



## Homoclinic solutions of singular differential equations with $\phi$ -Laplacian

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Received 8 March 2018, appeared 28 August 2018

Communicated by Josef Diblík

**Abstract.** A singular nonlinear initial value problem (IVP) with a  $\phi$ -Laplacian of the form

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \quad u(0) = u_0 \in [L_0, 0), \quad u'(0) = 0$$

is investigated on the half-line  $[0, \infty)$ . Here, function  $\phi$  is smooth and increasing on  $\mathbb{R}$  with  $\phi(0) = 0$ , function  $f$  is locally Lipschitz continuous with three zeros  $\phi(L_0) < 0 < \phi(L)$ , function  $p$  is smooth and increasing on  $(0, \infty)$ , and the problem is singular in the sense that  $p(0) = 0$  and  $1/p(t)$  may not be integrable on  $[0, 1]$ . The main result of the paper is the existence of homoclinic solutions defined as nondecreasing solutions  $u$  of the IVP satisfying  $\lim_{t \rightarrow \infty} u(t) = L$ .

**Keywords:** second order ODE, time singularity,  $\phi$ -Laplacian, homoclinic solution, half-line.

**2010 Mathematics Subject Classification:** 34A12, 34C37, 34D05.

### 1 Introduction

We investigate solutions of the initial value problem (IVP)

$$(p(t)\phi(u'(t)))' + p(t)f(\phi(u(t))) = 0, \quad t \in (0, \infty), \quad (1.1)$$

$$u(0) = u_0, \quad u'(0) = 0, \quad u_0 \in [L_0, 0), \quad (1.2)$$

where

$$\phi \in C^1(\mathbb{R}), \quad \phi'(x) > 0 \quad \text{for } x \in (\mathbb{R} \setminus \{0\}), \quad (1.3)$$

$$\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0, \quad (1.4)$$

$$L_0 < 0 < L, \quad f(\phi(L_0)) = f(0) = f(\phi(L)) = 0, \quad (1.5)$$

$$f \in \text{Lip}[\phi(L_0), \phi(L)], \quad xf(x) > 0 \quad \text{for } x \in ((\phi(L_0), \phi(L)) \setminus \{0\}), \quad (1.6)$$

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p'(t) > 0 \quad \text{for } t \in (0, \infty), \quad p(0) = 0. \quad (1.7)$$

In particular, we find additional conditions for  $p$ ,  $\phi$  and  $f$  which guarantee for some  $u_0 \in [L_0, 0)$  the existence of a nondecreasing solution of IVP (1.1), (1.2) converging to  $L$  for  $t \rightarrow \infty$ . Note that if we extend the function  $p$  in equation (1.1) from the half-line onto  $\mathbb{R}$  as an even function and assume that  $\phi$  is odd, then any solution  $u$  of IVP (1.1), (1.2) with  $\lim_{t \rightarrow \infty} u(t) = L$  fulfils  $\lim_{t \rightarrow -\infty} u(t) = L$ . Such solution  $u$  is called a *homoclinic solution*. This is a motivation for Definition 1.4. Due to condition (1.7) the function  $1/p(t)$  may not be integrable on  $[0, 1]$  and consequently equation (1.1) has a time singularity at  $t = 0$ . Problems of this type arise in hydrodynamics [10] or in the nonlinear field theory [7], where homoclinic solutions play an important role in the study of behaviour of corresponding differential models. The paper is a culmination of our previous research and results from [5] and [25], where other types of solutions of IVP (1.1), (1.2) have been studied.

Our first attempts in this subject have been made for the equation without  $\phi$ -Laplacian

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in (0, \infty),$$

with  $p \equiv q$  in [18–23] and for  $p \not\equiv q$  in [4, 6, 24, 26]. Other problems without  $\phi$ -Laplacian close to (1.1), (1.2) can be found in [1–3, 8, 12–14] and those with  $\phi$ -Laplacian in [9, 11, 15–17].

IVP (1.1), (1.2) can be transformed to the equivalent integral equation

$$u(t) = u_0 + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau))) \, d\tau \right) \, ds, \quad t \in [0, \infty). \quad (1.8)$$

Assumption (1.3) implies that  $\phi$  is locally Lipschitz continuous on  $\mathbb{R}$ , but if  $\phi'(0) = 0$ , then

$$\lim_{x \rightarrow 0} (\phi^{-1})'(x) = \infty,$$

and so  $\phi^{-1}$  does not fulfil the Lipschitz condition on intervals containing 0. If values of  $u$  are between  $L_0$  and  $L$ , we see that

$$\lim_{s \rightarrow 0^+} \frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u(\tau))) \, d\tau = 0.$$

Therefore  $\phi^{-1}$  in (1.8) is considered on an interval containing zero. Hence, in order to prove the uniqueness for IVP (1.1), (1.2) if  $\phi'(0) = 0$ , we need to use some new condition for  $\phi^{-1}$  instead of the Lipschitz one. Such cases of  $\phi$  have been considered in [5], where we have proved the existence of a unique solution of IVP (1.1), (1.2) for  $u_0 > L_0$  and  $\phi'(0) = 0$  under the assumption that  $\phi$  fulfils conditions

$$\limsup_{x \rightarrow 0^-} \left( -x (\phi^{-1})'(x) \right) < \infty, \quad \phi' \text{ is nonincreasing on } (-\infty, 0), \quad (1.9)$$

$$\limsup_{x \rightarrow 0^+} \left( x (\phi^{-1})'(x) \right) < \infty, \quad \phi' \text{ is nondecreasing on } (0, \infty). \quad (1.10)$$

**Example 1.1.** A typical model example is the  $\alpha$ -Laplacian  $\phi(x) = |x|^\alpha \operatorname{sgn} x$ ,  $x \in \mathbb{R}$ , where  $\alpha \geq 1$ . Then  $\phi'(x) = \alpha|x|^{\alpha-1}$  and conditions (1.3) and (1.4) are fulfilled. If  $\alpha > 1$ , then  $\phi'(0) = 0$ ,  $\phi'$  is nonincreasing on  $(-\infty, 0)$  and nondecreasing on  $(0, \infty)$ . Further,

$$\phi^{-1}(x) = |x|^{\frac{1}{\alpha}} \operatorname{sgn} x, \quad (\phi^{-1})'(x) = \frac{1}{\alpha} |x|^{\frac{1}{\alpha}-1}, \quad \lim_{x \rightarrow 0} (\phi^{-1})'(x) = \infty,$$

which yields that  $\phi^{-1}$  is not Lipschitz continuous at 0. Since

$$\lim_{x \rightarrow 0} x \left( \phi^{-1} \right)' (x) = \frac{1}{\alpha} \lim_{x \rightarrow 0} x |x|^{\frac{1}{\alpha}-1} = 0,$$

we see that the  $\alpha$ -Laplacian  $\phi(x) = |x|^\alpha \operatorname{sgn} x$  fulfils (1.9), (1.10).

If we take  $p(t) = t^\beta$ ,  $t \in [0, \infty)$ , where  $\beta > 0$ , then  $p$  fulfils (1.7). As an example of  $f$  satisfying conditions (1.5) and (1.6) we can take  $f(x) = x(x - \phi(L_0))(\phi(L) - x)$ ,  $x \in \mathbb{R}$ .

**Definition 1.2.** A function  $u \in C^1[0, \infty)$  with  $\phi(u') \in C^1(0, \infty)$  which satisfies equation (1.1) for every  $t \in (0, \infty)$  is called a *solution* of equation (1.1). If moreover  $u$  satisfies the initial conditions (1.2), then  $u$  is called a *solution* of IVP (1.1), (1.2).

**Remark 1.3.** Equation (1.1) has the constant solutions  $u(t) \equiv L$ ,  $u(t) \equiv 0$  and  $u(t) \equiv L_0$ .

**Definition 1.4.** Consider a solution  $u$  of IVP (1.1), (1.2) with  $u_0 \in [L_0, 0)$  and denote

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}.$$

If  $u_{\sup} < L$ , then  $u$  is called a *damped solution* of IVP (1.1), (1.2).

If  $u_{\sup} = L$  and  $u$  is nondecreasing (i.e.  $\lim_{t \rightarrow \infty} u(t) = L$ ), then  $u$  is called a *homoclinic solution* of IVP (1.1), (1.2).

The homoclinic solution is called a *regular homoclinic solution*, if  $u(t) < L$  for  $t \in [0, \infty)$  and a *singular homoclinic solution*, if there exists  $t_0 > 0$  such that  $u(t) = L$  for  $t \in [t_0, \infty)$ .

If  $u_{\sup} > L$ , then  $u$  is called an *escape solution* of IVP (1.1), (1.2).

Conditions giving the existence of damped solutions are published in [5] and those for the existence of escape solutions can be found in [25]. Our goal is to prove the existence of a homoclinic solution of IVP (1.1), (1.2) with some starting value  $u_0 \in [L_0, 0)$  provided some suitable additional conditions are fulfilled. The main result of the paper is contained in the next theorem.

