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SYMMETRIES FOR THOUGHT

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In memory of my Hungarian grandmother Clara.

Abstract. This paper covers several topics that involve symmetries of differential equations: from the connection between Lie symmetries and Jacobi last multiplier with a detour to the inverse problem of calculus of variations, to the importance of maximal group of Noether symmetries on the road from classical to quantum mechanics.

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1. INTRODUCTION

Symmetries are everywhere, yet symmetry is an ambiguous word and can be misleading. The symmetries that are the subject of this paper are the Lie symmetries admitted by a differential equation, those symmetries that generate a Lie algebra. Sophus Lie himself provided a link between his namesake symmetries and the solutions of a differential equation, or its conservation laws by means of the Jacobi last multiplier [29]. Lie symmetries are also related to the inverse problem of calculus of variations because of the properties of the Jacobi last multiplier. Since conservation laws can be obtained without even a Lagrangian, the importance of Noether symmetries seems to be diminished, yet we have recently linked them to the problem of quantization [11, 36–38].

In the next section, we will recall the main properties of the Jacobi last multiplier, its connection to Lie symmetries, and the inverse problem of calculus of variations, namely the possibility of finding one (or more) Lagrangians.

In section 3, the extraordinary journey taken by three Lie symmetries, i. e.,

$$\Gamma_1 = \partial_x - \partial_y, \quad \Gamma_2 = x\partial_x + y\partial_y, \quad \Gamma_3 = x^2\partial_x - y^2\partial_y, \tag{1.1}$$

a representation of $\mathrm{sl}(2, \mathbb{R})$, is narrated. It begins with the hydrodynamic equations as formulated by Riemann and ends with the Schrödinger equation. We anticipate that during this journey, more Lie symmetries will be picked up and consequently a representation of sl(3, $\mathbb{R}$) is obtained. Also the Schwarzian derivative [14] and Calogero's goldfish [4, 35] are encountered.

2. Lie symmetries, Jacobi last multiplier, Lagrangians

The method of the Jacobi last multiplier* [15–17] and [18]† provides a means to determine all the solutions of the partial differential equation

$$\mathcal{A}f = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i} = 0 \tag{2.1}$$

or its equivalent associated Lagrange's system

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \tag{2.2}$$

In fact, if one knows the Jacobi last multiplier and $n-2$ functionally independent solutions, then the last solution can be obtained by a quadrature. The Jacobi last multiplier M is given by‡

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = M\mathcal{A}f, \tag{2.3}$$

where

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = \det \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \omega_1}{\partial x_1} & & \dfrac{\partial \omega_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial \omega_{n-1}}{\partial x_1} & \cdots & \dfrac{\partial \omega_{n-1}}{\partial x_n} \end{pmatrix} = 0 \tag{2.4}$$

and $\omega_1, \ldots, \omega_{n-1}$ are $n-1$ solutions of (2.1) or, equivalently, first integrals of (2.2) independent of each other. This means that M is a function of the variables $(x_1, \ldots, x_n)$ and depends on the chosen $n-1$ solutions, in the sense that it varies as they vary. The essential properties of the Jacobi last multiplier were proven by Jacobi himself and they are:

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*Many authors have dealt with the Jacobi last multiplier, and an up-to-date (2004) nearly complete list can be found in [34]. It ranges from the 1871 paper by Laguerre [27] and the seminal 1874 paper by Lie [29] to the 2003 review paper by Berrone e Giacomini [2]. A missed reference in [34] is Section 2.11 in the 2001 book by Goriely [10].

†An English translation is now available [19].

‡M is not zero, yet connects two quantities that are zero.

