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M. Fakhar, Z. Soltani, and J. Zafarani

FIXED SETS AND ENDPOINTS OF SET-VALUED ASYMPTOTIC CONTRACTIONS IN METRIC SPACES

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Abstract. By introducing a new concept called "set-valued asymptotic contraction of final type" in metric spaces, the existence and uniqueness of compact fixed sets for such mappings have been obtained. Furthermore, by adding additional conditions, we prove the existence and uniqueness of endpoints for these maps.

1. INTRODUCTION AND PRELIMINARIES

In 2003, Kirk [8] introduced the notion of asymptotic contraction on metric spaces and proved a fixed point theorem for such contractions. Later on, Suzuki [11] introduced the notion of asymptotic contraction of Meir-Keeler and extended Kirk's fixed point theorem. Recently, Suzuki [12] introduced the concept of asymptotic contraction of final type for single valued mappings and proved the following result.

Theorem 1 (Suzuki [12]). *Let (X, d) be a complete metric space and let T be a continuous mapping on X. Assume that T is asymptotic contraction of final type (ACF, for short), i.e., the following contractions hold:*

(C1) $\lim_{\delta \to +0} \sup\{\limsup_{n \to \infty} d(T^n(x), T^n(y)) : d(x, y) < \delta\} = 0.$

(C2) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\varepsilon < d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $d(T^\nu(x), T^\nu(y)) \leq \varepsilon$.*

(C3) *For $x, y \in X$ with $x \neq y$, there exists $\nu \in \mathbb{N}$ such that*

$$d(T^\nu(x), T^\nu(y)) < d(x, y).$$

(C4) *For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that*

$$\epsilon < d(T^i(x), T^j(x)) < \epsilon + \delta \ \ implies \ \ d(T^\nu \circ T^i(x), T^\nu \circ T^j(x)) \leq \epsilon$$

for all $i, j \in \mathbb{N}$.

Then T has a unique fixed point $z \in X$. Moreover, $\lim_n T^n(x) = z$ holds for every $x \in X$.

Suzuki also mentioned that this result is the final generalization in some senses.

In recent years, many authors have studied the existence and uniqueness of fixed sets and endpoints for set-valued maps in metric spaces and topological spaces; see [1–7, 9, 10, 13–18] and references therein. Recently Fakhar [6] extended the Kirk's asymptotic contraction to set-valued maps and proved the existence and uniqueness of endpoints for those contractions which have the approximate endpoint property. Here, we introduce the notion of set-valued asymptotic contraction of final type in metric spaces and the existence and uniqueness of compact fixed set of these contractions are proved. Then, by using this result, we prove the existence and uniqueness of endpoint for such mappings which are topological contraction or they have the approximate endpoint property.

Let us introduce some definitions and facts which will be used in the sequel. Let X be a nonempty set and $T : X \to 2^X$ be a set-valued map with nonempty values. A subset A of X is said to be a fixed set of T, if $T(A) = A$. We denote the set of all fixed set of T with $Fix(T)$. An element $x \in X$ is said to be an endpoint (or stationary point or strict fixed point) of T, if $T(x) = \{x\}$. The set of all endpoints of T is denoted by $End(T)$. Suppose that X is a topological space, a set-valued map $T : X \to 2^X$ is said to be a topological contraction if for every nonempty compact subset A of X with $T(A) = A$, A is singleton, i.e., A is an endpoint of T. Let (X, d) be a metric space and $CB(X)$ and $K(X)$ denote the family of all nonempty closed and bounded subset of X and the family of all nonempty compact subset of X respectively. Then the Hausdorff metric on $CB(X)$ is given by

$$H(A, B) = \max\{e(A, B), e(B, A)\}$$

where

$$e(A, B) = \sup_{a \in A} d(a, B), e(B, A) = \sup_{b \in B} d(b, A) \quad \text{and} \quad d(a, B) = \inf_{b \in B} d(a, b).$$

It is well known that if (X, d) is a complete metric space, then the pair $(K(X), H)$ is a complete generalized metric space. We say that a set-valued map T on a metric space X has the approximate endpoint property if

$$\inf_{x \in X} \sup_{y \in Tx} d(x, y) = 0.$$

2. MAIN RESULTS

In this section we assume that (X, d) is a metric space. In the first step we introduce the set-valued asymptotic contraction of the final type in metric spaces as follows.

