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The Riesz-Kolmogorov theorem on metric spaces

Przemysław Górka and Anna Macios



THE RIESZ-KOLMOGOROV THEOREM ON METRIC SPACES

PRZEMYSŁAW GÓRKA AND ANNA MACIOS

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Abstract. We study precompact sets in $L^p(X, \mu)$, where (X, μ) is a metric measure space. Using Vitali and maximal function theorems we establish a full characterization of such sets.

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1. INTRODUCTION

Compact sets are important object of analysis' research because of their important properties and huge applications. Therefore, question about full characterization of compact or precompact sets (sets such that their closure is compact) in different spaces is very important. Riesz-Kolmogorov theorem gives necessary and sufficient conditions for precompactness of subset of $L^p(\mathbb{R}^n)$ (see [6, 10]). This theorem plays a fundamental role in analysis. Especially, it is frequently used in the theory of function spaces, e.g. Sobolev and Besov spaces.

The goal of the paper is to give a characterization of precompact sets in $L^p(X, \mu)$, where (X, μ) is a metric space with *doubling measure*, i.e positive Borel measure satisfying condition

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty$$

for all $x \in X$, $r > 0$ and some constant $C > 0$ called *doubling constant*. We shall denote the average of locally integrable function f over the measurable set A in the following manner

$$(f)_A = \frac{1}{\mu(A)} \int_A f d\mu.$$

In this note we would like to present the following observation.

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Theorem 1. *Let μ be a doubling measure such that*

$$h(r) := \inf\{\mu(B(x,r)) : x \in X\} > 0 \quad \text{for each } r > 0$$

and assume that $1 < p < \infty$. Let $x_0 \in X$, then the subset \mathcal{F} of $L^p(X, \mu)$ is relatively compact in $L^p(X, \mu)$ if and only if the following conditions are satisfied:

$$\mathcal{F} \quad \text{is bounded,} \quad (1.1)$$

$$\lim_{R \rightarrow \infty} \int_{X \setminus B(x_0, R)} |f(x)|^p d\mu(x) = 0, \quad \text{uniformly for } f \in \mathcal{F}, \quad (1.2)$$

$$\lim_{r \rightarrow 0} \int_X |f(x) - (f)_{B(x,r)}|^p d\mu(x) = 0 \quad \text{uniformly for } f \in \mathcal{F}. \quad (1.3)$$

We can find a bit similar theorem in [5], i.e. sufficient condition for precompactness of subset of $L^p(X, \mu)$ for $1 \leq p < \infty$, where (X, μ) is a metric space with finite measure. The proof of our theorem is obtained by appropriate modifications in the proof of theorem mentioned above [5]. We apply Vitali convergence theorem. Moreover, Lebesgue differentiation theorem as well as Hardy-Littlewood maximal function theorem will be used.

Finally, let us mention about some generalizations of the Riesz-Kolmogorov theorem. For instance, Weil [11] showed the compactness theorem in $L^p(G)$, where G is a locally compact group. Pego [8] (see also [2]) formulated Kolmogorov theorem for $p = 2$ in terms of the Fourier transform. There also exists a characterization of relatively compact subsets of general Banach spaces [9].

2. PROOF OF THEOREM

First of all we assume that conditions (1.1)-(1.3) hold. Let f_n be a sequence of elements from \mathcal{F} . We shall prove that f_n has a subsequence converging in $L^p(X, \mu)$. Since the set \mathcal{F} is bounded and the space $L^p(X, \mu)$ is reflexive, we may assume that f_n converges weakly to some $f \in L^p(X, \mu)$. We can represent the space X in the following manner

$$X = \bigcup_{n=1}^{\infty} B(x_0, n).$$

Now, we shall show that for each n

$$f_k \chi_{B(x_0, n)} \xrightarrow{k \rightarrow \infty} f \chi_{B(x_0, n)} \quad \text{in } L^p(X, \mu).$$

We apply Vitali convergence theorem (see e.g. [7]). We first prove that the sequence $f_k \chi_{B(x_0, n)}$ is p -equi-integrable. For this purpose we fix $\epsilon > 0$. By (1.3), there exists r such that for each k

$$\int_X |f_k(x) - (f_k)_{B(x,r)}|^p d\mu(x) \leq \frac{\epsilon}{2}.$$

Moreover, we fix a measurable set A such that

$$\mu(A) \leq \left(\frac{\varepsilon}{2 \sup_{k \in \mathbb{N}} \|f_k\|_{L^p(X)}} \right)^p h(r).$$