**Theorem 1.5** (Homoclinic solutions). *Let (1.3)–(1.7) and (2.2)–(2.4) hold. Further assume that*

$$\textit{there exists a right neighbourhood of } \phi(L_0), \textit{ where } f \textit{ is decreasing.} \quad (1.11)$$

*Then there exists  $u_0^* \in [L_0, \bar{B})$  such that a solution  $u_h$  of IVP (1.1), (1.2) with  $u_0 = u_0^*$  is homoclinic.*

Examples of graphs of homoclinic solutions and of a function  $f$  satisfying the conditions of Theorem 1.5 are in Figures 1.1–1.3.

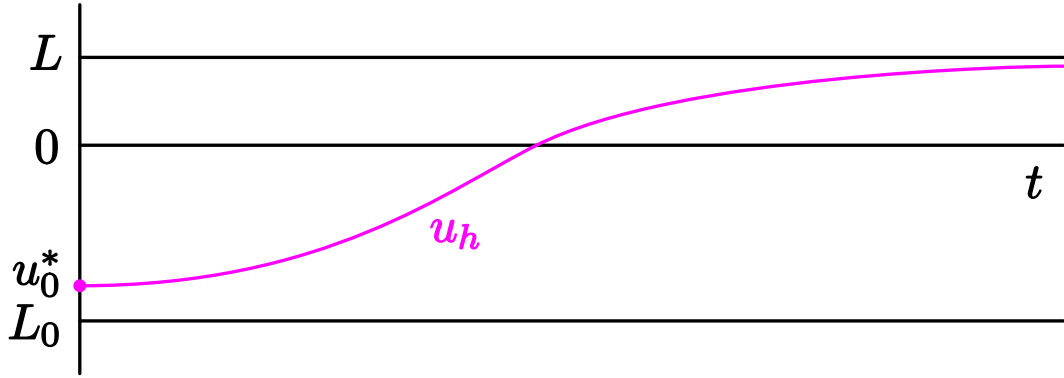
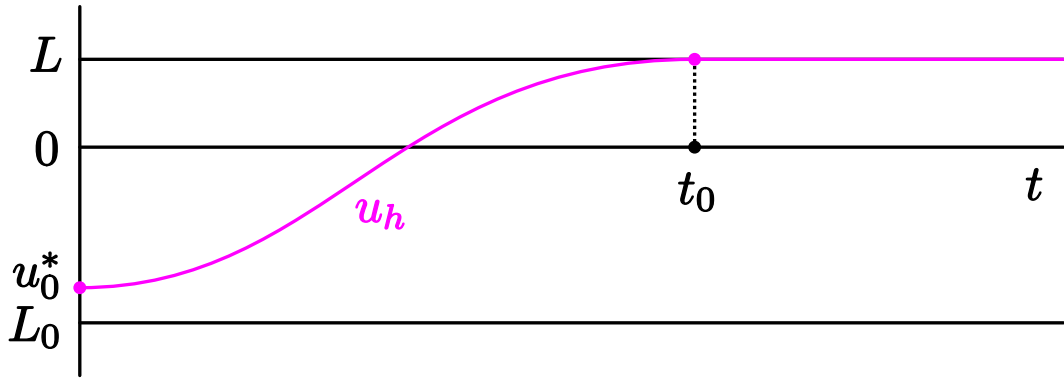
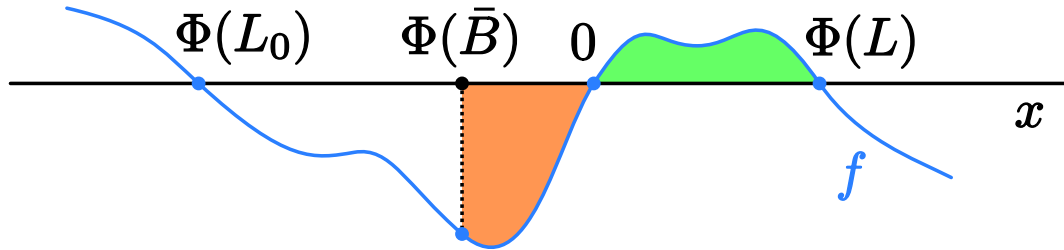
## 2 Auxiliary results

Here we present an overview of results from [5] and [25] which we need to get a homoclinic solution of IVP (1.1), (1.2). The first group consists of results about existence and uniqueness which follow from [5, Th. 4.1, Th. 5.1, Th. 5.4, Th. 6.5] and [25, Th. 4.7].

Since values of any homoclinic solution belong to  $[L_0, L]$ , we can assume without loss of generality

$$f(x) = 0 \quad \text{for } x \leq \phi(L_0), \quad x \geq \phi(L) \quad (2.1)$$

in our next investigation.

Figure 1.1: Graph of a regular homoclinic solution  $u_h$ Figure 1.2: Graph of a singular homoclinic solution  $u_h$ Figure 1.3: Graph of a function  $f$  (sizes of the orange and green areas are the same)

**Theorem 2.1** (Existence of solutions). Assume (1.3)–(1.7) and (2.1). Then, for each starting value  $u_0 \in [L_0, 0)$ , there exists a solution of IVP (1.1), (1.2).

**Theorem 2.2** (Damped solutions). Let (1.3)–(1.7) and (2.1) hold and let

$$\exists \bar{B} \in (L_0, 0): F(\bar{B}) = F(L), \quad \text{where} \quad F(x) = \int_0^x f(\phi(s)) \, ds, \quad x \in \mathbb{R}, \quad (2.2)$$

and

$$\lim_{t \rightarrow \infty} \frac{p'(t)}{p(t)} = 0. \quad (2.3)$$

Then every solution of IVP (1.1), (1.2) with the starting value  $u_0 \in [\bar{B}, 0)$  is damped.

Assume in addition that

$$\lim_{x \rightarrow 0} |x| \left( \phi^{-1} \right)'(x) < \infty, \quad (2.4)$$

and that  $u$  is a damped solution of IVP (1.1), (1.2) with the starting value  $u_0 \in (L_0, 0)$ . Then  $u$  is a unique solution of this IVP.

**Theorem 2.3** (Escape solutions). *Let (1.3)–(1.7) and (2.1)–(2.3) hold. Then there exist infinitely many escape solutions of IVP (1.1), (1.2) with starting values in  $[L_0, \bar{B})$ .*

*Assume in addition that (2.4) hold and that  $u$  is an escape solutions of IVP (1.1), (1.2) with the starting value  $u_0 \in (L_0, \bar{B})$ . Then  $u$  is a unique solution of this IVP.*

**Remark 2.4.** The uniqueness of damped and escape solutions is proved in [5, Th. 5.4, Th. 6.5] under the assumptions (1.9), (1.10). Using the arguments from the proof of Lemma 4.1, we see that the requirement of the monotonicity of  $\phi'$  can be omitted and (1.9), (1.10) can be replaced with (2.4).

**Remark 2.5.** If we assume that (1.3)–(1.7) and (2.1)–(2.3) hold and in addition that  $\phi'(0) > 0$ , then the condition

$$\phi^{-1} \in \text{Lip}_{\text{loc}}(\mathbb{R}), \quad (2.5)$$

is fulfilled. In this case, by Theorem 2.1 and [5, Th. 4.3], for each  $u_0 \in [L_0, 0)$  there exists a unique solution of IVP (1.1), (1.2). In particular for  $u_0 = L_0$  IVP (1.1), (1.2) has a unique (constant) solution.

**Example 2.6.** As an example of  $\phi$  satisfying (1.3), (1.4) and  $\phi'(0) > 0$  we choose

$$\phi(x) = \sinh(x) = (e^x - e^{-x})/2, \quad x \in \mathbb{R}.$$

The second group contains results about asymptotic behaviour of damped, escape and homoclinic solutions and can be reached from [5, L. 2.1b), L. 2.6, L. 2.8, L. 3.2, L. 3.4, L. 6.2, L. 6.3] and [25, L. 3.3, L. 3.4]. In particular, paper [5] mostly deals with damped solutions and proves their possible behaviour as illustrated in Figure 2.1. Paper [25] investigates escape solutions, proves their monotonicity and presents conditions guaranteeing their unboundedness. For graphs of escape solutions see Figure 2.2.