Definition 1. A set-valued map $T : X \to CB(X)$ is said to be set-valued asymptotic contraction of final type, if the following conditions hold:

(D1) $\lim_{\delta \to 0^+} \sup\{\limsup_{n \to \infty} H(T^n(x), T^n(y)) : d(x, y) < \delta\} = 0$.

(D2) For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $H(T^\nu(x), T^\nu(y)) < \varepsilon$.

(D3) For $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that
$$H(T^i(x), T^j(x)) < \varepsilon + \delta \quad \text{implies} \quad H(T^\nu \circ T^i(x), T^\nu \circ T^j(x)) < \varepsilon$$
for all $i, j \in \mathbb{N}$.

It's easy to see that, if $lim_{n \to \infty} H(T^n(x), T^n(y)) = 0$ for every $x, y \in X$ and $\{T^n(x)\}$ is a Cauchy sequence respect to Hausdorff metric, then T is set-valued asymptotic contraction of final type. According to this fact we give the following example of a set-valued asymptotic contraction of final type as follows.

Example 1. Let $X = \mathbb{R}$ and $T : X \to 2^X$ be defined as follows:
$$T(x) = \begin{cases} \{0\} & x < 0; \\ [1, 2 - \frac{x}{2}] & 0 \le x < 1; \\ \{3\} & 1 \le x \le 2; \\ [3, 4] & x > 2. \end{cases}$$

Since $T^n(x) = [3, 4]$ for $x \in X$ and $n \ge 4$. Therefore,
$$lim_{n \to \infty} H(T^n(x), T^n(y)) = 0$$
for every $x, y \in X$ and $\{T^n(x)\}$ is a Cauchy sequence respect to Hausdorff metric, then T is asymptotic contraction of final type.

Definition 2 ([6]). A set-valued map $T : X \to CB(X)$ is said to be a set-valued Kirk's asymptotic contraction, if there exists a continuous function φ from $[0, \infty)$ into itself and a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following:

(A1) $\varphi(0) = 0$.

(A2) $\varphi(r) < r$ for $r \in (0, \infty)$

(A3) $\{\varphi_n\}$ is uniformly convergent to φ

(A4) $H(T^n(x), T^n(y)) < \varphi_n(d(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$.

In the following result we show that any set-valued Kirk's asymptotic contraction is a set-valued asymptotic contraction of final type.

Proposition 1. *Suppose that T is a set-valued Kirk's asymptotic contraction on X. Then T is a set-valued asymptotic contraction of final type.*

Proof. Similar to Proposition 3 in [11], we put $E = \{d(x, y) : x, y \in X\}$ and define a sequence (ψ_n) of function from $[0, \infty)$ into itself by
$$\psi_n(t) = \begin{cases} \varphi_n(t) + \frac{t}{n} & \text{if } t \in E, \\ 0 & \text{if } t \notin E, \end{cases}$$

when the sequence (φ_n) is defined in Definition 2 corresponding to T.

Now, we prove the following conditions :

(B1) $\limsup_n \psi_n(\varepsilon) \le \varepsilon$ for all $\varepsilon \ge 0$,

(B2) For each $\varepsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \le \varepsilon$ for all $t \in [0, \varepsilon + \delta]$,

(B3) $H(T^n(x), T^n(y)) < \psi_n(d(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \ne y$.