(a) If one selects a different set of $n-1$ independent solutions $\eta_1, \ldots, \eta_{n-1}$ of equation (2.1), then the corresponding last multiplier N is linked to M by the relationship

$$N = M \frac{\partial(\eta_1, \ldots, \eta_{n-1})}{\partial(\omega_1, \ldots, \omega_{n-1})}.$$

(b) Given a non-singular transformation of variables

$$\tau : \quad (x_1, x_2, \ldots, x_n) \longrightarrow (x_1', x_2', \ldots, x_n'),$$

then the last multiplier M' of $\mathcal{A}'F = 0$ is given by:

$$M' = M \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(x_1', x_2', \ldots, x_n')},$$

where M obviously comes from the $n-1$ solutions of $\mathcal{A}F = 0$ which correspond to those chosen for $\mathcal{A}'F = 0$ through the inverse transformation τ^{-1}.

(c) One can prove that each multiplier M is a solution of the following linear partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0; \tag{2.5}$$

vice versa, every solution M of this equation is a Jacobi last multiplier.

(d) If one knows two Jacobi last multipliers M_1 and M_2 of equation (2.1), then their ratio is a solution ω of (2.1), or, equivalently, a first integral of (2.2). Naturally, the ratio may be quite trivial, namely a constant. *Vice versa*, the product of a multiplier M_1 times any solution ω yields another last multiplier $M_2 = M_1\omega$.

Since the existence of a solution/first integral is consequent upon the existence of symmetry, an alternative formulation in terms of symmetries was provided by Lie [29, 30]. A clear treatment of the formulation in terms of solutions/first integrals and symmetries is given by Bianchi [3]. If we know $n-1$ symmetries of (2.1)/(2.2), say

$$\Gamma_i = \sum_{j=1}^{n} \xi_{ij}(x_1, \ldots, x_n)\partial_{x_j}, \quad i = 1, \ldots, n-1, \tag{2.6}$$

a Jacobi last multiplier is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{pmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & \cdots & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{pmatrix}. \tag{2.7}$$

There is an obvious corollary to the results of Jacobi mentioned above. In the case that there exists a constant multiplier, the determinant is a first integral. This result is potentially very useful in the search for first integrals of systems of ordinary differential equations. In particular, if each component of the vector field of the equation of motion is missing the variable associated with that component, i. e., $\partial a_i/\partial x_i = 0$, the last multiplier is a constant, and any other Jacobi last multiplier is a first integral.

Another property of the Jacobi last multiplier is its (almost forgotten) relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = \phi(t, x, \dot{x}) \tag{2.8}$$

i. e. [18, Lecture 10], [52]

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} \tag{2.9}$$

where $M = M(t, x, \dot{x})$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log M) + \frac{\partial \phi}{\partial \dot{x}} = 0. \tag{2.10}$$

Then, equation (2.8) becomes the Euler–Lagrangian equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{2.11}$$

The proof is given by taking the derivative of (2.11) by $\dot{x}$ and showing that this yields (2.10). If one knows a Jacobi last multiplier, then L can be obtained by a double integration, namely,

$$L = \int \left(\int M \, \mathrm{d}\dot{x} \right) \mathrm{d}\dot{x} + \ell_1(t, x)\dot{x} + \ell_2(t, x), \tag{2.12}$$

where ℓ_1 and ℓ_2 are functions of t and x which have to satisfy a single partial differential equation related to (2.8) [40]. As it was shown in [40], ℓ_1, ℓ_2 are related to the gauge function $G = G(t, x)$. In fact, we may assume

$$\begin{aligned} \ell_1 &= \frac{\partial G}{\partial x}, \\ \ell_2 &= \frac{\partial G}{\partial t} + \ell_3(t, x), \end{aligned} \tag{2.13}$$

where ℓ_3 has to satisfy the partial differential equation mentioned and G is obviously arbitrary.

It was shown in [39] that Jacobi last multiplier yields the Lagrangian for any equation of even order*

$$u^{(2n)} = F(x, u, u', u'', \dots, u^{(2n-1)}), \tag{2.14}$$

since it can be derived from the following formula

$$M^{1/n} = \frac{\partial^2 L}{\partial (u^{(n)})^2}, \tag{2.15}$$

where M is the Jacobi last multiplier of equation (2.14) and L is its Lagrangian. This formula was given by Jacobi himself in [17] p. 364.

