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Solutions of a quadratic Volterra–Stieltjes integral equation in the class of functions converging at infinity

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Abstract. The paper deals with the study of the existence of solutions of a quadratic integral equation of Volterra–Stieltjes type. We are looking for solutions in the class of real functions continuous and bounded on the real half-axis $\mathbb{R}_+$ and converging to proper limits at infinity. The quadratic integral equations considered in the paper contain, as special cases, a lot of nonlinear integral equations such as Volterra–Chandrasekhar or Volterra–Wiener–Hopf equations, for example. In our investigations we use the technique associated with measures of noncompactness and the Darbo fixed point theorem. Particularly, we utilize a measure of noncompactness related to the class of functions in which solutions of the integral equation in question are looking for.

Keywords: space of continuous and bounded functions, variation of function, function of bounded variation, Riemann–Stieltjes integral, measure of noncompactness, Darbo fixed point theorem.

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1 Introduction

The theory of integral equations creates an important branch of nonlinear analysis. Both linear and nonlinear integral equations are applied in the description of several problems encountered in natural and exact sciences. Especially, a lot of problems of physics, mathematical physics, mechanics, engineering, electrochemistry, viscoelasticity, control theory, transport theory etc. can be modelled with help of integral equations of various types (cf. [11, 12, 14–16, 18–20], for instance).

Recently, a lot of interest has been directed to applications of the so-called fractional integral equations since those equations find a lot of applications in important real world topics connected with kinetic theory of gases, radiative transfer, in the theory of diffraction and so on (see [5] and references therein).

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Our goal in this paper is to consider the existence of solutions of a quadratic Volterra–Stieltjes integral equations in the class of real functions defined and continuous on the real half-axis and having finite limits at infinity. It is well-known that Volterra–Stieltjes integral equations generalize a lot of integral equations considered in nonlinear analysis [4,5,7,19]. On the other hand the so-called quadratic integral equations describe several events of the theory of radiative transfer, queuing theory, kinetic theory of gases and some others [3,4,11,15].

Thus, quadratic integral equations of Volterra–Stieltjes type link the theory of (ordinary) Volterra–Stieltjes integral equations and the theory of quadratic integral equations and enable us to generate results on the existence of solutions of nonlinear integral equations which contains both types of integral equations mentioned before.

As we pointed out above, in our considerations we will look for conditions guaranteeing the existence of solutions of quadratic integral equations in the class of functions which are defined, continuous on the interval $[0, \infty)$ and converging to finite limits at infinity. Similar investigations were conducted for nonlinear Volterra–Stieltjes integral equations in [7]. Thus, the investigations of this paper extend and generalize those carried out in [7].

Let us notice that in paper [7] we used the technique associated with compact integral operators and the Schauder fixed point principle. It turns out that those tools are no longer sufficient in the study conducted in this paper and this causes that we will use the technique connected with the theory of measures of noncompactness. More precisely, we will apply a measures of noncompactness in the space of functions continuous and bounded on the real half-axis $\mathbb{R}_+$ and associated with the class of functions converging to proper limits at infinity.

The results obtained in the paper generalize a lot of ones obtained earlier in papers [4,5,7]. Particularly, these results create an essential extension of those obtained in [7].

2 Notation, definitions and auxiliary facts

In this section we establish the notation which will be used throughout this paper and we recollect a few facts which will be utilized in our considerations.

By the symbol $\mathbb{R}$ we will denote the set of real numbers while $\mathbb{R}_+$ stands for the set of nonnegative real numbers i.e., $\mathbb{R}_+ = [0, \infty)$. If E is a Banach space with the norm $\|\cdot\|$ then the symbol $B(x, r)$ denotes the closed ball centered at x and with radius r. We will write B_r to denote the ball $B(\theta, r)$, where θ is the zero vector in E.

If X is an arbitrary subset of E then $\overline{X}$ stands for the closure of X and $\operatorname{Conv} X$ denotes the closed convex hull of X. We use the standard notation $X + Y$, λX to denote the classical algebraic operations on subset of E.

In what follows we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded sets in E and by $\mathfrak{N}_E$ its subfamily consisting of relatively compact sets.

We will accept the following axiomatic definition of the concept of a measure of noncompactness [8].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called *the measure of noncompactness* in the space E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \leqslant \mu(Y)$.

3° $\mu(\overline{X}) = \mu(X)$.

4° $\mu(\operatorname{Conv} X) = \mu(X)$.

5° $\mu(\lambda X + (1-\lambda)Y) \leqslant \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.

6° If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ appeared in axiom 1° is called *the kernel of the measure* μ. If $\ker\mu = \mathfrak{N}_E$ then the measure of noncompactness μ is called *full*.

It is worthwhile mentioning that the set X_∞ from axiom 6° is a member of the kernel $\ker\mu$. Indeed, it is a simple consequence of the inequality $\mu(X_\infty) \leqslant \mu(X_n)$ for any $n = 1, 2, \dots$ This yields that $\mu(X_\infty) = 0$ which means that $X_\infty \in \ker\mu$. This simple observation plays a crucial role in applications of the technique associated with measures of noncompactness.

Let us recall (cf. [8]) that the measure of noncompactness μ is called *sublinear* if it satisfies additionally the following conditions:

7° $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in \mathbb{R}$.

8° $\mu(X + Y) \leqslant \mu(X) + \mu(Y)$.

If it satisfies the condition

9° $\mu(X \cup Y) = \max\{\mu(X), \mu(Y)\}$

then it is referred to as *the measure with maximum property*.

A full and sublinear measure of noncompactness μ which has the maximum property is called *regular* [8].

One of the most important example of a measure of noncompactness is the function $\chi : \mathfrak{M}_E \to \mathbb{R}_+$ defined by the formula

$$\chi(X) = \inf\{\varepsilon > 0 : X \text{ has a finite } \varepsilon\text{-net in } E\}.$$

The function χ is called *the Hausdorff measure of noncompactness*. It can be shown that χ is a regular measure having some additional properties (cf. [8]).

Let us pay attention to the fact that measures of noncompactness are very useful in several applications [1, 6, 8, 9]. Especially, the following fixed point theorem, called the fixed point theorem of Darbo type [13] plays an essential role in applications.

Theorem 2.2. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E. Assume that $T : \Omega \to \Omega$ is a continuous operator and there exists a constant $k \in [0,1)$ such that $\mu(TX) \leqslant k\mu(X)$ for any nonempty subset X of Ω, where μ is a measure of noncompactness in E. Then T has at least one fixed point in the set Ω.*

It can be shown that the set $\operatorname{Fix} T$ of fixed points of the operator T belonging to Ω is a member of the kernel $\ker\mu$. This facts enables us to characterize solutions of considered operator equations (cf. [8]).

