}, \frac{c_{2} c_{3} k_{8} y_{0}\left(k_{7}+2 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)+2 k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+2 x_{0}\right)}{2 d_{2} x_{0}\left(k_{7}+k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+6 x_{0}\right)}\right)
\end{aligned}
$$

are the unique three equilibrium points of differential system (3.1) in $\Omega$.

### 3.3.1 Local dynamics and bifurcation at $p_{1}$

Lemma 3.15. If the parameters $k_{7}, k_{8}, b_{2}, k_{3}, c_{2}$ and $\rho$ satisfy the conditions

$$
\begin{align*}
& k_{7}=2 x_{0}, \quad k_{8}=x_{0} / 2, \quad b_{2}=\frac{27 k_{3}}{c_{3} \rho x_{0}}+y_{0}, \quad k_{3}=c_{2} c_{3} x_{0} y_{0}, \quad c_{2}=\frac{459^{2}-10358 \rho^{2}}{140049 \rho},  \tag{3.14}\\
& \rho<\frac{459}{\sqrt{10358}}, \quad d_{2}=d_{20}(\rho):=\frac{12349380771 \rho^{3}}{2\left(459^{2}-5171 \rho^{2}\right)\left(459^{2}-10358 \rho^{2}\right)},
\end{align*}
$$

then the equilibrium point

$$
p_{1}=\left(x_{0}, y_{0}, \frac{8 c_{3}\left(459^{2}-10358 \rho^{2}\right)^{2}\left(459^{2}-5171 \rho^{2}\right) y_{0}}{29401813269162243 \rho^{4}}\right)
$$

and the eigenvalues of the linear approximation at $p_{1}$ are

$$
\alpha=-\frac{13832 \rho^{3}}{2448 \rho^{2}+32234193} \quad \text { and } \quad \pm i
$$

Proof. The characteristic polynomial of the linear approximation $M_{p_{1}}$ of differential system (3.1), at the equilibrium point $p_{1}$ is $P(\lambda)=\operatorname{det}\left(\lambda I-M_{p_{1}}\right)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}$, where,

$$
\begin{aligned}
A_{1}= & -\frac{2\left(\frac{k_{3}}{c_{3}\left(b_{2}+y_{0}\right)}-\frac{k_{7} k_{8} x_{0}}{\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right)}\right)}{B_{3}}, \\
A_{2}= & \frac{\rho y_{0}\left(c_{2} c_{3} y_{0} B_{3} B_{2}+2 k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right)\left(k_{7}+k_{8}+2 x_{0}\right)\right)}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right)\left(B_{3}\right)^{2}}, \\
& +\frac{b_{2} \rho B_{2}\left(c_{2} c_{3} y_{0} B_{3}+2 k_{3}\right)+2 b_{2} d_{2} k_{3}\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right) B_{3}}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right)\left(B_{3}\right)^{2}}, \\
A_{3}= & \frac{4 b_{2} d_{2} k_{3} k_{7} k_{8} \rho x_{0}}{c_{3} y_{0}\left(b_{2}+y_{0}\right)\left(k_{7}+4 x_{0}\right)\left(k_{8}+4 x_{0}\right)\left(B_{3}\right)^{2}}, \\
B_{2}= & \left(k_{7}+k_{8}+4 x_{0}\right)\left(4 x_{0}\left(k_{7}+k_{8}\right)+k_{7} k_{8}+8 x_{0}^{2}\right), \\
B_{3}= & k_{7}+k_{8}+6 x_{0} .
\end{aligned}
$$

By hypothesis $k_{7}, k_{8}, b_{2}$ and $k_{3}$ satisfy (3.14), then $A_{1}, A_{2}$ and $A_{3}$ reduce to

$$
\begin{aligned}
& A_{1}=\frac{8 \rho^{2}}{459\left(27 c_{2}+2 \rho\right)}, \quad A_{2}=\frac{c_{2}\left(81 c_{2}\left(612 d_{2}+1729 \rho\right)+2 \rho\left(918 d_{2}+5179 \rho\right)\right)}{7803\left(27 c_{2}+2 \rho\right)}, \\
& A_{3}=\frac{16 c_{2} d_{2} \rho\left(27 c_{2}+\rho\right)}{7803\left(27 c_{2}+2 \rho\right)} .
\end{aligned}
$$