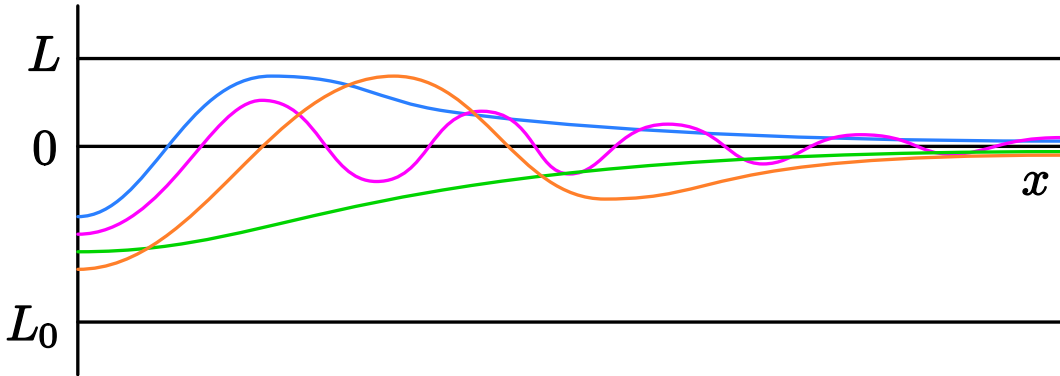


Figure 2.1: Graphs of damped solutions

**Theorem 2.7** (Starting value in  $(L_0, 0)$ ). *Let (1.3)–(1.7) and (2.1)–(2.3) hold and let  $u$  be a solution of IVP (1.1), (1.2) with the starting value  $u_0 \in (L_0, 0)$ . Then*

$$u(t) > L_0 \quad \text{and} \quad \exists \tilde{c} > 0 \quad \text{such that} \quad |u'(t)| \leq \tilde{c} \quad \text{for} \quad t \in (0, \infty). \quad (2.6)$$

*The constant  $\tilde{c}$  depends on  $L_0, L_1, \phi$  and  $f$  and does not depend on  $p$  and  $u$ .*

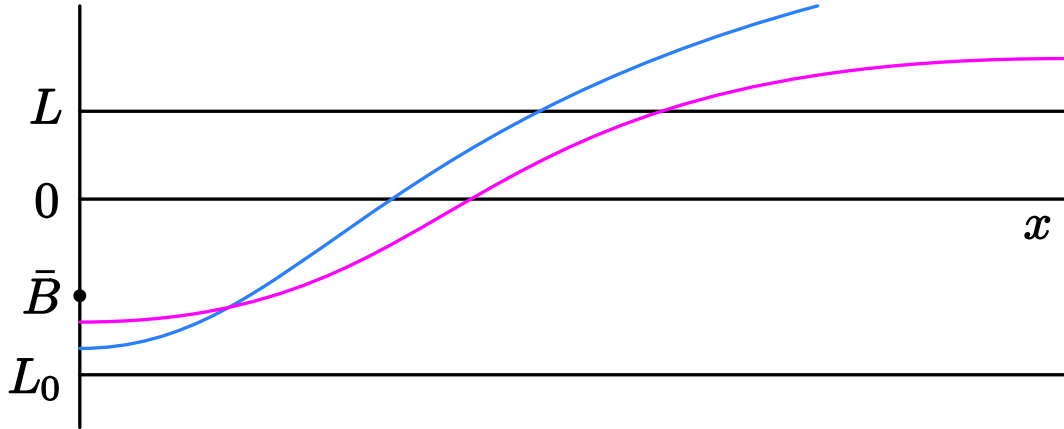


Figure 2.2: Graphs of escape solutions

1. Assume that  $u_{\text{sup}} < L$ , i.e.  $u$  is a damped solution.

- Let  $\theta > 0$  be the first zero of  $u$ . Then there exists  $\theta < a < b$  such that

$$u(a) \in (0, L), \quad u'(t) > 0 \quad \text{on } (0, a), \quad u'(a) = 0, \quad u'(t) < 0 \quad \text{on } (a, b). \quad (2.7)$$

- Let  $u < 0$  on  $[0, \infty)$ . Then

$$u'(t) > 0 \quad \text{for } t \in (0, \infty), \quad \lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0. \quad (2.8)$$

2. Assume that  $u_{\text{sup}} > L$ , i.e.  $u$  is an escape solution. Then

$$u'(t) > 0 \quad \text{for } t \in (0, \infty). \quad (2.9)$$

3. Assume that  $u_{\text{sup}} = L$ . Then there are two possibilities.

- $u(t) < L$  for  $t \in [0, \infty)$  which yields

$$u'(t) > 0 \quad \text{for } t \in (0, \infty), \quad \lim_{t \rightarrow \infty} u(t) = L, \quad \lim_{t \rightarrow \infty} u'(t) = 0, \quad (2.10)$$

and  $u$  is a regular homoclinic solution.

- There exists  $t_0 > 0$  such that  $u(t_0) = L$ ,  $u'(t_0) = 0$  which implies

$$u'(t) > 0 \quad \text{for } t \in (0, t_0), \quad (2.11)$$

and there exists a singular homoclinic solution  $v$ , where  $v = u$  on  $[0, t_0]$  and  $v = L$  on  $[t_0, \infty)$ .

Consider a solution  $u \not\equiv L_0$  of IVP (1.1), (1.2) with  $u_0 = L_0$ . Since  $L_0 < 0$ , there exists  $\varepsilon > 0$  such that  $u(t) < 0$  for  $t \in [0, \varepsilon]$ , and by (2.1),  $f(\phi(u(t))) \leq 0$  for  $t \in [0, \varepsilon]$ . Integrating (1.1) over  $[0, t]$  we get

$$p(t)\phi(u'(t)) = - \int_0^t p(s)f(\phi(u(s))) \, ds \geq 0, \quad t \in [0, \varepsilon].$$

Hence  $u'(t) \geq 0$  and  $u(t)$  is nondecreasing on  $[0, \varepsilon]$ . Consequently, since  $u \not\equiv L_0$ , there exists a maximal  $a_0 \geq 0$  such that

$$u(t) = L_0 \quad \text{on } [0, a_0] \quad \text{and } u \text{ is increasing in a right neighbourhood of } a_0. \quad (2.12)$$

The next theorem describes asymptotic behaviour of damped, homoclinic and escape solutions starting at  $L_0$ , which is the same as that of solutions with starting values greater than  $L_0$ .

**Theorem 2.8** (Starting value  $L_0$ ). *Let (1.3)–(1.7) and (2.1)–(2.3) hold and let  $u$  be a solution of IVP (1.1), (1.2) with the starting value  $u_0 = L_0$ . Further, let  $u \not\equiv L_0$  and let  $a_0 \geq 0$  be from (2.12). Then*

$$u(t) > L_0 \quad \text{and} \quad \exists \tilde{c} > 0 \quad \text{such that} \quad |u'(t)| \leq \tilde{c} \quad \text{for } t \in (a_0, \infty). \quad (2.13)$$

The constant  $\tilde{c}$  depends on  $L_0, L_1, \phi$  and  $f$  and does not depend on  $p$  and  $u$ .

1. Assume that  $u_{\text{sup}} < L$ , i.e.  $u$  is a damped solution.

- Let  $\theta > a_0$  be the first zero of  $u$ . Then there exist  $\theta < a < b$  such that

$$u(a) = (0, L), \quad u'(t) > 0 \quad \text{on } (a_0, a), \quad u'(a) = 0, \quad u'(t) < 0 \quad \text{on } (a, b). \quad (2.14)$$

- Let  $u < 0$  on  $[0, \infty)$ . Then

$$u'(t) > 0 \quad \text{on } (a_0, \infty), \quad \lim_{t \rightarrow \infty} u(t) = 0, \quad \lim_{t \rightarrow \infty} u'(t) = 0. \quad (2.15)$$

2. Assume that  $u_{\text{sup}} > L$ , i.e.  $u$  is an escape solution. Then

$$u'(t) > 0 \quad \text{for } t \in (a_0, \infty). \quad (2.16)$$

3. Assume that  $u_{\text{sup}} = L$ . Then there are two possibilities.

- $u(t) < L$  for  $t \in [0, \infty)$  which yields

$$u'(t) > 0 \quad \text{for } t \in (a_0, \infty), \quad \lim_{t \rightarrow \infty} u(t) = L, \quad \lim_{t \rightarrow \infty} u'(t) = 0, \quad (2.17)$$

and  $u$  is a regular homoclinic solution.

- There exists  $t_0 > a_0$  such that  $u(t_0) = L, u'(t_0) = 0$  which implies

$$u'(t) > 0 \quad \text{for } t \in (a_0, t_0), \quad (2.18)$$

and there exists a singular homoclinic solution  $v$ , where  $v = u$  on  $[0, t_0]$  and  $v = L$  on  $[t_0, \infty)$ .

### 3 Escape solution and damped solution start in $(L_0, 0)$

In this section we derive further needed properties of escape and homoclinic solutions.

Assume (1.3)–(1.7), (2.1)–(2.4) hold and define sets

$$\mathcal{M}_e = \{u_0 \in (L_0, 0) : u \text{ is an escape solution of IVP (1.1), (1.2)}\}, \quad (3.1)$$

$$\mathcal{M}_d = \{u_0 \in (L_0, 0) : u \text{ is a damped solution of IVP (1.1), (1.2)}\}. \quad (3.2)$$

By Theorem 2.2, the set  $\mathcal{M}_d$  is nonempty. In this section we assume that the set  $\mathcal{M}_e$  is also nonempty and prove that the sets  $\mathcal{M}_e$  and  $\mathcal{M}_d$  are open in  $(L_0, 0)$ . These properties of  $\mathcal{M}_e$  and  $\mathcal{M}_d$  are used in Section 5 in the proof of Theorem 1.5.