As we pointed out above the Hausdorff measure χ seems to be the most convenient and applicable measure of noncompactness. However, the use of χ requires to construct a formula expressing χ in connection with the structure of the Banach space E. It turns out that such formulas are only known in a few Banach spaces (cf. [1, 8, 9]). By these regards in practise we usually apply measures of noncompactness which are not full but which are subject to axioms of Definition 2.1. The use of such measures of noncompactness is very fruitful since it allows us to characterize solutions of various operator equations which are investigated with help of the technique of measures of noncompactness.

In what follows we will use a measure of noncompactness of such a type in a Banach space $BC(\mathbb{R}_+)$ consisting of functions $x : \mathbb{R}_+ \to \mathbb{R}$ which are continuous and bounded on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is equipped with the classical supremum norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$.

To construct the announced measure of noncompactness in the space $BC(\mathbb{R}_+)$ let us fix a nonempty and bounded subset X of the space $BC(\mathbb{R}_+)$ i.e., take $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Next, choose $T > 0$, $\varepsilon > 0$ and an arbitrary function $x \in X$. Let us define *the modulus of continuity* of the function x in the interval $[0, T]$ by putting:

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leqslant \varepsilon\}.$$

Further, we define the following quantities (cf. [8,10]):

$$\begin{aligned}\omega^T(X, \varepsilon) &= \sup\{\omega^T(x, \varepsilon) : x \in X\},\\ \omega_0^T(X) &= \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),\\ \omega_0(X) &= \lim_{T \to \infty} \omega_0^T(X).\end{aligned}$$

Next, we consider the quantity $b(X)$ defined in the following way

$$b(X) = \lim_{T \to \infty} \left\{ \sup_{x \in X} \{ \sup\{|x(t) - x(s)| : t, s \geqslant T\}\} \right\}.$$

Finally, we define the function $\mu = \mu(X)$ by putting

$$\mu(X) = \omega_0(X) + b(X). \tag{2.1}$$

It can be shown that μ is a measure of noncompactness in the space $BC(\mathbb{R}_+)$ which is sublinear and has the maximum property. The measure μ is not full [10].

We can show that the kernel $\ker \mu$ consists of all bounded subsets X of the space $BC(\mathbb{R}_+)$ such that functions from X are locally equicontinuous on $\mathbb{R}_+$ and tend to limits at infinity with the same rate, that means, functions from X tend to limits at infinity uniformly with respect to the set X.

In the sequel of this section we present a few facts concerning the concept of *the variation of a function* (cf. [2]). Thus, let us assume that x is a real function defined on a fixed interval $[a, b]$. Then the symbol $\bigvee_a^b x$ will denote the variation of the function x on the interval $[a, b]$. If $\bigvee_a^b x$ is finite we say that x is *of bounded variation* on the interval $[a, b]$. Similarly, if we have a function $u(t, s) = u : [a, b] \times [c, d] \to \mathbb{R}$, then by $\bigvee_{t=p}^q u(t, s)$ we denote the variation of the function $t \mapsto u(t, s)$ on the interval $[p, q] \subset [a, b]$, where s is a fixed number in $[c, d]$. Analogously, we define the quantity $\bigvee_{s=p}^q u(t, s)$.

Now, let us assume that x and φ are real functions defined on the interval $[a, b]$. Then, we can define *the Stieltjes integral* (in the Riemann–Stieltjes sense)

$$\int_a^b x(t) d\varphi(t) \tag{2.2}$$

of the function x with respect to the function φ, under appropriate assumptions on the functions x and φ (cf. [2,17]). If integral (2.2) does exists we say that x is Stieltjes integrable on the interval $[a,b]$ with respect to φ. For example, if we assume that x is continuous and φ is of bounded variation on $[a,b]$ then x is Stieltjes integrable on $[a,b]$ with respect to φ.

For other conditions guaranteeing the Stieltjes integrability we refer to [2,17].

The below quoted lemmas present the properties of the Stieltjes integral which will be utilized in our further considerations [2].

Lemma 2.3. *If x is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then*

$$\left|\int_a^b x(t)d\varphi(t)\right| \leqslant \int_a^b |x(t)|\, d\left(\bigvee_a^t \varphi\right).$$

Lemma 2.4. *Let x_1, x_2 be Stieltjes integrable on the interval $[a,b]$ with respect to a nondecreasing function φ such that $x_1(t) \leqslant x_2(t)$ for $t \in [a,b]$. Then*

$$\int_a^b x_1(t)d\varphi(t) \leqslant \int_a^b x_2(t)d\varphi(t).$$

Now, let us notice that we can also consider the Stieltjes integrals of the form

$$\int_a^b x(s)d_s g(t,s), \tag{2.3}$$

where $g : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_s indicates the integration with respect to the variable s. Details concerning the integral of form (2.3) will be provided later.

3 Main results

The main object of the study in this paper is the solvability of the quadratic Volterra–Stieltjes integral equation having the form

$$x(t) = a(t) + f\big(t, x(t)\big) \int_0^t v\big(t,s,x(s)\big) d_s K(t,s), \tag{3.1}$$

where $t \in \mathbb{R}_+$. We will consider Eq. (3.1) in the space $BC(\mathbb{R}_+)$ described in the previous section.

Our goal is to show that integral equation (3.1) has at least one solution in the space $BC(\mathbb{R}_+)$ which is convergent at infinity, obviously to a finite limit.

For our further purposes we denote by Δ the triangle $\Delta = \{(t,s) : 0 \leqslant s \leqslant t\}$.

Now, we formulate assumptions under which we will consider the solvability of Eq. (3.1). Namely, we impose the following assumptions.

(i) The function $a = a(t)$ is a member of the space $BC(\mathbb{R}_+)$ and there exists the limit $\lim_{t\to\infty} a(t)$.

(ii) The function $f(t,x) = f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $k(r) = k : \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing and continuous on $\mathbb{R}_+$ with the property $k(0) = 0$ and such that for each $r > 0$ the following inequality is satisfied

$$\big|f(t,x) - f(t,y)\big| \leqslant k(r)|x-y|$$

for all $x, y \in [-r,r]$ and for any $t \in \mathbb{R}_+$.