As in the proof of Lemma 2.3, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$ if and only if $A_{2}>0$ and

$$
\begin{equation*}
A_{1} A_{2}-A_{3}=0, \tag{3.15}
\end{equation*}
$$

where $\omega=\sqrt{A_{2}}$ and $\alpha=-A_{1}$. In this case,

$$
A_{1} A_{2}-A_{3}=\frac{8 c_{2} \rho\left(-669222 c_{2}^{2} d_{2}+81 c_{2} \rho\left(1729 \rho-306 d_{2}\right)+10358 \rho^{3}\right)}{3581577\left(27 c_{2}+2 \rho\right)^{2}} .
$$

Solving the equation (3.15) for the parameter $d_{2}$, we have that

$$
d_{2}=\frac{\rho^{2}\left(140049 c_{2}+10358 \rho\right)}{24786 c_{2}\left(27 c_{2}+\rho\right)} .
$$

Taking $c_{2}=\frac{459^{2}-10358 \rho^{2}}{140049 \rho}$, we have $A_{2}=1$. Since $c_{2}>0$, then the parameter $\rho$ must satisfy

$$
\rho<\frac{459}{\sqrt{10358}} .
$$

Therefore, the characteristic polynomial $P(\lambda)$ has a pair of purely imaginary roots $\pm i$ and a real root $\alpha=-\frac{13832 \rho^{3}}{2448 \rho^{2}+32234193}$. If $k_{7}, k_{8}, b_{2}, k_{3}, c_{2}, \rho$ and $d_{2}$ satisfy the relations given by (3.7), then

$$
p_{1}=\left(x_{0}, y_{0}, \frac{8 c_{3}\left(459^{2}-10358 \rho^{2}\right)^{2}\left(459^{2}-5171 \rho^{2}\right) y_{0}}{29401813269162243 \rho^{4}}\right) .
$$

Remark 3.16. If the assumptions of Lemma 3.15 are satisfied, then the linear approximation of the differential system (3.1) at $p_{1}$ has two purely imaginary eigenvalues when $d_{2}=d_{20}(\rho)$, hence, by continuity on the eigenvalues, the linear approximation of the differential system at $p_{1}$ has a pair of complex eigenvalues,

$$
\lambda\left(p_{1}, d_{2}, \rho\right)=\xi\left(p_{1}, d_{2}, \rho\right) \pm i \omega\left(p_{1}, d_{2}, \rho\right),
$$

when $d_{2}$ is in a neighborhood of $d_{20}(\rho)$.
In order to compute the first and second Lyapunov coefficients we make the following assignations

$$
c_{3}=1, \quad y_{0}=1, \quad \text { and } \quad x_{0}=1 .
$$

Applying the Kuznetsov formula and using the Mathematica software, we obtain the next result.

Lemma 3.17. If the assumptions given in Lemma 3.15 hold, then the eigenvalues of the linear approximation of the differential system (3.1) at the equilibrium point $p_{1}$ are $\alpha=-\frac{13832 \rho^{3}}{153\left(16 \rho^{2}+210681\right)}$ and $\pm i$. The first Lyapunov coefficient is

$$
\ell_{1}\left(p_{1}, d_{20}(\rho), \rho\right)=\frac{312947271 \rho^{5}\left(16 \rho^{2}+459^{2}\right) s_{3}(\rho)}{4\left(10358 \rho^{2}-459^{2}\right) s_{4}(\rho) s_{5}(\rho) s_{6}(\rho)}
$$