*We use a prime to indicate the derivative with respect to x.

We recall that Fels has proven [7] that the Lagrangian is unique in the case of fourth-order equation if it exists. In the case of equations of sixth and eighth order, the uniqueness was proven by Juráš [20].

In [47], Tonti provided a brief historical survey of the inverse problem of calculus of variations. He begins with the year 1887 when both Helmholtz [13] and Volterra [50] published their work. Unfortunately, no mention is made of Jacobi's work. That historical survey should have at least begun — if not with the year 1845 when Jacobi's paper [17] was published in Crelle's journal — with the year 1884 when Jacobi's Dynamics Lectures delivered at the University of Königsberg in the Winter Semester 1842–1843, were finally published [18] with a foreword by Weierstrass.

As pointed out by Tonti [47], many authors have dealt with the inverse problem of calculus of variations by either using a formal approach or an operatorial approach following on the steps of either Helmholtz or Volterra, for example, [1, 31, 43, 46, 48, 49] and many others.

We do not underestimate the research of these very distinguished authors. Yet, when feasible — a single equation, a non-dissipative system [41] — we prefer to follow Jacobi since his last multiplier has a direct link to conservation laws and symmetries that are the essential elements that, in our opinion, make the difference between a mathematical abstraction and a physical concreteness.

It is a matter for historians alike to find out why Darboux [5], Helmholtz [13], Koenigsberger himself* and many other successive authors, e. g., Douglas [6] and Havas† [12], never acknowledged the use of the Jacobi last multiplier in order to find Lagrangians of a second-order equation. Sonin did [45, p. 10], although very few authors [26] cite his 1886 Russian paper.

Volterra knew Jacobi's work, especially [18], which he mentions in 1887 in one of his earlier works [50, p. 280]. Therefore, Volterra knew Lecture 10 of [18] since he cites p. 78 of precisely this Lecture in [50]. Maybe he overlooked the following pages, especially p. 82, where Jacobi wrote his formula (2.9) that links the last multiplier to the Lagrangian of any second-order equation. Also, in his 1906 address at the Congress of Italian Naturalists [51], Volterra wrote: *una delle più celebri scoperte del matematico Jacobi, quella del principio dell'ultimo moltiplicatore.*‡

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*In 1902–1903, Koenigsberger wrote Helmholtz's biography [24] — which in 1906 was (abridged) translated into English with a Preface by Lord Kelvin [23] — after he wrote his 1901 book on Mechanics [21]. Neither book cites the connection between Jacobi last multiplier and Lagrangians. In 1904, Koenigsberger wrote Jacobi's biography [22] where the Jacobi last multiplier is extensively described.

†Havas even cites the book by Whittaker [52] but only in connection with the formulation of Lagrangian equations.

‡"One of the most celebrated discoveries by the mathematician Jacobi, that of the principle of the last multiplier." (tr. by MCN)

3. From Riemann to Schrödinger

In [8], it was found that the following linear second-order hyperbolic partial differential equation

$$(x+y)^2 u_{xy} + B_1(x+y)(u_x+u_y) + B_2 u = 0, \tag{3.1}$$

that encases the hydrodynamic equations as formulated by Riemann [8, 44] admits the following Lie symmetries for general values of the parameters B_1, B_2:

$$\Lambda_1 = \partial_x - \partial_y, \quad \Lambda_2 = x\partial_x + y\partial_y, \quad \Lambda_3 = x^2\partial_x - y^2\partial_y - B_1(x-y)u\partial_u,$$
$$\Lambda_4 = u\partial_u, \quad \Lambda_\infty = f(x,y)\partial_u, \tag{3.2}$$

where $f(x,y)$ is any solution to (3.1). We emphasize that Λ_4 and Λ_∞ are the symmetries that are always admitted by any linear homogeneous partial differential equation. It is obvious that taking the independent variables only in (3.2) allows the identification of $\Lambda_1, \Lambda_2, \Lambda_3$ with $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. We now assume that $y = y(x)$ and search for an ordinary differential equation of second order, say $y'' = F(x, y, y')$, that admits the three Lie symmetries (1.1). We obtain* a family of equations characterized by a parameter A, i. e.:

$$y'' = \frac{y'}{x+y}\left(A\sqrt{y'} + 2y' - 2\right). \tag{3.3}$$

It is known [28] that if we solve this equation for the parameter A, i. e.