(iii) The function $t \mapsto f(t,x)$ satisfies the Cauchy condition at infinity uniformly with respect to the variable x belonging to any bounded interval i.e., the following condition is satisfied

$$\forall_{r>0}\ \forall_{\varepsilon>0}\ \exists_{T>0}\ \forall_{t,s\geqslant T}\ \forall_{x\in[-r,r]} \quad \big|f(t,x)-f(s,x)\big| \leqslant \varepsilon. \tag{3.2}$$

(iv) The function $v(t,s,x) = v : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a continuous and nondecreasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\big|v(t,s,x)\big| \leq \phi(|x|)$$

for $(t,s) \in \Delta$ and $x \in \mathbb{R}$.

(v) The function v is uniformly continuous on the sets of the form $\Delta \times [-r,r]$, for any $r > 0$.

(vi) The function $K(t,s) = K : \Delta \to \mathbb{R}$ is continuous on Δ and $K(t,0) = 0$.

(vii) For any fixed $t > 0$ the function $s \mapsto K(t,s)$ has a bounded variation on the interval $[0,t]$ and the function $t \mapsto \bigvee_{s=0}^{t} K(t,s)$ is bounded on $\mathbb{R}_+$.

(viii) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$, $t_1 < t_2$, $t_2 - t_1 \leqslant \delta$, the following inequality holds

$$\bigvee_{s=0}^{t_1} \big[K(t_2,s) - K(t_1,s)\big] \leqslant \varepsilon.$$

Remark 3.1. Observe that from assumption (iii), on the basis of some classical facts from mathematical analysis, we conclude that for any fixed $x \in \mathbb{R}$ there exists a finite limit

$$\lim_{t\to\infty} f(t,x).$$

Particularly, there exists the finite limit $\lim_{t\to\infty} f(t,0)$. Hence in view of assumption (ii) we infer that the constant $\overline{F}$ defined as

$$\overline{F} = \sup\left\{|f(t,0)| : t \in \mathbb{R}_+\right\}$$

is finite.

Remark 3.2. Keeping in mind assumption (vii) we infer that $\overline{K} < \infty$, where $\overline{K}$ is the constant defined by the equality

$$\overline{K} = \sup\left\{\bigvee_{s=0}^{t} K(t,s) : t \in \mathbb{R}_+\right\}.$$

Now, we formulate our further assumptions.

(ix) The following equalities hold:

$$\lim_{T\to\infty}\left\{\sup\left[\bigvee_{\tau=s}^{t} K(t,\tau) : T \leqslant s < t\right]\right\} = 0,$$

$$\lim_{T\to\infty}\left\{\sup\left[\bigvee_{\tau=0}^{s} \big[K(t,\tau) - K(s,\tau)\big] : T \leqslant s < t\right]\right\} = 0,$$

$$\lim_{T\to\infty}\left\{\sup\left[\big|v(t,\tau,y) - v(s,\tau,y)\big| : t,s \geqslant T,\ \tau \in \mathbb{R}_+, \tau \leqslant s,\ \tau \leqslant t,\ y \in [-r,r]\right]\right\} = 0,$$

for each fixed $r > 0$.

(x) There exists a number r_0 satisfying the inequality
$$\|a\| + \overline{K}\big(rk(r) + \overline{F}\big)\phi(r) \leqslant r,$$
such that $\overline{K}k(r_0)\phi(r_0) < 1$.

Now we are prepared to formulate the main result of the paper concerning the solvability of Eq. (3.1).

Theorem 3.3. *Under the assumptions (i)–(x), there exists at least one solution $x = x(t)$ of Eq. (3.1) in the space $BC(\mathbb{R}_+)$ converging to a finite limit at infinity.*

Proof. For further purposes let us consider the operators F, V, Q defined on the space $BC(\mathbb{R}_+)$ in the following way:
$$\begin{aligned}(Fx)(t) &= f\big(t, x(t)\big),\\ (Vx)(t) &= \int_0^t v\big(t,s,x(s)\big)d_sK(t,s),\\ (Qx)(t) &= a(t) + (Fx)(t)(Vx)(t),\end{aligned} \tag{3.3}$$
for $t \in \mathbb{R}_+$. Obviously Eq. (3.1) can be written in the form $x(t) = (Qx)(t)$.

Now, let us fix arbitrarily a function $x \in BC(\mathbb{R}_+)$. We are going to show that the function Qx is continuous on the interval $\mathbb{R}_+$.

To this end fix arbitrarily $T > 0$ and $\varepsilon > 0$. Choose numbers $t, s \in [0, T]$ with $|t - s| \leqslant \varepsilon$. Without loss of generality we can assume that $s < t$. Then, taking into account Lemmas 2.3 and 2.4, we obtain:
$$\begin{aligned}\big|(Vx)(t) - (Vx)(s)\big| &\leqslant \left|\int_0^t v\big(t,\tau,x(\tau)\big)d_\tau K(t,\tau) - \int_0^s v\big(t,\tau,x(\tau)\big)d_\tau K(t,\tau)\right|\\
&\quad + \left|\int_0^s v\big(t,\tau,x(\tau)\big)d_\tau K(t,\tau) - \int_0^s v\big(t,\tau,x(\tau)\big)d_\tau K(s,\tau)\right|\\
&\quad + \left|\int_0^s v\big(t,\tau,x(\tau)\big)d_\tau K(s,\tau) - \int_0^s v\big(s,\tau,x(\tau)\big)d_\tau K(s,\tau)\right|\\
&\leqslant \left|\int_s^t v\big(t,\tau,x(\tau)\big)d_\tau K(t,\tau)\right| + \left|\int_0^s v\big(t,\tau,x(\tau)\big)d_\tau\big[K(t,\tau) - K(s,\tau)\big]\right|\\
&\quad + \left|\int_0^s \Big[v\big(t,\tau,x(\tau)\big) - v\big(s,\tau,x(\tau)\big)\Big]d_\tau K(s,\tau)\right|\\
&\leqslant \int_s^t \big|v\big(t,\tau,x(\tau)\big)\big|\, d_\tau\left(\bigvee_{p=0}^{\tau} K(t,p)\right)\\
&\quad + \int_0^s \big|v\big(t,\tau,x(\tau)\big)\big|\, d_\tau\left(\bigvee_{p=0}^{\tau} [K(t,p) - K(s,p)]\right)\\
&\quad + \int_0^s \big|v\big(t,\tau,x(\tau)\big) - v\big(s,\tau,x(\tau)\big)\big|\, d_\tau\left(\bigvee_{p=0}^{\tau} K(s,p)\right)\\
&\leqslant \phi(\|x\|)\int_s^t d_\tau\left(\bigvee_{p=0}^{\tau} K(t,p)\right) + \phi(\|x\|)\int_0^s d_\tau\left(\bigvee_{p=0}^{\tau} [K(t,p) - K(s,p)]\right)\\
&\quad + \int_0^s \omega^{1,T}_{\|x\|}(v,\varepsilon)d_\tau\left(\bigvee_{p=0}^{\tau} K(s,p)\right)\end{aligned} \tag{3.4}$$

where we denoted

$$\omega_\beta^{1,T}(v,\varepsilon) = \sup\Big\{\big|v(t,\tau,x) - v(s,\tau,x)\big| : t,s,\tau \in [0,T],\ |t-s| \leqslant \varepsilon,\ x \in [-\beta,\beta]\Big\},$$

for an arbitrary number $\beta > 0$.