**Lemma 3.1.** *Let (1.3)–(1.7) and (2.1)–(2.4) hold. Assume that  $B \in [L_0, 0), B_n \in (L_0, 0)$  for  $n \in \mathbb{N}$  and*

$$\lim_{n \rightarrow \infty} B_n = B.$$

Further, let  $u_n$  be a solution of IVP (1.1), (1.2) with  $u_0 = B_n, n \in \mathbb{N}$ , and let  $u$  be a damped solution or an escape solution of IVP (1.1), (1.2) with  $u_0 = B$ . Then for each  $b > 0$

$$\lim_{n \rightarrow \infty} u_n(t) = u(t) \quad \text{uniformly on } [0, b]. \quad (3.3)$$

*Proof.* Since each  $u_n$  fulfils (1.1) we get after integration

$$u_n(t) = B_n + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u_n(\tau))) d\tau \right) ds, \quad t \in [0, \infty), n \in \mathbb{N}. \quad (3.4)$$

Choose an arbitrary  $b > 0$ . By (2.6) the sequence  $\{u_n\}$  is bounded and equicontinuous on  $[0, b]$  and by the Arzelà–Ascoli Theorem there exists a subsequence  $\{u_k\} \subset \{u_n\}$  which uniformly converges on  $[0, b]$  to a continuous function  $v$ . Hence the limit  $v$  fulfils

$$v(t) = B + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(v(\tau))) d\tau \right) ds, \quad t \in [0, b].$$

So,  $v$  is a solution of IVP (1.1), (1.2) with  $u_0 = B$ . By Theorems 2.2 or 2.3, we get  $u = v$  on  $[0, b]$  and (3.3) follows.  $\square$

**Lemma 3.2.** *Let (1.3)–(1.7) and (2.1)–(2.4) hold. Then the set  $\mathcal{M}_e$  from (3.1) is open in  $(L_0, 0)$ .*

*Proof.* Let us choose an arbitrary  $B \in \mathcal{M}_e$ . Then the corresponding solution  $u$  of IVP (1.1), (1.2) with  $u_0 = B$  is an escape solution and so there exists  $b > 0$  such that  $u(b) > L$ .

Assume that in any neighbourhood of  $B$  there exist starting values of solutions which are not escape solutions. Then we get a sequence  $\{B_n\} \subset (L_0, 0)$  converging to  $B$  and a corresponding sequence  $\{u_n\}$  of solutions of IVP (1.1), (1.2) with  $u_0 = B_n$  satisfying (3.3). In addition  $u_n \leq L$  on  $[0, \infty)$  for  $n \in \mathbb{N}$ . By (3.3) we get  $u(b) \leq L$ , a contradiction. Therefore for each  $B \in \mathcal{M}_e$  there exists a neighbourhood of  $B$  belonging to  $\mathcal{M}_e$ .  $\square$

**Lemma 3.3.** *Let (1.3)–(1.7) and (2.1)–(2.4) hold. Then the set  $\mathcal{M}_d$  from (3.2) is open in  $(L_0, 0)$ .*

*Proof.* Let us choose an arbitrary  $B \in \mathcal{M}_d$ . Then the corresponding solution  $u$  of IVP (1.1), (1.2) with  $u_0 = B$  is a damped solution.

Assume that in any neighbourhood of  $B$  there exist starting values of solutions which are not damped solutions. By Theorem 2.7 we get a sequence  $\{B_n\} \subset (L_0, 0)$  converging to  $B$  and a corresponding sequence  $\{u_n\}$  of nondecreasing solutions of IVP (1.1), (1.2) with  $u_0 = B_n$  satisfying (3.3). Therefore  $u$  is also nondecreasing.

1. Assume that  $u$  has a zero  $\theta > 0$ . By (2.7) there exist  $\theta < a < b$  such that  $u(a) \in (0, L)$  and  $u$  is decreasing on  $[a, b]$ , a contradiction.

2. Assume that  $u < 0$  on  $[0, \infty)$ . Then (1.1) yields

$$\phi'(u'(t))u'(t)u''(t) + \frac{p'(t)}{p(t)}\phi(u'(t))u'(t) + f(\phi(u(t)))u'(t) = 0, \quad t > 0. \quad (3.5)$$

Integrating (3.5) from 0 to  $t > 0$  and using (2.2) and (2.8) we get

$$\int_0^{u'(t)} x\phi'(x) dx + \int_0^t \frac{p'(s)}{p(s)}\phi(u'(s))u'(s) ds = F(B) - F(u(t)), \quad t > 0, \quad (3.6)$$

and for  $t \rightarrow \infty$

$$\int_0^\infty \frac{p'(s)}{p(s)}\phi(u'(s))u'(s) ds = F(B) \in (0, \infty). \quad (3.7)$$

Consequently there exist  $b > 0$  and  $\eta > 0$  such that

$$\int_b^\infty \frac{p'(s)}{p(s)}\phi(u'(s))u'(s) ds < \eta < \frac{F(L)}{3}. \quad (3.8)$$



Hence (3.7) and (3.8) give

$$\int_0^b \frac{p'(s)}{p(s)} \phi(u'(s)) u'(s) ds > F(B) - \eta. \quad (3.9)$$

(i) Assume for each  $n \in \mathbb{N}$ , that starting value  $B_n$  can be chosen such that the corresponding solution  $u_n$  is not a singular homoclinic solution. So,  $u_n$  is either escape or regular homoclinic solution, and for  $n \in \mathbb{N}$ , we get similarly as in (3.6)

$$\int_0^{u'_n(t)} x \phi'(x) dx + \int_0^t \frac{p'(s)}{p(s)} \phi(u'_n(s)) u'_n(s) ds = F(B_n) - F(u_n(t)), \quad t > 0, \quad (3.10)$$

and so

$$F(u_n(t)) < F(B_n) - \int_0^b \frac{p'(s)}{p(s)} \phi(u'_n(s)) u'_n(s) ds, \quad t > b. \quad (3.11)$$

Using (3.9) we derive an estimation for the integral in (3.11) as follows. We have

$$\int_0^b \frac{p'(s)}{p(s)} \phi(u'_n(s)) u'_n(s) ds > \int_0^b \frac{p'(s)}{p(s)} (\phi(u'_n(s)) u'_n(s) - \phi(u'(s)) u'(s)) ds + F(B) - \eta,$$

and due to (3.6) and (3.10),

$$\begin{aligned} & \int_0^b \frac{p'(s)}{p(s)} (\phi(u'_n(s)) u'_n(s) - \phi(u'(s)) u'(s)) ds \\ &= F(B_n) - F(B) + F(u(b)) - F(u_n(b)) + \int_{u'_n(b)}^{u'(b)} x \phi'(x) dx. \end{aligned}$$

Therefore, (3.11) yields

$$F(u_n(t)) < |F(u(b)) - F(u_n(b))| + \left| \int_{u'_n(b)}^{u'(b)} x \phi'(x) dx \right| + \eta, \quad t > b.$$

By (3.3) and (3.4),

$$\lim_{n \rightarrow \infty} F(u_n(b)) = F(u(b)), \quad \lim_{n \rightarrow \infty} u'_n(b) = u'(b),$$

and so if  $n$  is sufficiently large, then

$$F(u_n(t)) < 3\eta < F(L), \quad t > b.$$

By (2.2) the function  $F(x)$  is increasing for  $x \in (0, \infty)$ , and so if  $0 < u_n(t)$  then  $u_n(t) < F^{-1}(3\eta) < L$  for  $t > b$ . Consequently, since  $u_n$  is increasing on  $[0, \infty)$  we have  $u_n < F^{-1}(3\eta) < L$  on  $[0, \infty)$  which contradicts the assumption that  $u_n$  is an escape or regular homoclinic solution.

(ii) Let  $B_n$ ,  $n \in \mathbb{N}$ , be such that the corresponding solutions  $u_n$ ,  $n \in \mathbb{N}$ , are singular homoclinic solutions. According to Theorem 2.7 there exists a sequence  $\{t_n\} \subset (0, \infty)$  such that

$$u'_n(t) > 0, \quad t \in (0, t_n), \quad u'_n(t_n) = 0, \quad u_n(t) = L, \quad t \in [t_n, \infty), \quad n \in \mathbb{N}. \quad (3.12)$$

Let there exist  $c \in (0, \infty)$  such that  $t_n \leq c$ , and hence  $u_n(c) = L$ ,  $n \in \mathbb{N}$ . Then (3.3) yields  $u(c) = L$ . Since we assumed that  $u < 0$  on  $[0, \infty)$ , we get a contradiction. Therefore there exists a subsequence  $\{t_k\} \subset \{t_n\}$  going to  $\infty$ , and for  $b$  from (3.8) we get  $t_k > b$  for  $k \geq k_0$ , with a sufficiently large  $k_0$ . Similarly as in (3.10) and (3.11) we derive

$$\int_0^{u'_k(t)} x \phi'(x) dx + \int_0^t \frac{p'(s)}{p(s)} \phi(u'_k(s)) u'_k(s) ds = F(B_k) - F(u_k(t)), \quad t \in (0, t_k), \quad k \geq k_0,$$

$$F(u_k(t)) < F(B_k) - \int_0^b \frac{p'(s)}{p(s)} \phi(u'_k(s)) u'_k(s) ds, \quad t \in (b, t_k), k \geq k_0. \quad (3.13)$$

We derive the estimation of the integral in (3.13) as in (i) and get for a sufficiently large  $k$  the estimate  $u_k(t_k) \leq F^{-1}(3\eta) < L$  contrary to (3.12).