$$A = \frac{y''(x+y) - 2y'^2 + 2y'}{y'\sqrt{y'}}, \tag{3.4}$$

and then derive once with respect to x, a third-order equation is obtained, i. e.

$$y''' = \frac{3y''^2}{2y'}, \tag{3.5}$$

that admits a six-dimensional Lie symmetry algebra [9] generated by

$$\partial_x, \quad x\partial_x, \quad x^2\partial_x, \quad \partial_y, \quad y\partial_y, \quad y^2\partial_y. \tag{3.6}$$

Equation (3.5) contains the Schwarzian derivative [14] and thus, it is connected to both a linear second-order equation and a Riccati equation. In particular, the transformation [14]

$$y = \int \frac{1}{r^2}\,dx + a_3 \tag{3.7}$$

yields $r'' = 0$, so that $r = a_1 x + a_2$, and the general solution to (3.3) is

$$y = -\frac{1}{a_1(a_1 x + a_2)} + a_3, \quad a_3 = \frac{2a_2 - A}{2a_1}, \tag{3.8}$$

where a_1 and a_2 are arbitrary parameters.

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*We employ our own *ad hoc* interactive REDUCE programs [33] in order to find Lie symmetries.

The calculation of the Jacobi last multiplier requires that the differential equation (3.3) be written as a system of first-order equations (2.2), i. e.

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}y}{y'} = \frac{\mathrm{d}y'}{\frac{y'}{x+y}\left(A\sqrt{y'} + 2y' - 2\right)}, \tag{3.9}$$

then we can determine three Jacobi last multipliers for equation (3.3) by using the determinant (2.7) and two of the symmetries (1.1), i. e.

$$\Delta_{12} = \det\begin{pmatrix} 1 & y' & \frac{y'}{x+y}\left(A\sqrt{y'} + 2y' - 2\right) \\ 1 & -1 & 0 \\ x & y & 0 \end{pmatrix}, \tag{3.10}$$

$$\Delta_{13} = \det\begin{pmatrix} 1 & y' & \frac{y'}{x+y}\left(A\sqrt{y'} + 2y' - 2\right) \\ 1 & -1 & 0 \\ x^2 & -y^2 & -2y'(x+y) \end{pmatrix}, \tag{3.11}$$

and

$$\Delta_{23} = \det\begin{pmatrix} 1 & y' & \frac{y'}{x+y}\left(A\sqrt{y'} + 2y' - 2\right) \\ x & y & 0 \\ x^2 & -y^2 & -2y'(x+y) \end{pmatrix}. \tag{3.12}$$

Therefore, the three Jacobi last multipliers are:

$$JLM_{1,2} = \frac{1}{\Delta_{12}} = \frac{1}{y'(A\sqrt{y'} + 2y' - 2)}, \tag{3.13}$$

$$JLM_{1,3} = \frac{1}{\Delta_{13}} = \frac{1}{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}, \tag{3.14}$$

$$JLM_{2,3} = \frac{1}{\Delta_{23}} = \frac{1}{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}. \tag{3.15}$$

From property (d) in Section 2, we have that the ratio of two Jacobi last multipliers is a first integral of equation (3.3) and, therefore, three first integrals can be obtained, i. e.

$$Int_1 = \frac{JLM_{1,2}}{JLM_{1,3}} = \frac{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}{y'(A\sqrt{y'} + 2y' - 2)},$$

$$Int_2 = \frac{JLM_{1,2}}{JLM_{2,3}} = \frac{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}{y'(A\sqrt{y'} + 2y' - 2)},$$

$$Int_3 = \frac{JLM_{1,3}}{JLM_{2,3}} = \frac{y'(2x^2y' - 2y^2 - Axy\sqrt{y'})}{y'(4y - Ay\sqrt{y'} + 4xy' + Ax\sqrt{y'})}.$$