Further, from estimate (3.4) we get:

$$\begin{aligned}\big|(Vx)(t) - (Vx)(s)\big| &\leqslant \phi(\|x\|) \bigvee_{\tau=s}^{t} K(t,\tau) \\ &\quad + \phi(\|x\|) \bigvee_{\tau=0}^{s} \big[K(t,\tau) - K(s,\tau)\big] + \omega_{\|x\|}^{1,T}(v,\varepsilon) \bigvee_{\tau=0}^{s} K(s,\tau).\end{aligned} \tag{3.5}$$

Hence, keeping in mind that $\omega_{\|x\|}^{1,T}(v,\varepsilon) \to 0$ as $\varepsilon \to 0$ and taking into account assumptions (vii), (viii) and Lemmas 2.3 and 2.4, we infer that the function Vx is continuous on the interval $[0,T]$. Since T was chosen arbitrarily this yields the continuity of the function Vx on the interval $\mathbb{R}_+$.

On the other hand let us observe that from assumption (ii) stems the following estimate

$$\begin{aligned}\big|(Fx)(t) - (Fx)(s)\big| &\leqslant \big|f\big(t,x(t)\big) - f\big(t,x(s)\big)\big| + \big|f\big(t,x(s)\big) - f\big(s,x(s)\big)\big| \\ &\leqslant k\big(\|x\|\big)\,|x(t) - x(s)| + \omega_{\|x\|}^{1,T}(f,\varepsilon)\end{aligned} \tag{3.6}$$

where we denoted

$$\omega_\beta^{1,T}(f,\varepsilon) = \sup\Big\{\big|f(t,x) - f(s,x)\big| : t,s \in [0,T],\ |t-s| \leqslant \varepsilon,\ x \in [-\beta,\beta]\Big\}$$

for an arbitrary $\beta > 0$.

It is obvious that $\omega_\beta^{1,T}(f,\varepsilon) \to 0$ as $\varepsilon \to 0$ which is a consequence of assumption (iii). Thus, joining this fact with estimate (3.6) we conclude that the function Fx is continuous on the interval $[0,T]$. The arbitrariness of T implies the continuity of the function Fx on the interval $\mathbb{R}_+$.

Finally, taking into account representation (3.3) and assumption (i) we conclude that the function Qx is continuous on $\mathbb{R}_+$.

Further on, for a fixed function $x \in BC(\mathbb{R}_+)$ and for arbitrary $t \in \mathbb{R}_+$, in virtue of our assumptions and Lemmas 2.3 and 2.4, we obtain:

$$\begin{aligned}\big|(Qx)(t)\big| &\leqslant \big|a(t)\big| + \Big|f\big(t,x(t)\big)\Big| \left|\int_0^t v\big(t,\tau,x(\tau)\big)d_\tau K(t,\tau)\right| \\ &\leqslant \|a\| + \Big[\big|f\big(t,x(t)\big) - f(t,0)\big| + \big|f(t,0)\big|\Big] \int_0^t \Big|v\big(t,\tau,x(\tau)\big)\Big| d_\tau \left(\bigvee_{p=0}^{\tau} K(t,p)\right) \\ &\leqslant \|a\| + \Big[k\big(\|x\|\big)\big|x(t)\big| + \overline{F}\Big] \int_0^t \phi\big(\|x\|\big) d_\tau \left(\bigvee_{p=0}^{\tau} K(t,p)\right) \\ &\leqslant \|a\| + \Big[\|x\| k\big(\|x\|\big) + \overline{F}\Big] \phi\big(\|x\|\big) \bigvee_{\tau=0}^{t} K(t,\tau) \\ &\leqslant \|a\| + \Big[\|x\| k\big(\|x\|\big) + \overline{F}\Big] \phi\big(\|x\|\big) \overline{K}.\end{aligned}$$

The above estimate shows that the function Qx is bounded on the interval $\mathbb{R}_+$ and yields the inequality

$$\|Qx\| \leqslant \|a\| + \Big[\|x\|k(\|x\|) + \overline{F}\Big]\overline{K}\phi(\|x\|). \tag{3.7}$$

Observe that linking the continuity of the function Qx with is boundedness established above we infer that the operator Q transforms the space $BC(\mathbb{R}_+)$ into itself. Moreover, in view of estimate (3.7) and assumption (x) we deduce that there exists a number $r_0 > 0$ such that Q maps the ball B_{r_0} (in the space $BC(\mathbb{R}_+)$) into itself. Apart from this we have that $\overline{K}k(r_0)\phi(r_0) < 1$.

Now, we show that the operator Q is continuous on the ball B_{r_0}. To this end fix $\varepsilon > 0$ and take arbitrary functions $x, y \in B_{r_0}$ such that $\|x - y\| \leqslant \varepsilon$. Then, in view of imposed assumptions, for an arbitrary number $t \in \mathbb{R}_+$, we get

$$\begin{aligned}\big|(Fx)(t) - (Fy)(t)\big| &= \Big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\Big| \\ &\leqslant k(r_0)\Big|x(t) - y(t)\Big|.\end{aligned}$$

Hence we infer that

$$\|Fx - Fy\| \leqslant k(r_0)\|x - y\| \leqslant k(r_0)\varepsilon. \tag{3.8}$$

Thus, the operator F is continuous on the ball B_{r_0}.