It is obvious that (ψ_n) satisfies (B1) and (B3). Let us prove (B2). Fix $\varepsilon > 0$. Since $\varphi(\varepsilon) < \varepsilon$ and φ is continuous, we can choose $\delta > 0$ such that $\delta < \frac{(\varepsilon - \varphi(\varepsilon))}{2}$ and $\varphi(t) - \varphi(\varepsilon) < \frac{(\varepsilon - \varphi(\varepsilon))}{2}$ for $t \in [\varepsilon - \delta, \varepsilon + \delta]$. Also for $t \in (0, \varepsilon - \delta]$, we have $\varphi(t) < t < \varepsilon - \delta$. For such δ, we also choose $\nu \in \mathbb{N}$ such that

$$\frac{(\varepsilon + \delta)}{\nu} < \frac{\delta}{2} \quad \text{and} \quad \sup\{ |\text{'} \circ (t) - \text{'}(t)| : t \in E \} < \frac{\text{ffi}}{2}.$$

We fix $t \in (0, \varepsilon + \delta]$. In the case of $t \notin E$, we have $\psi_\nu(t) = 0 < \varepsilon$. For the case $t \in E$ and $t \in (0, \varepsilon - \delta]$, we have

$$\psi_\nu(t) = \varphi_\nu(t) + \frac{t}{\nu} \le \varphi(t) + \frac{\delta}{2} + \frac{\delta}{2} \le \varepsilon - \delta + \frac{2\delta}{2} = \varepsilon.$$

Also, in the case $t \in E$ and $t \in [\varepsilon - \delta, \varepsilon + \delta]$, we have

$$
\begin{aligned}
\psi_\nu(t) &= \varphi_\nu(t) + \frac{t}{\nu} \\
&\le \varphi(t) + \frac{\delta}{2} + \frac{\delta}{2} \\
&\le \varphi(\varepsilon) + \frac{(\varepsilon - \varphi(\varepsilon))}{2} + \frac{2\delta}{2} \\
&\le \varphi(\varepsilon) + \frac{2(\varepsilon - \varphi(\varepsilon))}{2} = \varepsilon,
\end{aligned}
$$

which completes our assertion. Now, we prove T fulfills the conditions (D1), (D2) and (D3). Since

$$
\begin{aligned}
\lim_{\delta \to 0^+} \sup\{ &\limsup_{n \to \infty} H(T^n(x), T^n(y)) : d(x, y) < \delta \} \\
&\le \lim_{\delta \to 0^+} \sup\{ \limsup_{n \to \infty} \psi_n(t) : t < \delta \} \\
&\le \lim_{\delta \to 0^+} = 0,
\end{aligned}
$$

hence we obtain (D1). Let $\varepsilon > 0$, from (B2), there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \le \varepsilon$ for all $t \in [0, \varepsilon + \delta]$. Then for each $x, y \in X$ with $d(x, y) < \varepsilon + \delta$, from (B3), we have

$$H(T^\nu(x), T^\nu(y)) < \psi_\nu(d(x, y)) \le \varepsilon,$$

which implies (D2).

For (D3), let $x \in X$ such that $H(T^i(x), T^j(x)) < \varepsilon + \delta$ for all $i, j \in \mathbb{N}$. So

$$e(T^i(x), T^j(x)) < \varepsilon + \delta$$

and $e(T^j(x), T^i(x)) < \varepsilon + \delta$. By the first inequality, we have

$$\forall z \in T^i(x), \ \exists w \in T^j(x) \quad \text{such that} \quad d(z, w) < \delta + \varepsilon.$$

Therefore, by (D2), for all $z \in T^i(x)$, there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z), T^\nu(w)) < \varepsilon,$$

that deduces $e(T^\nu(z), T^\nu(w)) < \varepsilon$, so for all $z \in T^i(x)$, $e(T^\nu(z), T^\nu(T^j(x))) < \varepsilon$. Then, $e(T^\nu(T^i(x)), T^\nu(T^j(x))) < \varepsilon$. By the same argument, we have

$$e(T^\nu(T^j(x)), T^\nu(T^i(x))) < \varepsilon.$$

Therefore,

$$H(T^\nu(T^i(x)), T^\nu(T^j(x))) < \varepsilon,$$

which implies (D3). □

Example 1 and Theorem 9 of [12] and Proposition 3 of [11], show that set-valued asymptotic contraction of the final type is strictly weaker than set-valued Kirk's asymptotic contraction.

Now, we give the main result as follows.