Three Lagrangians $L_{1,2}$, $L_{1,3}$, and $L_{2,3}$ can be obtained from (2.12) and the Jacobi last multipliers $JLM_{1,2}, JLM_{1,3}, JLM_{2,3}$, respectively. Then, applying Noether's theorem [32] yields that each of them admits only one Lie symmetry among those in (1.1) as Noether symmetry, and they are different from each other. In particular, we obtain that $L_{1,2}$ admits Γ_1, $L_{1,3}$ admits Γ_2, and $L_{2,3}$ admits Γ_3. Since the expressions of those three Lagrangians are very long, we do not write them down here. They are available in REDUCE format upon request.

If $A = 0$, then equation (3.3) becomes:

$$y'' = \frac{2y'}{x+y}\left(y' - 1\right) \tag{3.16}$$

and admits five further Lie symmetries, namely,

$$\begin{gathered} \Gamma_4 = \frac{1}{x+y}\left(\partial_x + \partial_y\right), \quad \Gamma_5 = \frac{1}{x+y}\left(y\partial_x - x\partial_y\right), \\ \Gamma_6 = \frac{xy}{x+y}\left(\partial_x + \partial_y\right), \quad \Gamma_7 = \frac{xy}{x+y}\left(x\partial_x - y\partial_y\right), \\ \Gamma_8 = \frac{xy}{x+y}\left(x^2\partial_x + y^2\partial_y\right). \end{gathered} \tag{3.17}$$

This equation is related to Calogero's goldfish [4], namely it is equation (24) in [35] with $a_{12} = -1$ and $w_2 = -y$.

The eight Lie operators $\Gamma_i, i = 1, 8$ generate an eight-dimensional Lie symmetry algebra sl$(3, \mathbb{R})$ which implies that equation (3.16) is linearizable by means of a point transformation [30]. In order to find the linearizing transformation, we have to look for an abelian intransitive subalgebra of sl$(3, \mathbb{R})$ and, following Lie's classification of two-dimensional algebras in the real plane [30], we have to transform it into the canonical form

$$\partial_{\tilde{y}}, \qquad \tilde{x}\partial_{\tilde{y}} \tag{3.18}$$

with $\tilde{y}$ and $\tilde{x}$ the new dependent and independent variables, respectively. We have found two* such subalgebras, although they are obviously related since there exists only one abelian intransitive subalgebra of sl$(3, \mathbb{R})$ [53]. One such subalgebra is that generated by Γ_4 and $\Gamma_{10} = 2\Gamma_5 - \Gamma_1$ while the other subalgebra is that generated by Γ_8 and $\Gamma_9 = -2\Gamma_7 + \Gamma_3$. Then, it is easy to derive that

$$\{\Gamma_4, \Gamma_{10}\} \implies \tilde{x} = y - x, \quad \tilde{y} = -xy, \tag{3.19}$$

while

$$\{\Gamma_8, \Gamma_9\} \implies \tilde{x} = \frac{x-y}{xy}, \quad \tilde{y} = -\frac{1}{xy}. \tag{3.20}$$

*Actually, there is also a third one, i. e., that generated by Γ_7 and $\Gamma_1 - \Gamma_5$. The further analysis of this case is analogous to the other two and we omit it here at the request of an anonymous Referee.

Equation (3.16) becomes

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\tilde{x}^2} = 0 \tag{3.21}$$

by means of both transformations.