Then, using Minkowski and Hölder inequality we get

$$\begin{aligned} & \left(\int_A |f_k \chi_{B(x_0, n)}(x)|^p d\mu(x) \right)^{\frac{1}{p}} \leq \left(\int_A |f_k(x)|^p d\mu(x) \right)^{\frac{1}{p}} \\ & \leq \left(\int_A |f_k(x) - (f_k)_{B(x, r)}|^p d\mu(x) \right)^{\frac{1}{p}} + \left(\int_A |(f_k)_{B(x, r)}|^p d\mu(x) \right)^{\frac{1}{p}} \\ & \leq \frac{\varepsilon}{2} + \left(\int_A \frac{1}{(\mu(B(x, r)))^p} \left| \int_{B(x, r)} f_k(y) d\mu(y) \right|^p d\mu(x) \right)^{\frac{1}{p}} \\ & \leq \frac{\varepsilon}{2} + \left(\int_A \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |f_k(y)|^p d\mu(y) d\mu(x) \right)^{\frac{1}{p}} \\ & \leq \frac{\varepsilon}{2} + \left(\frac{\mu(A)}{h(r)} \right)^{\frac{1}{p}} \sup_{k \in \mathbb{N}} \|f_k\|_{L^p(X)} < \varepsilon. \end{aligned}$$

This ends the proof of p -equi-integrability.

Our next claim is that $f_k \chi_{B(x_0, n)} \xrightarrow[k \rightarrow \infty]{\mu} f \chi_{B(x_0, n)}$ in measure. Let us take $\varepsilon > 0$.

We note

$$\begin{aligned} & \{x \in X : |f_k \chi_{B(x_0, n)}(x) - f \chi_{B(x_0, n)}(x)| > \varepsilon\} = \{x \in B(x_0, n) : |f_k(x) - f(x)| > \varepsilon\} \\ & \subset \left\{ x \in B(x_0, n) : |f_k(x) - (f_k)_{B(x, r)}| > \frac{\varepsilon}{3} \right\} \\ & \quad \cup \left\{ x \in B(x_0, n) : |(f_k)_{B(x, r)} - (f)_{B(x, r)}| > \frac{\varepsilon}{3} \right\} \\ & \quad \cup \left\{ x \in B(x_0, n) : |f(x) - (f)_{B(x, r)}| > \frac{\varepsilon}{3} \right\} \\ & = A_1 \cup A_2 \cup A_3. \end{aligned}$$

It suffices to prove that for each $\delta > 0$, there exists K such that for each $k \geq K$, the inequality

$$\mu(A_1) + \mu(A_2) + \mu(A_3) < \delta$$

holds. Using Markov's inequality we get

$$\begin{aligned}\mu(A_1) &\leq \left(\frac{3}{\varepsilon}\right)^p \int_{B(x_0, n)} |f_k(x) - (f_k)_{B(x, r)}|^p d\mu(x) \\ &\leq \left(\frac{3}{\varepsilon}\right)^p \int_X |f_k(x) - (f_k)_{B(x, r)}|^p d\mu(x).\end{aligned}$$

Therefore, by assumption (1.3), there exists r_0 such that for each $r \leq r_0$ we get

$$\mu(A_1) < \frac{\delta}{3}.$$

Next, we consider the set A_2 . We conclude from $f_k \xrightarrow[k \rightarrow \infty]{w} f$ that for each $x \in B(x_0, n)$ and $r > 0$

$$(f_k)_{B(x, r)} \rightarrow (f)_{B(x, r)}.$$

This gives $(f_k)_{B(x, r)} \xrightarrow{\mu} (f)_{B(x, r)}$ in measure on $B(x_0, n)$. By the definition, there exists $K(r)$ such that for each $k \geq K(r)$ the inequality $\mu(A_2) < \frac{\delta}{3}$ holds.

It remains to consider the set A_3 . Using Markov's inequality we have

$$\mu(A_3) \leq D_r := \left(\frac{3}{\varepsilon}\right)^p \int_{B(x_0, n)} |f(x) - (f)_{B(x, r)}|^p d\mu(x).$$

Thanks to Lebesgue differentiation theorem (see [4]) we get that $\lim_{r \rightarrow 0} f_{B(x, r)} = f(x)$ a.e. on X . Direct calculations leads us to the following estimates

$$\begin{aligned}|f(x) - f_{B(x, r)}|^p &\leq (|f(x)| + |(f)_{B(x, r)}|)^p \leq 2^{p-1} (|f(x)|^p + |(f)_{B(x, r)}|^p) \\ &\leq 2^{p-1} (|f(x)|^p + |\mathcal{M}(f)(x)|^p),\end{aligned}$$

where $\mathcal{M}(f)$ is a maximal function of f . By virtue of the Hardy-Littlewood maximal function theorem (see [4]) we have $\|\mathcal{M}(f)\|_{L^p(X, \mu)} \leq C(p) \|f\|_{L^p(X, \mu)}$, where $p > 1$. Therefore, since $|f|^p$ is integrable, the Lebesgue dominated convergence theorem gives

$$\lim_{r \rightarrow 0} D_r = \left(\frac{3}{\varepsilon}\right)^p \int_{B(x_0, n)} \lim_{r \rightarrow 0} |f(x) - (f)_{B(x, r)}|^p d\mu(x) = 0$$

and hence there exists r_1 , such that for $r \leq r_1$ we have

$$\left(\frac{3}{\varepsilon}\right)^p \int_{B(x_0, n)} |f(x) - (f)_{B(x, r)}|^p d\mu(x) < \frac{\delta}{3}.$$