We have proved that for each  $B \in \mathcal{M}_d$  there exists a neighbourhood of  $B$  belonging to  $\mathcal{M}_d$ .  $\square$

## 4 Escape solution and damped solution start at $L_0$

If we have not an escape solution of IVP (1.1), (1.2) starting at  $u_0 > L_0$ , we need some further properties of escape and damped solutions starting at  $L_0$ .

Assume (1.3)–(1.7), (1.11) and (2.1)–(2.4) hold and denote by  $\mathcal{S}$  the set of all damped, escape and homoclinic solutions of IVP (1.1), (1.2) with the starting value  $u_0 = L_0$ . Let  $u_e$  be an escape solution and  $u_d \neq L_0$  be a damped solution of IVP (1.1), (1.2). In this section we assume that

$$u_e, u_d \in \mathcal{S}. \quad (4.1)$$

According to (1.11) there exists  $C \in (L_0, 0)$  such that  $f$  is decreasing on  $[\phi(L_0), \phi(C)]$ . By Theorem 2.8 there exist minimal  $\gamma_d > 0$  and minimal  $\gamma_e > 0$  such that  $u_d(\gamma_d) = C$  and  $u_e(\gamma_e) = C$ . Let us put

$$\gamma_0 = \min\{\gamma_e, \gamma_d\}. \quad (4.2)$$

**Lemma 4.1.** *Let (1.3)–(1.7), (1.11) and (2.1)–(2.4) hold. Then for each  $\gamma \geq \gamma_0$  there exists a unique solution  $u_\gamma \in \mathcal{S}$  satisfying  $u_\gamma(\gamma) = C$ . Further there exists  $a_\gamma \in [0, \gamma)$  such that*

$$u_\gamma(t) = L_0 \quad \text{on } [0, a_\gamma], \quad u_\gamma(t) \in (L_0, C) \quad \text{on } (a_\gamma, \gamma). \quad (4.3)$$

*Proof.* The existence follows from Lemma 4.6 in [25] where it is proved by the lower and upper functions method. Theorem 2.8 yields (4.3). It remains to prove the uniqueness.

*Step 1.* Let us show that

$$\gamma_0 \leq \gamma_1 < \gamma_2 \implies a_{\gamma_1} \leq a_{\gamma_2}. \quad (4.4)$$

Assume on the contrary that  $a_{\gamma_1} > a_{\gamma_2}$ , so the graphs of  $u_{\gamma_1}$  and  $u_{\gamma_2}$  intersect and there exists  $\xi \in (a_{\gamma_1}, \gamma_1)$  such that

$$u_{\gamma_1}(t) = u_{\gamma_2}(t) = L_0 \quad \text{on } [0, a_{\gamma_2}], \quad u_{\gamma_1}(t) < u_{\gamma_2}(t) \quad \text{on } (a_{\gamma_2}, \xi), \quad u_{\gamma_1}(\xi) = u_{\gamma_2}(\xi) \in (L_0, C).$$

Consequently,

$$u'_{\gamma_1}(\xi) \geq u'_{\gamma_2}(\xi). \quad (4.5)$$

On the other hand, since  $f$  is decreasing on  $[\phi(L_0), \phi(C)]$  we get due to (1.3)  $-f(\phi(u_{\gamma_2}(t))) > -f(\phi(u_{\gamma_1}(t)))$  for  $t \in (a_{\gamma_2}, \xi)$ . Since  $u_{\gamma_i}$  satisfy (1.1), we get by integration over  $[0, \xi]$

$$\phi(u'_{\gamma_i}(\xi)) = -\frac{1}{p(\xi)} \int_0^\xi p(s) f(\phi(u_{\gamma_i}(s))) ds, \quad i = 1, 2,$$

so  $\phi(u'_{\gamma_2}(\xi)) > \phi(u'_{\gamma_1}(\xi))$  and  $u'_{\gamma_2}(\xi) > u'_{\gamma_1}(\xi)$ , contrary to (4.5).

*Step 2.* Now, assume that for some  $\gamma \geq \gamma_0$  there exist two different solution  $u_1, u_2 \in \mathcal{S}$  such that  $u_1(\gamma) = u_2(\gamma) = C$ . Similarly as in Step 1 we get that the graphs of  $u_1$  and  $u_2$  cannot intersect. Therefore there exists an interval  $(\tau_0, \tau_1) \subset (0, \gamma)$  such that  $u_1 > u_2$  on  $(\tau_0, \tau_1)$  and

$u_1(\tau_0) = u_2(\tau_0)$ ,  $u'_1(\tau_0) = u'_2(\tau_0)$ ,  $u_1(\tau_1) = u_2(\tau_1)$ . Then  $u'_1(\tau_1) \leq u'_2(\tau_1)$ . Since  $u_1, u_2$  are solutions of (1.1), it holds

$$(p(t)\phi(u'_i(t)))' + p(t)f(\phi(u_i(t))) = 0, \quad t \in (0, \infty), \quad i = 1, 2. \quad (4.6)$$

Integrating (4.6) over  $[\tau_0, \tau_1]$  we have

$$p(\tau_1)\phi(u'_i(\tau_1)) = p(\tau_0)\phi(u'_i(\tau_0)) - \int_{\tau_0}^{\tau_1} p(s)f(\phi(u_i(s))) \, ds, \quad i = 1, 2,$$

which implies  $u'_2(\tau_1) < u'_1(\tau_1)$ , a contradiction. We have proved that

$$u_1(t) = u_2(t) \quad \text{for } t \in [0, \gamma]. \quad (4.7)$$

*Step 3.* Integrating (4.6) over  $[\gamma, t]$  we get for  $t > \gamma$

$$\phi(u'_i(t)) = \frac{p(\gamma)}{p(t)}\phi(u'_i(\gamma)) - \frac{1}{p(t)} \int_{\gamma}^t p(s)f(\phi(u_i(s))) \, ds =: A_i(t), \quad i = 1, 2,$$

and so

$$u'_i(t) = \phi^{-1}(A_i(t)), \quad u_i(t) = C + \int_{\gamma}^t \phi^{-1}(A_i(s)) \, ds, \quad i = 1, 2.$$

By Theorem 2.8 there exist  $\beta > \gamma$  and  $c_0, \tilde{c}$  such that

$$0 < c_0 \leq u'_i(t) \leq \tilde{c}, \quad u_i(t) \in (L_0, L), \quad t \in [\gamma, \beta], \quad i = 1, 2. \quad (4.8)$$

Then  $0 < \phi(c_0) < A_i(t) \leq \phi(\tilde{c})$  for  $t \in [\gamma, \beta]$ ,  $i = 1, 2$ . Therefore, due to (1.3), there exists a Lipschitz constant  $\Lambda_{\phi^{-1}}$  of the function  $\phi^{-1}$  on the interval  $[\phi(c_0), \phi(\tilde{c})]$  such that

$$|u'_1(t) - u'_2(t)| \leq \Lambda_{\phi^{-1}}|A_1(t) - A_2(t)|, \quad |u_1(t) - u_2(t)| \leq \Lambda_{\phi^{-1}} \int_{\gamma}^t |A_1(s) - A_2(s)| \, ds, \quad t \in [\gamma, \beta].$$

In addition, by (1.3) and (1.6) we can find Lipschitz constants  $\Lambda_{\phi}$  and  $\Lambda_f$  of the functions  $\phi$  and  $f$  on the intervals  $[L_0, L]$  and  $[\phi(L_0), \phi(L)]$ , respectively. Hence, by (1.7), (4.7) and (4.8),

$$\begin{aligned} |A_1(t) - A_2(t)| &\leq \frac{1}{p(t)} \int_{\gamma}^t p(s) |f(\phi(u_2(s))) - f(\phi(u_1(s)))| \, ds \\ &\leq \Lambda_f \Lambda_{\phi} \int_{\gamma}^t |u_2(s) - u_1(s)| \, ds, \quad t \in [\gamma, \beta]. \end{aligned}$$

This implies

$$|u_1(t) - u_2(t)| \leq \Lambda_{\phi^{-1}} \Lambda_f \Lambda_{\phi} (\beta - \gamma) \int_{\gamma}^t |u_1(s) - u_2(s)| \, ds, \quad t \in [\gamma, \beta],$$

and the Gronwall lemma yields

$$u_1(t) = u_2(t) \quad \text{for } t \in [\gamma, \beta]. \quad (4.9)$$

Let  $\beta^*$  be a supremum of all such  $\beta$  satisfying (4.8). Let us denote  $\rho(t) := u_1(t) - u_2(t)$ . Then by (4.7) and (4.9)

$$\rho(t) = 0 \quad \text{for } t \in [0, \beta^*]. \quad (4.10)$$

If  $\beta^* = \infty$ , then  $u_1 = u_2$  on  $[0, \infty)$  and the uniqueness is proved.