where

$$
\begin{aligned}
s_{3}(\rho)= & 170742882058653810693863735296 \rho^{14} \\
& +492262972296675528436161659630208 \rho^{12} \\
& -108161454636720612200055862850486688 \rho^{10} \\
& -211670758978607800175029461991756871226 \rho^{8} \\
& -9485999858302719829966772948728271221506 \rho^{6} \\
& +89659805291097747554775289194107377312320 \rho^{4} \\
& +14441392841936945229748638034904616202601802 \rho^{2} \\
& -184062861428630673823768217741646402096430491, \\
s_{4}(\rho)= & 45900865178296384 \rho^{8}+5567524388989669584 \rho^{6} \\
& +147414277124069435267049 \rho^{4}+246951858447854967628881 \rho^{2} \\
& +31522559050648267281936, \\
s_{5}(\rho)= & 16\left(2989441 \rho^{4}+374544 \rho^{2}+9863663058\right) \rho^{2}+1039043198361249, \\
s_{6}(\rho)= & 32\left(5978882 \rho^{4}+187272 \rho^{2}+4931831529\right) \rho^{2}+1039043198361249 .
\end{aligned}
$$

Corollary 3.18. If the assumptions given in Lemma 3.15 hold, then there exists a unique real number $0<\rho_{0}<\frac{459}{\sqrt{10358}}$ such that the first Lyapunov coefficient $\ell_{1}\left(p_{1}, d_{20}\left(\rho_{0}\right), \rho_{0}\right)=0$.

Proof. By Lemma 3.17, the first Lyapunov coefficient $\ell_{1}\left(p_{1}, d_{20}(\rho), \rho\right)=0$ if and only if $s_{3}(\rho)=$ 0 . According to Descartes rule of signs, there are 1 or 3 positive real numbers $\rho$ such that $s_{3}(\rho)=0$. Indeed, numerically this equation has three positive solutions, but only $\rho_{0}(\approx$ $3.71999)$ is less than $\frac{459}{\sqrt{10358}}$.

Lemma 3.19 (Bautin regularity condition). If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, then the map $\left(d_{2}, \rho\right) \mapsto\left(\xi\left(p_{1}, d_{2}, \rho\right), \ell_{1}\left(p_{1}, d_{2}(\rho), \rho\right)\right)$ is regular at $\left(d_{0}, \rho_{0}\right)$, where $\xi\left(p_{1}, d_{2}, \rho\right)$ is given in Remark 3.16 and $d_{0}:=d_{20}\left(\rho_{0}\right)$.

Proof. By hypothesis, the linear approximation of the differential system (3.1) at $p_{1}$ depends only on the parameters $d_{2}$ and $\rho$. Let $M_{p_{1}}\left(d_{2}, \rho\right)$ be this linear approximation. By Lemma 3.15, the complex numbers $\pm i$ are eigenvalues of $M_{p_{1}}\left(d_{0}, \rho_{0}\right)$. Hence, the real part of the complex eigenvalues of $M_{p_{1}}\left(d_{0}, \rho_{0}\right)$ is

$$
\xi\left(p_{1}, d_{0}, \rho_{0}\right)=0
$$

Let $\mathbf{p}$ and $\mathbf{q}$ be eigenvectors of $M_{p_{1}}\left(d_{0}, \rho_{0}\right)$ for the corresponding eigenvalues $-i$ and $i$, respectively, such that

$$
\overline{\mathbf{q}}^{t r} \cdot \mathbf{q}=1 \quad \text { and } \quad \overline{\mathbf{p}}^{t r} \cdot \mathbf{q}=1
$$

