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FINITE GROUPS IN WHICH SOME PARTICULAR SUBGROUPS ARE TI-SUBGROUPS

JIANGTAO SHI AND CUI ZHANG

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Abstract. We prove that G is a group in which all noncyclic subgroups are TI-subgroups if and only if all noncyclic subgroups of G are normal in G. Moreover, we classify groups in which all subgroups of even order are TI-subgroups.

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1. INTRODUCTION

All groups in this paper are considered to be finite. Let G be a group and M a subgroup of G. It is known that M is called a TI-subgroup of G if $M \cap M^g = 1$ or M for any $g \in G$.

In [5], Walls studied groups in which all subgroups are TI-subgroups. In [1], Guo, Li and Flavell classified groups in which all abelian subgroups are TI-subgroups. As a generalization of [1], we showed in [3] that G is a group in which all nonabelian subgroups are TI-subgroups if and only if all nonabelian subgroups of G are normal in G, and we showed in [4] that G is a group in which all nonnilpotent subgroups are TI-subgroups if and only if all nonnilpotent subgroups of G are normal in G.

In this paper, as a further generalization of above references, we first classify groups in which all noncyclic subgroups are TI-subgroups. We call such groups NCTI-groups. For NCTI-groups, we have the following result, the proof of which is given in Section 2.

Theorem 1. *A group G is an NCTI-group if and only if all noncyclic subgroups of G are normal in G.*

Remark 1. It is easy to show that any NCTI-group is solvable but it might not be supersolvable. For example, the alternating group A_4 is an NCTI-group but A_4 is nonsupersolvable.

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We say that a group G has nontrivial even order if $|G|$ is even but $|G| \neq 2$. In the following, we will classify groups in which all subgroups of even order are TI-subgroups. We call such groups EOTI-groups. For EOTI-groups, we have the following result, the proof of which is given in Section 3.

Theorem 2. *A group G is a EOTI-group if and only if one of the following statements holds:*

(1) *all subgroups of G of nontrivial even order are normal in G.*

(2) *$G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel Z_p and complement $\langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$.*

Remark 2. We can easily get that any EOTI-group is solvable. But the alternating group A_4 shows that a EOTI-group might not be supersolvable.

Arguing as in proof of Theorem 2, we can obtain the following three results, which are generalizations of Theorem 1, [3] and [4] respectively. Here we omit their proofs.

Theorem 3. *A group G is a group in which all noncyclic subgroups of even order are TI-subgroups if and only if all noncyclic subgroups of G of even order are normal in G.*

Theorem 4. *A group G is a group in which all nonabelian subgroups of even order are TI-subgroups if and only if all nonabelian subgroups of G of even order are normal in G.*

Theorem 5. *A group G is a group in which all nonnilpotent subgroups of even order are TI-subgroups if and only if all nonnilpotent subgroups of G of even order are normal in G.*

2. Proof of Theorem 1

Proof. The sufficiency part is evident, we only need to prove the necessity part.

Let G be an NCTI-group. Assume that G has at least one nonnormal noncyclic subgroup. Choose M as a nonnormal noncyclic subgroup of G of largest order. Then, for any noncyclic subgroup K of G, we have that K is normal in G if $M < K$.

We claim that $M = N_G(M)$. Otherwise, assume that $M < N_G(M)$. Let L be a subgroup with $M < L \leq N_G(M)$ such that M has prime index in L. By the choice of M, we have that L is normal in G. Since M is not normal in G, $N_G(M) < G$. Take $h \in G \backslash N_G(M)$. We have $M^h < L^h = L$. Since M is a normal maximal subgroup of L and $M^h \neq M$, we have $L = MM^h$. Note that M is a TI-subgroup of G, which implies that $M \cap M^h = 1$. Then $|L| = |MM^h| = |M||M^h|$. It follows that $|M| = |M^h| = \frac{|L|}{|M|} = |L : M|$ is a prime, this contradicts that M is noncyclic. Thus $M = N_G(M)$.

Then, by the hypothesis, $M \cap M^a = 1$ for every $a \in G \backslash N_G(M) = G \backslash M$. We have that G is a Frobenius group with M being a Frobenius complement. Since G

is an NCTI-group, it follows that all nonabelian subgroups of G are TI-groups. By [3], we have that all nonabelian subgroups of G are normal in G. Then M is abelian. It follows that every Sylow subgroup of M must be cyclic by [2, Theorem 10.5.6]. This implies that M is cyclic, a contradiction. Thus G has no nonnormal noncyclic subgroup.