Theorem 2. *Let (X, d) be a complete metric space. Suppose that T is a set-valued asymptotic contraction of final type. Assume that T is continuous and compact valued. Then T has a unique fixed set $A \in K(X)$. Moreover, $\lim_n H(T^n(B), A) = 0$ holds for every $B \in K(X)$.*

Proof. We consider function $F : K(X) \to K(X)$ defined by $F(A) = T(A) = \cup_{x \in A} T(x)$ for all $A \in K(X)$. Since T is continuous and compact valued, then F is well defined. Also, the continuity of T implies the continuity of F. We will claim that F also fulfills condition (D1), (D2) and (D3). Fix $\varepsilon > 0$. From (D1) for T, there exists $\delta > 0$ such that

$$\sup\{\limsup_{n \to \infty} H(T^n(x), T^n(y)) : d(x, y) < \delta\} < \varepsilon.$$

So for each $x, y \in X$ such that $d(x, y) < \delta$, there exists $m_0 > 1$ such that for $n > m_0$, $H(T^n(x), T^n(y)) < \varepsilon$. Let $A, B \in K(X)$ such that $H(A, B) < \delta$, we show that $H(F^n(A), F^n(B)) < \varepsilon$. If $H(A, B) < \delta$, then $e(A, B) < \delta$ and $e(B, A) < \delta$. So by the first inequality,

$$\forall z \in A, \ \exists w \in B \quad \text{such that} \quad d(z, w) < \delta,$$

It follows that for $n > m_0$, $H(T^n(z), T^n(w)) < \varepsilon$. Thus, $e(T^n(z), T^n(w)) < \varepsilon$. Therefore, $e(T^n(z), T^n(B)) < \varepsilon$ for all $z \in A$. We deduce that

$$e(T^n(A), T^n(B)) < \varepsilon.$$

From $e(B, A) < \delta$, by the same argument, we obtain $e(T^n(B), T^n(A)) < \varepsilon$. Hence, $H(T^n(A), T^n(B)) < \varepsilon$ and so,

$$\sup\{\limsup_{n\to\infty} H(T^n(A), T^n(B)) : H(A, B) < \delta\} < \varepsilon.$$

Let $\varepsilon > 0$, from condition (D2), there exists $\delta > 0$ such that for $x, y \in X$ with $d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $H(T^\nu(x), T^\nu(y)) \leq \varepsilon$. Let $A, B \in K(X)$ such that $H(A, B) < \delta + \varepsilon$, we show that $H(F^\nu(A), F^\nu(B)) < \varepsilon$. If $H(A, B) < \delta + \varepsilon$, then $e(A, B) < \delta + \varepsilon$ and $e(B, A) < \delta + \varepsilon$. So by the first inequality,

$$\forall z \in A, \ \exists w \in B \quad \text{such that} \quad d(z, w) < \delta + \varepsilon.$$

Hence, there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z), T^\nu(w)) < \varepsilon.$$

Thus, $e(T^\nu(z), T^\nu(w)) < \varepsilon$, it follows that, for all $z \in A$, $e(T^\nu(z), T^\nu(B)) < \varepsilon$, we deduce that $e(T^\nu(A), T^\nu(B)) < \varepsilon$. From $e(B, A) < \delta + \varepsilon$, by the same argument, we obtain $e(T^\nu(B), T^\nu(A)) < \varepsilon$. Hence, $H(T^\nu(A), T^\nu(B)) < \varepsilon$ and so

$$H(F^\nu(A), F^\nu(B)) < \varepsilon,$$

which shows that F satisfies condition (D2).

Now, from condition (D3), there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that for $x \in X$ with

$$H(T^i(x), T^j(x)) < \varepsilon + \delta \ \text{ implies } \ H(T^\nu \circ T^i(x), T^\nu \circ T^j(x)) \leq \varepsilon \ \text{ for all } \ i, j \in \mathbb{N}.$$

Let $A \in K(X)$ such that $H(F^i(A), F^j(A)) < \varepsilon + \delta$, we will show that

$$H(F^\nu \circ (A), F^\nu \circ (A)) < \varepsilon.$$

Since $H(T^i(A), T^j(A)) < \varepsilon + \delta$, then

$$e(T^i(A), T^j(A)) < \varepsilon + \delta$$

and

$$e(T^i(A), T^j(A)) < \varepsilon + \delta.$$

So by the first inequality,

$$\forall z \in T^i(A), \ \exists w \in T^j(A) \quad \text{such that} \quad d(z, w) < \delta + \varepsilon.$$