By Lemma 2.7 and taking into account the values of $\mathbf{q}, \overline{\mathbf{p}}, \frac{\partial M_{p_{1}}\left(d_{2}, \rho\right)}{\partial d_{2}}$ and $\frac{\partial M_{p_{1}}\left(d_{2}, \rho\right)}{\partial \rho}$, we obtain

$$
\begin{gather*}
\frac{\partial \xi}{\partial d_{2}}\left(d_{0}, \rho_{0}\right)=\frac{8\left(459^{2}-10358 \rho_{0}^{2}\right)^{2}\left(459^{2}-5171 \rho_{0}^{2}\right)}{46683 Q_{5}},  \tag{3.16}\\
Q_{5}=32\left(5978882 \rho_{0}^{4}+187272 \rho_{0}^{2}+4931831529\right) \rho_{0}^{2}+1039043198361249 \\
\frac{\partial \xi}{\partial \rho}\left(d_{0}, \rho_{0}\right)=-\frac{1058148 \rho_{0}^{2}\left(53561218 \rho_{0}^{4}+3271665249 \rho_{0}^{2}-133159451283\right)}{\left(5171 \rho_{0}^{2}-210681\right) Q_{7}},  \tag{3.17}\\
Q_{7}=\left(32\left(5978882 \rho_{0}^{4}+187272 \rho_{0}^{2}+4931831529\right) \rho_{0}^{2}+1039043198361249\right) .
\end{gather*}
$$

Numerically, taking $\rho_{0} \approx 9.76907$, we have that $\operatorname{Re}\left(\frac{\partial \mathrm{C}_{1}}{\partial d_{2}}\left(d_{0}, \rho_{0}\right)\right) \approx-0.60234$ and $\operatorname{Re}\left(\frac{\partial \mathbf{C}_{1}}{\partial \rho}\left(d_{0}, \rho_{0}\right)\right) \approx 14.78256$, where $\mathbf{C}_{1}\left(d_{2}, \rho\right)$ is as in the proof of Lemma 2.8. Then by (3.16), (3.17) and the analogous formula of (2.13) given in Lemma 2.8,

$$
\operatorname{det}\left(\begin{array}{ll}
\frac{\partial \tilde{\xi}}{\partial d_{2}}\left(d_{0}, \rho_{0}\right) & \frac{\partial \tilde{\xi}}{\partial \rho}\left(d_{0}, \rho_{0}\right) \\
\frac{\partial \ell_{1}}{\partial d_{2}}\left(d_{0}, \rho_{0}\right) & \frac{\partial \ell_{1}}{\partial \rho}\left(d_{0}, \rho_{0}\right)
\end{array}\right) \approx-0.00291 .
$$

Hence, the map $\left(d_{2}, \rho\right) \mapsto\left(\xi\left(p_{1}, d_{2}, \rho\right), \ell_{1}\left(p_{1}, d_{2}, \rho\right)\right)$ is regular at $\left(d_{0}, \rho_{0}\right)$.
Theorem 3.20. If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, then the differential system (3.1) exhibits an Andronov-Hopf bifurcation at

$$
p_{1}=\left(1,1, \frac{8\left(459^{2}-10358 \rho^{2}\right)^{2}\left(459^{2}-5171 \rho^{2}\right)}{2940181326916243 \rho^{4}}\right)
$$

with respect to the parameter $d_{2}$ and its critical bifurcation value is $d_{20}(\rho)$, where $\rho \in(0,459 / \sqrt{10358})$ and $\rho \neq \rho_{0}$. Moreover, if $\rho<\rho_{0}$ the bifurcation is subcritical and if $\rho>\rho_{0}$ the bifurcation is supercritical.

Proof. From Lemma 3.15, the linearization $M_{p_{1}}\left(d_{2}, \rho\right)$ of differential system (3.1) at $p_{1}$ has a negative real eigenvalue and a pair of purely imaginary eigenvalues if $d_{2}=d_{20}(\rho)$. From Lemma 3.19, the derivative of the real part of the complex eigenvalues

$$
\begin{gathered}
\frac{\partial \xi}{\partial d_{2}}\left(d_{20}(\rho), \rho\right)=\frac{8\left(459^{2}-10358 \rho^{2}\right)^{2}\left(459^{2}-5171 \rho^{2}\right)}{46683 Q_{8}} \\
Q_{8}=32\left(5978882 \rho^{4}+187272 \rho^{2}+4931831529\right) \rho^{2}+1039043198361249
\end{gathered}
$$

which is positive if $\rho \in(0,459 / \sqrt{10358})$, and hence the transversality condition holds. Lemma 3.17 and Corollary 3.18 imply that the first Lyapunov coefficient is negative if $\rho>\rho_{0}$, and is positive if $\rho<\rho_{0}$, (see Figure 3.1). Then the hypotheses of Andronov-Hopf bifurcation theorem hold and we conclude the proof.