Step 4. Let  $\beta^* < \infty$ . Since  $\rho \in C^1[0, \infty)$ , it holds  $\rho(\beta^*) = 0$ ,  $\rho'(\beta^*) = 0$  due to (4.10), and  $u_1, u_2$  reach  $L$  at  $\beta^*$  or  $u'_1, u'_2$  reach 0 at  $\beta^*$ .

(i) Let  $u_1(\beta^*) = u_2(\beta^*) = L$  and  $u'_1(\beta^*) = u'_2(\beta^*) > 0$ . Then  $u_1$  and  $u_2$  are escape solutions, and by (2.1), we obtain by integration of (4.6) over  $[\beta^*, t]$

$$\phi(u'_1(t)) = \frac{p(\beta^*)}{p(t)} \phi(u'_1(\beta^*)) = \phi(u'_2(t)), \quad t \geq \beta^*.$$

Therefore  $u'_1 = u'_2$  on  $[\beta^*, \infty)$  and

$$u_1(t) = L + \int_{\beta^*}^t \phi^{-1} \left( \frac{p(\beta^*)}{p(s)} \phi(u'_1(\beta^*)) \right) ds = u_2(t), \quad t \geq \beta^*. \quad (4.11)$$

(ii) Let  $u_1(\beta^*) = u_2(\beta^*) = L$  and  $u'_1(\beta^*) = u'_2(\beta^*) = 0$ . Then  $u_1$  and  $u_2$  are singular homoclinic solutions, and

$$u_1(t) = L = u_2(t), \quad t \geq \beta^*. \quad (4.12)$$

To summarize, in the both cases (i) and (ii) the uniqueness is proved.

(iii) Let  $u_1(\beta^*) = u_2(\beta^*) < L$  and  $u'_1(\beta^*) = u'_2(\beta^*) = 0$ . Then  $u_1$  and  $u_2$  are damped solutions and by Theorem 2.8 there exists  $b > \beta^*$  such that  $u_1, u_2$  are decreasing and positive on  $[\beta^*, b]$ . Therefore

$$\min\{f(\phi(u_i(t))) : t \in [\beta^*, b]\} =: K_{\min} > 0, \quad \max\{f(\phi(u_i(t))) : t \in [\beta^*, b]\} =: K_{\max} < \infty.$$

Integrating (4.6) over  $[\beta^*, t]$ , we get for  $t > \beta^*$

$$\phi(u'_i(t)) = -\frac{1}{p(t)} \int_{\beta^*}^t p(s) f(\phi(u_i(s))) ds =: A_i^*(t), \quad i = 1, 2,$$

and hence

$$-K_{\max} \int_{\beta^*}^t \frac{p(s)}{p(t)} ds \leq A_i^*(t) \leq -K_{\min} \int_{\beta^*}^t \frac{p(s)}{p(t)} ds, \quad t \in (\beta^*, b]. \quad (4.13)$$

Consequently, there exists a function  $K$  with

$$K_{\min} \leq K(t) \leq K_{\max} \quad \text{for } t \in (\beta^*, b],$$

such that

$$|\phi^{-1}(A_1^*(t)) - \phi^{-1}(A_2^*(t))| \leq (\phi^{-1})' \left( -K(t) \int_{\beta^*}^t \frac{p(s)}{p(t)} ds \right) |A_1^*(t) - A_2^*(t)|, \quad t \in (\beta^*, b].$$

Due to (2.4), there exists  $K_\phi > 0$  such that

$$0 < |x| (\phi^{-1})'(x) \leq K_\phi, \quad x \in [-1, 0), \quad (4.14)$$

and since  $K$  is bounded, there exists  $\delta \in (\beta^*, b)$  such that

$$-1 \leq -K(t) \int_{\beta^*}^t \frac{p(s)}{p(t)} ds < 0, \quad t \in (\beta^*, \delta].$$

Clearly, for  $x = -K(t) \int_{\beta^*}^t \frac{p(s)}{p(t)} ds$  in (4.14), we obtain

$$0 < K(t) \int_{\beta^*}^t \frac{p(s)}{p(t)} ds (\phi^{-1})' \left( -K(t) \int_{\beta^*}^t \frac{p(s)}{p(t)} ds \right) \leq K_\phi, \quad t \in (\beta^*, \delta]. \quad (4.15)$$

Denote

$$\rho(t) := \max\{|u_1(s) - u_2(s)| : s \in [\beta^*, t]\}, \quad t \in [\beta^*, \delta].$$

Since

$$u_i(t) = u_i(\beta^*) + \int_{\beta^*}^t \phi^{-1}(A_i^*(s)) \, ds, \quad i = 1, 2,$$

and

$$|A_1^*(t) - A_2^*(t)| \leq \frac{1}{p(t)} \int_{\beta^*}^t p(s) |f(\phi(u_2(s))) - f(\phi(u_1(s)))| \, ds \leq \rho(t) \Lambda_f \Lambda_\phi \int_{\beta^*}^t \frac{p(s)}{p(t)} \, ds,$$

we get by (4.15)

$$\rho(t) \leq \frac{K_\phi}{K_{\min}} \Lambda_f \Lambda_\phi \int_{\beta^*}^t \rho(s) \, ds, \quad t \in [\beta^*, \delta].$$

The Gronwall lemma yields

$$u_1(t) = u_2(t) \quad \text{for } t \in [\beta^*, \delta]. \quad (4.16)$$

Modifying and repeating the arguments from Steps 3–5 we get the uniqueness in case (iii).  $\square$

Define sets

$$\Gamma_e = \{\gamma \in [\gamma_0, \infty) : u_\gamma \in \mathcal{S} \text{ is an escape solution and } u_\gamma(\gamma) = C\}, \quad (4.17)$$

$$\Gamma_d = \{\gamma \in [\gamma_0, \infty) : u_\gamma \in \mathcal{S} \text{ is a damped solution and } u_\gamma(\gamma) = C\}. \quad (4.18)$$

According to (4.1) the sets  $\Gamma_e, \Gamma_d$  are nonempty. We prove that these sets are open in  $[\gamma_0, \infty)$ , which we need in the proof in Section 5.

**Lemma 4.2.** *Let (1.3)–(1.7), (1.11) and (2.1)–(2.4) hold. For  $n \in \mathbb{N}$  consider  $\gamma_n \in (\gamma_0, \infty)$  and  $u_{\gamma_n} \in \mathcal{S}$  with  $u_{\gamma_n}(\gamma_n) = C$ . Assume that*

$$\lim_{n \rightarrow \infty} \gamma_n = \gamma \in [\gamma_0, \infty).$$

Then for each  $b > \gamma$

$$\lim_{n \rightarrow \infty} u_{\gamma_n}(t) = u_\gamma(t) \quad \text{uniformly on } [0, b], \quad u_\gamma \in \mathcal{S} \quad \text{and} \quad u_\gamma(\gamma) = C. \quad (4.19)$$

*Proof.* Since each  $u_{\gamma_n}$  fulfils (1.1) we get after integration

$$u_{\gamma_n}(t) = L_0 + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(u_{\gamma_n}(\tau))) \, d\tau \right) \, ds, \quad t \in [0, \infty), \quad n \in \mathbb{N}. \quad (4.20)$$

Choose an arbitrary  $b > \gamma$ . By (2.13) the sequence  $\{u_{\gamma_n}\}$  is bounded and equicontinuous on  $[0, b]$  and by the Arzelà-Ascoli Theorem there exists a subsequence  $\{u_{\gamma_k}\} \subset \{u_{\gamma_n}\}$  which uniformly converges on  $[0, b]$  to a continuous function  $v$ . Hence the limit  $v$  fulfils  $v(\gamma) = C$  and

$$v(t) = L_0 + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(v(\tau))) \, d\tau \right) \, ds, \quad t \in [0, b].$$

So,  $v \in \mathcal{S}$ . By Lemma 4.1 we get  $v = u_\gamma$  on  $[0, b]$ , and (4.19) follows.  $\square$

**Remark 4.3.** Consider  $u_\gamma$  and the sequence  $\{u_{\gamma_n}\}$  from Lemma 4.2. Since  $u_\gamma(\gamma) = C$  and  $u_{\gamma_n}(\gamma_n) = C$ , we have  $u_\gamma \not\equiv L_0$  and  $u_{\gamma_n} \not\equiv L_0$ ,  $n \in \mathbb{N}$ . So, according to (2.12) and (4.19), there exist maximal  $a_0 \in [0, \gamma)$  and  $a_n \in [0, \gamma_n)$  such that

$$u_\gamma(t) = L_0 \quad \text{for } t \in [0, a_0], \quad u_{\gamma_n}(t) = L_0 \quad \text{for } t \in [0, a_n], \quad n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} a_n = a_0. \quad (4.21)$$

**Lemma 4.4.** Let (1.3)–(1.7), (1.11) and (2.1)–(2.4) hold. Then the set  $\Gamma_e$  from (4.17) is open in  $[\gamma_0, \infty)$ .