We can also determine many Jacobi last multipliers for equation (3.16) by using the determinant (2.7) with two of the Lie point symmetries Γ_j, $j = 1, \ldots, 10$. For example:

$$\begin{aligned}
JLM_{1,2} &= \frac{1}{2y'(y'-1)},\\
JLM_{1,3} &= \frac{1}{4y'(xy'+y)},\\
JLM_{2,3} &= \frac{1}{2y'(x^2y'-y^2)},\\
JLM_{1,8} &= \frac{(x+y)^2}{(y^2-x^2y')(x^2y'^2+y^2y'+4xyy'+x^2y'+y^2)}\\
JLM_{8,9} &= JLM_{8,3} = \frac{(x+y)^2}{(y^2-x^2y')^3},\\
JLM_{4,10} &= JLM_{4,5} = \frac{(x+y)^2}{(1-y')^3},\\
JLM_{1,9} &= JLM_{3,10} = \frac{1}{2}JLM_{4,8} = \frac{1}{2}\frac{(x+y)^2}{(y'-1)(y^2-x^2y')(xy'+y)},\\
JLM_{4,6} &= -\frac{(x+y)^2}{(1-y')^2(xy'+y)},\\
JLM_{3,5} &= \frac{(x+y)^2}{(y'-1)(y^2-x^2y')^2}.
\end{aligned}$$

We observe that

$$\Delta_{67} = \Delta_{68} = \Delta_{78} = 0.$$

Then, it is easy to show [42] that $\{\Gamma_6, \Gamma_7, \Gamma_8\}$ are a representation of the complete symmetry group of equation (3.16), i.e. the group of the symmetries which completely specifies the differential equation under consideration [25].

We can now derive many Lagrangians for equation (3.16) from (2.12). In particular, the Lagrangian

$$L_{4,10} = \frac{1}{2}\frac{(x+y)^2}{1-y'} \tag{3.22}$$

corresponds to the Jacobi last multiplier $JLM_{4,10}$, and admits five Noether symmetries and corresponding first integrals, i. e.

$$\begin{aligned}
\Gamma_4 &\implies \frac{xy'+y}{1-y'}\\
\Gamma_5 &\implies -\frac{(y'-2)x^2y'-2xyy'-(2y'-1)y^2}{(2(y'-1)^2}\\
-3\Gamma_6+2\Gamma_2 &\implies \frac{(y^2-x^2y')(y+xy')}{(3(y'-1)^2}\\
\Gamma_9 &\implies -\frac{(y^2-x^2y'2)^2}{4(y'-1)^2}\\
\Gamma_1 &\implies \frac{y'(x+y)^2}{(y'-1)^2}.
\end{aligned} \tag{3.23}$$

Also the Lagrangian

$$L_{8,9}=\frac{1}{2}\frac{(x+y)^2}{x^4(y^2-x^2y')} \tag{3.24}$$

admits five Noether symmetries and corresponding first integrals, i. e.

$$\begin{aligned}
\Gamma_{10} &\implies \frac{(y'-1)^2}{4(y^2-x^2y')^2}\\
-3\Gamma_6+\Gamma_2 &\implies \frac{(y+xy')(y'-1)}{3(y^2-x^2y')^2}\\
\Gamma_7 &\implies \frac{(y+xy')^2}{2(y^2-x^2y')^2}\\
\Gamma_8 &\implies \frac{y+xy'}{x^2y'-y^2}\\
\Gamma_3 &\implies \frac{y'(x+y)^2}{(y^2-x^2y')^2}.
\end{aligned} \tag{3.25}$$

Since equation (3.16) is related to the free-particle equation (3.21), then we can derive the corresponding Schrödinger equation by means of the five Noether symmetries admitted by the Lagrangian $L_{4,10}$ [11, 36–38] in spite of the fact that the transformation (3.19) is not linear.