In order to show the Bautin bifurcation, we compute the second Lyapunov coefficient $\ell_{2}$. Applying the Kuznetsov formula, and using the Mathematica software, we obtain that the second Lyapunov coefficient $\ell_{2}\left(p_{1}, d_{20}(\rho), \rho\right)$, of the differential system (3.1) at the equilibrium point $p_{1}$ takes the value $\ell_{2}\left(p_{1}, d_{20}\left(\rho_{0}\right), \rho_{0}\right) \approx-26718.1$.

Lemma 3.21 (Second Lyapunov coefficient). If we have the assumptions given in Lemma 3.15, then the second Lyapunov coefficient of differential system (3.1) at the equilibrium point $p_{1}$, $\ell_{2}\left(p_{1}, d_{20}\left(\rho_{0}\right), \rho_{0}\right) \neq 0$.

From Corollary 3.18, Lemma 3.19 and Lemma 3.21 we have the necessary and sufficient conditions to apply the Bautin bifurcation theorem. Therefore we obtain the following.

Theorem 3.22. If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, then the differential system (3.1) exhibits a Bautin bifurcation at $p_{1}$, with respect to the parameters $d_{2}$ and $\rho$ and its critical bifurcation value is $\left(d_{20}\left(\rho_{0}\right), \rho_{0}\right)$.

From Theorems 3.20 and 3.22 we have shown the existence of limit cycles in $\Omega$ to differential system (3.1) near to $p_{1}$. Now, we will analyze the local dynamics at equilibrium point $p_{2}$.

### 3.3.2 Local dynamics and bifurcation at $p_{2}$

In this subsection we assume that the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, and $x_{0}=y_{0}=c_{3}=1$. In the same way of the previous subsection, we obtain the next results relative to $p_{2}$.

Lemma 3.23. If

$$
d_{2}=d_{21}(\rho):=\frac{42 \rho\left(39818709-152092 \rho^{2}\right)\left(1629662 \rho^{2}+12430179\right)}{5447429\left(459^{2}-5171 \rho^{2}\right)\left(459^{2}-10358 \rho^{2}\right)}
$$

then the equilibrium point $p_{2}$ of the differential system (3.1) is given by

$$
p_{2}=\left(2,1, \frac{3869893\left(459^{2}-10358 \rho^{2}\right)^{2}\left(459^{2}-5171 \rho^{2}\right)}{5882058 \rho^{2}\left(39818709-152092 \rho^{2}\right)\left(1629662 \rho^{2}+12430179\right)}\right)
$$

and the eigenvalues of the linear approximation of system (3.1) at $p_{2}$ are

$$
\alpha=-\frac{\rho\left(1629662 \rho^{2}+12430179\right)}{5967\left(16 \rho^{2}+210681\right)} \quad \text { and } \quad \pm \frac{4 \sqrt{39818709-152092 \rho^{2}}}{221 \sqrt{38779}} i .
$$