Next, keeping in mind our assumptions, we obtain:

$$\begin{aligned}\big|(Vx)(t) - (Vy)(t)\big| &\leqslant \left|\int_0^t \big[v\big(t, \tau, x(\tau)\big) - v\big(t, \tau, y(\tau)\big)\big]d_\tau K(t, \tau)\right| \\ &\leqslant \int_0^t \big|v\big(t, \tau, x(\tau)\big) - v\big(t, \tau, y(\tau)\big)\big| d_\tau \left(\bigvee_{p=0}^{\tau} K(t, p)\right) \\ &\leqslant \int_0^t \omega_{r_0}^3(v, \varepsilon) d_\tau \left(\bigvee_{p=0}^{\tau} K(t, p)\right) \\ &\leqslant \omega_{r_0}^3(v, \varepsilon) \bigvee_{\tau=0}^{t} K(t, \tau) \leq \overline{K}\omega_{r_0}^3(v, \varepsilon),\end{aligned} \tag{3.9}$$

where we denoted

$$\omega_{r_0}^3(v, \varepsilon) = \sup\Big\{\big|v(t, \tau, x) - v(t, \tau, y)\big| : t, \tau \in \mathbb{R}_+, \ \ x, y \in [-r_0, r_0], \ |x - y| \leqslant \varepsilon\Big\}.$$

Notice that $\omega_{r_0}^3(f, \varepsilon) \to 0$ as $\varepsilon \to 0$ which follows immediately from assumption (v).

Now, combining estimates (3.8), (3.9) and representation (3.3) we infer that the operator Q transforms continuously the ball B_{r_0} into itself.

In what follows let us fix an arbitrary nonempty subset X of the ball B_{r_0}. Next, fix numbers $T > 0$ and $\varepsilon > 0$. Further, take a function $x \in X$ and choose $t, s \in [0, T]$ such that $|t - s| \leqslant \varepsilon$. Without loss of generality we can assume that $s < t$. Then, in view of previously obtained estimate (3.5), we get

$$\begin{aligned}\big|(Vx)(t) - (Vx)(s)\big| &\leqslant \phi(r_0) \bigvee_{\tau=s}^{t} K(t, \tau) \\ &\quad + \phi(r_0) \bigvee_{\tau=0}^{s} \big[K(t, \tau) - K(s, \tau)\big] + \omega_{r_0}^{1,T}(v, \varepsilon) \bigvee_{\tau=0}^{s} K(s, \tau).\end{aligned} \tag{3.10}$$

Now, let us define two auxiliary functions $M(\varepsilon)$ and $N(\varepsilon)$ by putting:

$$M(\varepsilon) = \sup\left\{ \bigvee_{\tau=0}^{t_1} \left[K(t_2,\tau) - K(t_1,\tau)\right] : t_1,t_2 \in \mathbb{R}_+,\ t_1 < t_2,\ t_2 - t_1 \leqslant \varepsilon \right\},$$

$$N(\varepsilon) = \sup\left\{ \bigvee_{\tau=t_1}^{t_2} K(t_2,\tau) : t_1,t_2 \in \mathbb{R}_+,\ t_1 < t_2,\ t_2 - t_1 \leqslant \varepsilon \right\}.$$

Observe that in virtue of assumption (viii) and Lemma 2.4 we have that $M(\varepsilon) \to 0$ and $N(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, from estimate (3.10) we obtain

$$\omega^T(Vx,\varepsilon) \leqslant \phi(r_0)\big(M(\varepsilon) + N(\varepsilon)\big) + \overline{K}\omega_{r_0}^{1,T}(v,\varepsilon). \tag{3.11}$$

Further, utilizing (3.6), we arrive at the following estimate

$$\omega^T(Fx,\varepsilon) \leqslant k(r_0)\omega^T(x,\varepsilon) + \omega_{r_0}^{1,T}(f,\varepsilon), \tag{3.12}$$

where the symbol $\omega_{r_0}^{1,T}(f,\varepsilon)$ was introduced earlier.

Finally for $x \in X$ and for $t,s \in [0,T]$, $s < t$, $t - s \leqslant \varepsilon$, on the basis of estimates (3.11), (3.12), representation (3.3) and earlier obtained evaluations, we get:

$$\begin{aligned}
\big|(Qx)(t) - (Qx)(s)\big| &\leqslant |a(t) - a(s)| + \big|(Fx)(t)(Vx)(t) - (Fx)(s)(Vx)(s)\big| \\
&\leqslant |a(t) - a(s)| + \big|(Fx)(t)\big|\big|(Vx)(t) - (Vx)(s)\big| \\
&\quad + \big|(Vx)(s)\big|\big|(Fx)(t) - (Fx)(s)\big| \\
&\leqslant \omega^T(a,\varepsilon) + \Big[\big|f(t,x(t)) - f(t,0)\big| + \big|f(t,0)\big|\Big]\omega^T(Vx,\varepsilon) \\
&\quad + \phi(r_0)\bigvee_{\tau=0}^{t} K(t,\tau)\omega^T(Fx,\varepsilon) \\
&\leqslant \omega^T(a,\varepsilon) + \big(r_0k(r_0) + \overline{F}\big)\omega^T(Vx,\varepsilon) + \phi(r_0)\overline{K}\omega^T(Fx,\varepsilon) \\
&\leqslant \omega^T(a,\varepsilon) + \big(r_0k(r_0) + \overline{F}\big)\Big\{\phi(r_0)\big(M(\varepsilon) + N(\varepsilon)\big) + \overline{K}\omega_{r_0}^{1,T}(v,\varepsilon)\Big\} \\
&\quad + \overline{K}\phi(r_0)\Big\{k(r_0)\omega^T(x,\varepsilon) + \omega_{r_0}^{1,T}(f,\varepsilon)\Big\}.
\end{aligned}$$

The above estimate yields the following one:

$$\begin{aligned}
\omega^T(QX,\varepsilon) &\leqslant \omega^T(a,\varepsilon) + \big(r_0k(r_0) + \overline{F}\big)\Big\{\phi(r_0)\big(M(\varepsilon) + N(\varepsilon)\big) + \overline{K}\omega_{r_0}^{1,T}(v,\varepsilon)\Big\} \\
&\quad + \overline{K}\phi(r_0)\Big\{k(r_0)\omega^T(X,\varepsilon) + \omega_{r_0}^{1,T}(f,\varepsilon)\Big\}.
\end{aligned} \tag{3.13}$$

Hence, keeping in mind earlier established properties of the quantities $\omega_{r_0}^{1,T}(v,\varepsilon)$, $\omega_{r_0}^{1,T}(f,\varepsilon)$, $M(\varepsilon)$, $N(\varepsilon)$ and taking into account the fact that $\omega^T(a,\varepsilon) \to 0$ as $\varepsilon \to 0$, from estimate (3.13) we obtain

$$\omega_0^T(QX) \leqslant \overline{K}\phi(r_0)k(r_0)\omega_0^T(X).$$

Consequently, we get

$$\omega_0(QX) \leqslant \overline{K}\phi(r_0)k(r_0)\omega_0(X). \tag{3.14}$$

In what follows let us assume that x is an arbitrary function from the set X, $X \subset B_{r_0}$, and $t,s \in \mathbb{R}_+$ are such that $t,s \geqslant T$. Similarly as previously we may assume that $s < t$.