*Proof.* Let us choose an arbitrary  $\gamma \in \Gamma_e$ . Then the corresponding solution  $u_\gamma \in \mathcal{S}$  with  $u_\gamma(\gamma) = C$  is an escape solution and so there exists  $b > \gamma$  such that  $u_\gamma(b) > L$ .

Assume that there exist a sequence  $\{\gamma_n\} \subset (\gamma_0, \infty)$  converging to  $\gamma$  and a corresponding sequence of non-escape solutions  $\{u_{\gamma_n}\} \subset \mathcal{S}$  with  $u_{\gamma_n}(\gamma_n) = C$ . By Lemma 4.2, the sequence  $\{u_{\gamma_n}\}$  uniformly converges to  $u_\gamma$  on  $[0, b]$ . Since  $u_{\gamma_n}(b) \leq L$ , we get  $u_\gamma \leq L$ , a contradiction. Therefore for each  $\gamma \in \Gamma_e$  there exists a neighbourhood of  $\gamma$  in  $[\gamma_0, \infty)$  belonging to  $\Gamma_e$ .  $\square$

**Lemma 4.5.** Let (1.3)–(1.7), (1.11) and (2.1)–(2.4) hold. Then the set  $\Gamma_d$  from (4.18) is open in  $[\gamma_0, \infty)$ .

*Proof.* Let us choose an arbitrary  $\gamma \in \Gamma_d$ . Then the corresponding solution  $u_\gamma \in \mathcal{S}$  with  $u_\gamma(\gamma) = C$  is a damped solution.

Assume that there exist a sequence  $\{\gamma_n\} \subset (\gamma_0, \infty)$  converging to  $\gamma$  and a corresponding sequence of non-damped solutions  $\{u_{\gamma_n}\} \subset \mathcal{S}$  with  $u_{\gamma_n}(\gamma_n) = C$ . Due to Remark 4.3, the conditions (4.21) hold. By Theorem 2.8, each  $u_{\gamma_n}$  is nondecreasing on  $[0, \infty)$ . By Lemma 4.2 the sequence  $\{u_{\gamma_n}\}$  uniformly converges to  $u_\gamma$  on  $[0, b]$  for any  $b > \gamma$ . Therefore  $u_\gamma$  is nondecreasing on  $[0, \infty)$ .

1. Assume that  $u_\gamma$  has a zero  $\theta > a_0$ . By (2.14) there exist  $\theta < a < b$  such that  $u_\gamma(a) \in (0, L)$  and  $u_\gamma$  is decreasing on  $[a, b]$ , a contradiction.

2. Assume that  $u_\gamma < 0$  on  $[0, \infty)$ . Then (1.1) yields

$$\phi'(u'_\gamma(t))u'_\gamma(t)u''_\gamma(t) + \frac{p'(t)}{p(t)}\phi(u'_\gamma(t))u'_\gamma(t) + f(\phi(u_\gamma(t)))u'_\gamma(t) = 0, \quad t > a_0. \quad (4.22)$$

Integrating (4.22) from  $a_0$  to  $t > a_0$ , using (2.2), (2.15) and arguing as in the proof of Lemma 3.3, we get

$$\int_0^{u'_\gamma(t)} x\phi'(x) dx + \int_{a_0}^t \frac{p'(s)}{p(s)}\phi(u'_\gamma(s))u'_\gamma(s) ds = F(L_0) - F(u_\gamma(t)), \quad t > a_0, \quad (4.23)$$

and for  $t \rightarrow \infty$

$$\int_{a_0}^\infty \frac{p'(s)}{p(s)}\phi(u'_\gamma(s))u'_\gamma(s) ds = F(L_0) \in (0, \infty). \quad (4.24)$$

Consequently there exist  $b > a_0$ ,  $b > a_n$ ,  $n \in \mathbb{N}$ , and  $\eta > 0$  such that

$$\int_b^\infty \frac{p'(s)}{p(s)}\phi(u'_\gamma(s))u'_\gamma(s) ds < \eta < \frac{F(L)}{3}. \quad (4.25)$$

Hence (4.24) and (4.25) give

$$\int_{a_0}^b \frac{p'(s)}{p(s)}\phi(u'_\gamma(s))u'_\gamma(s) ds > F(L_0) - \eta. \quad (4.26)$$

(i) Assume for each  $n \in \mathbb{N}$ , that  $\gamma_n \in (\gamma_0, \infty)$  can be chosen such that the corresponding solution  $u_{\gamma_n}$  is not a singular homoclinic solution. So,  $u_{\gamma_n}$  is either escape or regular homoclinic solution, and for  $n \in \mathbb{N}$ , we get similarly as in (4.23)

$$\int_0^{u'_{\gamma_n}(t)} x\phi'(x) dx + \int_{a_n}^t \frac{p'(s)}{p(s)} \phi(u'_{\gamma_n}(s)) u'_{\gamma_n}(s) ds = F(L_0) - F(u_{\gamma_n}(t)), \quad t > a_n, \quad (4.27)$$

and so

$$F(u_{\gamma_n}(t)) < F(L_0) - \int_{a_n}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma_n}(s)) u'_{\gamma_n}(s) ds, \quad t > b. \quad (4.28)$$

Using (4.26) we derive an estimation for the integral in (4.28) as follows. We have

$$\begin{aligned} & \int_{a_n}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma_n}(s)) u'_{\gamma_n}(s) ds \\ & > \int_{a_n}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma_n}(s)) u'_{\gamma_n}(s) ds - \int_{a_0}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma}(s)) u'_{\gamma}(s) ds + F(L_0) - \eta, \end{aligned}$$

and due to (4.23) and (4.27),

$$\begin{aligned} & \int_{a_n}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma_n}(s)) u'_{\gamma_n}(s) ds - \int_{a_0}^b \frac{p'(s)}{p(s)} \phi(u'_{\gamma}(s)) u'_{\gamma}(s) ds \\ & = F(u_{\gamma}(b)) - F(u_{\gamma_n}(b)) + \int_{u'_{\gamma_n}(b)}^{u'_{\gamma}(b)} x\phi'(x) dx. \end{aligned}$$

Therefore, (4.28) yields

$$F(u_{\gamma_n}(t)) < |F(u_{\gamma}(b)) - F(u_{\gamma_n}(b))| + \left| \int_{u'_{\gamma_n}(b)}^{u'_{\gamma}(b)} x\phi'(x) dx \right| + \eta, \quad t > b.$$

By (4.19) and (4.20),

$$\lim_{n \rightarrow \infty} F(u_{\gamma_n}(b)) = F(u_{\gamma}(b)), \quad \lim_{n \rightarrow \infty} u'_{\gamma_n}(b) = u'_{\gamma}(b),$$

and so if  $n$  is sufficiently large, then

$$F(u_{\gamma_n}(t)) < 3\eta < F(L), \quad t > b.$$

We get a contradiction as in part (i) of the proof of Theorem 3.3.

(ii) Let  $\gamma_n$ ,  $n \in \mathbb{N}$ , be such that the corresponding solutions  $u_{\gamma_n}$ ,  $n \in \mathbb{N}$ , are singular homoclinic solutions. According to Theorem 2.8, for  $n \in \mathbb{N}$ , there exists a  $t_n \in (a_n, \infty)$  such that

$$u'_{\gamma_n}(t) > 0, \quad t \in (a_n, t_n), \quad u'_{\gamma_n}(t_n) = 0, \quad u_{\gamma_n}(t) = L, \quad t \in [t_n, \infty), \quad n \in \mathbb{N}.$$

Then we argue similarly as in part (ii) of the proof of Theorem 3.3 (working on  $(a_k, t_k)$  instead of  $(0, t_k)$ ) and derive a contradiction.

We have proved that for each  $\gamma \in \Gamma_d$  there exists a neighbourhood of  $\gamma$  in  $[\gamma_0, \infty)$  belonging to  $\Gamma_d$ .  $\square$

## 5 Proof of Theorem 1.5

Having the results from Section 3 and Section 4 we are ready to prove Theorem 1.5.

*Proof.* First, assume (2.1).