Finally, taking $\tilde{r} = \min\{r_0, r_1\}$, we obtain the inequality

$$\mu(\{x \in X : |f_k \chi_{B(x_0, n)}(x) - f \chi_{B(x_0, n)}(x)| > \varepsilon\}) < \delta$$

for each $k \geq K(\tilde{r})$. Now, we show that

$$f_k \xrightarrow[k \rightarrow \infty]{} f \quad \text{in } L^p(X, \mu).$$

Let us fix $\varepsilon > 0$. Since $f \in L^p(X, \mu)$ and by (1.2), there exists N such that

$$\int_{X \setminus B(x_0, N)} |f_k(x) - f(x)|^p d\mu(x) \leq \frac{\varepsilon}{2}.$$

Hence,

$$\begin{aligned} \|f_k - f\|_{L^p(X, \mu)}^p &= \int_X |f_k \chi_{B(x_0, N)}(x) - f \chi_{B(x_0, N)}(x)|^p d\mu(x) \\ &\quad + \int_{X \setminus B(x_0, N)} |f_k(x) - f(x)|^p d\mu(x) \leq \varepsilon. \end{aligned}$$

This ends the proof of relatively compactness of \mathcal{F} .

Now, we shall show the converse. We assume that the family \mathcal{F} is relatively compact. Hence, the boundedness is straightforward. To establish condition (1.2), let us fix $\varepsilon > 0$ and let $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ be an ε -cover of \mathcal{F} . For each $k = 1, \dots, n$ we can select $g_k \in U_k$ and $R_k > 0$ such that

$$\int_{X \setminus B(x_0, R_k)} |g_k(x)|^p d\mu(x) < \varepsilon.$$

Let us fix $f \in \mathcal{F}$. Since \mathcal{U} is an ε -cover of \mathcal{F} , there exists $1 \leq k \leq n$ such that $f \in U_k$. Thus,

$$\int_X |f(x) - g_k(x)|^p d\mu(x) < \varepsilon$$

and for $R = \max\{R_i : 1 \leq i \leq n\}$ we get

$$\begin{aligned} \int_{X \setminus B(x_0, R)} |f(x)|^p d\mu(x) &\leq 2^{p-1} \int_{X \setminus B(x_0, R)} |f(x) - g_k(x)|^p d\mu(x) \\ &\quad + 2^{p-1} \int_{X \setminus B(x_0, R)} |g_k(x)|^p d\mu(x) < 2^p \varepsilon. \end{aligned}$$

So condition (1.2) holds.

Using g_k choosen as above, we obtain:

$$\begin{aligned} \int_X |f(x) - (f)_{B(x, r)}|^p d\mu(x) &\leq 2^{p-1} \int_X |f(x) - g_k(x)|^p d\mu(x) \\ &\quad + 2^{2(p-1)} \int_X |g_k(x) - (g_k)_{B(x, r)}|^p d\mu(x) \\ &\quad + 2^{2(p-1)} \int_X |(g_k)_{B(x, r)} - (f)_{B(x, r)}|^p d\mu(x) \\ &= 2^{p-1} I_1 + 2^{2(p-1)} I_2 + 2^{2(p-1)} I_3. \end{aligned}$$

We easily get $I_1 < \varepsilon$. Now, using the Hardy-Littlewood maximal function theorem we obtain

$$\begin{aligned} I_3 &= \int_X \left| \frac{1}{\mu(B(x,r))} \int_{B(x,r)} (g_k(y) - f(y)) d\mu(y) \right|^p d\mu(x) \\ &\leq \int_X |\mathcal{M}(g_k - f)(x)|^p d\mu(x) \leq C(p)^p \int_X |g_k(x) - f(x)|^p d\mu(x) \leq C(p)^p \varepsilon. \end{aligned}$$

Now it remains to prove that I_2 is arbitrary small. By Lebesgue differentiation theorem we obtain that

$$\lim_{r \rightarrow 0} |g_k(x) - (g_k)_{B(x,r)}| = 0, \quad \text{a.e. on } X$$

Moreover,

$$|g_k(x) - (g_k)_{B(x,r)}|^p \leq 2^{p-1} (|g_k(x)|^p + |\mathcal{M}(g_k)(x)|^p).$$

Since $|g_k|^p$ and $|\mathcal{M}(g_k)|^p$ are integrable, we can apply Lebesgue dominated convergence theorem. Thus,

$$I_2 = \int_X |g_k(x) - (g_k)_{B(x,r)}|^p d\mu(x) < \varepsilon$$

and the proof is complete.

Let us remark that if μ is doubling and $\text{diam} X < \infty$ or μ is continuous with respect to the metric ρ (see [1, 3]) and X is compact, then for each $r > 0$, $h(r)$ is positive.

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Authors' addresses

Przemysław Górka

Warsaw University of Technology, Department of Mathematics and Information Sciences, Ul. Koszykowa 75, 00-662 Warsaw, Poland

E-mail address: P.Gorka@mini.pw.edu.pl

Anna Macios

Warsaw University of Technology, Department of Mathematics and Information Sciences, Ul. Koszykowa 75, 00-662 Warsaw, Poland