For further purposes let us define the following auxiliary functions:

$$U(T) = \sup\left\{ \bigvee_{\tau=s}^{t} K(t,\tau) : T \leqslant s < t \right\},$$
$$W(T) = \sup\left\{ \bigvee_{\tau=0}^{s} \left[K(t,\tau) - K(s,\tau)\right] : T \leqslant s < t \right\}.$$

Moreover, for an arbitrary fixed $R > 0$ let us put:

$$Y_R(T) = \sup\left\{ \left|f(t,x) - f(s,x)\right| : T \leqslant s < t, x \in [-R,R] \right\},$$
$$Z_R(T) = \sup\left\{ \left|v(t,\tau,x) - v(s,\tau,x)\right| : t,s \geqslant T, \tau \in \mathbb{R}_+, \tau \leqslant s, \tau \leqslant t, x \in [-R,R] \right\}.$$

Notice that in view of assumption (ix) we have that $U(T) \to 0$ and $W(T) \to 0$ as $T \to \infty$. Moreover, $Z_R(T) \to 0$ as $T \to \infty$ for any fixed $R > 0$. Finally, $Y_R(T) \to 0$ as $T \to \infty$ which is a consequence of assumption (iii).

Next, arguing similarly as in (3.5) and (3.6), for $T \leqslant s < t$ and for $x \in X$ we obtain

$$\begin{aligned}
&\left|(Qx)(t) - (Qx)(s)\right| \\
&\quad \leqslant |a(t) - a(s)| + \Big[\left|f(t,x(t)) - f(t,0)\right| + \left|f(t,0)\right|\Big]\left|(Vx)(t) - (Vx)(s)\right| \\
&\qquad + \phi(r_0) \bigvee_{\tau=0}^{s} K(s,\tau)\left|(Fx)(t) - (Fx)(s)\right| \\
&\quad \leqslant |a(t) - a(s)| + \left(r_0 k(r_0) + \overline{F}\right)\Big\{\phi(r_0) \bigvee_{\tau=s}^{t} K(t,\tau) + \phi(r_0) \bigvee_{\tau=0}^{s} \left[K(t,\tau) - K(s,\tau)\right] \\
&\qquad + \sup\Big[\left|v(t,\tau,x) - v(s,\tau,x)\right| : \tau \in \mathbb{R}_+, T \leqslant s < t, x \in [-r_0,r_0]\Big] \bigvee_{\tau=0}^{s} K(s,\tau)\Big\} \\
&\qquad + \phi(r_0) \bigvee_{\tau=0}^{s} K(s,\tau)\Big\{k(r_0)\left|x(t) - x(s)\right| + \sup\Big[\left|f(t,x) - f(s,x)\right| : T \leqslant s < t,\ x \in [-r_0,r_0]\Big]\Big\} \\
&\quad \leqslant \sup\Big\{|a(t) - a(s)| : T \leqslant s < t\Big\} \\
&\qquad + \left(r_0 k(r_0) + \overline{F}\right)\Big\{\phi(r_0) U(T) + \phi(r_0) W(T) + \overline{K} Z_{r_0}(T)\Big\} \\
&\qquad + \phi(r_0)\overline{K}\Big\{k(r_0) \sup\Big[\left|x(t) - x(s)\right| : T \leqslant s < t\Big] + Y_{r_0}(T)\Big\}.
\end{aligned}$$

Now, passing with $T \to \infty$ and utilizing the above indicated properties of the quantities $U(T)$, $W(T)$, $Y_{r_0}(T)$ and $Z_{r_0}(T)$, from the above estimate and assumption (i) we derive the following inequality

$$b(QX) \leqslant \overline{K}\phi(r_0)k(r_0)b(X), \tag{3.15}$$

where the quantity $b(X)$ was defined in Section 2.

Finally, combining inequalities (3.14) and (3.15) we deduce the following estimate

$$\mu(QX) \leqslant \overline{K}\phi(r_0)k(r_0)\mu(X), \tag{3.16}$$

where μ is the measure of noncompactness in the space $BC(\mathbb{R}_+)$ defined by formula (2.1).

Now taking into account estimate (3.16), the second part of assumption (x) and applying Theorem 2.2 we complete the proof. □

Let us observe that taking in Eq. (3.1) $f(t,x) \equiv 1$ we obtain the Volterra–Stieltjes integral equation of the form

$$x(t) = a(t) + \int_0^t v(t,s,x(s))\,d_sK(t,s), \tag{3.17}$$

for $t \in \mathbb{R}_+$. Thus, Eq. (3.17) is a special case of Eq. (3.1).

Let us notice that Eq. (3.17) was investigated in details in [7]. By these regards the results obtained in this paper create the generalizations of those form [7].

It is also worthwhile mentioning that in [7] there were discussed a lot of other special cases of Eq. (3.17).

4 Final remarks and an example

At the beginning of this section we recall a few remarks from paper [7] which can be also adapted to Eq. (3.1) considered in this paper.

First of all let us indicate a condition being convenient in applications and guaranteeing that the function $K = K(t,s)$ appearing in Eq. (3.1) satisfies assumption (viii) of Theorem 3.3. That assumption is crucial in our considerations (cf. also [5]).

To this end assume, similarly as before, that $K : \Delta \to \mathbb{R}$. Then, the mentioned condition can be formulated in the following way.

(viii$'$) For arbitrary $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$ the function $s \mapsto K(t_2,s) - K(t_1,s)$ is nondecreasing (nonincreasing) on the interval $[0,t_1]$.

It can be shown [5] that if the function $K(t,s)$ satisfies assumptions (viii$'$) and (vi) then for arbitrarily fixed $s \in \mathbb{R}_+$ the function $t \mapsto K(t,s)$ is nondecreasing (nonincreasing) on the interval $[s,\infty]$. Moreover, under assumptions (viii$'$) and (vi) the function K satisfies assumption (viii).

Remark 4.1. It is worthwhile noticing [7] that under assumptions (vi) and (viii$'$) the second equality in assumption (ix) can be replaced by the following requirement:

$$(\mathrm{xi}_1)\quad \lim_{T\to\infty}\left\{\sup\left[K(t,s) - K(s,s) : T \leqslant s < t\right]\right\} = 0$$

in the case when we assume in (viii$'$) that the function $s \mapsto K(t_2,s) - K(t_1,s)$ is nondecreasing. In the case when we assume that the mentioned function is nonincreasing then the second equality in (ix) can be replaced by the following requirement:

$$(\mathrm{xi}_2)\quad \lim_{T\to\infty}\left\{\sup\left[K(s,s) - K(t,s) : T \leq s < t\right]\right\} = 0.$$

Now, we are going to illustrate our existence result contained in Theorem 3.3 by an example.