Indeed, we have determined a linear parabolic partial differential equation

$$\phi_{xx}+2\phi_{xy}+\phi_{yy}+2\mathrm{i}\frac{-x(x+y)^2+\mathrm{i}}{x+y}\phi_x+2\mathrm{i}\frac{y(x+y)^2+\mathrm{i}}{x+y}\phi_y=0 \tag{3.26}$$

that admits the following Lie symmetries

$$\begin{gathered}\Gamma_4, \quad \Gamma_5 + \mathrm{i}xy\phi\partial_\phi, \quad -3\Gamma_6 + 2\Gamma_2 + \frac{1}{4}\left(x - y + \mathrm{i}x^2y^2\right)\phi\partial_\phi, \quad \Gamma_9, \\ \Gamma_1 + \mathrm{i}xy\phi\partial_\phi, \quad \phi\partial_\phi, \quad \alpha(x, y)\partial_\phi,\end{gathered} \tag{3.27}$$

where $\alpha(x, y)$ is any solution to (3.26). It is obvious that the Noether symmetries (3.23) admitted by the Lagrangian $L_{4,10}$ of equation (3.16) correspond to the five Lie symmetries of equation (3.26). We can also transform the parabolic equation (3.26) into its canonical form with the help of MAPLE, i. e.

$$2\mathrm{i}\psi_t + \frac{1}{(2q - t)^2}\psi_{qq} + 2\frac{\mathrm{i}q(2q - t)^2 - 1}{(2q - t)^3}\psi_q = 0 \tag{3.28}$$

with $q = y, t = y - x$. A further trivial substitution, i. e.

$$\psi = \sqrt{2q - t}\exp\left(-\frac{\mathrm{i}q^2}{6}(4q - 3t)\right)\Psi,$$

eliminates the first derivative ψ_q in (3.28) and yields

$$2\mathrm{i}\Psi_t + \frac{1}{(2q - t)^2}\Psi_{qq} - \left(\frac{2\mathrm{i}}{2q - t} + \frac{3}{(2q - t)^4}\right)\Psi = 0. \tag{3.29}$$

Another linear parabolic partial differential equation can be derived from the Lagrangian $L_{8,9}$ and its admitted five Noether symmetries (3.25), i. e.

$$\begin{aligned}&\frac{x^4}{y^4}\phi_{xx} + 2\frac{x^2}{y^2}\phi_{xy} + \phi_{yy} + \frac{2}{y^5(x + y)}\left(\mathrm{i}x^2 + 2\mathrm{i}xy + \mathrm{i}y^2 + x^4y + 2x^3y^2\right)\phi_x \\ &\qquad + \frac{2}{xy^4(x + y)}\left(-\mathrm{i}x^2 - 2\mathrm{i}xy - \mathrm{i}y^2 + 2x^2y^3 + xy^4\right)\phi_y = 0\end{aligned} \tag{3.30}$$

that admits the following Lie symmetries

$$\begin{gathered}\Gamma_{10} + \frac{\mathrm{i} + xy(y - x)}{4x^2y^2}\phi\partial_\phi, \quad -3\Gamma_6 + \Gamma_2, \quad \Gamma_7, \quad \Gamma_8, \quad \Gamma_3 - \frac{\mathrm{i}}{xy}\phi\partial_\phi, \\ \phi\partial_\phi, \quad \beta(x, y)\partial_\phi,\end{gathered} \tag{3.31}$$

where $\beta(x, y)$ is any solution to (3.30). We can also transform the parabolic equation (3.30) into its canonical form with the help of MAPLE, i. e.

$$2\mathrm{i}\psi_t + \frac{q^6}{(2 + tq)^2}\psi_{qq} + \frac{2q^2}{(2 + tq)^3}\left(q^3(3 + tq) + \mathrm{i}(2 + tq)^2\right)\psi_q = 0 \tag{3.32}$$

with $q = x, t = \frac{x-y}{xy}$. A further trivial substitution, i. e.

$$\psi = \frac{\sqrt{2 + qt}}{q^{3/2}}\exp\left(\frac{\mathrm{i}}{6q^3}(3tq + 4)\right)\Psi,$$

eliminates the first derivative ψ_q in (3.32) and yields

$$2\mathrm{i}\Psi_t + \frac{q^6}{(2+qt)^2}\Psi_{qq} + q\left(\frac{2\mathrm{i}}{2+qt} - \frac{3q^3}{(2+qt)^4}\right)\Psi = 0. \tag{3.33}$$

The journey is over.

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