Hence, from condition (D2), there exists $\nu \in \mathbb{N}$ such that

$$H(T^\nu(z), T^\nu(w)) < \varepsilon.$$

Thus, $e(T^\nu(z), T^\nu(w)) < \varepsilon$. It follows that $e(T^\nu(z), T^\nu(T^j(A))) < \varepsilon$ for all $z \in T^i(A)$, which deduces that $e(T^\nu(T^i(A)), T^\nu(T^j(A))) < \varepsilon$. From $e(T^i(A), T^j(A)) < \varepsilon + \delta$, by the same argument, we obtain

$$e(T^\nu(T^j(A)), T^\nu(T^i(A))) < \varepsilon.$$

Hence, $H(T^\nu(T^i(A)), T^\nu(T^j(A))) < \varepsilon$ and so

$$H(F^\nu \circ F^i(A)), F^\nu \circ F^j(A))) < \varepsilon,$$

which completes our claim. Therefore, F is a single-valued asymptotic contraction of final type. Hence, by Theorem 1, F has a unique fixed point $A \in K(X)$, i.e., $T(A) = F(A) = A$. Moreover, $\lim_n H(F^n(B), A) = 0$ holds for every $B \in K(X)$. $\square$

As a consequence of Proposition 1 and Theorem 2 we obtain the following result.

Corollary 1. *Let (X, d) be a complete metric space and T be a set-valued Kirk's asymptotic contraction on X. If T is continuous and compact valued, then T has a compact fixed set.*

Theorem 3. *Let (X, d) be a complete metric space and T satisfying in the conditions of the above theorem. Suppose that T satisfies one of the following conditions:*

   (i) *T has the approximate endpoint property.*
  (ii) *T is a topological contraction.*
Then T has the unique endpoint x_0.

Proof. By Theorem 2, there is a compact fixed set $A \in K(X)$ such that $T(A) = A$. Let T have the approximate endpoint property. Then

$$C_n = \{x \in A : H(x, T(x)) = \sup_{y \in Tx} d(x, y)\} \neq \varnothing,$$

for each $n \in \mathbb{N}$. It is clear that for each $n \in \mathbb{N}$, $C_n \supseteq C_{n+1}$. Since the map $x \to H(x, T(x))$ is continuous, we have that C_n is closed, so C_n is compact. Therefore, $\cap_{n \in \mathbb{N}} C_n = C \neq \varnothing$. Thus, for $x \in C$, $H(x, T(x)) = \sup_{y \in Tx} d(x, y) = 0$, and so $T(x) = \{x\}$. If T is a topological contraction then by definition, A is a singleton that is there exists $x_0 \in X$ such that $T(x_0) = \{x_0\} = A$. Since T has a unique fixed set A in $K(X)$, then T has a unique endpoint. $\square$

By the above theorem and Proposition 1, we have the following result.

Corollary 2. *Let (X, d) be a complete metric space and T be a set-valued Kirk's asymptotic contraction. Assume that T is continuous and compact valued. Suppose that T satisfies one of the following conditions:*

   (i) *T has the approximate endpoint property.*
  (ii) *T is a topological contraction.*
Then T has the unique endpoint x_0.

Remark 1. Since continuous set-valued map with compact valued and compact domain is uniformly continuous, the above corollary extends Theorem 2.3 in [6].

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Authors' addresses

M. Fakhar
Department of Mathematics, University of Isfahan, Isfahan, 81745-163, Iran

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P. O. Box : 19395-5746, Tehran, Iran

E-mail address: `fakhar@sci.ui.ac.ir`

Z. Soltani

Department of Mathematics, Faculty of Science, University of Kashan, Kashan, 87317-51167, Iran
E-mail address: `z.soltani@kashanu.ac.ir`

J. Zafarani

Department of Mathematics, Sheikhbahaee University and University of Isfahan, Isfahan, Iran
E-mail address: `jzaf@zafarani.ir`