Example 4.2. We consider the quadratic integral equation of Volterra–Hammerstain type having the form

$$x(t) = \frac{\alpha t}{t+1} + \beta \sin\left(\frac{t^2 + x^2(t)}{t^2+1}\right) \int_0^t \frac{ste^{-t} + \frac{s}{t^2+1}x^2(s)}{1+s^2+t^2}\,ds, \tag{4.1}$$

for $t \in \mathbb{R}_+$, where $\alpha > 0$ and $\beta > 0$ are some constants. Observe that Eq. (4.1) can be written in the form of the quadratic integral equation of Volterra–Stieltjes type

$$x(t) = \frac{\alpha t}{t+1} + \beta \sin\left(\frac{t^2 + x^2(t)}{t^2+1}\right) \int_0^t \left(ste^{-t} + \frac{s}{t^2+1}x^2(s)\right) d_sK(t,s), \tag{4.2}$$

where the function $K(t,s)$ has the form

$$K(t,s) = \frac{1}{\sqrt{1+t^2}} \arctan \frac{s}{\sqrt{1+t^2}} \tag{4.3}$$

for $(t,s) \in \Delta = \{(t,s) : 0 \leqslant s \leqslant t\}$.

Indeed, it is easy to check that for function (4.3) we have

$$d_sK(t,s) = \frac{\partial K(t,s)}{\partial s} ds = \frac{1}{1+s^2+t^2} ds.$$

Let us notice that Eq. (4.2) is a particular case of Eq. (3.1) if we put

$$\begin{aligned} a(t) &= \frac{\alpha t}{t+1}, \\ f(t,x) &= \beta \sin\left(\frac{t^2+x^2}{t^2+1}\right), \\ v(t,s,x) &= ste^{-t} + \frac{s}{t^2+1}x^2 \end{aligned}$$

and if the function K has the form (4.3).

In what follows we show that the above indicated functions being the components of Eq. (4.1) satisfy assumptions of Theorem 3.3.

At the beginning let us note that the function function $a = a(t)$ satisfies assumption (i) and $\|a\| = \alpha$.

In order to show that the function $f = f(t,x)$ satisfies assumption (ii) observe that f is continuous on the set $\mathbb{R}_+ \times \mathbb{R}$. Next, fix arbitrary $r > 0$ and take $t \in \mathbb{R}_+$ and $x, y \in [-r, r]$. Then we have

$$\begin{aligned} |f(t,x) - f(t,y)| &\leqslant \beta \left| \frac{t^2+x^2}{t^2+1} - \frac{t^2+y^2}{t^2+1} \right| = \beta \frac{|x^2-y^2|}{t^2+1} \\ &\leqslant \beta |x+y| \, |x-y| \leqslant 2\beta r \, |x-y| \, . \end{aligned}$$

Hence we see that the function f satisfies assumption (ii) if we put $k(r) = 2\beta r$.

Further on let us notice that for $x \in \mathbb{R}$ arbitrarily fixed we get

$$\lim_{t\to\infty} f(t,x) = \lim_{t\to\infty} \beta \sin\left(\frac{t^2}{t^2+1} + \frac{x^2}{t^2+1}\right) = \beta \sin 1,$$

and the above limit is uniform with respect to x belonging to an arbitrary interval of the form $[-r, r]$, where $r > 0$. This means that assumption (iii) is satisfied.

The above statement can be proved also immediately. Indeed, fix arbitrarily $r > 0$ and assume that $x \in [-r, r]$. Then for arbitrary fixed $T > 0$ and for $t, s \geqslant T$ we obtain

$$\begin{aligned}
\left|f(t,x) - f(s,x)\right| &\leqslant \beta \left| \frac{t^2 + x^2}{t^2 + 1} - \frac{s^2 + x^2}{s^2 + 1} \right| \leqslant \beta \frac{\left(x^2+1\right)\left|t^2 - s^2\right|}{\left(t^2+1\right)\left(s^2+1\right)} \\
&\leqslant \beta \left(r^2+1\right) \left| \frac{t^2}{\left(t^2+1\right)\left(s^2+1\right)} - \frac{s^2}{\left(t^2+1\right)\left(s^2+1\right)} \right| \\
&\leqslant \beta \left(r^2+1\right) \left(\frac{t^2}{\left(t^2+1\right)\left(s^2+1\right)} + \frac{s^2}{\left(t^2+1\right)\left(s^2+1\right)} \right) \\
&\leqslant \beta \left(r^2+1\right) \left(\frac{1}{s^2+1} + \frac{1}{t^2+1} \right) \leqslant 2\beta \left(r^2+1\right) \frac{1}{T^2+1}.
\end{aligned}$$

Hence we infer that inequality (3.2) from assumption (iii) is satisfied if we choose $T > 0$ big enough.

Subsequently, let us note that the function $v = v(t,s,x)$ is continuous on the set $\Delta \times \mathbb{R}$. Moreover, taking arbitrary $(t,s) \in \Delta$ and $x \in \mathbb{R}$ we get

$$\left|v(t,s,x)\right| \leqslant t^2 e^{-t} + \frac{t}{t^2+1}|x|^2 \leqslant \frac{4}{e^2} + \frac{1}{2}|x|^2.$$

Thus the function v satisfies the inequality from assumption (iv) if we take $\phi(r) = \frac{4}{e^2} + \frac{1}{2}r^2$.

Summing up we see that the function v satisfies assumption (iv).

It is also easily seen that function $v = v(t,s,x)$ is uniformly continuous on each set of the form $\Delta \times [-r, r]$ for $r > 0$. This allows us to infer that v satisfies assumption (v).

Obviously it is easy to notice that the function $K = K(t,s)$ defined by (4.3) satisfies assumption (vi).

To show that the function $K(t,s)$ satisfies assumption (vii) let us note that in view of the inequality

$$\frac{\partial K(t,s)}{\partial s} = \frac{1}{1 + s^2 + t^2} > 0$$

we infer that the function $s \mapsto K(t,s)$ is increasing on every interval of the form $[0,t]$. Since $K(t,s)$ is bounded on the triangle Δ this implies that the function $s \mapsto K(t,s)$ is of bounded variation on the interval $[0,t]$.