Lemma 3.24. If we have the assumptions given in Lemma 3.23, the first Lyapunov coefficient at $p_{2}$ is

$$
\ell_{1}\left(p_{2}, d_{21}(\rho), \rho\right)=-\frac{15877775277 \sqrt{247} \rho^{5}\left(16 \rho^{2}+210681\right)\left(152092 \rho^{2}-39818709\right) \sigma_{3}(\rho)}{314 \sqrt{6251537313-23878444 \rho^{2}}\left(10358 \rho^{2}-210681\right) \sigma_{4}(\rho) \sigma_{5}(\rho) \sigma_{6}(\rho)},
$$

where

$$
\begin{aligned}
\sigma_{3}(\rho)= & 2674229435224414678826952483987392727006431186560 \rho^{14} \\
& +805184870865477880770429208091467190331922797852960 \rho^{12} \\
& -83325105178769624726247845367896567277949050385850640 \rho^{10} \\
& -3990833474133489275436613965587927698093938169581979104 \rho^{8} \\
& +1893941697027008827574154049507129272853933667749875352320 \rho^{6} \\
& -103337284251342816394773872075621179984010459502936989864334 \rho^{4} \\
& +2045265511014986633744532946163166674530762170351157348636599 \rho^{2} \\
& -13861057129694950151794339970517627657596357576143945686701523,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{4}(\rho)= & 102987383148633964 \rho^{6}+1523727465817145724 \rho^{4}-296450186139603299565 \rho^{2} \\
& +82460396686124879118144, \\
\sigma_{5}(\rho)= & 102988745581469548 \rho^{6}+1559250610762430844 \rho^{4}-69618757972976491437 \rho^{2} \\
& +20615099171531219779536,
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{6}(\rho)= & 8930934742886284720660 \rho^{8}-29375711674972586158626756 \rho^{6} \\
& +7535731011874228165827361581 \rho^{4}+1121586858755371780236256518 \rho^{2} \\
& +7624308163261898438530822053
\end{aligned}
$$

Corollary 3.25. There exists a unique real number $0<\rho_{1}<\frac{459}{\sqrt{10358}}$ such that the first Lyapunov coefficient $\ell_{1}\left(p_{2}, d_{21}\left(\rho_{1}\right), \rho_{1}\right)=0$. Indeed $\rho_{1} \approx 4.36757$.

Lemma 3.26 (Bautin regularity condition). The map $\left(d_{2}, \rho\right) \mapsto\left(\xi\left(p_{2}, d_{2}, \rho\right), \ell_{1}\left(p_{2}, d_{2}, \rho\right)\right)$ is regular at $\left(d_{1}, \rho_{1}\right)$, where $\xi\left(p_{2}, d_{2}, \rho\right)$ is as in Lemma 3.19 and $d_{1}:=d_{21}\left(\rho_{1}\right)$.

Theorem 3.27. The differential system (3.1) exhibits an Andronov-Hopf bifurcation at $p_{2}$ with respect to the parameter $d_{2}$ and its critical bifurcation value is $d_{21}(\rho)$, where $\rho \in(0,459 / \sqrt{10358})$ and $\rho \neq \rho_{1}$. Moreover, if $\rho<\rho_{1}$ the bifurcation is subcritical and if $\rho>\rho_{1}$ the bifurcation is supercritical.

In order to show a Bautin bifurcation, we compute the second Lyapunov coefficient.
Lemma 3.28 (Second Lyapunov coefficient). If we have the assumptions given in Lemma 3.23, then the second Lyapunov coefficient of differential system (3.1), $\ell_{2}\left(p_{2}, d_{21}\left(\rho_{1}\right), \rho_{1}\right)$ is negative. Indeed $\ell_{2}\left(p_{2}, d_{21}\left(\rho_{1}\right), \rho_{1}\right) \approx-161.216$.

We summarize the results in the following theorem.
Theorem 3.29. The differential system (3.1) exhibits a Bautin bifurcation at $p_{2}$, with respect to the parameters $d_{2}$ and $\rho$ and its critical bifurcation value is $\left(d_{21}\left(\rho_{1}\right), \rho_{1}\right)$.