*Step 1.* Consider the sets  $\mathcal{M}_e$  and  $\mathcal{M}_d$  from (3.1) and (3.2), respectively, and assume that  $\mathcal{M}_e$  is nonempty. By Theorem 2.2 the set  $\mathcal{M}_d$  is also nonempty. Further, by Lemmas 3.2 and 3.3 the sets  $\mathcal{M}_e$  and  $\mathcal{M}_d$  are open in  $(L_0, 0)$ . Therefore the set  $\mathcal{M}_h := (L_0, 0) \setminus \{\mathcal{M}_e \cup \mathcal{M}_d\}$  is nonempty. Consequently, there exists at least one starting value  $u_0^* \in (L_0, 0)$  which does not belong to  $\mathcal{M}_e \cup \mathcal{M}_d$  and hence a solution  $u_h$  of IVP (1.1), (1.2) with  $u_0 = u_0^*$  satisfies

$$\sup\{u_h(t) : t \in [0, \infty)\} = L.$$

According to Definition 1.4,  $u_h$  is a homoclinic solution. By Theorem 2.2, every solution of IVP (1.1), (1.2) with a starting value  $u_0 \in [\bar{B}, 0)$  is a damped solution, and hence  $u_0^* \in (L_0, \bar{B})$ . See Figure 5.1.

*Step 2.* If  $\mathcal{M}_e$  is empty, then no escape solution of IVP (1.1), (1.2) has its starting value  $u_0$  greater than  $L_0$ . In this case we consider the set  $\mathcal{S}$  of all damped, escape and homoclinic solutions of IVP (1.1), (1.2) with the starting value  $u_0 = L_0$ . Theorem 2.3 guarantee the existence of infinitely many escape solutions in  $\mathcal{S}$ . Choose one of them and denote it by  $u_e$ .

Consider a sequence  $\{B_n\} \subset (0, L_0)$  converging to  $L_0$  and a sequence  $\{u_n\}$  of solutions of IVP (1.1), (1.2) with starting values  $B_n$ . By (2.13), for each  $b > 0$ , the sequence  $\{u_n\}$  is bounded and equicontinuous on  $[0, b]$ , and by the Arzelà–Ascoli theorem there exists a subsequence  $\{u_k\} \subset \{u_n\}$  uniformly converging on  $[0, b]$  to a continuous function  $v$ . Since  $u_n$  fulfils (3.4),  $v$  satisfies

$$v(t) = L_0 + \int_0^t \phi^{-1} \left( -\frac{1}{p(s)} \int_0^s p(\tau) f(\phi(v(\tau))) d\tau \right) ds, \quad t \in [0, b].$$

By (2.1),  $f$  is bounded and hence  $v$  can be extended to  $[0, \infty)$  as a solution of equation (1.1). Consequently  $v \in \mathcal{S}$ .

If  $v$  is an escape solution, then similarly as in the proof of Lemma 3.2 we deduce for a sufficiently large  $k$  that  $u_k$  is also an escape solution. But  $u_k \notin \mathcal{S}$  because  $B_k > L_0$ , and so we have a contradiction. Therefore  $v$  cannot be an escape solution.

If  $v \equiv L_0$ , we get as in the proof of Theorem 4.7 in [25] that there exist escape solutions in the sequence  $\{u_n\}$ , which yields a contradiction as before.

Assume that  $v \neq L_0$  is a damped solution and denote it by  $u_d$ . Then (4.1) holds and we consider the nonempty sets  $\Gamma_e$  and  $\Gamma_d$  from (4.17) and (4.18), respectively. By Lemmas 4.4 and 4.5 the sets  $\Gamma_e$  and  $\Gamma_d$  are open in  $[\gamma_0, \infty)$ . Therefore the set  $\Gamma_h := [\gamma_0, \infty) \setminus \{\Gamma_e \cup \Gamma_d\}$  is nonempty. Consequently, there exists at least one value  $\gamma_h \in (\gamma_0, \infty)$  which does not belong to  $\Gamma_e \cup \Gamma_d$  and hence a solution  $u_h \in \mathcal{S}$  with  $u_h(\gamma_h) = C$  satisfies

$$\sup\{u_h(t) : t \in [0, \infty)\} = L.$$

According to Definition 1.4,  $u_h$  is a homoclinic solution.

Finally, if  $v$  is not a damped solution, it has to be a homoclinic solution and we can put  $v = u_h$ . See Figure 5.2.

*Step 3.* To summarize, we have proved that IVP (1.1), (1.2) has a homoclinic solution  $u_h$  for some  $u_0 = u_0^* \in (L_0, \bar{B})$  - in Step 1 or for  $u_0 = L_0$  - in Step 2. This was proved under



assumption (2.1). So, it remains to show that assumption (2.1) can be omitted. It is clear that if (2.1) is not fulfilled, we can define an auxiliary function  $\tilde{f}$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in [\phi(L_0), \phi(L)], \\ 0 & \text{for } x < \phi(L_0), x > \phi(L), \end{cases} \quad (5.1)$$

and consider an auxiliary equation which has the form

$$(p(t)\phi(u'(t)))' + p(t)\tilde{f}(\phi(u(t))) = 0, \quad t \in (0, \infty). \quad (5.2)$$

Since  $\tilde{f}$  satisfies (1.5), (1.6) and (2.1), we know, according to Steps 1 and 2, that IVP (5.2), (1.2) has a homoclinic solution  $u_h$  for some  $u_0 \in [L_0, \bar{B})$ . By Definition 1.4, it holds  $L_0 \leq u_h(t) \leq L$  for  $t \in [0, \infty)$ . Consequently  $\tilde{f}(\phi(u_h(t))) = f(\phi(u_h(t)))$  for  $t \in [0, \infty)$ , and hence  $u_h$  is a homoclinic solution of IVP (1.1), (1.2).  $\square$

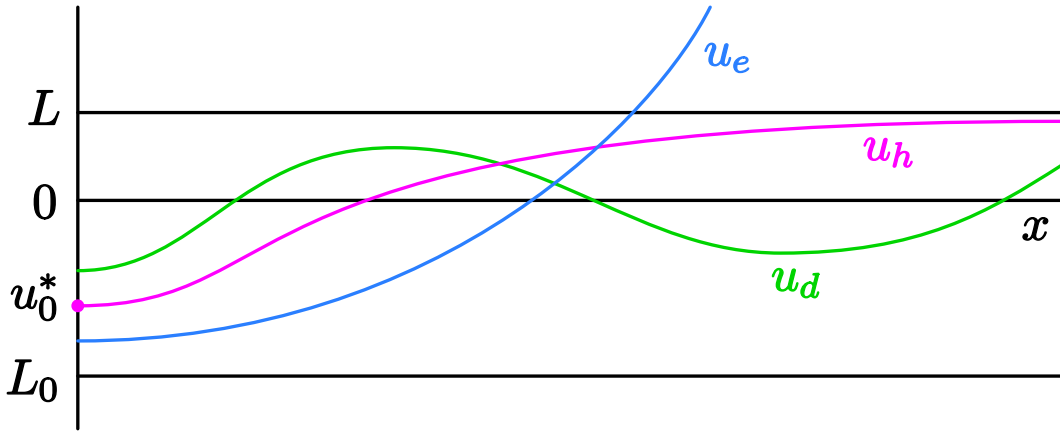


Figure 5.1: Graphs of damped, homoclinic and escape solutions  $u_d, u_h, u_e$

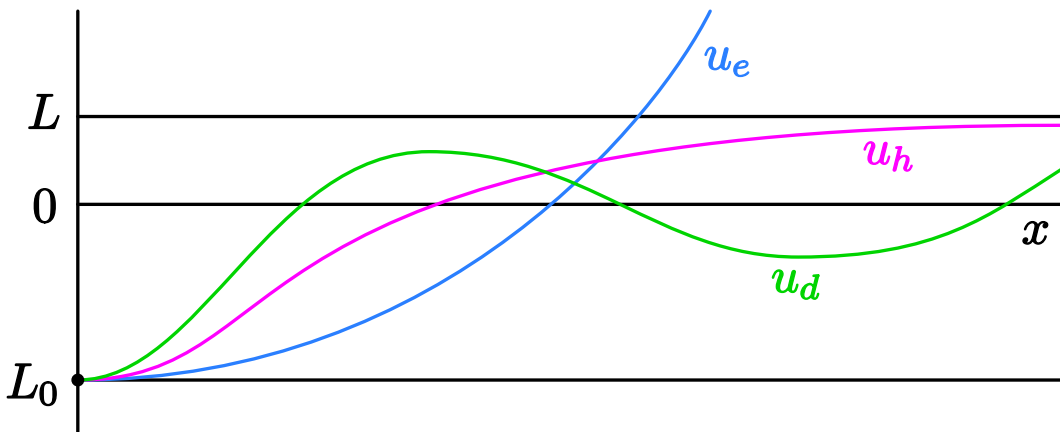


Figure 5.2: Graphs of damped, homoclinic and escape solutions  $u_d, u_h, u_e$

**Remark 5.1.** By Remark 2.5, if  $\phi$  in Theorem 1.5 fulfils in addition  $\phi'(0) > 0$ , then each homoclinic solution of IVP (1.1), (1.2) has its starting value greater than  $L_0$ .

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