Moreover, we have

$$\bigvee_{s=0}^{t} K(t,s) = K(t,t) - K(t,0) = \frac{1}{\sqrt{1+t^2}} \arctan \frac{t}{\sqrt{1+t^2}} \leqslant \frac{\pi}{4\sqrt{1+t^2}} \leqslant \frac{\pi}{4}.$$

This shows that there is satisfied assumption (vii) and we have that $\overline{K} \leqslant \frac{\pi}{4}$. In what follows we will accept that $\overline{K} = \frac{\pi}{4}$.

In order to verify assumption (viii) it is sufficient to show that the function $K(t,s)$ satisfies assumption (viii′). Thus, fix arbitrarily $t_1, t_2 \in \mathbb{R}_+$ with $t_1 < t_2$. Consider the function $s \mapsto K(t_2,s) - K(t_1,s)$. In view of the equality

$$K(t_2,s) - K(t_1,s) = \frac{1}{\sqrt{1+t_2^2}} \arctan \frac{s}{\sqrt{1+t_2^2}} - \frac{1}{\sqrt{1+t_1^2}} \arctan \frac{s}{\sqrt{1+t_1^2}}$$

we derive the following assertion

$$\frac{\partial}{\partial s}\big(K(t_2,s) - K(t_1,s)\big) = \frac{t_1^2 - t_2^2}{\left(1+s^2+t_1^2\right)\left(1+s^2+t_2^2\right)} < 0.$$

This yields that the function $s \mapsto K(t_2, s) - K(t_1, s)$ is nonincreasing on $\mathbb{R}_+$. Particularly, it is nonincreasing on the interval $[0, t_1]$. This means that the function $K(t, s)$ satisfies assumption (viii′). Taking into account the fact that K satisfies assumption (vi), we conclude that the function $K = K(t, s)$ satisfies assumption (viii).

Now, keeping in mind that the function $s \mapsto K(t, s)$ is increasing on the interval $[0, t]$, we get

$$\begin{aligned}\bigvee_{\tau=s}^{t} K(t,\tau) &= K(t,t) - K(t,s) \\ &= \frac{1}{\sqrt{1+t^2}} \arctan \frac{t}{\sqrt{1+t^2}} - \frac{1}{\sqrt{1+t^2}} \arctan \frac{s}{\sqrt{1+t^2}} \\ &\leqslant \frac{1}{\sqrt{1+t^2}} \arctan \frac{t}{\sqrt{1+t^2}}.\end{aligned}$$

Obviously the above estimate implies that the function K satisfies the first equality from assumption (ix).

In order to verify the second equality from assumption (ix) let us notice that in view of the above established facts and Remark 4.1 it is sufficient to show that there is satisfied assumption (xi_2). Thus, taking $T \leqslant s < t$, we get

$$\begin{aligned}K(s,s) - K(t,s) &= \frac{1}{\sqrt{1+s^2}} \arctan \frac{s}{\sqrt{1+s^2}} - \frac{1}{\sqrt{1+t^2}} \arctan \frac{s}{\sqrt{1+t^2}} \\ &\leqslant \frac{1}{\sqrt{1+s^2}} \arctan \frac{s}{\sqrt{1+s^2}} \leqslant \frac{\pi}{4} \frac{1}{\sqrt{1+s^2}} \leqslant \frac{\pi}{4} \frac{1}{\sqrt{1+T^2}}.\end{aligned}$$

Obviously the above estimate implies that for the function $K = K(t, s)$ the second equality from assumption (ix) holds.

Now, we intend to check the last equality from assumption (ix).

To this end fix $r > 0$, $T > 0$ and take $t, s, \tau \in \mathbb{R}_+$ such that $s, t \geqslant T$, $\tau \leqslant s$, $\tau \leqslant t$ and $y \in [-r, r]$. Without loss of generality we may assume that $T \geqslant 2$.

Then, we derive the following estimate:

$$\begin{aligned}\left|v(t,\tau,y) - v(s,\tau,y)\right| &= \left|\tau t e^{-t} + \frac{\tau}{t^2+1} x^2 - \tau s e^{-s} - \frac{\tau}{s^2+1} x^2\right| \\ &\leqslant \tau\left|t e^{-t} - s e^{-s}\right| + \tau \left|\frac{1}{t^2+1} - \frac{1}{s^2+1}\right| |x|^2 \\ &\leqslant \tau\left(t e^{-t} + s e^{-s}\right) + \tau \left(\frac{1}{t^2+1} + \frac{1}{s^2+1}\right) r^2 \\ &= \tau t e^{-t} + \tau s e^{-s} + \left(\frac{\tau}{t^2+1} + \frac{\tau}{s^2+1}\right) r^2 \\ &\leqslant t^2 e^{-t} + s^2 e^{-s} + \left(\frac{t}{t^2+1} + \frac{s}{s^2+1}\right) r^2 \\ &\leqslant 2T^2 e^{-T} + \frac{2T}{T^2+1} r^2.\end{aligned}$$

From the above estimate we conclude that the third equality from assumption (ix) is satisfied.

In order to proceed to assumption (x) let us first observe that $f(t, 0) = \beta \sin \frac{t^2}{t^2+1}$. Hence, we have

$$\overline{F} = \sup\left\{|f(t,0)| : t \in \mathbb{R}_+\right\} = \beta \sin 1.$$

Now, let us consider the first inequality from assumption (x). Taking into account the above established facts we can write that inequality in the form

$$\alpha + \frac{\pi}{4}\left(2\beta r^2 + \beta \sin 1\right)\left(\frac{4}{e^2} + \frac{1}{2}r^2\right) \leqslant r. \tag{4.4}$$

Putting, for example, $r = 1$ and assuming that $\alpha < 1$, we can rewrite inequality (4.4) in the form

$$\frac{\pi}{4}\beta\left(2 + \sin 1\right)\left(\frac{4}{e^2} + \frac{1}{2}\right) \leqslant 1 - \alpha.$$

Hence, we get

$$\beta \leqslant \frac{4(1-\alpha)}{\pi\left(2 + \sin 1\right)\left(\frac{4}{e^2} + \frac{1}{2}\right)}. \tag{4.5}$$

Thus assumption (x) will be satisfied if we choose $\alpha < 1$, $r_0 = 1$ and if we take β such that inequality (4.5) holds. Obviously, inequality (4.5) yields the second inequality from assumption (x) which has the following form in our situation

$$\beta < \frac{2}{\pi\left(\frac{4}{e^2} + \frac{1}{2}\right)}.$$

Thus, on the basis of Theorem 3.3 we conclude that under suitable constraints concerning the constants α and β integral equation (4.1) (or (4.2)) has a solution in the space $BC(\mathbb{R}_+)$ which converges to a finite limit at infinity.

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