□

3. PROOF OF THEOREM 2

Proof. (1) We first prove the necessity part. Let G be a EOTI-group. Assume that G has at least one nonnormal subgroup of nontrivial even order. Let Q be any nonnormal subgroup of G of nontrivial even order. Choose L as a nonnormal subgroup of G of nontrivial even order of largest order such that $Q \leq L$.

We claim that $L = N_G(L)$. Assume that $L < N_G(L)$. Let K be a subgroup with $L < K \leq N_G(L)$ such that L has prime index in K. By the choice of L, we have that K is normal in G. Let y be an element of $G \backslash N_G(L)$. Then $L \cap L^y = 1$ by the hypothesis. It follows that $K = LL^y$ since L is a normal maximal subgroup of K. Then $|L| = |L^y| = |K : L|$ is a prime, a contradiction. Thus $L = N_G(L)$.

It follows that G is a Frobenius group with complement L. Let N be the Frobenius kernel of G. We have $G = N \rtimes L$.

We claim that L is a maximal subgroup of G. If L is not a maximal subgroup of G. Let M be a maximal subgroup of G such that $L < M$. It is obvious that M also has nontrivial even order. By the choice of L, we have that M is normal in G. Since G is a Frobenius group, it follows that either $M \leq N$ or $N < M$. If $M \leq N$, then $L < N$, a contradiction. If $N < M$, then $G = N \rtimes L \leq M$, again a contradiction. Thus L is a maximal subgroup of G.

It follows that N is a minimal normal subgroup of G. Since N is nilpotent by [2, Theorem 10.5.6], we can assume that $N = {Z_p}^m$, where $m \geq 1$ is a positive integer. Let d be an element of L of order 2. Since G is a Frobenius group and N is abelian, we have $e^d = e^{-1}$ for every $1 \neq e \in N$. Thus $\langle e \rangle \rtimes \langle d \rangle$ is a subgroup of G.

We claim that $\langle e \rangle \rtimes \langle d \rangle$ is normal in G. Note that $|\langle e \rangle \rtimes \langle d \rangle| = 2p$, where p is an odd prime. If $\langle e \rangle \rtimes \langle d \rangle$ is not normal in G. Arguing as the subgroup Q, there exists a nonnormal subgroup T of G of nontrivial even order of largest order such that $\langle e \rangle \rtimes \langle d \rangle \leq T$. And we have that T is also a Frobnius complement of G. But $N \cap T \geq \langle e \rangle \neq 1$, a contradiction. Thus $\langle e \rangle \rtimes \langle d \rangle$ is normal in G.

Since $\langle e \rangle \rtimes \langle d \rangle \nleq N = {Z_p}^m$, we have ${Z_p}^m < \langle e \rangle \rtimes \langle d \rangle$. It follows that $m = 1$. Thus $N = Z_p$ is cyclic. By the N/C-theorem, since $C_G(N) = N$ it follows that L is isomorphic to a subgroup of $\mathrm{Aut}(Z_p) \cong Z_{p-1}$. Then L is a cyclic group of order $2n$ for some $n > 1$. Assume that $L = \langle g \rangle$. We have $G = Z_p \rtimes \langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$.

(2) Now we prove the sufficiency part.

(i) If all subgroups of G of nontrivial even order are normal in G, then G is obviously a EOTI-group.

(ii) Assume that $G = Z_p \rtimes \langle g \rangle$ is a Frobenius group with kernel Z_p and complement $\langle g \rangle$, where p is an odd prime and $o(g) = 2n$ for some $n > 1$. Let S be any subgroup of G of even order. It is obvious that $S \cap Z_p = 1$ or Z_p.

If $S \cap Z_p = 1$, we have $S \leq \langle g \rangle^f$ for some $f \in G$. Obviously S is not normal in G. Then $N_G(S) = \langle g \rangle^f$. We have $S \cap S^x \leq \langle g \rangle^f \cap (\langle g \rangle^f)^x = 1$ for every $x \in G \backslash \langle g \rangle^f = G \backslash N_G(S)$.

If $S \cap Z_p = Z_p$, then $Z_p < S$. We have $S = S \cap G = S \cap (Z_p \rtimes \langle g \rangle) = Z_p \rtimes (S \cap \langle g \rangle)$. Then S is normal in G.

It follows that S is always a TI-subgroup of G whenever $S \cap Z_p = 1$ or Z_p. Then G is also a EOTI-group.

□

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Authors' addresses

Jiangtao Shi

Yantai University, School of Mathematics and Information Science, 264005 Yantai, China

E-mail address: shijt@math.pku.edu.cn

Cui Zhang

Technical University of Denmark, Department of Applied Mathematics and Computer Science, DK-2800 Lyngby, Denmark

E-mail address: cuizhang2008@gmail.com