### 3.3.3 Local dynamics at $p_{3}$

Theorem 3.30. If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, then, the differential system (3.1) does not exhibit a Hopf bifurcation at the equilibrium point

$$
p_{3}=\left(\frac{5}{4}, 1, \frac{94\left(210681-10358 \rho^{2}\right)}{30950829 d_{2} \rho}\right) .
$$

Proof. A necessary condition for a differential system to exhibit an Andronov-Hopf bifurcation at an equilibrium point is that the characteristic polynomial of its linear approximation has a pair of purely imaginary roots. According to the proof of Lemma 2.3, the characteristic polynomial $P(\lambda)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3}$ has a pair of purely imaginary roots $\pm i \omega$ and a real root $\alpha$ if and only if $A_{2}>0$ and

$$
A_{1} A_{2}-A_{3}=0
$$

where $\omega=\sqrt{A_{2}}$ and $\alpha=-A_{1}$. By hypothesis, if $M_{p_{3}}$ is the linear approximation of differential system (3.1) at $p_{3}$ then

$$
P(\lambda)=\operatorname{det}\left(\lambda I-M_{p_{3}}\right)=\lambda^{3}+A_{1} \lambda^{2}+A_{2} \lambda+A_{3},
$$

where,

$$
\begin{aligned}
& A_{1}=-\frac{\rho\left(417 c_{2}+10 \rho\right)}{663\left(27 c_{2}+2 \rho\right)}, \quad A_{2}=\frac{2 c_{2}\left(81 c_{2}\left(10387 d_{2}+14070 \rho\right)+\rho\left(31161 d_{2}+84655 \rho\right)\right)}{146523\left(27 c_{2}+2 \rho\right)} \\
& A_{3}=-\frac{470 c_{2} d_{2} \rho\left(27 c_{2}+\rho\right)}{146523\left(27 c_{2}+2 \rho\right)} .
\end{aligned}
$$

Since $c_{2}=\frac{459^{2}-10358 \rho^{2}}{140049 \rho}>0$, when $0<\rho<\frac{459}{\sqrt{10358}}$, we have that

$$
\begin{gathered}
A_{1} A_{2}-A_{3}=-\frac{2 c_{2} \rho Q_{9}}{97144749\left(27 c_{2}+2 \rho\right)^{2}}<0, \\
Q_{9}=237259854 c_{2}^{2} d_{2}+475242390 c_{2}^{2} \rho+8787402 c_{2} d_{2} \rho+46697835 c_{2} \rho^{2}+846550 \rho^{3} .
\end{gathered}
$$

Therefore, the differential system (3.1) does not exhibit an Andronov-Hopf bifurcation at the equilibrium point $p_{3}=\left(\frac{5}{4}, 1, \frac{94\left(210681-10358 \rho^{2}\right)}{30950829 d_{2} \rho}\right)$.

Theorem 3.31. If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15, then the equilibrium point $p_{3}=\left(\frac{5}{4}, 1, \frac{94\left(210681-10358 \rho^{2}\right)}{30950829 d_{2} \rho}\right)$ of differential system (3.1) is locally unstable.

Proof. From Theorem 3.30, the characteristic polynomial $P(\lambda)$ has three sign changes in its coefficients. By the Descartes rule of signs, we have that there exists at least one positive eigenvalue for the linearization at $p_{3}$. Then $p_{3}$ is unstable.

### 3.3.4 Simultaneous periodic orbits at $p_{1}$ and $p_{2}$

If the parameters $k_{3}, k_{7}, k_{8}, b_{2}, c_{2}$ and $\rho$ satisfy the relations (3.14) of Lemma 3.15 , then according to Theorem 3.20 the differential system (3.1) exhibits an Andronov-Hopf bifurcation at $p_{1}$, with respect to the parameter $d_{2}$, with critical bifurcation value $d_{2}=d_{20}(\rho)$. By Theorem 3.27 the differential system (3.1) exhibits an Andronov-Hopf bifurcation at $p_{2}$, with respect to the parameter $d_{2}$, with critical bifurcation value $d_{2}=d_{21}(\rho)$. In order to find a parameter value where the differential system exhibits a simultaneous Andronov-Hopf bifurcation we solve the equation

$$
d_{21}(\rho)-d_{20}(\rho)=0
$$

The unique solution in the interval $\left(0, \frac{459}{\sqrt{10358}}\right)$ is

$$
\rho_{*}:=8262 \sqrt{\frac{2478}{38779 \sqrt{10580386691137}+126081609691}} \approx 0.81892
$$

The Figure 3.1 (a), shows the graph of critical bifurcation value in terms of $\rho$ for each equilibrium point and its intersection at $\left(\rho_{*}, d_{2,0}\left(\rho_{*}\right)\right)$ Therefore the differential system (3.1) exhibits a simultaneous Andronov-Hopf bifurcation at $p_{1}$ and $p_{2}$, with respect to the parameter $d_{2}$, with critical bifurcation value

$$
\begin{aligned}
d_{20}\left(\rho_{*}\right) & =d_{21}\left(\rho_{*}\right) \\
& =\frac{1593299484 \sqrt{2478(38779 \sqrt{10580386691137}+126081609691)}}{76178503957 \sqrt{10580386691137}+248078464264819189} \approx 0.08032
\end{aligned}
$$

Since $\rho_{*}<\rho_{0}<\rho_{1}$, by Theorems 3.20 and 3.27, this simultaneous Andronov-Hopf bifurcation is subcritical at $p_{1}$ and $p_{2}$ (the Figure 3.1 (b) shows the first Lyapunov coefficient corresponding to $p_{1}$ or $p_{2}$ ). In this case, the limit cycle bifurcating from $p_{1}$ and the limit cycle bifurcating from $p_{2}$ are unstable. By Theorems 3.22 and 3.29 the differential systems exhibits a Bautin bifurcation, then there are two limit cycles bifurcating from $p_{1}$ or $p_{2}$, where one is stable and the other is unstable.

(b) First Lyapunov coefficient at $p_{1}$ and $p_{2}$

Example 3.32. Taking $a_{1}=$ 22.7118, $a_{2}=984.723, b_{1}=5.75, b_{2}=1.10108, c_{1}=0.0060473$, $c_{2}=0.0164717, c_{3}=1$ and $\rho=4.4>\rho_{1}>\rho_{0}$, then $d_{20}=468.666, d_{21}=49.021$ and the parameters involved in the differential system (3.1) satisfy the hypothesis established in the Subsection 3.3. Hence we have three equilibrium points $p_{1}, p_{2}$ and $p_{3}$. The first Lyapunov coefficient at $p_{1}$ and $p_{2}$ are

$$
\ell_{1}\left(p_{1}, d_{20}\right)=-1.24243, \quad \ell_{1}\left(p_{2}, d_{21}\right)=-0.0135894
$$

In this case, we have two stable limit cycle each one bifurcating from the equilibrium points $p_{1}$ and $p_{2}$. In the Figure 3.2 we show two trajectories whose $\omega$-limit are the stable periodic orbits, where $d_{2}=d_{20}+1 / 100$.

## 4 Conclusion

When the prey has a linear growth, the differential system has only one equilibrium point in the positive octant $\Omega$ and around this point appear an stable periodic orbit generated by an Andronov-Hopf bifurcation or a Bautin bifurcation. On the other hand, if the growth of the prey is logistic, the differential system can have even three equilibrium points in $\Omega$. In particular, when there is only one equilibrium point in $\Omega$, it is not hyperbolic. When there are two equilibria, one is not hyperbolic and the other exhibits an limit cycle generated by an Andronov-Hopf bifurcation or Bautin bifurcation. In the case, when there are three equilibrium points, two of them can present Andronov-Hopf and Bautin bifurcation, in fact they can appear simultaneously. Thus the differential system exhibits bi-stability. The other equilibrium point is always unstable. This analysis shows that the condition to have coexistence of the three populations is better in the logistic growth.


Figure 3.2: Phase space of differential system (3.1) with three equilibria and two limit cycles.

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